

Shadowing, internal chain transitivity and α -limit sets

Good, Chris; Mitchell, Joel; Meddaugh, Jonathan

DOI:

[10.1016/j.jmaa.2020.124291](https://doi.org/10.1016/j.jmaa.2020.124291)

License:

Creative Commons: Attribution-NonCommercial-NoDerivs (CC BY-NC-ND)

Document Version

Peer reviewed version

Citation for published version (Harvard):

Good, C, Mitchell, J & Meddaugh, J 2020, 'Shadowing, internal chain transitivity and α -limit sets', *Journal of Mathematical Analysis and Applications*, vol. 491, no. 1, 124291. <https://doi.org/10.1016/j.jmaa.2020.124291>

[Link to publication on Research at Birmingham portal](#)

General rights

Unless a licence is specified above, all rights (including copyright and moral rights) in this document are retained by the authors and/or the copyright holders. The express permission of the copyright holder must be obtained for any use of this material other than for purposes permitted by law.

- Users may freely distribute the URL that is used to identify this publication.
- Users may download and/or print one copy of the publication from the University of Birmingham research portal for the purpose of private study or non-commercial research.
- User may use extracts from the document in line with the concept of 'fair dealing' under the Copyright, Designs and Patents Act 1988 (?)
- Users may not further distribute the material nor use it for the purposes of commercial gain.

Where a licence is displayed above, please note the terms and conditions of the licence govern your use of this document.

When citing, please reference the published version.

Take down policy

While the University of Birmingham exercises care and attention in making items available there are rare occasions when an item has been uploaded in error or has been deemed to be commercially or otherwise sensitive.

If you believe that this is the case for this document, please contact UBIRA@lists.bham.ac.uk providing details and we will remove access to the work immediately and investigate.

Shadowing, internal chain transitivity and α -limit sets

Chris Good¹

School of Mathematics, University of Birmingham, Birmingham, B15 2TT, UK

Jonathan Meddaugh¹

Department of Mathematics, Baylor University, Waco, TX 76798-7328, USA

Joel Mitchell*

School of Mathematics, University of Birmingham, Birmingham, B15 2TT, UK

Abstract

Let $f: X \rightarrow X$ be a continuous map on a compact metric space X and let α_f , ω_f and ICT_f denote the set of α -limit sets, ω -limit sets and nonempty closed internally chain transitive sets respectively. We show that if the map f has shadowing then every element of ICT_f can be approximated (to any prescribed accuracy) by both the α -limit set and the ω -limit set of a full-trajectory. Furthermore, if f is additionally expansive then every element of ICT_f is equal to both the α -limit set and the ω -limit set of a full-trajectory. In particular this means that shadowing guarantees that $\overline{\alpha_f} = \overline{\omega_f} = ICT_f$ (where the closures are taken with respect to the Hausdorff topology on the space of compact sets), whilst the addition of expansivity entails $\alpha_f = \omega_f = ICT_f$. We progress by introducing novel variants of shadowing which we use to characterise both maps for which $\overline{\alpha_f} = ICT_f$ and maps for which $\alpha_f = ICT_f$.

Keywords: Shadowing, α -limit set, ω -limit set, Internally chain transitive, Expansive, Pseudo-orbit

2020 MSC: 37B99, 37C50

1. Introduction

Let $f: X \rightarrow X$ be a dynamical system, so that f is a continuous map on the compact metric space X . Given a point $x \in X$, its ω -limit set is the set of accumulation points of the sequence $x, f(x), f^2(x), \dots$. Calculating the ω -limit set of a given point is often relatively easy. Conversely one may ask if a given set is an ω -limit set: this can be

*Corresponding author

Email addresses: `c.good@bham.ac.uk` (Chris Good), `jonathan_meddaugh@baylor.edu` (Jonathan Meddaugh), `jsm140@bham.ac.uk` (Joel Mitchell)

¹The first and second author gratefully acknowledge support from the European Union through funding the H2020-MSCA-IF-2014 project ShadOmIC (SEP-210195797).

1 quite difficult to answer. As such, various authors have either studied, or attempted
 2 to characterise, the set of all ω -limit sets, denoted here by ω_f , in a variety of settings.
 3 For example, ω -limit sets of continuous maps of the closed unit interval I have been
 4 completely characterised in [1, 13]: the authors show that a nonempty subset E of I
 5 is an ω -limit set of some continuous map f if and only if E is either a closed, nowhere
 6 dense set, or a union of finitely many non-degenerate closed intervals. Furthermore, it
 7 has been shown that ω_f is closed (with respect to the Hausdorff topology) for maps
 8 of the circle [42], the interval [8] and other finite graphs [32]. It is known [29] that
 9 every ω -limit set is *internally chain transitive*: briefly a set $A \subseteq X$ is internally chain
 10 transitive if for any $a, b \in A$ and any $\varepsilon > 0$ there exists a finite sequence $\langle x_0, x_1, \dots, x_n \rangle$
 11 in A such that $x_0 = a$, $x_n = b$ and $d(f(x_i), x_{i+1}) < \varepsilon$ for each i . We denote the set of
 12 nonempty closed internally chain transitive sets by ICT_f . The map f is said to have
 13 *shadowing* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for any sequence $\langle x_i \rangle_{i=0}^\infty$ with
 14 $d(f(x_i), x_{i+1}) < \delta$ for each i , there is a point $z \in X$ such that $d(f^i(z), x_i) < \varepsilon$ for
 15 each i . In this case we say z *shadows* or ε -*shadows* the sequence $\langle x_i \rangle_{i=0}^\infty$. Shadowing
 16 has both numerical and theoretical importance and has been studied extensively in a
 17 variety of settings; in the context of Axiom A diffeomorphisms [9], in numerical analysis
 18 [14, 15, 37], as an important factor in stability theory [40, 43, 47], in understanding the
 19 structure of ω -limit sets and Julia sets [5, 6, 7, 10, 33], and as a property in and of itself
 20 [16, 24, 26, 31, 35, 38, 40, 44]. A variety of variants of shadowing have also been studied
 21 including, for example, ergodic, thick and Ramsey shadowing [11, 12, 19, 21, 36], limit,
 22 or asymptotic, shadowing [4, 27, 41], s -limit shadowing [4, 27, 31], orbital shadowing
 23 [23, 34, 39, 41], and inverse shadowing [15, 25, 30].

24 Of particular importance to us is a result of Meddaugh and Raines [33] who establish
 25 that, for maps with shadowing, $\overline{\omega_f} = ICT_f$. More recently, using novel variants of
 26 shadowing, Good and Meddaugh [23] precisely characterised maps for which $\overline{\omega_f} = ICT_f$
 27 and $\omega_f = ICT_f$.

28 Whilst the ω -limit set of a point can be thought of as its *target* - it is where the point
 29 *ends up* - an α -limit set concerns where a point came from - its source, so to speak.
 30 However, whilst the definition of an ω -limit set is fairly natural, giving an appropriate
 31 definition of an α -limit set is less straightforward. This is because a point may have
 32 multiple points in its preimage (or indeed, if the map is not surjective, it may have
 33 empty preimage). Various approaches to this difficulty have been taken; these will be
 34 discussed in more detail in Section 3. We follow the approach taken in [2] and [29],
 35 by refraining from defining such sets for individual points, but rather defining them
 36 for *backward trajectories*. Given a point $x \in X$ an infinite sequence $\langle x_i \rangle_{i \leq 0}$ is called a
 37 *backward trajectory* of x if $f(x_i) = x_{i+1}$ for all $i \leq -1$ and $x_0 = x$. The α -limit set of
 38 $\langle x_i \rangle_{i \leq 0}$ is the set of accumulation points of this sequence. We denote the set of all α -limit
 39 sets by α_f . Although α -limit sets have not been studied quite as extensively as their ω
 40 counterparts, interest in them has been growing (see, for example, [2, 17, 18, 28, 29]).

41 As with ω -limit sets, it is known that α -limit sets are internally chain transitive [29].
 42 In this paper we seek to provide a characterisation of maps for which α_f and ICT_f
 43 coincide. We start with the preliminaries in Section 2. Section 3 is a standalone section
 44 in which we briefly explain the various types of α -limit sets that have been studied in the
 45 literature. In Section 4 we show that, for maps with shadowing, for any $\varepsilon > 0$ and any
 46 $A \in ICT_f$ there is a full trajectory whose α -limit set and ω -limit set both lie within ε of
 47 A (with respect to the Hausdorff distance). Furthermore, we show that the addition of

1 expansivity entails that there is a full trajectory whose limit sets equal A . In particular
2 this means that for maps with shadowing $\overline{\alpha_f} = \overline{\omega_f} = ICT_f$, whilst the addition of
3 expansivity means that $\alpha_f = \omega_f = ICT_f$. We progress in Section 5 by introducing novel
4 types of shadowing which we use to characterise both maps for which $\overline{\alpha_f} = ICT_f$ and
5 maps for which $\alpha_f = ICT_f$, complementing the work of the first and second author in
6 [23].

7 2. Preliminaries

8 A *dynamical system* is a pair (X, f) consisting of a compact metric space X and a
9 continuous function $f: X \rightarrow X$. We say the *positive orbit of x under f* is the set of
10 points $\{x, f(x), f^2(x), \dots\}$; we denote this set by $\text{Orb}_f^+(x)$. A *backward trajectory* of the
11 point x is a sequence $\langle x_i \rangle_{i \leq 0}$ for which $f(x_i) = x_{i+1}$ for all $i \leq -1$ and $x_0 = x$. We say a
12 bi-infinite sequence $\langle x_i \rangle_{i \in \mathbb{Z}}$ is a *full orbit* (of each x_i) if $f(x_i) = x_{i+1}$ for each $i \in \mathbb{Z}$. We
13 emphasise that a full orbit of a point need not be unique. Note further that we do not
14 assume that the map f is a surjection. (NB. Because we will be particularly concerned
15 with backward accumulation points of individual trajectories, for clarity we will say that
16 a point which does not have an infinite backward trajectory does not have a full orbit.
17 Whenever we say *full orbit*, we mean a bi-infinite trajectory.)

18 For a sequence $\langle x_i \rangle_{i > N}$ in X , where $N \geq -\infty$, we define its ω -limit set, denoted
19 $\omega(\langle x_i \rangle_{i > N})$, or simply $\omega(\langle x_i \rangle)$, to be the set of accumulation points of the positive tail of the
20 sequence. Formally:

$$\omega(\langle x_i \rangle) = \bigcap_{M \in \mathbb{N}} \overline{\{x_n \mid n > M\}}.$$

21 For $x \in X$, we define the ω -limit set of x : $\omega(x) := \omega(\langle f^n(x) \rangle_{n=0}^\infty)$. In similar fashion, for
22 a sequence $\langle x_i \rangle_{i < N}$ in X , where $N \leq \infty$, we define its α -limit set, denoted $\alpha(\langle x_i \rangle_{i < N})$, or
23 simply $\alpha(\langle x_i \rangle)$, to be the set of accumulation points of the negative tail of the sequence.
24 Formally:

$$\alpha(\langle x_i \rangle) = \bigcap_{M \in \mathbb{N}} \overline{\{x_n \mid n < -M\}}.$$

25 We denote by ω_f the set of all ω -limit sets of points in X . We denote by α_f the set of
26 all α -limit sets of full trajectories in (X, f) . Note that since X is compact it follows that
27 elements of α_f and ω_f are closed, compact and nonempty.

28 We denote by 2^X the hyperspace of nonempty compact subsets of X . This is a
29 (compact) metric space in its own right with the *Hausdorff metric* induced by the metric
30 d . For $A, B \in 2^X$ the *Hausdorff distance* between A and B is given by

$$d_H(A, A') = \inf\{\varepsilon > 0 \mid A \subseteq B_\varepsilon(A') \text{ and } A' \subseteq B_\varepsilon(A)\}.$$

31 Note that, as collections of nonempty compact sets, α_f and ω_f are subsets of 2^X .

32 A set $A \subseteq X$ is said to be *invariant* if $f(A) \subseteq A$. It is *strongly invariant* if $f(A) = A$.
33 A nonempty closed set A is *minimal* if $\omega(x) = A$ for all $x \in A$.

34 A finite or infinite sequence $\langle x_i \rangle_{i=0}^N$ is said to be an ε -chain if $d(f(x_i), x_{i+1}) < \varepsilon$ for
35 all indices $i < N$. If $N = \infty$ then we say the sequence is an ε -pseudo-orbit. A set A is
36 *internally chain transitive* if for any pair of points $a, b \in A$ and any $\varepsilon > 0$ there exists a
37 finite ε -chain $\langle x_i \rangle_{i=0}^N$ in A with $x_0 = a$, $x_N = b$ and $N \geq 1$. We denote by ICT_f the set of

1 all nonempty closed internally chain transitive sets. Notice that $ICT_f \subseteq 2^X$. Meddaugh
 2 and Raines [33] establish the following result.

3 **Lemma 2.1.** [33] *Let (X, f) be a dynamical system. Then ICT_f is closed in 2^X .*

4 Hirsch *et al.* [29] show that the α -limit set (resp. ω -limit set) of any pre-compact
 5 backward (resp. forward) trajectory is internally chain transitive. Since our setting is a
 6 compact metric space all α - and ω - limit sets are internally chain transitive. We formulate
 7 this as Lemma 2.2 below.

8 **Lemma 2.2.** [29] *Let (X, f) be a dynamical system. Then $\alpha_f \subseteq ICT_f$ and $\omega_f \subseteq ICT_f$.*

9 *Remark 2.3.* When one first encounters positive and negative limit sets of trajectories,
 10 it is natural to ask (for a surjective map) if every ω -limit set is also an α -limit set, along
 11 with the converse. The following is an example of a homeomorphism for which neither is
 12 true. Take two copies of the interval and embed them side by side in the plane (i.e. one
 13 on the left and one on the right). Snake one infinite line between them which has each
 14 interval as an accumulation set - akin to how the topologist's sine curve approaches the
 15 y -axis. Define a continuous map as follows: Let every point on each of the two intervals
 16 be fixed whilst points on the line move continuously along it, away from the left interval
 17 and towards the right. It follows that the left interval is the α -limit set of the unique
 18 backward trajectory of any point on the line, whilst the right left interval is the ω -limit
 19 set of any point on the line. However it is clear that the left interval is not an ω -limit
 20 set, whilst the right interval is not an α -limit set.

21 *Remark 2.4.* As stated in [2], a minimal set is both an ω -limit set and an α -limit set.

22 Whilst it may be the case that $\alpha_f \neq \omega_f$, it is true that every α -limit set contains the
 23 ω -limit set of every one of its points and, similarly, every ω -limit set contains an α -limit
 24 set of a backward trajectory of each of its points. To show this we recall the well-known
 25 fact that the ω -limit sets in compact systems are strongly invariant (e.g. [20, Theorem
 26 3.1.9]). The same is true of the α -limit sets of backward trajectories (e.g. [2, Lemma 1]).

27 **Proposition 2.5.** *Let $x, y \in X$ and suppose that $\langle z_i \rangle_{i \leq 0}$ is a backward trajectory of a
 28 point $z = z_0 \in X$. Then:*

- 29 1. *If $x \in \alpha(\langle z_i \rangle)$ then $\overline{\text{Orb}_f^+(x)} \subseteq \alpha(\langle z_i \rangle)$.*
 30 2. *If $y \in \omega(x)$ then there is a backward trajectory $\langle y_i \rangle_{i \leq 0}$, with $y_0 = y$, which lies in
 31 $\omega(x)$ and such that $\alpha(\langle y_i \rangle) \subseteq \omega(x)$.*

32 *Proof.* Condition (1) is immediate from the fact that α -limit sets are closed and invariant
 33 under f .

34 Now suppose $y \in \omega(x)$ and let $y_0 = y$. Since ω -limit sets are strongly invariant y has
 35 a preimage in $\omega(x)$, call it y_{-1} . This itself has a preimage in $\omega(x)$; call it y_{-2} . Continuing
 36 in this manner gives a backward trajectory $\langle y_i \rangle_{i \leq 0}$ of y which lies in $\omega(x)$. The result
 37 now follows by observing that $\omega(x)$ is closed. \square

38 *Remark 2.6.* In [28] the author proves condition (1) in Proposition 2.5 holds for interval
 39 maps.

1 A point x is said to ε -shadow a sequence $\langle x_i \rangle_{i=0}^{\infty}$ if $d(f^i(x), x_i) < \varepsilon$ for all $i \in \mathbb{N}_0$.
 2 We say the system (X, f) has the *shadowing property*, or simply shadowing, if for every
 3 $\varepsilon > 0$ there exists $\delta > 0$ such that every δ -pseudo-orbit is ε -shadowed.

4 **Definition 2.7.** Suppose that (X, f) is a dynamical system.

- 5 1. The sequence $\langle x_i \rangle_{i \leq 0}$ is a *backward δ -pseudo-orbit* if $d(f(x_i), x_{i+1}) < \delta$ for each
 6 $i \leq -1$.
- 7 2. The sequence $\langle x_i \rangle_{i \in \mathbb{Z}}$ is a *two-sided δ -pseudo-orbit* if $d(f(x_i), x_{i+1}) < \delta$ for each
 8 $i \in \mathbb{Z}$.
- 9 3. The system (X, f) has *backward shadowing* if for any $\varepsilon > 0$ there exists $\delta > 0$ such
 10 that for any backward δ -pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there exists a backward trajectory
 11 $\langle z_i \rangle_{i \leq 0}$ such that $d(x_i, z_i) < \varepsilon$ for all $i \leq 0$.
- 12 4. The system (X, f) has *two-sided shadowing* if for any $\varepsilon > 0$ there exists $\delta > 0$ such
 13 that for any two-sided δ -pseudo-orbit $\langle x_i \rangle_{i \in \mathbb{Z}}$ there exists a full trajectory $\langle z_i \rangle_{i \in \mathbb{Z}}$
 14 such that $d(x_i, z_i) < \varepsilon$ for all $i \in \mathbb{Z}$.

15 A sequence $\langle x_i \rangle_{i=0}^{\infty}$ is called an *asymptotic pseudo-orbit* if $d(f(x_i), x_{i+1}) \rightarrow 0$ as $i \rightarrow$
 16 ∞ . Similarly a sequence $\langle x_i \rangle_{i \leq 0}$ is a *backward asymptotic pseudo-orbit* if $d(f(x_i), x_{i+1}) \rightarrow$
 17 0 as $i \rightarrow -\infty$. Finally a sequence $\langle x_i \rangle_{i \in \mathbb{Z}}$ is called a *two-sided asymptotic pseudo-orbit* if
 18 $d(f(x_i), x_{i+1}) \rightarrow 0$ as $i \rightarrow \pm\infty$.

19 The system (X, f) has *s-limit shadowing* if, in addition to having shadowing, for
 20 any $\varepsilon > 0$ there exists $\delta > 0$ such that for any asymptotic δ -pseudo orbit $\langle x_i \rangle_{i=0}^{\infty}$ there
 21 exists $z \in X$ which *asymptotically ε -shadows* $\langle x_i \rangle_{i=0}^{\infty}$ (i.e. $d(f^i(z), x_i) \rightarrow 0$ as $i \rightarrow \infty$
 22 and $d(f^i(z), x_i) < \varepsilon$ for all $i \in \mathbb{N}_0$). The system has *two-sided s-limit shadowing* if, in
 23 addition to two-sided shadowing, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any
 24 two-sided asymptotic δ -pseudo orbit $\langle x_i \rangle_{i \in \mathbb{Z}}$ there exists a full trajectory $\langle z_i \rangle_{i \in \mathbb{Z}}$ which
 25 asymptotically ε -shadows $\langle x_i \rangle_{i \in \mathbb{Z}}$ (i.e. $d(f^i(z), x_i) \rightarrow 0$ as $i \rightarrow \pm\infty$ and $d(f^i(z), x_i) < \varepsilon$
 26 for all $i \in \mathbb{Z}$).

27 2.1. Shift spaces

28 Given a finite set Σ considered with the discrete topology, the *one-sided full shift with*
 29 *alphabet* Σ consists of the set of infinite sequences in Σ , that is $\Sigma^{\mathbb{N}_0}$, which we consider
 30 with the product topology. This forms a dynamical system with the *shift map* σ , given
 31 by

$$\sigma(\langle a_i \rangle_{i \geq 0}) = \langle a_{i+1} \rangle_{i \geq 0}.$$

32 A *one-sided shift space* is some compact strongly invariant (under σ) subset of some
 33 one-sided full shift.

34 In similar fashion, the *two-sided full shift with alphabet* Σ consists of the set of bi-
 35 infinite sequences in Σ , that is $\Sigma^{\mathbb{Z}}$, which we consider with the product topology. As
 36 before, this forms a dynamical system with the *shift map* σ , which we define by saying
 37 that, for each $i \in \mathbb{Z}$,

$$\pi_i(\sigma(\langle a_i \rangle_{i \in \mathbb{Z}})) = a_{i+1},$$

38 where π_i is the projection map for each i . A *two-sided shift space* is some compact
 39 strongly invariant (under σ) subset of some two-sided full shift. If (X, σ) is a two-sided

1 shift space and $x = \langle a_i \rangle_{i \in \mathbb{Z}} \in X$ then we refer to the sequences $\langle a_i \rangle_{i \geq 0}$ and $\langle a_i \rangle_{i \leq 0}$ as
 2 the *right-tail* and *left-tail* of x respectively.

3 Given an alphabet Σ , a word in Σ is a finite sequence $a_0 a_1 \dots a_m$, made up of elements
 4 of Σ . Let \mathcal{F} be a finite set of words in Σ . The *one-sided shift of finite type associated with*
 5 \mathcal{F} is the dynamical system $(X_{\mathcal{F}}, \sigma)$ where $X_{\mathcal{F}}$ is the set of all infinite sequences which
 6 do not contain any occurrence of any word from \mathcal{F} . The *two-sided shift of finite type*
 7 *associated with \mathcal{F}* is the dynamical system $(Z_{\mathcal{F}}, \sigma)$ where $Z_{\mathcal{F}}$ is the set of all bi-infinite
 8 sequences which do not contain any occurrence of any word from \mathcal{F} . A shift space (X, σ)
 9 is said to be a *one-sided (resp. two-sided) shift of finite type* if there exists a finite set of
 10 words \mathcal{F} such that $X = X_{\mathcal{F}}$ (resp. $X = Z_{\mathcal{F}}$).

11 If (X, σ) is a one-sided shift space, $x = \langle a_i \rangle_{i \geq 0} \in X$ and $n \in \mathbb{N}_0$, we refer to the word
 12 $a_0 a_1 \dots a_n$ as an *initial segment* of x . In similar fashion, if (X, σ) is a two-sided shift
 13 space and $x = \langle a_i \rangle_{i \in \mathbb{Z}} \in X$ and $n \in \mathbb{N}_0$, we refer to the word $a_{-n} \dots a_{-1} a_0 a_1 \dots a_n$ as a
 14 *central segment* of x . In the two-sided case, when writing out an element of X in full we
 15 use a “.” to indicate the position of the middle of the central segment:

$$x = \dots a_{-3} a_{-2} a_{-1} \cdot a_0 a_1 a_2 a_3 \dots$$

16 The following two theorems concerning limit sets in shift spaces are folklore.

17 **Theorem 2.8.** *Let (X, σ) be a one-sided shift space. Let $x, y \in X$. Then $y \in \omega(x)$
 18 if and only if every initial segment of y occurs infinitely often in x . Given a backward
 19 trajectory $\langle x_i \rangle_{i \leq 0}$ consider the backward infinite sequence $\langle a_i \rangle_{i \leq 0}$ where $a_i = \pi_0(x_i)$.
 20 Then $y \in \alpha(\langle x_i \rangle)$ if and only if every initial segment of y occurs infinitely often in
 21 $\langle a_i \rangle_{i \leq 0}$.*

22 **Theorem 2.9.** *Let (X, σ) be a two-sided shift space. Let $x, y \in X$. Then $y \in \omega(x)$ if
 23 and only if every central segment of y occurs infinitely often in the right-tail of x . Given
 24 a backward trajectory $\langle x_i \rangle_{i \leq 0}$ then $y \in \alpha(\langle x_i \rangle)$ if and only if every central segment of y
 25 occurs infinitely often in the left-tail of x_0 .*

26 For those wanting more information about shift systems, [20, Chapter 5] provides a
 27 thorough introduction to the topic.

28 As stated in Lemma 2.2, α_f and ω_f are both subsets of ICT_f . Example 2.10 gives a
 29 surjective shift space (X, σ) where $\alpha_\sigma, \omega_\sigma$ and ICT_σ are all distinct, complementing the
 30 discussion in Remark 2.3.

31 **Example 2.10.** *Let $x = 1010^2 10^3 \dots$, and $y = 2020^2 20^3 \dots$. Let*

$$P(x) = \{30^n 30^{n-1} \dots 30x \mid n \in \mathbb{N}\}.$$

32 *Take*

$$X = \overline{\bigcup_{z \in P(x)} \text{Orb}_\sigma^+(z) \cup \text{Orb}_\sigma^+(y) \cup \{0^n y \mid n \in \mathbb{N}\}},$$

33 *where the closure is taken with regard to the one-sided full shift on the alphabet $\{0, 1, 2, 3\}$.
 34 Considering the system (X, σ) , $\alpha_\sigma \neq \omega_\sigma \neq ICT_\sigma$. Furthermore $\alpha_\sigma \not\subseteq \omega_\sigma$ and $\omega_\sigma \not\subseteq \alpha_\sigma$.*

35 In Example 2.10, $\omega(x) = \{0^\infty, 0^n 10^\infty \mid n \geq 0\}$ and $\omega(y) = \{0^\infty, 0^n 20^\infty \mid n \geq 0\}$. It
 36 is easy to see that the only other ω -limit set is $\{0^\infty\}$. Thus

$$\omega_\sigma = \{\{0^\infty\}, \{0^\infty, 0^n 10^\infty \mid n \geq 0\}, \{0^\infty, 0^n 20^\infty \mid n \geq 0\}\}.$$

1 Meanwhile

$$\alpha_\sigma = \{\{0^\infty\}, \{0^\infty, 0^n 30^\infty \mid n \geq 0\}\}.$$

2 Finally whilst $ICT_\sigma \supseteq \alpha_\sigma \cup \omega_\sigma$ it additionally contains $\{0^\infty, 0^n 10^\infty, 0^n 20^\infty \mid n \geq 0\}$,
 3 $\{0^\infty, 0^n 10^\infty, 0^n 30^\infty \mid n \geq 0\}$, $\{0^\infty, 0^n 20^\infty, 0^n 30^\infty \mid n \geq 0\}$ and
 4 $\{0^\infty, 0^n 10^\infty, 0^n 20^\infty, 0^n 30^\infty \mid n \geq 0\}$. Hence $\alpha_\sigma \neq \omega_\sigma \neq ICT_\sigma$, $\alpha_\sigma \not\subseteq \omega_\sigma$ and $\omega_\sigma \not\subseteq \alpha_\sigma$.

5 3. Various notions of negative limit sets

6 In the previous section we defined what we mean by the term α -limit set: it was
 7 defined for backward sequences. Meanwhile the definition of an ω -limit set was extended
 8 to individual points. This was done in the only natural way: any given point only has
 9 one forward orbit. If one wishes to define the α -limit set of a point, say x , the best way
 10 forward is less obvious; there are multiple approaches one might reasonably take when
 11 defining negative limit sets of points. In this standalone section we give a brief outline
 12 of several different approaches taken in the literature and give two examples which serve
 13 to illustrate their differences.

14 For homeomorphisms one can define α -limit sets (or negative limit sets) in precisely
 15 the same way as ω -limit sets. With non-invertible maps, however, a seemingly natural
 16 definition is less obvious. One approach is to take the set of accumulation points of the
 17 sequence of sets $f^{-k}(\{x\})$: this is done in [17] and [18]. Call this Approach 1 (A1). Two
 18 further approaches are motivated by considering the accumulation points of backward
 19 trajectories of the point in question. One might say that y is in the negative limit set of
 20 a point x if there exists a sequence $\langle y_i \rangle_{i=0}^\infty$ such that $y_i \in \text{Orb}_f^+(y_{i+1})$ for each i , $x = y_0$
 21 and $\lim_{i \rightarrow \infty} y_i = y$: that is, the negative limit set of x is the union of all accumulation
 22 points of backward trajectories from x . In [28] the author defines this set as the *special*
 23 *α -limit set of x* and examines them for interval maps. These sets are investigated in [46]
 24 and [45] for graph maps and dendrites. Call this Approach 2 (A2). The final approach,
 25 A3, used in [28], is to say y is in the α -limit set of a point x if there exists a sequence
 26 $\langle y_i \rangle_{i=1}^\infty$ and a strictly increasing sequence $\langle n_i \rangle_{i=1}^\infty$ such that $f^{n_i}(y_i) = x$ for each i and
 27 $\lim_{i \rightarrow \infty} y_i = y$. Clearly this set contains the one given by A2. The converse is not true
 28 (see Example 3.2).

29 By means of demonstrating some of the differences A1-3 yield we provide the following
 30 two examples.

31 **Example 3.1.** Define a map $f: [-1, 1] \rightarrow [-1, 1]$ by

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \in [-1, -1/2), \\ 0 & \text{if } x \in [-1/2, 1/2), \\ 2x - 1 & \text{if } x \in [1/2, 1]. \end{cases}$$

32 The graph of this function may be seen in Figure 1.

33 In Example 3.1, under A1 the negative limit set of 0 can be seen to be the whole
 34 interval $[-1, 1]$. Under A2 and A3 the negative limit set of 0 is simply $\{-1, 0, 1\}$. Notice
 35 that the negative limit set of any backward trajectory from 0 will be either $\{-1\}$ or $\{0\}$
 36 or $\{1\}$.

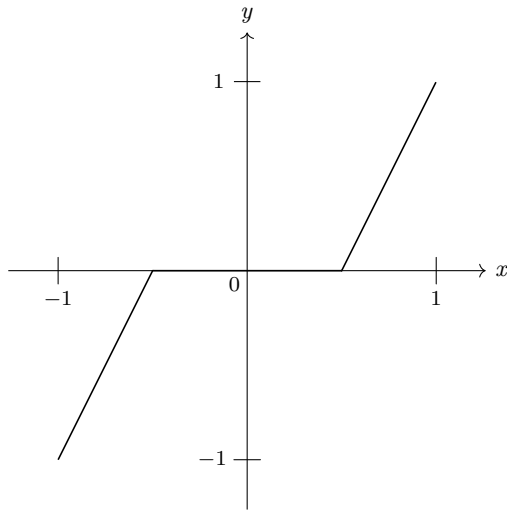


Figure 1: Example 3.1

- 1 **Example 3.2.** Define a map $f: [-1, 2] \rightarrow [-1, 2]$ by

$$f(x) = \begin{cases} 2x + 2 & \text{if } x \in [-1, 0), \\ 2 - 2x & \text{if } x \in [0, 1), \\ 2x - 2 & \text{if } x \in [1, 2]. \end{cases}$$

- 2 The graph of this function may be seen in Figure 2.

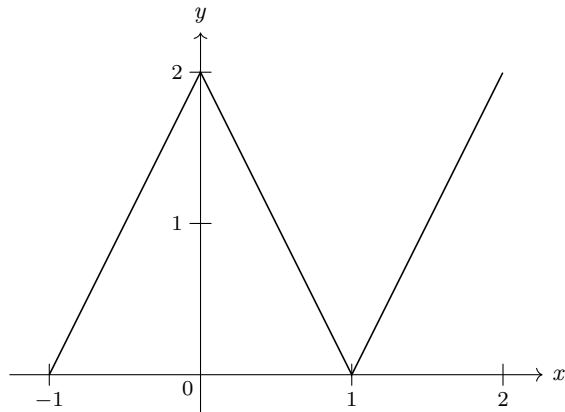


Figure 2: Example 3.2

- 3 In Example 3.2, under A2 the negative limit set of 0 is $\{2/3, 2\}$. Consider the back-
 4 ward trajectory of 0 given by the increasing sequence $\langle x_i \rangle_{i \geq 0}$, where $x_0 = 0$ and $x_1 = 1$,
 5 $x_2 = \frac{3}{2}$, $x_3 = \frac{7}{4}$ This sequence approaches 2. However each point x_i in this sequence
 6 has a preimage y_i in the interval $[-1, 0)$. Each of these y_i thereby eventually map onto 0

1 but they do not themselves have preimages. Furthermore, if $f^n(y_i) = 0$ and $f^m(y_{i+1}) = 0$
2 then by construction $m > n$. This, together with the fact that $\lim_{i \rightarrow \infty} y_i = 0$ implies
3 that 0 is in the negative limit set of itself under A3. Under A3 the negative limit set of 0
4 is $\{0, 2/3, 2\}$. (NB. Hero [28] provides an example illustrating this same difference. For
5 Hero, 0 would be an α -limit point of itself but not a special α -limit point of itself: these
6 would only be $2/3$ and 2 .)

7 As stated previously, in this paper we will not define α -limit sets of individual points,
8 instead we focus on the accumulation points of individual backward trajectories. Note
9 that this is the approach taken in [2] and [29].

10 4. Shadowing, ICT and α_f

11 The following lemma is a recent observation of the authors *et al.* (see [22]).

12 **Lemma 4.1.** [22] *Let (X, f) be a dynamical system. If f has shadowing then it has*
13 *backward shadowing and two-sided shadowing. If f is onto then all three properties are*
14 *equivalent.*

15 **Theorem 4.2.** *Let (X, f) be a dynamical system with shadowing. Then for any $\varepsilon > 0$*
16 *and any $A \in ICT_f$ there is a full trajectory $\langle x_i \rangle_{i \in \mathbb{Z}}$ such that*

- 17 1. $d_H(\omega(x_0), A) < \varepsilon$
- 18 2. $d_H(\alpha(\langle x_i \rangle), A) < \varepsilon$.

19 *In particular every element of ICT_f is in both $\overline{\alpha_f}$ and $\overline{\omega_f}$.*

20 **Remark 4.3.** Before proving Theorem 4.2, we observe that for any $A \in ICT_f$, for each
21 $\eta > 0$ and for each $a \in A$ there exists a finite η -chain $\langle a = a_0, a_1, \dots, a_m = a \rangle$ in A which
22 is η -dense in A , i.e. $\bigcup_{i=0}^{m-1} B_\eta(a_i) \supseteq A$ and for each $i \in \{0, \dots, m-1\}$, $a_i \in A$.

23 *Proof of Theorem 4.2.* Let $A \in ICT_f$ and let $\varepsilon > 0$ be given. By Lemma 4.1 there exists
24 $\delta > 0$ such that every two-sided δ -pseudo-orbit is $\varepsilon/2$ -shadowed by a full orbit. We will
25 construct a two-sided asymptotic δ -pseudo-orbit in A which is η -dense in A for all $\eta > 0$.
26 To this end, let $l \in \mathbb{N}$ be such that $1/2^l < \delta$. Pick $b \in A$. For each $k \in \mathbb{N}_0$ choose a
27 finite $1/2^{l+k}$ -chain $\langle a_{k,0} = b, a_{k,1}, a_{k,2}, \dots, a_{k,m_k} \rangle$ in A which is $1/2^{l+k}$ -dense in A and such
28 that $d(f(a_{k,m_k}), b) < 1/2^{l+k}$. (Here we are simply using the observation in Remark 4.3.)
29 Concatenation of these chains now gives us an asymptotic δ -pseudo-orbit in A :

$$\langle a_{0,0}, a_{0,1}, a_{0,2}, \dots, a_{0,m_0}, a_{1,0}, a_{1,1}, a_{1,2}, \dots, a_{1,m_1}, \dots, a_{k,0}, a_{k,1}, a_{k,2}, \dots, a_{k,m_k}, \dots \rangle.$$

30 We can now extend this into a two-sided asymptotic δ -pseudo-orbit in A by ‘running
31 backwards’ through the δ -chains:

$$\langle \dots, a_{2,0}, a_{2,1}, \dots, a_{2,m_2}, a_{1,0}, a_{1,1}, \dots, a_{1,m_1}, a_{0,0}, a_{0,1}, \dots, a_{0,m_0}, a_{1,0}, a_{1,1}, \dots, a_{1,m_1}, \dots \rangle.$$

32 We call this two-sided asymptotic δ -pseudo-orbit φ . In order to simplify notation we
33 now denote the k^{th} coordinate of φ by a_k , so that, for example, $a_0 = a_{0,0}$ is the 0^{th}

1 coordinate of φ and $a_{-1} = a_{1 \cdot m_1}$ is the $(-1)^{\text{th}}$ coordinate of φ . With this revised
 2 notation $\varphi = \langle a_i \rangle_{i \in \mathbb{Z}}$. From the construction of φ it follows that

$$A = \bigcap_{n \geq 0} \overline{\{a_i \mid i \geq n\}},$$

3 and

$$A = \bigcap_{n \leq 0} \overline{\{a_i \mid i \leq n\}}.$$

4 Let $\langle x_i \rangle_{i \in \mathbb{Z}}$ be a full trajectory such that $d(x_i, a_i) < \varepsilon/2$ for all $i \in \mathbb{Z}$. We claim
 5 that $d_H(\alpha(\langle x_i \rangle), A) < \varepsilon$. Indeed, pick $a \in A$. Then there is a decreasing sequence
 6 $\langle i_n \rangle_{n \in \mathbb{N}}$ of negative integers such that $a = \lim_{n \rightarrow \infty} a_{i_n}$. Thus there is $N \in \mathbb{N}$ such that
 7 $d(a, a_{i_n}) < \varepsilon/3$ for all $n > N$. Since $d(x_{i_n}, a_{i_n}) < \varepsilon/2$ for all $n \in \mathbb{N}$, it follows that
 8 $x_{i_n} \in \overline{B_{\frac{5\varepsilon}{6}}(a)}$ for $n > N$. By compactness the sequence $\langle x_{i_n} \rangle_{n > N}$ has a limit point
 9 $z \in \overline{B_{\frac{5\varepsilon}{6}}(a)}$: in particular $d(z, a) < \varepsilon$. Hence $z \in \alpha(\langle x_i \rangle)$ and

$$A \subseteq \bigcup_{y \in \alpha(\langle x_i \rangle)} B_\varepsilon(y). \quad (1)$$

10 Now take $z \in \alpha(\langle x_i \rangle)$. Then there is a decreasing sequence $\langle i_n \rangle_{n \in \mathbb{N}}$ of negative
 11 integers such that $z = \lim_{n \rightarrow \infty} x_{i_n}$. Let $k \in \mathbb{N}$ be such that $d(z, x_{i_k}) < \varepsilon/2$. By shadowing
 12 $d(a_{i_k}, x_{i_k}) < \varepsilon/2$. By the triangle inequality $d(z, a_{i_k}) < \varepsilon$. Since $a_{i_k} \in A$ it follows that

$$\alpha(\langle x_i \rangle) \subseteq \bigcup_{a \in A} B_\varepsilon(a). \quad (2)$$

13 By Equations (1) and (2) it follows that $d_H(\alpha(\langle x_i \rangle_{i \in \mathbb{Z}}), A) < \varepsilon$.

14 The fact that $d_H(\omega(x_0), A) < \varepsilon$ follows by similar argument. \square

15 The following example shows that the converse to Theorem 4.2 is false.

16 **Example 4.4.** Define a map $f: [-1, 1] \rightarrow [-1, 1]$ by

$$f(x) = \begin{cases} (x+1)^2 - 1 & \text{if } x \in [-1, 0), \\ x^2 & \text{if } x \in [0, 1]. \end{cases}$$

17 Then f does not have shadowing but $ICT_f = \alpha_f = \omega_f$. The graph of this function may
 18 be seen in Figure 3.

19 In Example 4.4, it is easy to see that $ICT_f = \alpha_f = \omega_f = \{\{-1\}, \{0\}, \{1\}\}$. However f
 20 does not have shadowing. Let $\varepsilon = 1/3$. For any $\delta > 0$ we can construct a δ -pseudo-orbit
 21 which is not ε -shadowed. Indeed, fix $\delta > 0$ and let $n > 1$ be such that $1/n < \delta$. Now
 22 pick $z \in (2/3, 1)$ such that $1/n \in \text{Orb}_f^+(z)$. Let $m \in \mathbb{N}$ be such that $f^m(z) = 1/n$. Now let
 23 $k \in \mathbb{N}$ be such that $f^k(-1/n) \in (-1, -3/4)$. Then

$$\langle z, f(z), \dots, f^m(z), 0, -1/n, f(-1/n), \dots, f^k(-1/n) \rangle$$

24 is a finite δ -pseudo orbit. Suppose x ε -shadows this pseudo-orbit. Then $x \in B_\varepsilon(z) \subseteq$
 25 $(1/3, 1]$. But $[0, 1]$ is strongly invariant under f , hence $\text{Orb}_f^+(x) \subseteq [0, 1]$. Since $(-1, -3/4) \cap$
 26 $B_\varepsilon([0, 1]) = \emptyset$ this is a contradiction: f does not exhibit shadowing.

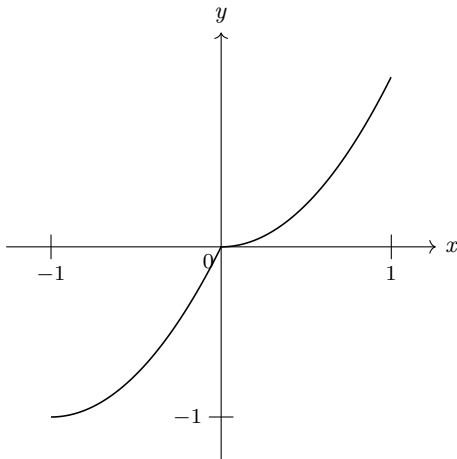


Figure 3: Example 4.4

1 **Corollary 4.5.** *Let (X, f) be a dynamical system with shadowing. Then $\overline{\alpha_f} = \overline{\omega_f} =$*
 2 *ICT_f .*

3 *Proof.* By Lemma 2.2, $ICT_f \supseteq \alpha_f$ and $ICT_f \supseteq \omega_f$. The result now follows immediately
 4 from Theorem 4.2. \square

5 *Remark 4.6.* The fact that $\overline{\omega_f} = ICT_f$ for systems with shadowing has been proved
 6 previously by Meddaugh and Raines in [33].

7 Since ICT_f is always closed in the hyperspace 2^X (see Lemma 2.1), we also get the
 8 following corollary.

9 **Corollary 4.7.** *Let (X, f) be a dynamical system for which $\alpha_f = ICT_f$. Then α_f is*
 10 *closed.*

11 Theorem 4.2 suggests the following question: when is it the case that every element
 12 of ICT_f is both the α -limit set and the ω -limit set of the same full trajectory? The next
 13 result gives a sufficient condition for this to be the case.

14 **Theorem 4.8.** *Let (X, f) be a dynamical system with two-sided s-limit shadowing. Then*
 15 *for any $A \in ICT_f$ there is a full trajectory $\langle x_i \rangle_{i \in \mathbb{Z}}$ such that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle) = A$. In*
 16 *particular $\alpha_f = \omega_f = ICT_f$.*

17 *Proof.* Let $A \in ICT_f$ and let $\varepsilon > 0$ be given. By two-sided s-limit shadowing there
 18 exists $\delta > 0$ such that every two-sided asymptotic δ -pseudo-orbit is asymptotically $\varepsilon/2$ -
 19 shadowed by a full trajectory (without loss of generality we assume $\delta < \varepsilon/2$).

20 Now follow the construction of the two-sided asymptotic δ -pseudo orbit $\langle a_i \rangle_{i \in \mathbb{Z}}$ in the
 21 proof of Theorem 4.2. Recall that

$$A = \bigcap_{n \geq 0} \overline{\{a_i \mid i \geq n\}},$$

1 and

$$A = \bigcap_{n \leq 0} \overline{\{a_i \mid i \leq n\}}.$$

2 Let $\langle x_i \rangle_{i \in \mathbb{Z}}$ be a full trajectory such that

- 3 1. $d(x_i, a_i) < \varepsilon/2$ for all $i \in \mathbb{Z}$,
- 4 2. $\lim_{i \rightarrow \pm\infty} d(x_i, a_i) = 0$.

5 It follows that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle) = A$. The fact that $\alpha_f = \omega_f = ICT_f$ now follows from
6 Lemma 2.2. \square

7 *Remark 4.9.* We did not use the fact that $\langle x_i \rangle_{i \in \mathbb{Z}}$ $\varepsilon/2$ -shadows $\langle a_i \rangle_{i \in \mathbb{Z}}$ in the proof of
8 Theorem 4.8. Therefore, we could replace the hypothesis of “two-sided s-limit shadowing”
9 with the weaker condition: “there exists $\delta > 0$ such that for any two-sided asymptotic δ -
10 pseudo-orbit $\langle y_i \rangle_{i \in \mathbb{Z}}$ there exists a full trajectory $\langle z_i \rangle_{i \in \mathbb{Z}}$ such that $\lim_{i \rightarrow \pm\infty} d(y_i, z_i) = 0$.”

11 A system (X, f) is *expansive* if there exists $\eta > 0$ (referred to as an *expansivity*
12 *constant*) such that given any two distinct full trajectories $\langle x_i \rangle_{i \in \mathbb{Z}}$ and $\langle y_i \rangle_{i \in \mathbb{Z}}$ there
13 exists $i \in \mathbb{Z}$ such that $d(x_i, y_i) \geq \eta$. In [5] the first author *et al.* showed that an
14 expansive map has shadowing if and only if it has s-limit shadowing. We extended that
15 result in [22] to show that an expansive map has shadowing if and only if it has two-sided
16 s-limit shadowing. Combining this result with Theorem 4.8, we immediately obtain the
17 following.

18 **Theorem 4.10.** *Let (X, f) be a dynamical system with shadowing. If f is expansive*
19 *then for any $A \in ICT_f$ there is a full trajectory $\langle x_i \rangle_{i \in \mathbb{Z}}$ such that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle) = A$.*
20 *In particular $\alpha_f = \omega_f = ICT_f$.*

21 **Corollary 4.11.** *Let (X, σ) be a shift of finite type (whether one- or two- sided). Then*
22 *for any $A \in ICT_\sigma$ there is a full trajectory $\langle x_i \rangle_{i \in \mathbb{Z}}$ such that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle) = A$. In*
23 *particular $\alpha_\sigma = \omega_\sigma = ICT_\sigma$.*

24 *Proof.* Shifts of finite type are precisely the shift systems that exhibit shadowing [47].
25 By Theorem 4.10 it now suffices to note that all shift spaces are expansive. \square

26 *Remark 4.12.* Corollary 4.11 enhances a result of Barwell *et al.* [3] who show that $ICT_\sigma =$
27 ω_σ for shifts of finite type.

28 4.1. A remark on γ -limit sets

29 At this point we digress from our main topic to make a brief foray into γ -limit sets.
30 First introduced by Hero [28] who studied them for interval maps, γ -limit sets have
31 since been further examined by Sun *et al.* in [46] and [45] for graph maps and dendrites
32 respectively. The γ -*limit set* of a point x , denoted $\gamma(x)$, is defined by saying that, for
33 any $y \in X$, $y \in \gamma(x)$ if and only if $y \in \omega(x)$ and there exists a sequence $\langle y_i \rangle_{i=1}^\infty$ in X
34 and a strictly increasing sequence $\langle n_i \rangle_{i=1}^\infty$ in \mathbb{N} such that $f^{n_i}(y_i) = x$ for each i and
35 $\lim_{i \rightarrow \infty} y_i = y$. Note that it is possible that $\gamma(x) = \emptyset$. We denote by γ_f the set of all
36 γ -limit sets of (X, f) .

1 *Remark 4.13.* Whilst we have refrained from defining the α -limit set of a point, if one
2 were to use Hero's definition of such (see Section 3), then it would follow that $\gamma(x) =$
3 $\alpha(x) \cap \omega(x)$.

4 *Remark 4.14.* For a dynamical system (X, f) , if f is a homeomorphism it is easy to
5 see that, for any $x \in X$, $\gamma(x) = \alpha(\langle x_i \rangle) \cap \omega(x)$, where $\langle x_i \rangle_{i \leq 0}$ is the unique backward
6 trajectory of x .

7 Unlike α - and ω - limit sets, γ -limit sets are not necessarily internally chain transitive.
8 The example below demonstrates this.

9 **Example 4.15.** Let (X, σ) be the full two-sided shift with alphabet $\{0, 1, 2\}$. Consider
10 the point x :

$$x = \dots 0^n 1^n 0^{n-1} 1^{n-1} \dots 0^2 1^2 0 1 \cdot 0^2 2 1^2 2 0^3 2 1^3 \dots 0^n 2 1^n \dots$$

11 Then $\gamma(x)$ is not internally chain transitive.

12 In Example 4.15, let $\langle x_i \rangle_{i \leq 0}$ be the unique backward trajectory of x . By Theorem
13 2.9 we can observe that:

$$\alpha(\langle x_i \rangle) = \{0^\infty, 1^\infty, \sigma^n(0^\infty \cdot 1^\infty) \mid n \in \mathbb{Z}\},$$

14

$$\omega(x) = \{0^\infty, 1^\infty, \sigma^n(0^\infty 2 \cdot 1^\infty) \mid n \in \mathbb{Z}\}.$$

15 Since σ is a homeomorphism, by Remark 4.14,

$$\gamma(x) = \{0^\infty, 1^\infty\}.$$

16 It is obvious that $\gamma(x)$ is not internally chain transitive.

17 Example 4.15 notwithstanding, every γ -limit set is closed and contained in a single
18 *chain component* of the dynamical system, i.e. for each $\varepsilon > 0$ and for all $a, b \in \gamma(x)$ there
19 is an ε -chain from a to b in X (as opposed to in $\gamma(x)$).

20 **Proposition 4.16.** Let (X, f) be a dynamical system. For any $x \in X$, $\gamma(x)$ is closed
21 and contained in a single chain component of (X, f) .

22 *Proof.* If $\gamma(x) = \emptyset$ then the closedness holds and chain transitivity is vacuous.

23 Let $a, b \in \gamma(x)$. Let $\delta > 0$ be given. Let $y \in X$ be such that $d(f(y), f(a)) < \delta$
24 and there exists $n > 1$ such that $f^n(y) = x$: such a point exists by the continuity of f
25 combined with the fact that $a \in \gamma(x)$. Now let $m \in \mathbb{N}$ be such that $d(f^m(x), b) < \delta$. It
26 follows that $\langle a, f(y), f^2(y), \dots, f^n(y) = x, f(x), f^2(x), \dots, f^{m-1}(x), b \rangle$ is a δ -chain from
27 a to b .

28 Now suppose $z \in \overline{\gamma(x)}$. Then there is a sequence $\langle y_i \rangle_{i=1}^\infty$ in $\gamma(x)$ such that $\lim_{i \rightarrow \infty} y_i =$
29 z . Note that, since $\omega(x)$ is closed and $y_i \in \omega(x)$ for each i it follows that $z \in \omega(x)$. Now,
30 for each $i \in \mathbb{N}$, let $z_i \in B_{1/i}(y_i)$ and $n_i \in \mathbb{N}$ be such that $f^{n_i}(z_i) = x$ and $\langle n_i \rangle_{i=1}^\infty$ is an
31 increasing sequence. Then, as $\lim_{i \rightarrow \infty} z_i = z$, it follows that $z \in \gamma(x)$. \square

32 Using theorems 4.8 and 4.10 we obtain the following corollaries concerning the nonempty
33 closed internally chain transitive sets in systems with two-sided s -limit shadowing.

34 **Corollary 4.17.** If (X, f) is a system with two-sided s -limit shadowing then $ICT_f \subseteq \gamma_f$.

1 *Proof.* Let $A \in ICT_f$. By Theorem 4.8 there is a full trajectory $\langle x_i \rangle_{i \in \mathbb{Z}}$ through $x_0 = x$
2 such that $\alpha(\langle x_i \rangle) = \omega(x) = A$. Notice that $\gamma(x) \subseteq \omega(x)$ by definition. Since $\alpha(\langle x_i \rangle) =$
3 $\omega(x)$, and $\langle x_i \rangle_{i \leq 0}$ is a backward trajectory of x , it follows that $\gamma(x) = A$. Hence $ICT_f \subseteq$
4 γ_f . \square

5 **Corollary 4.18.** *If (X, f) is an expansive system with shadowing then $ICT_f \subseteq \gamma_f$.*

6 5. Characterising $\overline{\alpha_f} = ICT_f$ and $\alpha_f = ICT_f$

7 In [23] the authors characterise systems for which $\overline{\omega_f} = ICT_f$ and $\omega_f = ICT_f$ in
8 terms of novel shadowing properties. In this section we show that the natural backward
9 analogues of these shadowing properties characterise when $\overline{\alpha_f} = ICT_f$ and $\alpha_f = ICT_f$.
10 We also demonstrate by way of examples that, in contrast to the shadowing property,
11 there is no general entailment between the backward and forward versions of these types
12 of shadowing.

13 In [23] it is shown that the property of $\overline{\omega_f} = ICT_f$ is characterised by a variation
14 on shadowing the authors term *cofinal orbital shadowing*. A system $f: X \rightarrow X$ has the
15 cofinal orbital shadowing property if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any
16 δ -pseudo-orbit $\langle x_i \rangle_{i=0}^\infty$ there exists a point $z \in X$ such that for any $K \in \mathbb{N}$ there exists
17 $N \geq K$ such that

$$d_H(\overline{\{f^{N+i}(z)\}_{i=0}^\infty}, \overline{\{x_{N+i}\}_{i=0}^\infty}) < \varepsilon.$$

18 The authors additionally demonstrate that this form of shadowing is equivalent to one
19 which seems *prima facie* stronger: the *eventual strong orbital shadowing property*. A
20 system $f: X \rightarrow X$ has the eventual strong orbital shadowing property if for all $\varepsilon > 0$
21 there exists $\delta > 0$ such that for any δ -pseudo-orbit $\langle x_i \rangle_{i=0}^\infty$ there exist a point $z \in X$ and
22 $K \in \mathbb{N}$ such that

$$d_H(\overline{\{f^{N+i}(z)\}_{i=0}^\infty}, \overline{\{x_{N+i}\}_{i=0}^\infty}) < \varepsilon$$

23 for all $N \geq K$.

24 **Definition 5.1.** A system $f: X \rightarrow X$ has the *backward cofinal orbital shadowing property*
25 if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any backward δ -pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there
26 exists a backward trajectory $\langle z_i \rangle_{i \leq 0}$ such that for any $K \in \mathbb{N}$ there exists $N \geq K$ such
27 that

$$d_H(\overline{\{z_{i-N}\}_{i \leq 0}}, \overline{\{x_{i-N}\}_{i \leq 0}}) < \varepsilon.$$

28 **Definition 5.2.** A system $f: X \rightarrow X$ has the *backward eventual strong orbital shadowing*
29 *property* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any backward δ -pseudo-orbit
30 $\langle x_i \rangle_{i \leq 0}$ there exists a backward trajectory $\langle z_i \rangle_{i \leq 0}$ and there exists $K \in \mathbb{N}$ such that

$$d_H(\overline{\{z_{i-N}\}_{i \leq 0}}, \overline{\{x_{i-N}\}_{i \leq 0}}) < \varepsilon$$

31 for all $N \geq K$.

32 **Theorem 5.3.** *Let (X, f) be a dynamical system. The following are equivalent:*

- 33 1. *f has the backward cofinal orbital shadowing property;*
- 34 2. *f has the backward eventual strong orbital shadowing property;*

1 3. $\overline{\alpha_f} = ICT_f$.

2 *Proof.* From the definitions it is easy to see that (2) \implies (1). We will show (1) \implies (3)
 3 and that (3) \implies (2).

4 Suppose that f has the backward cofinal orbital shadowing property. Recall that
 5 $\overline{\alpha_f} \subseteq ICT_f$, hence it will suffice to show $ICT_f \subseteq \overline{\alpha_f}$. Let $A \in ICT_f$. Let $\varepsilon > 0$ be given.
 6 It will suffice to show there exists $B \in \alpha_f$ with $d_H(A, B) < \varepsilon$. Let $\delta > 0$ correspond to
 7 $\varepsilon/2$ for cofinal orbital shadowing. Now, follow the construction of the sequence $\langle a_i \rangle_{i \in \mathbb{Z}}$ in
 8 Theorem 4.2 (but for $\varepsilon/2$ and δ as here) and let $x_i = a_i$ for all $i \leq 0$. Recall that this
 9 means

$$A = \alpha(\langle x_i \rangle_{i \leq 0}).$$

10 Let $\langle z_i \rangle_{i \leq 0}$ be given by backward cofinal orbital shadowing so that for any $K \in \mathbb{N}$ there
 11 exists $N \geq K$ such that

$$d_H(\overline{\{z_{i-N}\}_{i \leq 0}}, \overline{\{x_{i-N}\}_{i \leq 0}}) < \varepsilon/2.$$

12 Notice that in particular this means that

$$d_H(\alpha(\langle x_i \rangle), \alpha(\langle z_i \rangle)) < \varepsilon.$$

13 Since $\alpha(\langle x_i \rangle_{i \leq 0}) = A$ it follows that $A \in \overline{\alpha_f}$.

14 Now suppose that (X, f) does not have backward eventual strong orbital shadowing
 15 and let $\varepsilon > 0$ witness this. (We will show $ICT_f \neq \overline{\alpha_f}$.) This means that for each $n \in \mathbb{N}$
 16 there is a backward $1/2^n$ -pseudo-orbit $\langle x_i^n \rangle_{i \leq 0}$ such that for any backward orbit $\langle z_i \rangle_{i \leq 0}$
 17 and any $K \in \mathbb{N}$ there exists $N \geq K$ with

$$d_H(\overline{\{z_{i-N}\}_{i \leq 0}}, \overline{\{x_{i-N}^n\}_{i \leq 0}}) \geq \varepsilon.$$

18 It follows that, in particular, for each backward orbit $\langle z_i \rangle_{i \leq 0}$ and any $n \in \mathbb{N}$

$$d_H(\alpha(\langle z_i \rangle), \alpha(\langle x_i^n \rangle)) \geq \varepsilon/2. \tag{3}$$

19 For each $n \in \mathbb{N}$ let $A_n = \alpha(\langle x_i^n \rangle_{i \leq 0})$. The sequence of compact sets $\langle A_n \rangle_{n \in \mathbb{N}}$ has a
 20 convergent subsequence which converges in the hyperspace 2^X . Without loss of generality
 21 we may assume the sequence itself is convergent; let A be its limit. We claim $A \in ICT_f$
 22 but that $A \notin \overline{\alpha_f}$.

23 Let $a, b \in A$ and let $\xi > 0$ be arbitrary. By the uniform continuity of f , there exists
 24 $\eta > 0$ such that for any $x, y \in X$ if $d(x, y) < \eta$ then $d(f(x), f(y)) < \xi/2$. Without loss of
 25 generality take $\eta < \xi/2$. Let $M \in \mathbb{N}$ be such that $1/2^M < \eta/3$ and $d_H(A_M, A) < \eta/3$. Now
 26 take $K \in \mathbb{N}$ such that

$$d_H(\overline{\{x_{i-K}^M\}_{i \leq 0}}, A_M) < \eta/3.$$

27 Thus

$$d_H(\overline{\{x_{i-K}^M\}_{i \leq 0}}, A) < 2\eta/3.$$

Let $m \in \mathbb{N}$ be such that $d(x_{-m-K}^M, b) < 2\eta/3$ and let $l > m$ be such that $d(x_{-l-K}^M, a) < 2\eta/3$. Let $y_0 = a$ and $y_{l-m} = b$. For each $j \in 1, \dots, l-m-1$ pick $y_j \in A$ with

$d(y_j, x_{-l-K+j}^M) < 2\eta/3$. We claim $\langle y_0, y_1, \dots, y_{l-k} \rangle$ is a ξ -chain from a to b . Indeed, for $j \in \{0, \dots, l-m-1\}$

$$\begin{aligned} d(f(y_j), y_{j+1}) &\leq d(f(y_j), f(x_{-l-K+j}^M)) + d(f(x_{-l-K+j}^M), x_{-l-K+j+1}^M) \\ &\quad + d(x_{-l-K+j+1}^M, y_{j+1}) \\ &\leq \xi/2 + 1/2^M + 2\eta/3 \\ &\leq \xi/2 + \eta/3 + 2\eta/3 \\ &\leq \xi. \end{aligned}$$

1 Since a and b were chosen arbitrarily in A we have that A is internally chain transitive.
2 Thus, since A is nonempty and closed, $A \in ICT_f$.

3 Suppose for a contradiction that $A \in \overline{\alpha_f}$. Then there exists a backward trajectory
4 $\langle z_i \rangle_{i \leq 0} \in X$ such that $d_H(\alpha(\langle z_i \rangle_{i \leq 0}), A) < \varepsilon/4$. Let $M \in \mathbb{N}$ be such that $d_H(A_M, A) < \varepsilon/4$.
5 Then $d_H(\alpha(\langle z_i \rangle_{i \leq 0}), A_M) < \varepsilon/2$, which contradicts Equation 3. Therefore $A \in ICT_f \setminus \overline{\alpha_f}$.
6 Thus $\overline{\alpha_f} \neq ICT_f$. \square

7 *Remark 5.4.* Unlike with shadowing (see Lemma 4.1), none of the shadowing properties
8 in Theorem 5.3 imply their forward analogues (nor vice-versa). To see this, by Theorem
9 5.3 and [23, Theorem 13], it suffices to give an example where $\alpha_f = ICT_f$ but $\overline{\omega_f} \neq ICT_f$
10 and an example where $\omega_f = ICT_f$ but $\overline{\alpha_f} \neq ICT_f$. Examples 5.5 and 5.6 provide this.

11 **Example 5.5.** Let $x = 1010^2 10^3 \dots$. Take

$$X = \overline{\text{Orb}_\sigma^+(x) \cup \{0^n x \mid n \in \mathbb{N}\}},$$

12 where the closure is taken with regard to the one-sided full shift on the alphabet $\{0, 1\}$,
13 and consider the system (X, σ) . Then $ICT_\sigma = \omega_\sigma \neq \overline{\alpha_\sigma}$.

14 In Example 5.5, $\omega(x) = \{0^\infty, 0^n 10^\infty \mid n \geq 0\}$. It is easy to see that the only other
15 ω -limit set is $\{0^\infty\}$. Thus

$$\omega_\sigma = \{\{0^\infty\}, \{0^\infty, 0^n 10^\infty \mid n \geq 0\}\}.$$

16 Meanwhile

$$\alpha_\sigma = \{\{0^\infty\}\}.$$

17 Observe that $\overline{\alpha_\sigma} = \alpha_\sigma$. Finally $ICT_\sigma = \omega_\sigma \neq \overline{\alpha_\sigma}$. Hence the system has cofinal orbital
18 shadowing and eventual strong orbital shadowing by [23, Theorem 13] but the system
19 does not have their backward analogues by Theorem 5.3.

20 **Example 5.6.** Take

$$X = \overline{\{\sigma^k(10^n 10^{n-1} \dots 10^2 10^\infty) \mid k, n \in \mathbb{N}\}},$$

21 where the closure is taken with regard to the one-sided full shift on the alphabet $\{0, 1\}$,
22 and consider the system (X, σ) . Then $ICT_\sigma = \alpha_\sigma \neq \overline{\omega_\sigma}$.

23 In Example 5.6 it is easily observed that

$$\omega_\sigma = \{\{0^\infty\}\}.$$

1 Meanwhile

$$\alpha_\sigma = \{\{0^\infty\}, \{0^\infty, 0^n 10^\infty \mid n \geq 0\}\}.$$

2 Observe that $\overline{\omega_\sigma} = \omega_\sigma$. Finally $ICT_\sigma = \alpha_\sigma \neq \overline{\omega_\sigma}$. Hence the system (X, σ) has backward
3 cofinal orbital shadowing and backward eventual strong orbital shadowing by Theorem
4 5.3 but it does not have their forward analogues by [23, Theorem 13].

5 In [23, Theorem 22] the authors show that the property of $\omega_f = ICT_f$ is characterised
6 by several equivalent asymptotic variants of shadowing: These are *asymptotic orbital*
7 *shadowing*, *asymptotic strong orbital shadowing* and *orbital limit shadowing*. The system
8 (X, f) has then has the *asymptotic orbital shadowing* property if for any asymptotic
9 pseudo-orbit $\langle x_i \rangle_{i \geq 0}$ there exists a point $z \in X$ such that for any $\varepsilon > 0$ there exists
10 $N \in \mathbb{N}$ such that

$$d_H(\overline{\{x_{N+i}\}_{i \geq 0}}, \overline{\{f^{N+i}(z)\}_{i \geq 0}}) < \varepsilon.$$

11 The system has the *asymptotic strong orbital shadowing* property if for any asymptotic
12 pseudo-orbit $\langle x_i \rangle_{i \geq 0}$ there exists a point $z \in X$ such that for any $\varepsilon > 0$ there exists
13 $K \in \mathbb{N}$ such that

$$d_H(\overline{\{x_{N+i}\}_{i \geq 0}}, \overline{\{f^{N+i}(z)\}_{i \geq 0}}) < \varepsilon$$

14 for all $N \geq K$. Finally, the system has the *orbital limit shadowing* property, as introduced
15 by Pilyugin [41], if for any asymptotic pseudo-orbit $\langle x_i \rangle_{i \geq 0}$ there exists a point $z \in X$
16 such that $\omega(z) = \omega(\langle x_i \rangle)$.

17 Before characterising $\omega_f = ICT_f$ by these notions of shadowing, the authors [23] note
18 that *asymptotic shadowing*, also known as limit shadowing, is sufficient but not necessary
19 for $\omega_f = ICT_f$: a system has asymptotic shadowing if for each asymptotic pseudo-orbit
20 $\langle x_i \rangle_{i \geq 0}$ there exists a point $z \in X$ such that

$$\lim_{i \rightarrow \infty} d(f^i(z), x_i) = 0.$$

21 As with other shadowing variants, asymptotic shadowing has a backward analogue.

22 **Definition 5.7.** A system $f: X \rightarrow X$ has the *backward asymptotic shadowing property*
23 if for each backward asymptotic pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there exists a backward trajectory
24 $\langle z_i \rangle_{i \leq 0}$ such that

$$\lim_{i \rightarrow -\infty} d(z_i, x_i) = 0.$$

25 We shall see (Corollary 5.13) that backward asymptotic shadowing is sufficient for
26 $\alpha_f = ICT_f$, however it is not necessary. The irrational rotation of the circle satisfies
27 $\alpha_f = ICT_f$ (as a minimal map, both are equal to $\{X\}$) however it fails to have backward
28 asymptotic shadowing. To see this one can observe that for any irrational rotation f of
29 the circle, the inverse function f^{-1} is also an irrational rotation of the circle. It thereby
30 suffices to note that no irrational rotation of the circle has asymptotic shadowing [41].

31 **Definition 5.8.** A system $f: X \rightarrow X$ has the *backward asymptotic orbital shadowing*
32 *property* if for each backward asymptotic pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there exists a backward
33 trajectory $\langle z_i \rangle_{i \leq 0}$ such that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d_H(\overline{\{z_{i-N}\}_{i \leq 0}}, \overline{\{x_{i-N}\}_{i \leq 0}}) < \varepsilon.$$

1 **Definition 5.9.** A system $f: X \rightarrow X$ has the *backward asymptotic strong orbital shadow-*
 2 *ing property* if for each backward asymptotic pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there exists a backward
 3 trajectory $\langle z_i \rangle_{i \leq 0}$ such that for any $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that

$$d_H(\overline{\{z_{i-N}\}_{i \leq 0}}, \overline{\{x_{i-N}\}_{i \leq 0}}) < \varepsilon$$

4 for all $N \geq K$.

5 The following is a backward version of the orbital limit shadowing property, studied
 6 by Pilyugin *et al.* [41].

7 **Definition 5.10.** A system $f: X \rightarrow X$ has the *backward orbital limit shadowing property*
 8 if for each backward asymptotic pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there exists a backward trajectory
 9 $\langle z_i \rangle_{i \leq 0}$ such that

$$\alpha(\langle z_i \rangle) = \alpha(\langle x_i \rangle).$$

10 As mentioned previously, Hirsch *et al.* [29] showed that the α -limit set (resp. ω -limit
 11 set) of any backward (resp. forward) pre-compact trajectory is internally chain transitive.
 12 In the same paper, the authors show that the ω -limit set of any pre-compact asymptotic
 13 pseudo-orbit is internally chain transitive [29, Lemma 2.3]. Whilst we omit the proof,
 14 the same is true of pre-compact backward asymptotic pseudo-orbits. We formulate this
 15 as Lemma 5.11 below.

16 **Lemma 5.11.** [29] *Let (X, f) be a dynamical system where X is a (not necessarily*
 17 *compact) metric space. The α -limit set (resp. ω -limit set) of any backward (resp. forward)*
 18 *pre-compact asymptotic pseudo-orbit is internally chain transitive. In particular, when*
 19 *X is compact, all such limit sets are in ICT_f .*

20 **Theorem 5.12.** *Let (X, f) be a dynamical system. The following are equivalent:*

- 21 1. $\alpha_f = ICT_f$;
- 22 2. f has the backward orbital limit shadowing property;
- 23 3. f has the backward asymptotic orbital shadowing property;
- 24 4. f has the backward asymptotic strong orbital shadowing property.

25 *Proof.* Clearly (4) \implies (3). It is also easy to see that (2) \implies (4). We will show
 26 (3) \implies (1) \implies (2).

27 To this end, suppose that f has backward asymptotic orbital shadowing. Let $A \in$
 28 ICT_f . Form a backward asymptotic pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ by following the construction
 29 as in the proof of Theorem 4.2 and taking $x_i = a_i$ for all $i \leq 0$. (We may ignore the ε
 30 and δ in the construction, we can simply take $l = 0$.) Recall that this means

$$A = \bigcap_{n \leq 0} \overline{\{x_i \mid i \leq n\}},$$

31 or equivalently,

$$A = \alpha(\langle x_i \rangle).$$

1 Let $\langle z_i \rangle_{i \leq 0}$ be given by backward asymptotic orbital shadowing. Now let $\varepsilon > 0$ be given
 2 and let $N \in \mathbb{N}$ be such that

$$d_H(\alpha(\langle z_i \rangle), \overline{\{z_{i-N}\}_{i \leq 0}}) < \varepsilon/3,$$

3

$$d_H(\overline{\{z_{i-N}\}_{i \leq 0}}, \overline{\{x_{i-N}\}_{i \leq 0}}) < \varepsilon/3,$$

4 and

$$d_H(\overline{\{x_{i-N}\}_{i \leq 0}}, \alpha(\langle x_i \rangle)) < \varepsilon/3.$$

5 By the triangle inequality it follows that $d_H(\alpha(\langle z_i \rangle), A) < \varepsilon$. Since $\varepsilon > 0$ was picked
 6 arbitrarily this implies that $A = \alpha(\langle z_i \rangle)$. Hence $ICT_f \subseteq \alpha_f$. By Lemma 2.2 we have
 7 $\alpha_f \subseteq ICT_f$, thus (1) holds.

8 Now suppose that $\alpha_f = ICT_f$. Let $\langle x_i \rangle_{i \leq 0}$ be a backward asymptotic pseudo-orbit.
 9 By Lemma 5.11 $\alpha(\langle x_i \rangle) \in ICT_f$. Since $\alpha_f = ICT_f$ there exists a backward trajec-
 10 tory $\langle z_i \rangle_{i \leq 0}$ with $\alpha(\langle z_i \rangle) = \alpha(\langle x_i \rangle)$. Hence f has the backward orbital limit shadowing
 11 property, i.e. (2) holds. \square

12 **Corollary 5.13.** *If (X, f) has backward asymptotic shadowing then $\alpha_f = ICT_f$.*

13 *Proof.* By Theorem 5.12 it suffices to note that backward asymptotic shadowing implies
 14 backward orbital limit shadowing. \square

15 *Remark 5.14.* Combining theorems 5.3 and 5.12 we have that if α_f is closed then the
 16 following are equivalent:

- 17 1. f has the backward orbital limit shadowing property;
- 18 2. f has the backward eventual strong orbital shadowing property;
- 19 3. f has the backward asymptotic (strong) orbital shadowing property;
- 20 4. f has the backward cofinal orbital shadowing property.

21 *Remark 5.15.* Examples 5.5 and 5.6, together with Theorem 5.12 and [23, Theorem
 22 22], show that, unlike shadowing (see Lemma 4.1), neither the backward orbital limit
 23 shadowing property nor the backward asymptotic orbital shadowing nor the backward
 24 asymptotic strong orbital shadowing is equivalent to its forward analogue.

25 References

- 26 [1] S. J. Agronsky, A. M. Bruckner, J. G. Ceder, and T. L. Pearson. The structure of ω -limit sets for
 27 continuous functions. *Real Anal. Exchange*, 15(2):483–510, 1989/90.
- 28 [2] Francisco Balibrea, Gabriela Dvorníková, Marek Lampart, and Piotr Oprocha. On negative limit
 29 sets for one-dimensional dynamics. *Nonlinear Anal.*, 75(6):3262–3267, 2012.
- 30 [3] Andrew Barwell, Chris Good, Robin Knight, and Brian E. Raines. A characterization of ω -limit
 31 sets in shift spaces. *Ergodic Theory Dynam. Systems*, 30(1):21–31, 2010.
- 32 [4] Andrew D. Barwell, Chris Good, and Piotr Oprocha. Shadowing and expansivity in subspaces.
 33 *Fund. Math.*, 219(3):223–243, 2012.
- 34 [5] Andrew D. Barwell, Chris Good, Piotr Oprocha, and Brian E. Raines. Characterizations of ω -limit
 35 sets in topologically hyperbolic systems. *Discrete Contin. Dyn. Syst.*, 33(5):1819–1833, 2013.
- 36 [6] Andrew D. Barwell, Jonathan Meddaugh, and Brian E. Raines. Shadowing and ω -limit sets of
 37 circular Julia sets. *Ergodic Theory Dynam. Systems*, 35(4):1045–1055, 2015.

- 1 [7] Andrew D. Barwell and Brian E. Raines. The ω -limit sets of quadratic Julia sets. *Ergodic Theory*
2 *Dynam. Systems*, 35(2):337–358, 2015.
- 3 [8] Alexander Blokh, A. M. Bruckner, P. D. Humke, and J. Smítal. The space of ω -limit sets of a
4 continuous map of the interval. *Trans. Amer. Math. Soc.*, 348(4):1357–1372, 1996.
- 5 [9] R. Bowen. Markov partitions for Axiom A diffeomorphisms. *Amer. J. Math.*, 92:725–747, 1970.
- 6 [10] Rufus Bowen. ω -limit sets for axiom A diffeomorphisms. *J. Differential Equations*, 18(2):333–339,
7 1975.
- 8 [11] Will Brian and Piotr Oprocha. Ultrafilters and Ramsey-type shadowing phenomena in topological
9 dynamics. *Israel J. Math.*, 227(1):423–453, 2018.
- 10 [12] William R. Brian, Jonathan Meddaugh, and Brian E. Raines. Chain transitivity and variations of
11 the shadowing property. *Ergodic Theory Dynam. Systems*, 35(7):2044–2052, 2015.
- 12 [13] Andrew M. Bruckner and Jaroslav Smítal. The structure of ω -limit sets for continuous maps of the
13 interval. *Math. Bohem.*, 117(1):42–47, 1992.
- 14 [14] Robert M. Corless. Defect-controlled numerical methods and shadowing for chaotic differential
15 equations. *Phys. D*, 60(1-4):323–334, 1992. Experimental mathematics: computational issues in
16 nonlinear science (Los Alamos, NM, 1991).
- 17 [15] Robert M. Corless and S. Yu. Pilyugin. Approximate and real trajectories for generic dynamical
18 systems. *J. Math. Anal. Appl.*, 189(2):409–423, 1995.
- 19 [16] Ethan M. Coven, Ittai Kan, and James A. Yorke. Pseudo-orbit shadowing in the family of tent
20 maps. *Trans. Amer. Math. Soc.*, 308(1):227–241, 1988.
- 21 [17] Ethan M. Coven and Zbigniew Nitecki. Nonwandering sets of the powers of maps of the interval.
22 *Ergodic Theory Dynamical Systems*, 1(1):9–31, 1981.
- 23 [18] Hongfei Cui and Yiming Ding. The α -limit sets of a unimodal map without homtervals. *Topology*
24 *Appl.*, 157(1):22–28, 2010.
- 25 [19] Dawoud Ahmadi Dastjerdi and Maryam Hosseini. Sub-shadowings. *Nonlinear Anal.*, 72(9-10):3759–
26 3766, 2010.
- 27 [20] Jan de Vries. *Topological dynamical systems*, volume 59 of *De Gruyter Studies in Mathematics*. De
28 Gruyter, Berlin, 2014. An introduction to the dynamics of continuous mappings.
- 29 [21] Abbas Fakhari and F. Helen Ghane. On shadowing: ordinary and ergodic. *J. Math. Anal. Appl.*,
30 364(1):151–155, 2010.
- 31 [22] Chris Good, Sergio Macías, Jonathan Meddaugh, Joel Mitchell, and Joe Thomas. Expansivity and
32 unique shadowing, 2020. arXiv: 2002.11199.
- 33 [23] Chris Good and Jonathan Meddaugh. Orbital shadowing, internal chain transitivity and ω -limit
34 sets. *Ergodic Theory Dynam. Systems*, 38(1):143–154, 2018.
- 35 [24] Chris Good and Jonathan Meddaugh. Shifts of finite type as fundamental objects in the theory of
36 shadowing. *Invent. Math.*, 220(3):715–736, 2020.
- 37 [25] Chris Good, Joel Mitchell, and Joe Thomas. On inverse shadowing. *Dynamical Systems*, 0(0):1–9,
38 2020.
- 39 [26] Chris Good, Joel Mitchell, and Joe Thomas. Preservation of shadowing in discrete dynamical
40 systems. *J. Math. Anal. Appl.*, 485(1):123767, 39, 2020.
- 41 [27] Chris Good, Piotr Oprocha, and Mate Puljiz. Shadowing, asymptotic shadowing and s-limit shad-
42 owing. *Fund. Math.*, 244(3):287–312, 2019.
- 43 [28] Michael W. Hero. Special α -limit points for maps of the interval. *Proc. Amer. Math. Soc.*,
44 116(4):1015–1022, 1992.
- 45 [29] Morris W. Hirsch, Hal L. Smith, and Xiao-Qiang Zhao. Chain transitivity, attractivity, and strong
46 repellers for semidynamical systems. *J. Dynam. Differential Equations*, 13(1):107–131, 2001.
- 47 [30] Keonhee Lee. Continuous inverse shadowing and hyperbolicity. *Bull. Austral. Math. Soc.*, 67(1):15–
48 26, 2003.
- 49 [31] Keonhee Lee and Kazuhiro Sakai. Various shadowing properties and their equivalence. *Discrete*
50 *Contin. Dyn. Syst.*, 13(2):533–540, 2005.
- 51 [32] Jie-Hua Mai and Song Shao. Spaces of ω -limit sets of graph maps. *Fund. Math.*, 196(1):91–100,
52 2007.
- 53 [33] Jonathan Meddaugh and Brian E. Raines. Shadowing and internal chain transitivity. *Fund. Math.*,
54 222(3):279–287, 2013.
- 55 [34] Joel Mitchell. Orbital shadowing, ω -limit sets and minimality. *Topology Appl.*, 268:106903, 7, 2019.
- 56 [35] Helena E. Nusse and James A. Yorke. Is every approximate trajectory of some process near an
57 exact trajectory of a nearby process? *Comm. Math. Phys.*, 114(3):363–379, 1988.
- 58 [36] Piotr Oprocha. Shadowing, thick sets and the Ramsey property. *Ergodic Theory Dynam. Systems*,
59 36(5):1582–1595, 2016.

- 1 [37] D. W. Pearson. Shadowing and prediction of dynamical systems. *Math. Comput. Modelling*, 34(7-
2 8):813–820, 2001.
- 3 [38] Timothy Pennings and Jeffrey Van Eeuwen. Pseudo-orbit shadowing on the unit interval. *Real*
4 *Anal. Exchange*, 16(1):238–244, 1990/91.
- 5 [39] S. Yu. Pilyugin, A. A. Rodionova, and K. Sakai. Orbital and weak shadowing properties. *Discrete*
6 *Contin. Dyn. Syst.*, 9(2):287–308, 2003.
- 7 [40] Sergei Yu. Pilyugin. *Shadowing in dynamical systems*, volume 1706 of *Lecture Notes in Mathemat-*
8 *ics*. Springer-Verlag, Berlin, 1999.
- 9 [41] Sergei Yu. Pilyugin. Sets of dynamical systems with various limit shadowing properties. *J. Dynam.*
10 *Differential Equations*, 19(3):747–775, 2007.
- 11 [42] David Pokluda. On the transitive and ω -limit points of the continuous mappings of the circle. *Arch.*
12 *Math. (Brno)*, 38(1):49–52, 2002.
- 13 [43] Clark Robinson. Stability theorems and hyperbolicity in dynamical systems. In *Proceedings of the*
14 *Regional Conference on the Application of Topological Methods in Differential Equations (Boulder,*
15 *Colo., 1976)*, volume 7, pages 425–437, 1977.
- 16 [44] Kazuhiro Sakai. Various shadowing properties for positively expansive maps. *Topology Appl.*,
17 131(1):15–31, 2003.
- 18 [45] Taixiang Sun, Yalin Tang, Guangwang Su, Hongjian Xi, and Bin Qin. Special α -limit points and
19 γ -limit points of a dendrite map. *Qual. Theory Dyn. Syst.*, 17(1):245–257, 2018.
- 20 [46] TaiXiang Sun, HongJian Xi, and HaiLan Liang. Special α -limit points and unilateral γ -limit points
21 for graph maps. *Sci. China Math.*, 54(9):2013–2018, 2011.
- 22 [47] Peter Walters. On the pseudo-orbit tracing property and its relationship to stability. In *The*
23 *structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D.,*
24 *1977)*, volume 668 of *Lecture Notes in Math.*, pages 231–244. Springer, Berlin, 1978.