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TWO BIFURCATION SETS ARISING FROM THE BETA TRANSFORMATION WITH A HOLE AT 0

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Abstract. Given $\beta \in (1,2]$, the β-transformation $T_\beta : x \mapsto \beta x \pmod 1$ on the circle $[0,1)$ with a hole $[0,t)$ was investigated by Kalle et al. (2019). They described the set-valued bifurcation set

$$\mathscr{E}_\beta := \{t \in [0,1) : K_\beta(t') \neq K_\beta(t)\ \forall t' > t\},$$

where $K_\beta(t) := \{x \in [0,1) : T_\beta^n(x) \geq t\ \forall n \geq 0\}$ is the survivor set. In this paper we investigate the dimension bifurcation set

$$\mathscr{B}_\beta := \{t \in [0,1) : \dim_H K_\beta(t') \neq \dim_H K_\beta(t)\ \forall t' > t\},$$

where $\dim_H$ denotes the Hausdorff dimension. We show that if $\beta \in (1,2]$ is a multinacci number then the two bifurcation sets $\mathscr{B}_\beta$ and $\mathscr{E}_\beta$ coincide. Moreover we give a complete characterization of these two sets. As a corollary of our main result we prove that for β a multinacci number we have $\dim_H(\mathscr{E}_\beta \cap [t,1]) = \dim_H K_\beta(t)$ for any $t \in [0,1)$. This confirms a conjecture of Kalle et al. for β a multinacci number.

1. Introduction

Given $\beta \in (1,2]$, the β-transformation T_β on the circle $\mathbb{R}/\mathbb{Z} \sim [0,1)$ is defined by

$$T_\beta : [0,1) \to [0,1); \quad x \mapsto \beta x \pmod 1.$$

Following the pioneering work of Rényi [11] and Parry [9] there has been a great interest in the study of T_β. In general, the system $\Phi_\beta = ([0,1), T_\beta)$ does not admit a Markov partition (cf. [12]), this makes describing the dynamics of Φ_β more challenging.

When $\beta = 2$, Urbański considered in [14, 15] the open dynamical system under the doubling map T_2 with a hole at zero. More precisely, for $t \in [0,1)$ let

$$K_2(t) := \{x \in [0,1) : T_2^n(x) \geq t\ \forall\, n \geq 0\}.$$

Here we use a slightly different definition of $K_2(t)$ from that by Urbański. By [14, Theorem 1 and Corollary 1] it follows that the dimension function $t \mapsto \eta_2(t) := \dim_H K_2(t)$ is a Devil's staircase on $[0,1)$, that is (i) η_2 is decreasing and continuous on $[0,1)$; (ii) η_2 is locally constant almost everywhere on $[0,1)$; and (iii) η_2 is not constant on $[0,1)$. Here and throughout the paper $\dim_H$ denotes the Hausdorff dimension. Moreover, Urbański investigated the bifurcation sets

$$\mathscr{E}_2 := \left\{t \in [0,1) : K_2(t') \neq K_2(t)\ \forall\, t' > t\right\} \quad \text{and} \quad \mathscr{B}_2 := \left\{t \in [0,1) : \eta_2(t') \neq \eta_2(t)\ \forall\, t' > t\right\}.$$

Clearly, $\mathscr{B}_2 \subseteq \mathscr{E}_2$. It can be easily deduced from the proof of Theorem 1 in [14] that $\mathscr{B}_2 = \mathscr{E}_2$, and its topological closure $\overline{\mathscr{B}_2}$ is a *Cantor set*, i.e., a non-empty compact set that has neither

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isolated nor interior points. Furthermore, the following local dimension property was shown to hold: $\lim_{r\to 0}\dim_H(\mathscr{E}_2\cap(t-r,t+r))=\eta_2(t)$ for all $t\in\mathscr{E}_2$. Recently, Carminati and Tiozzo in [1] showed that the local Hölder exponent of the dimension function η_2 at any $t\in\mathscr{E}_2$ equals $\eta_2(t)$.

Inspired by the work of Urbański [14, 15], Kalle et al. in [6] considered the analogous problem for the β-transformation with a hole $[0,t)$. More precisely, for $t\in[0,1)$ they investigated the survivor set

$$K_\beta(t):=\left\{x\in[0,1):T_\beta^n(x)\ge t\ \forall\, n\ge 0\right\},$$

and showed that the dimension function $t\mapsto\dim_H K_\beta(t)$ is also a Devil's staircase on $[0,1)$. Furthermore, they characterized the *set-valued bifurcation set*

$$\mathscr{E}_\beta:=\left\{t\in[0,1):K_\beta(t')\ne K_\beta(t)\ \forall\, t'>t\right\},$$

and proved that $\mathscr{E}_\beta$ is a Lebesgue null set of full Hausdorff dimension for any $\beta\in(1,2)$. Note that the bifurcation set $\mathscr{E}_\beta$ defined here coincides with the set

$$E_\beta^+:=\left\{t\in[0,1):T_\beta^n(t)\ge t\ \forall n\ge 0\right\}$$

in [6]. Interestingly, they showed that $\mathscr{E}_\beta$ contains infinitely many isolated points for Lebesgue almost every $\beta\in(1,2)$. This is in contrast to the case where $\beta=2$ and $\mathscr{E}_2$ has no isolated points. For β-transformation with an arbitrary hole we refer to the work of Clark [2]. We also mention that the study of bifurcation sets plays an important role in one-dimensional dynamics (cf. [5]).

Since for each $\beta\in(1,2)$ the dimension function $\eta_\beta:t\mapsto\dim_H K_\beta(t)$ is a Devil's staircase, it is natural to consider the *dimension bifurcation set*

$$\mathscr{B}_\beta:=\left\{t\in[0,1):\eta_\beta(t')\ne\eta_\beta(t)\ \forall\, t'>t\right\}.$$

This set records those t for which the dimension function η_β has a 'change' within any right neighborhood. Since η_β is continuous, $\mathscr{B}_\beta$ cannot have isolated points. On the other hand, the set-valued bifurcation set $\mathscr{E}_\beta$ contains (infinitely many) isolated points for Lebesgue almost every $\beta\in(1,2)$. So in general we cannot expect the coincidence of the two bifurcation sets $\mathscr{B}_\beta$ and $\mathscr{E}_\beta$. That being said, in this paper we show that if β is a multinacci number, i.e., the unique root in $(1,2)$ of the equation

$$x^{m+1}=x^m+x^{m-1}+\cdots+x+1$$

for some $m\in\mathbb{N}$, then the two bifurcation sets indeed coincide. Importantly, if β is a multinacci number then its quasi-greedy expansion of 1 is of the form $((1^m0)^\infty)$. This property will be useful in our analysis. Here for $\beta\in(1,2]$ the *quasi-greedy* β-expansion $\delta(\beta)=\delta_1(\beta)\delta_2(\beta)\dots$ of 1 is the lexicographically largest zero-one sequence not ending with an infinite string of zeros and satisfying $1=\sum_{i=1}^\infty\delta_i(\beta)/\beta^i$ (see Section 2 for more details). Furthermore, throughout the paper we will use lexicographical order '$\prec,\preccurlyeq,\succ$' and '$\succcurlyeq$' between sequences and words.

When $\beta\in(1,2)$ is a multinacci number, the following result for the set-valued bifurcation set $\mathscr{E}_\beta$ was established in [6, Theorems C and D]. We record it here for later use.

Theorem 1.1 ([6]). *Let $\beta\in(1,2]$ be a multinacci number. Then the topological closure $\overline{\mathscr{E}_\beta}$ is a Cantor set. Furthermore,* $\max\overline{\mathscr{E}_\beta}=1-1/\beta$.

In order to give a complete description of the dimension bifurcation set $\mathscr{B}_\beta$ we introduce a class of basic intervals.

Definition 1.2. Let $\beta \in (1,2]$. A word $s_1 \dots s_m$ is called *β-Lyndon* if

$$s_{i+1} \dots s_m \succ s_1 \dots s_{m-i} \quad \forall\, 1 \le i < m, \quad \text{and} \quad \sigma^n((s_1 \dots s_m)^\infty) \prec \delta(\beta) \quad \forall\, n \ge 0.$$

Accordingly, an interval $[t_L, t_R) \subset [0,1)$ is called a *β-Lyndon interval* if there exists a β-Lyndon word $s_1 \dots s_m$ such that

$$t_L = \sum_{i=1}^{m} \frac{s_i}{\beta^i} \quad \text{and} \quad t_R = \frac{\beta^m}{\beta^m - 1} \cdot t_L.$$

Here we mention that in Definition 1.2 the left endpoint $t_L = (s_1 \dots s_m 0^\infty)_\beta$ has a finite β-expansion and the right endpoint $t_R = ((s_1 \dots s_m)^\infty)_\beta$ has a periodic β-expansion, see Section 2 for more explanations.

We will show that the β-Lyndon intervals are pairwise disjoint for all $\beta \in (1,2]$, and when β is multinacci they cover the interval $[0, 1 - 1/\beta)$ up to a Lebesgue null set. The latter statement can be seen as a consequence of our main result for the coincidence of the two bifurcation sets, which we state below.

Theorem 1. *Let $\beta \in (1,2]$ be a multinacci number. Then*

$$\begin{aligned} \mathscr{B}_\beta = \mathscr{E}_\beta &= \left[0, 1 - \frac{1}{\beta}\right) \setminus \bigcup [t_L, t_R) \\ &= \left\{ t \in [0,1) : \lim_{r \to 0} \dim_H(\mathscr{B}_\beta \cap (t, t+r)) = \dim_H K_\beta(t) > 0 \right\}, \end{aligned}$$

where the union is taken over all pairwise disjoint β-Lyndon intervals.

By Theorem 1 it follows that the topological closure $[t_L, t_R]$ of each β-Lyndon interval is indeed a maximal interval where the dimension function η_β is constant. As a corollary of Theorem 1 we confirm a conjecture of [6] for β a multinacci number.

Corollary 2. *If $\beta \in (1,2]$ is a multinacci number, then*

$$\dim_H(\mathscr{E}_\beta \cap [t,1]) = \dim_H K_\beta(t) \quad \forall\, t \in [0,1).$$

The rest of the paper is organized as follows. In Section 2 we recall some properties from symbolic dynamics and the dimension formula for the survivor set $K_\beta(t)$. The proof of Theorem 1 and Corollary 2 will be given in Section 3. In Section 4 we make some remarks and point out that the method of proof for Theorem 1 can be applied to some other special values of $\beta \in (1,2]$.

2. Preliminaries and β-Lyndon intervals

Given $\beta \in (1,2]$, for each $x \in I_\beta := [0, 1/(\beta - 1)]$ there exists a sequence $(d_i) = d_1 d_2 \dots \in \{0,1\}^{\mathbb{N}}$ such that

$$x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} =: ((d_i))_\beta.$$

The sequence (d_i) is called a *β-expansion* of x. Sidorov [13] showed that for $\beta \in (1,2)$ Lebesgue almost every $x \in I_\beta$ has a continuum of β-expansions. This is rather different from the case when $\beta = 2$ where every number in $I_2 = [0,1]$ has a unique dyadic expansion except for countably many points that have precisely two expansions. Given $x \in I_\beta$, among all of its β-expansions let

$$b(x,\beta) = (b_i(x,\beta))$$

be the *greedy* β-expansion of x, i.e., the lexicographically largest β-expansion of x. Such a sequence always exists and is generated by the orbit of x under the map T_β. Let σ be the *left-shift* on $\{0,1\}^{\mathbb{N}}$ defined by $\sigma((c_i)) = (c_{i+1})$. Then $b(T_\beta(x),\beta) = \sigma(b(x,\beta))$ for any $x \in [0,1)$. Similarly, for $x \in (0, 1/(\beta-1)]$ let

$$a(x,\beta) = (a_i(x,\beta))$$

be the *quasi-greedy* β-expansion of x (cf. [3]), which is the lexicographically largest β-expansion of x not ending with 0^∞. Here for a word $\mathbf{c}$ we denote by $\mathbf{c}^\infty := \mathbf{c}\mathbf{c}\cdots$ the periodic sequence with periodic block $\mathbf{c}$. Throughout the paper we will use the lexicographic order between sequences and words in the usual way. For example, for two sequences $(c_i), (d_i) \in \{0,1\}^{\mathbb{N}}$ we write $(c_i) \prec (d_i)$ if $c_1 < d_1$, or there exists $n > 1$ such that $c_1 \ldots c_{n-1} = d_1 \ldots d_{n-1}$ and $c_n < d_n$. Furthermore, for two words $\mathbf{c}, \mathbf{d}$ we say $\mathbf{c} \prec \mathbf{d}$ if $\mathbf{c}0^\infty \prec \mathbf{d}0^\infty$.

For $\beta \in (1,2]$ recall that

$$\delta(\beta) = \delta_1(\beta)\delta_2(\beta)\ldots$$

is the quasi-greedy β-expansion of 1, i.e., $\delta(\beta) = a(1,\beta)$. The following lexicographic characterizations of $\delta(\beta)$ and the greedy expansion $b(x,\beta)$ are essentially due to Parry [9] (see also [4]).

Lemma 2.1. (i) *The map $\beta \mapsto \delta(\beta)$ is a strictly increasing bijection from $(1,2]$ onto the set of sequences $(\delta_i) \in \{0,1\}^{\mathbb{N}}$ not ending with 0^∞ and satisfying*

$$\sigma^n((\delta_i)) \preccurlyeq (\delta_i) \quad \forall\ n \geq 0.$$

(ii) *Let $\beta \in (1,2]$. Then the map $x \mapsto b(x,\beta)$ is a strictly increasing bijection from $[0,1)$ onto the set of all sequences $(b_i) \in \{0,1\}^{\mathbb{N}}$ satisfying*

$$\sigma^n((b_i)) \prec \delta(\beta) \quad \forall\ n \geq 0.$$

(iii) *For any $\beta \in (1,2)$ the sequence $b(1,\beta) = (b_i)$ satisfies $\sigma^n((b_i)) \prec \delta(\beta)\ \forall\ n \geq 1$.*

For $\beta \in (1,2]$ let $[t_L, t_R)$ be a β-Lyndon interval generated by a β-Lyndon word $s_1 \ldots s_m$. Then by Definition 1.2 and Lemma 2.1 (ii) it follows that

$$b(t_L,\beta) = s_1 \ldots s_m 0^\infty \quad \text{and} \quad b(t_R,\beta) = (s_1 \ldots s_m)^\infty.$$

Lemma 2.2. *For any $\beta \in (1,2]$ the β-Lyndon intervals are pairwise disjoint.*

Proof. Let $[t_L, t_R)$ and $[t'_L, t'_R)$ be two β-Lyndon intervals generated by the β-Lyndon words $s_1 \ldots s_p$ and $s'_1 \ldots s'_q$, respectively. Suppose on the contrary that $[t_L, t_R) \cap [t'_L, t'_R) \neq \emptyset$. Without loss of generality we assume $t_L < t'_L < t_R$. Then by Definition 1.2 and Lemma 2.1(ii) it follows that

$$s_1 \ldots s_p 0^\infty \prec s'_1 \ldots s'_q 0^\infty \prec (s_1 \ldots s_p)^\infty.$$

This implies

$$q > p, \quad s'_1 \ldots s'_p = s_1 \ldots s_p \quad \text{and} \quad s'_{p+1} \ldots s'_q 0^\infty \prec (s_1 \ldots s_p)^\infty.$$

Write $q = Np + r$ with $N \geq 1$ and $0 < r \leq p$. So, either there exists $1 \leq k < N$ such that

$$s'_{p+1} \ldots s'_{kp} = (s_1 \ldots s_p)^{k-1} \quad \text{and} \quad s'_{kp+1} \ldots s'_{(k+1)p} \prec s_1 \ldots s_p,$$

or

$$s'_{p+1}\dots s'_{Np}=(s_1\dots s_p)^{N-1}\quad\text{and}\quad s'_{Np+1}\dots s'_q\preccurlyeq s_1\dots s_{q-Np}.$$

Using $s'_1\dots s'_p=s_1\dots s_p$ we conclude in both cases that

$$s'_{j+1}\dots s'_q\preccurlyeq s'_1\dots s'_{q-j}\quad\text{for some } j\in\{p,p+1,\dots,q-1\}.$$

This is not possible by the definition of a β-Lyndon word. □

To describe the Hausdorff dimension of the survivor set

$$K_\beta(t)=\left\{x\in[0,1):T_\beta^n(x)\ge t\ \forall n\ge 0\right\},$$

we recall from [8, Chapter 4] the definition of topological entropy for a symbolic set. For a set $X\subset\{0,1\}^{\mathbb{N}}$, its *topological entropy* is defined to be

$$h(X)=\liminf_{n\to\infty}\frac{\log\#B_n(X)}{n},$$

where $B_n(X)$ is the set of all length n prefixes of sequences from X.

The following characterization of the set-valued bifurcation set $\mathscr{E}_\beta$ was implicitly given in [14] (see also [6, Proposition 2.3]). Furthermore, the Hausdorff dimension of $K_\beta(t)$ was implicitly given by Raith in [10], and was recently explicitly presented in [6, Equation (2.6)].

Proposition 2.3. (i) *Let $\beta\in(1,2]$. Then*

$$\mathscr{E}_\beta=\left\{t\in[0,1):T_\beta^n(t)\ge t\ \forall n\ge 0\right\}.$$

(ii) *Let $\beta\in(1,2]$ and $t\in[0,1)$. Then the Hausdorff dimension of $K_\beta(t)$ is given by*

$$\dim_H K_\beta(t)=\frac{h(\widetilde{K}_\beta(t))}{\log\beta},$$

where $\widetilde{K}_\beta(t):=\left\{(x_i)\in\{0,1\}^{\mathbb{N}}:b(t,\beta)\preccurlyeq\sigma^n((x_i))\preccurlyeq\delta(\beta)\ \forall n\ge 0\right\}$. Furthermore, the dimension function $\eta_\beta:t\mapsto\dim_H K_\beta(t)$ is a Devil's staircase, i.e., η_β is a non-constant, decreasing and continuous function which is locally constant almost everywhere in $[0,1)$.

3. Proof of Theorem 1

In this section we will prove Theorem 1. First we show that the dimension bifurcation set $\mathscr{B}_\beta$ coincides with the set-valued bifurcation set $\mathscr{E}_\beta$, we then derive a complete characterization of these sets via the β-Lyndon intervals. The proof heavily relies upon the transitivity of the symbolic survivor set $\widetilde{K}_\beta(t)$ (see Lemma 3.2 below).

Proposition 3.1. *Let $\beta\in(1,2)$ be a multinacci number. Then*

$$\mathscr{B}_\beta=\mathscr{E}_\beta=\left[0,1-\frac{1}{\beta}\right)\setminus\bigcup[t_L,t_R),$$

where the union is taken over all β-Lyndon intervals.

Observe by Lemma 2.2 that the β-Lyndon intervals are pairwise disjoint. In fact the closed β-Lyndon intervals $\{[t_L,t_R]\}$ are also pairwise disjoint. So by Proposition 3.1 it follows that each closed β-Lyndon interval is a maximal interval where the dimension function η_β is constant.

The proof of Proposition 3.1 will be split into several lemmas. We fix a multinacci number $\beta\in(1,2)$ with $\delta(\beta)=(1^m0)^\infty$ for some $m\ge 1$. In view of Proposition 2.3 it is necessary to investigate the symbolic survivor set

$$\widetilde{K}_\beta(t)=\left\{(x_i)\in\{0,1\}^{\mathbb{N}}: b(t,\beta)\preccurlyeq\sigma^n((x_i))\preccurlyeq\delta(\beta)\ \forall n\ge 0\right\}.$$

Lemma 3.2. *Let $\beta\in(1,2)$ with $\delta(\beta)=(1^m0)^\infty$, and let $[t_L,t_R)\subset[0,1-1/\beta)$ be a β-Lyndon interval. Then the set-valued map $t\mapsto\widetilde{K}_\beta(t)$ is constant on $[t_L,t_R]$, and the set $\widetilde{K}_\beta(t_R)$ is a transitive subshift of finite type.*

Proof. Suppose $[t_L,t_R)$ is a β-Lyndon interval generated by $s_1\dots s_p$. First we claim that

$$\sigma^n((x_i))\succcurlyeq s_1\dots s_p0^\infty\ \forall n\ge 0\quad\Longleftrightarrow\quad\sigma^n((x_i))\succcurlyeq(s_1\dots s_p)^\infty\ \forall n\ge 0. \tag{3.1}$$

Since $(s_1\dots s_p)^\infty\succ s_1\dots s_p0^\infty$, the implication '$\Longleftarrow$' in (3.1) is obvious. For the reverse implication we assume $\sigma^n((x_i))\prec(s_1\dots s_p)^\infty$ for some $n\ge0$. Then there exists $\ell\ge 0$ such that

$$x_{n+1}\dots x_{n+\ell p}=(s_1\dots s_p)^\ell\quad\text{and}\quad x_{n+\ell p+1}\dots x_{n+(\ell+1)p}\prec s_1\dots s_p.$$

This yields $\sigma^{n+\ell p}((x_i))\prec s_1\dots s_p0^\infty$, completing the proof of '$\Longrightarrow$' in (3.1).

Take $t\in[t_L,t_R]$. Then by Lemma 2.1(ii) it follows that

$$\widetilde{K}_\beta(t_R)\subseteq\widetilde{K}_\beta(t)\subseteq\widetilde{K}_\beta(t_L).$$

Observe that $\delta(\beta)=(1^m0)^\infty$ for some $m\in\mathbb{N}$. Then

$$\begin{aligned}\widetilde{K}_\beta(t_L)&=\{(x_i): s_1\dots s_p0^\infty\preccurlyeq\sigma^n((x_i))\preccurlyeq(1^m0)^\infty\ \forall n\ge0\}\\&=\{(x_i):(s_1\dots s_p)^\infty\preccurlyeq\sigma^n((x_i))\preccurlyeq(1^m0)^\infty\ \forall n\ge 0\}=\widetilde{K}_\beta(t_R).\end{aligned}\tag{3.2}$$

So, the set-valued map $t\mapsto\widetilde{K}_\beta(t)$ is constant on $[t_L,t_R]$. Furthermore, $\widetilde{K}_\beta(t_R)$ is a subshift of finite type with the set of forbidden blocks given by

$$\mathscr{F}=\left\{c_1\dots c_k\in\{0,1\}^k: c_1\dots c_k0^\infty\prec s_1\dots s_p0^\infty\ \text{or}\ c_1\dots c_k0^\infty\succ(1^m0)^\infty\right\},$$

where $k=\max\{p,m+1\}$. It remains to prove the transitivity of $\widetilde{K}_\beta(t_R)$.

Since $[t_L,t_R)\subset[0,1-\frac{1}{\beta})$, by Lemma 2.1 (ii) it follows that $b(t_R,\beta)\prec b(1-\frac{1}{\beta},\beta)$, which gives

$$(s_1\dots s_p)^\infty\prec 01^m0^\infty. \tag{3.3}$$

Arbitrarily fix an admissible word $\varepsilon=\varepsilon_1\dots\varepsilon_k$ and an admissible sequence $\gamma=\gamma_1\gamma_2\dots$ in $\widetilde{K}_\beta(t_R)$. We will construct a word ν such that $\varepsilon\nu\gamma\in\widetilde{K}_\beta(t_R)$. Observe that $\sigma^n((s_1\dots s_p)^\infty)\prec(1^m0)^\infty$ for all $n\ge0$. Thus, there exists a large integer N such that

$$\sigma^n((s_1\dots s_p)^\infty)\prec(1^m0)^N0^\infty\quad\text{for all}\quad n\ge 0. \tag{3.4}$$

Denote by $(\delta_i):=\delta(\beta)=(1^m0)^\infty$. Note that $\varepsilon_{i+1}\dots\varepsilon_k\preccurlyeq\delta_1\dots\delta_{k-i}$ for all $0\le i<k$. Let $i_0\in\{0,1,\dots,k-1\}$ be the smallest index such that

$$\varepsilon_{i_0+1}\dots\varepsilon_k=\delta_1\dots\delta_{k-i_0}.$$

If such an index i_0 does not exist, then we put $i_0=k$. In either case there exists a word μ such that $\varepsilon\mu=\varepsilon_1\dots\varepsilon_{i_0}(1^m0)^N$. Since $\gamma\preccurlyeq(1^m0)^\infty$, there exists $q\in\{0,1,\dots,m\}$ such that

γ begins with $\gamma_1 \dots \gamma_{q+1} = 1^q 0$. We emphasize here that if $q = 0$ then γ begins with digit 0. Now we claim that

$$\varepsilon\mu 1^{m-q}\gamma = \varepsilon_1 \dots \varepsilon_{i_0}(1^m 0)^{N+1}\gamma_{q+2}\gamma_{q+3}\dots \in \widetilde{K}_\beta(t_R),$$

or equivalently,

$$(s_1 \dots s_p)^\infty \preccurlyeq \sigma^n(\varepsilon\mu 1^{m-q}\gamma) \preccurlyeq (1^m 0)^\infty \quad \text{for all } n \geq 0. \tag{3.5}$$

First we prove the second inequality in (3.5). By the definition of i_0 it follows that $\sigma^n(\varepsilon\mu 1^{m-q}\gamma) \prec \delta(\beta) = (1^m 0)^\infty$ holds for all $0 \leq n < i_0$. Furthermore, since $\gamma \in \widetilde{K}_\beta(t_R)$, the second inequality in (3.5) also holds for $n \geq |\varepsilon| + |\mu| + m - q$. Here for a word $\mathbf{c}$ we denote its length by $|\mathbf{c}|$. For the remaining n we observe that $\sigma^{i_0}(\varepsilon\mu 1^{m-q}\gamma) = (1^m 0)^{N+1}\gamma_{q+2}\gamma_{q+3}\dots$ and $\gamma_{q+2}\gamma_{q+3}\dots \in \widetilde{K}_\beta(t_R)$. So it is easy to verify that

$$\sigma^n(\varepsilon\mu 1^{m-q}\gamma) \preccurlyeq (1^m 0)^\infty \quad \text{for all } i_0 \leq n < |\varepsilon| + |\mu| + m - q.$$

This proves the second inequality in (3.5).

For the first inequality in (3.5) we observe that $\varepsilon\mu 1^{m-q}\gamma = \varepsilon_1 \dots \varepsilon_{i_0}(1^m 0)^N 1^m \gamma_{q+1}\gamma_{q+2}\dots$ and $\gamma_{q+1}\gamma_{q+2}\dots \in \widetilde{K}_\beta(t_R)$. Then by (3.3) it follows that

$$\sigma^n(\varepsilon\mu 1^{m-q}\gamma) \succcurlyeq (s_1 \dots s_p)^\infty \quad \text{for all } n \geq i_0.$$

If $i_0 = 0$, then we are done. Otherwise, we take $0 \leq n < i_0$. Since $\varepsilon_1 \dots \varepsilon_{i_0}$ is an admissible word in $\widetilde{K}_\beta(t_R)$, we have

$$\varepsilon_{n+1} \dots \varepsilon_{i_0} \succcurlyeq t_1 \dots t_{i_0 - n},$$

where $(t_i) := (s_1 \dots s_p)^\infty$. The first inequality in (3.5) now holds by (3.4), which tells us that

$$(1^m 0)^N 1^m \gamma_{q+1}\gamma_{q+2}\dots \succ t_{i_0-n+1}t_{i_0-n+2}\dots.$$

This completes the proof of our claim.

Since ε and γ are chosen arbitrarily, it follows that $\widetilde{K}_\beta(t_R)$ is transitive. $\square$

Remark 3.3.
- The fact that $\tilde{K}_\beta(t_R)$ is a subshift of finite type can also be deduced from [7].
- The proof of Lemma 3.2 can be adjusted to prove the more general case with $\beta > 2$ with $\delta(\beta) = (M^m k)^\infty$, where $M = \lceil \beta \rceil - 1$ and $k \in \{0, 1, \dots, M-1\}$. The transitivity property of $\tilde{K}_\beta(t_R)$ holds only for t_R sufficiently close to 0.

To prove the coincidence of $\mathscr{B}_\beta$ and $\mathscr{E}_\beta$ we still need the following inequalities.

Lemma 3.4. *Let $(t_1 \dots t_N)^\infty \in \{0,1\}^{\mathbb{N}}$ be a periodic sequence with period $N \geq 2$. If*

$$\sigma^n((t_1 \dots t_N)^\infty) \succcurlyeq (t_1 \dots t_N)^\infty \quad \forall\, n \geq 0,$$

then

$$t_{j+1} \dots t_N \succ t_1 \dots t_{N-j} \quad \forall\, 1 \leq j < N.$$

Proof. Note that $N \geq 2$ is the period of $(t_1 \dots t_N)^\infty$, and

$$\sigma^n((t_1 \dots t_N)^\infty) \succcurlyeq (t_1 \dots t_N)^\infty \quad \forall\, n \geq 0. \tag{3.6}$$

Then $t_1 = 0$ and $t_N = 1$. Taking the reflection on both sides of (3.6) it follows that

$$\sigma^n((\overline{t_1 \dots t_N})^\infty) \preccurlyeq (\overline{t_1 \dots t_N})^\infty \quad \text{for all } n \geq 0.$$

Here for a word $c_1 \dots c_k \in \{0,1\}^k$ its reflection is defined by $\overline{c_1 \dots c_k} := (1-c_1)(1-c_2)\dots(1-c_k)$. By Lemma 2.1(i) it follows that $(\overline{t_1 \dots t_N})^\infty$ is the quasi-greedy expansion of 1 for some base $\beta' \in (1,2]$, i.e., $\delta(\beta') = (\overline{t_1 \dots t_N})^\infty$. Since N is the period of the sequence $\delta(\beta')$, the greedy β'-expansion of 1 is given by

$$b(1,\beta') = \overline{t_1 \dots t_{N-1}}\,10^\infty.$$

So, by Lemma 2.1 (iii) it follows that

$$\overline{t_{j+1}\dots t_N} \prec \overline{t_{j+1}\dots t_{N-1}}\,1 \preccurlyeq \overline{t_1 \dots t_{N-j}} \quad \text{for all } 1 \le j < N.$$

Then the lemma follows by taking the reflection in the above equation. □

Now we prove the coincidence of the two bifurcation sets.

Lemma 3.5. *Let $\beta \in (1,2)$ with $\delta(\beta) = (1^m0)^\infty$. Then $\mathscr{E}_\beta = \mathscr{B}_\beta$.*

Proof. By the definition of the two bifurcation sets it is easy to see that $\mathscr{B}_\beta \subset \mathscr{E}_\beta$. So in the following we prove $\mathscr{E}_\beta \subset \mathscr{B}_\beta$.

Let $t \in \mathscr{E}_\beta$ with its greedy β-expansion $b(t,\beta) = (t_i)$. Then by Theorem 1.1 we have $t \le 1 - 1/\beta < 1/\beta$. This gives $t_1 = 0$. By Lemmas 2.1 (ii) and Proposition 2.3 (i) it follows that

$$\sigma^n((t_i)) \succcurlyeq (t_i) \quad \text{for all } n \ge 0.$$

Let $N \ge 1$ be the smallest index such that $\sigma^N((t_i)) = (t_i)$. If such an integer N does not exist, then we set $N = \infty$. In the following we will prove $t \in \mathscr{B}_\beta$ by considering the following two cases: (I) $N < \infty$; and (II) $N = \infty$.

Case (I). $N < \infty$. We claim that $t_1 \dots t_N$ is a β-Lyndon word. If $N = 1$, then $(t_i) = t_1^\infty = 0^\infty$. It is easy to check that $t_1 = 0$ is a β-Lyndon word. In the following we assume $N \ge 2$. Since $\sigma^N((t_i)) = (t_i)$, we have $(t_i) = (t_1 \dots t_N)^\infty$. Note that (t_i) is the greedy β-expansion of t. Then by Lemma 2.1 (ii) it follows that

$$\sigma^n((t_1 \dots t_N)^\infty) \prec \delta(\beta) \quad \text{for all } n \ge 0.$$

Note that $\sigma^n((t_1 \dots t_N)^\infty) \succcurlyeq (t_1 \dots t_N)^\infty$. Then by Lemma 3.4 and the definition of N, it follows that

$$t_{j+1} \dots t_N \succ t_1 \dots t_{N-j} \quad \text{for all } 1 \le j < N.$$

So by Definition 1.2 we establish the claim.

Hence, $t = ((t_1 \dots t_N)^\infty)_\beta = t_R$ is the right endpoint of a β-Lyndon interval generated by $t_1 \dots t_N$. By Lemma 3.2 it follows that $\widetilde{K}_\beta(t)$ is a transitive subshift of finite type. Observe that for any $t' > t$ we have

$$\widetilde{K}_\beta(t') \subset \widetilde{K}_\beta(t) \quad \text{and} \quad (t_1 \dots t_N)^\infty \in \widetilde{K}_\beta(t) \setminus \widetilde{K}_\beta(t').$$

Recall by [8, Corollary 4.4.9] that for any transitive subshift of finite type, any proper subshift has strictly smaller topological entropy. Therefore,

$$h(\widetilde{K}_\beta(t')) < h(\widetilde{K}_\beta(t)) \quad \text{for any} \quad t' > t.$$

By Proposition 2.3 (ii) this yields $\eta_\beta(t') < \eta_\beta(t)$ for any $t' > t$. So $t \in \mathscr{B}_\beta$.

Case (II). $N = \infty$. Then $\sigma^n((t_i)) \succ (t_i)$ for all $n \ge 1$. So (t_i) is not periodic. Observe that (t_i) begins with digit 0, and

$$\sigma^n((t_i)) \prec (1^m0)^\infty \quad \text{for all } n \ge 0.$$

So there exists a subsequence (m_k) of positive integers such that for any $k \geq 1$ we have $t_{m_k} = 0$, and the word $t_1 \dots t_{m_k}^+ := t_1 \dots t_{m_k - 1}1$ does not contain $m+1$ consecutive ones. Then by noting $t_1 = 0$ it follows that

$$\sigma^n((t_1 \dots t_{m_k}^+)^\infty) \prec (1^m 0)^\infty \quad \forall\, n \geq 0.$$

Since $\sigma^n((t_i)) \succcurlyeq (t_i)$ for all $n \geq 0$, by Definition 1.2 it follows that $t_1 \dots t_{m_k}^+$ is a β-Lyndon word for any $k \geq 1$. Let $s_k := ((t_1 \dots t_{m_k}^+)^\infty)_\beta$. Then s_k is the right endpoint of a β-Lyndon interval generated by $t_1 \dots t_{m_k}^+$. Furthermore, s_k strictly decreases to $t = ((t_i))_\beta$ as $k \to \infty$.

So, for any $t' > t$ we can find k such that $s_k \in (t, t')$. By the same arguments as in the proof of Case (I) for s_k we conclude that

$$\eta_\beta(t') < \eta_\beta(s_k) \leq \eta_\beta(t).$$

So $t \in \mathscr{B}_\beta$, completing the proof. □

Finally, we describe the bifurcation sets via the β-Lyndon intervals.

Lemma 3.6. *Let $\beta \in (1, 2]$ with $\delta(\beta) = (1^m 0)^\infty$. Then*

$$\left[0, 1 - \frac{1}{\beta}\right) \setminus \bigcup [t_L, t_R) \subset \mathscr{E}_\beta.$$

Proof. Take $t \in [0, 1 - 1/\beta) \setminus \mathscr{E}_\beta$ with its greedy β-expansion (t_i). Then $t_1 = 0$. Since $t \notin \mathscr{E}_\beta$, by Proposition 2.3 (i) there exists a smallest positive integer N such that $T_\beta^N(t) < t$, which implies

$$t_{N+1} t_{N+2} \dots \prec (t_i). \tag{3.7}$$

We claim that $t_1 \dots t_N$ is a β-Lyndon word. Clearly, if $N = 1$ then $t_1 = 0$ is a β-Lyndon word. In the following we assume $N \geq 2$. By Definition 1.2 it suffices to prove

$$t_{j+1} \dots t_N \succ t_1 \dots t_{N-j} \quad \text{for all } 1 \leq j < N, \tag{3.8}$$

and

$$\sigma^n((t_1 \dots t_N)^\infty) \prec (1^m 0)^\infty \quad \text{for all } n \geq 0. \tag{3.9}$$

First we prove (3.8). By the definition of N in (3.7) it follows that

$$t_{j+1} t_{j+2} \dots \succcurlyeq (t_i) \quad \text{for all } 1 \leq j < N, \tag{3.10}$$

which implies $t_{j+1} \dots t_N \succcurlyeq t_1 \dots t_{N-j}$ for all $1 \leq j < N$. Suppose $t_{j+1} \dots t_N = t_1 \dots t_{N-j}$ for some $j \in \{1, 2, \dots, N-1\}$. Applying (3.7) and then (3.10) it follows that

$$t_{j+1} t_{j+2} \dots = t_1 \dots t_{N-j} t_{N+1} t_{N+2} \dots \prec t_1 \dots t_{N-j} t_1 t_2 \dots \preccurlyeq (t_i),$$

leading to a contradiction with the minimality of N. This proves (3.8).

To prove (3.9) we observe that $\delta(\beta) = (1^m 0)^\infty$ and (t_i) is the greedy β-expansion of t. Then by Lemma 2.1 (ii) it follows that $t_1 \dots t_N$ cannot contain $m+1$ consecutive ones. Since $t_1 = 0$, we have

$$\sigma^n((t_1 \dots t_N)^\infty) \preccurlyeq (1^m 0)^\infty \quad \text{for all } n \geq 0.$$

So to prove (3.9) it remains to prove that $\sigma^n((t_1 \dots t_N)^\infty) \neq (1^m 0)^\infty$ for any $n \geq 0$. Suppose the equality $\sigma^n((t_1 \dots t_N)^\infty) = (1^m 0)^\infty$ holds for some $n \geq 0$. Then by using $t_1 = 0$ it follows that

$$t_1 \dots t_{m+1} = 01^m.$$

This implies $b(t,\beta) = (t_i) \succcurlyeq 01^m0^\infty = b(1-1/\beta,\beta)$. By Lemma 2.1 (ii) we have $t \ge 1-1/\beta$, leading to a contradiction. This establishes (3.9).

By the claim there exists a β-Lyndon interval $[t_L, t_R)$ generated by $t_1 \dots t_N$. Furthermore, by (3.7) it follows that

$$\begin{aligned}(t_i) = t_1 \dots t_N t_{N+1} t_{N+2} \dots &\prec t_1 \dots t_N t_1 t_2 \dots = (t_1 \dots t_N)^2 t_{N+1} t_{N+2} \dots \\ &\prec (t_1 \dots t_N)^2 t_1 t_2 \dots = (t_1 \dots t_N)^3 t_{N+1} t_{N+2} \dots \\ &\dots \\ &\preccurlyeq (t_1 \dots t_N)^\infty.\end{aligned}$$

Therefore, $t_1 \dots t_N 0^\infty \preccurlyeq (t_i) \prec (t_1 \dots t_N)^\infty$, which gives $t \in [t_L, t_R)$ by Lemma 2.1 (ii). This completes the proof. □

Proof of Proposition 3.1. By Lemmas 3.5 and 3.6 it suffices to prove

$$\mathscr{B}_\beta \subset \left[0, 1 - \frac{1}{\beta}\right) \setminus \bigcup [t_L, t_R).$$

Note by Lemma 3.5 and Theorem 1.1 that $\mathscr{B}_\beta = \mathscr{E}_\beta \subset [0, 1-1/\beta]$. In fact we have $\mathscr{E}_\beta \subset [0, 1-1/\beta)$. Observe that $b(1-1/\beta,\beta) = 01^m0^\infty$. Then $T_\beta^{m+1}(1-1/\beta) < 1-1/\beta$. By Proposition 2.3 (i) this implies $1-1/\beta \notin \mathscr{E}_\beta$. Hence, $\mathscr{E}_\beta \subset [0, 1-1/\beta)$.

In the following it remains to prove $\mathscr{B}_\beta \cap \bigcup [t_L, t_R) = \emptyset$. Take a β-Lyndon interval $[t_L, t_R)$. If $t \in [t_L, t_R)$, then by (3.2) it follows that

$$\widetilde{K}_\beta(t) = \widetilde{K}_\beta(t_L) = \widetilde{K}_\beta(t_R),$$

which gives $\eta_\beta(t') = \eta_\beta(t) = \eta_\beta(t_L)$ for all $t' \in (t, t_R)$. So, $t \notin \mathscr{B}_\beta$. □

As a consequence of Proposition 3.1 and Theorem 1.1 it follows that for $\beta \in (1,2]$ a multinacci number the β-Lyndon intervals cover $[0, 1-1/\beta)$ up to a Lebesgue null set.

Corollary 3.7. *Let $\beta \in (1,2]$ be a multinacci number.*

- (i) *The union of all β-Lyndon intervals covers $[0, 1-1/\beta)$ up to a Lebesgue null set. Furthermore, for any $t \in \mathscr{B}_\beta$ and any $r > 0$ the interval $(t, t+r)$ contains infinitely many β-Lyndon intervals.*
- (ii) *$\eta_\beta(t) > 0$ if and only if $t < 1-1/\beta$.*

Proof. Note that $\mathscr{E}_\beta$ is a Lebesgue null set which, by Theorem 1.1, has no isolated points. Then (i) follows from Proposition 3.1 which tells us that $\bigcup [t_L, t_R) = [0, 1-1/\beta) \setminus \mathscr{E}_\beta$. For (ii) it can be deduced from Proposition 3.1 and Theorem 1.1 that $\sup \mathscr{B}_\beta = 1-1/\beta$ and $1-1/\beta \notin \mathscr{B}_\beta$. □

Now we turn to investigate the local dimension of the bifurcation set $\mathscr{B}_\beta$.

Lemma 3.8. *Let $\beta \in (1,2]$ with $\delta(\beta) = (1^m0)^\infty$. Then*

$$\lim_{r\to 0} \dim_H(\mathscr{B}_\beta \cap (t, t+r)) = \dim_H K_\beta(t) > 0 \quad \forall\ t \in \mathscr{B}_\beta.$$

Proof. Take $t \in \mathscr{B}_\beta$. By Proposition 3.1 we have $t < 1 - 1/\beta$, and then by Corollary 3.7 (ii) it gives $\eta_\beta(t) = \dim_H K_\beta(t) > 0$. Note by Proposition 3.1 and Proposition 2.3 (i) that

$$\mathscr{B}_\beta \cap (t, t+r) = \mathscr{E}_\beta \cap (t, t+r) \subseteq K_\beta(t) \quad \text{for any } r > 0.$$

Then $\lim_{r\to 0} \dim_H(\mathscr{B}_\beta \cap (t, t+r)) \le \eta_\beta(t)$. So it remains to prove

$$\lim_{r\to 0} \dim_H(\mathscr{B}_\beta \cap (t, t+r)) \ge \eta_\beta(t). \tag{3.11}$$

We prove this now by considering the following two cases: (I) $t = t_R$ is the right endpoint of a β-Lyndon interval; (II) $t \in [0, 1-1/\beta) \setminus \bigcup [t_L, t_R]$.

Case (I). Suppose $t = t_R$ is the right endpoint of a β-Lyndon interval. Let $(t_i) = (t_1 \dots t_p)^\infty$ be the greedy β-expansion of t_R. Note that $t_R \in \mathscr{B}_\beta$. Then by Corollary 3.7 (i) there exists a sequence $(t_R^{(n)}) \subset \mathscr{B}_\beta$ such that each $t_R^{(n)}$ is a right endpoint of a β-Lyndon interval and $t_R^{(n)} \searrow t_R$ as $n \to \infty$. Fix $r > 0$. Then we can find a large integer N satisfying

$$t_R^{(n)} \in (t_R, t_R + r) \quad \text{for all } n \ge N.$$

Furthermore, since $b(t_R, \beta) = (t_1 \dots t_p)^\infty$, by Lemma 2.1 (ii) it follows that for each $n \ge N$ there exists an integer k_n such that the greedy β-expansion $b(t_R^{(n)}, \beta)$ of $t_R^{(n)}$ satisfies

$$b(t_R^{(n)}, \beta) \succ (t_1 \dots t_p)^{k_n} 1^\infty. \tag{3.12}$$

Observe by Proposition 3.1 and Proposition 2.3 (i) that

$$\mathscr{B}_\beta = \mathscr{E}_\beta = \{((s_i))_\beta : (s_i) \preccurlyeq \sigma^n((s_i)) \prec (1^m 0)^\infty \ \forall n \ge 0\}.$$

So by using $t_R \in \mathscr{B}_\beta$, (3.12) and Lemma 2.1 (ii) it follows that for any $n \ge N$,

$$\begin{aligned} &\left\{ \left((t_1 \dots t_p)^{k_n} x_1 x_2 \dots \right)_\beta : x_1 \dots x_p = t_1 \dots t_p,\ (x_i) \in \widetilde{K}_\beta(t_R^{(n)}) \right\} \\ &\subseteq \mathscr{B}_\beta \cap [t_R, t_R^{(n)}) \\ &\subseteq \mathscr{B}_\beta \cap [t_R, t_R + r). \end{aligned} \tag{3.13}$$

Note by Lemma 3.2 that $\widetilde{K}_\beta(t_R^{(n)})$ is a transitive subshift of finite type. Then by (3.13) it follows that

$$\dim_H(\mathscr{B}_\beta \cap (t_R, t_R + r)) \ge \dim_H K_\beta(t_R^{(n)}) = \eta_\beta(t_R^{(n)}) \quad \text{for all } n \ge N.$$

Letting $n \to \infty$ and by the continuity of η_β (see Proposition 2.3 (ii)) we obtain that

$$\dim_H(\mathscr{B}_\beta \cap (t_R, t_R + r)) \ge \eta_\beta(t_R).$$

Since $r > 0$ was given arbitrary, letting $r \to 0$ we conclude that

$$\lim_{r\to 0} \dim_H(\mathscr{B}_\beta \cap (t_R, t_R + r)) \ge \eta_\beta(t_R). \tag{3.14}$$

Case (II). $t \in [0, 1 - \frac{1}{\beta}) \setminus \bigcup [t_L, t_R]$. Then by Corollary 3.7 (i) there exists a sequence $(t_R^{(k)})$ such that each $t_R^{(k)}$ is the right endpoint of a β-Lyndon interval, and $t_R^{(k)} \searrow t$ as $k \to \infty$. So, for any $r > 0$ there exists a sufficiently large integer k such that $t_R^{(k)} \in (t, t+r)$. By (3.14) with t_R replaced by $t_R^{(k)}$ it follows that for any $\varepsilon > 0$ there exists $r_k > 0$ such that $(t_R^{(k)}, t_R^{(k)} + r_k) \subset (t, t+r)$ and

$$\dim_H(\mathscr{B}_\beta \cap (t, t+r)) \ge \dim_H(\mathscr{B}_\beta \cap (t_R^{(k)}, t_R^{(k)} + r_k)) \ge \eta_\beta(t_R^{(k)}) - \varepsilon.$$

Letting $r \to 0$, and then $t_R^{(k)} \to t$, we conclude by the continuity of η_β that

$$\lim_{r\to 0} \dim_H(\mathscr{B}_\beta \cap (t, t+r)) \geq \eta_\beta(t) - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we obtain $\lim_{r\to 0} \dim_H(\mathscr{B}_\beta \cap (t, t+r)) \geq \eta_\beta(t)$. This, together with (3.14), proves (3.11). □

Proof of Theorem 1. Let $\beta \in (1, 2)$ with $\delta(\beta) = (1^m 0)^\infty$. By Lemma 2.2, Proposition 3.1 and Lemma 3.8 it suffices to prove

$$\left\{ t \in [0,1) : \lim_{r\to 0} \dim_H(\mathscr{B}_\beta \cap (t, t+r)) = \eta_\beta(t) > 0 \right\} \subset \mathscr{B}_\beta. \tag{3.15}$$

Take $t \in [0, 1) \setminus \mathscr{B}_\beta$. Then by Proposition 3.1 we have $t \in [1 - 1/\beta, 1)$ or $t \in [t_L, t_R)$ for some β-Lyndon interval. If $t \geq 1 - 1/\beta$, then $\eta_\beta(t) = 0$ by Corollary 3.7 (ii). If $t \in [t_L, t_R)$, then by Proposition 3.1 there exists $r > 0$ such that $\mathscr{B}_\beta \cap (t, t+r) = \emptyset$. This completes the proof. □

Proof of Corollary 2. Note by Proposition 3.1 that $\mathscr{E}_\beta \subset [0, 1 - 1/\beta)$. So if $t \geq 1 - 1/\beta$, then clearly the result holds by Corollary 3.7 (ii). Now let $t \in [0, 1-1/\beta)$. Observe by Proposition 2.3 (i) that $\mathscr{E}_\beta \cap [t, 1] \subset K_\beta(t)$. So it suffices to prove

$$\dim_H(\mathscr{E}_\beta \cap [t, 1]) \geq \dim_H K_\beta(t). \tag{3.16}$$

If $t \in [0, 1 - 1/\beta) \setminus [t_L, t_R)$, then (3.16) follows by Lemma 3.8. If $t \in [t_L, t_R)$, then we still have (3.16) by using Lemma 3.8 that

$$\dim_H(\mathscr{E}_\beta \cap [t, 1]) \geq \dim_H(\mathscr{E}_\beta \cap [t_R, 1]) \geq \dim_H K_\beta(t_R) = \dim_H K_\beta(t),$$

where the last equality holds by (3.2). □

4. Final remarks

The main results obtained in this paper can be easily modified to study the following analogous bifurcation sets:

$$\mathscr{E}'_\beta := \left\{ t \in [0,1) : K_\beta(t') \neq K_\beta(t) \ \forall t' \neq t \right\},$$
$$\mathscr{B}'_\beta := \left\{ t \in [0,1) : \dim_H K_\beta(t') \neq \dim_H K_\beta(t) \ \forall t' \neq t \right\}.$$

If $\beta \in (1, 2]$ is a multinacci number, one can show that

$$\begin{aligned}\mathscr{B}'_\beta = \mathscr{E}'_\beta &= \left[0, 1 - \frac{1}{\beta}\right) \setminus \bigcup [t_L, t_R] \\ &= \left\{ t \in [0,1) : \lim_{r\to 0} \dim_H(\mathscr{E}_\beta \cap (t-r, t)) = \lim_{r\to 0} \dim_H(\mathscr{E}_\beta \cap (t, t+r)) = \dim_H K_\beta(t) > 0 \right\},\end{aligned}$$

where the union is taken over all pairwise disjoint closed β-Lyndon intervals.

Observe that the main result Theorem 1 holds under the assumption that $\beta \in (1, 2]$ is a multinacci number, i.e., $\delta(\beta) = (1^m 0)^\infty$ for some $m \in \mathbb{N}$. The method used in this paper can be adapted to show that Theorem 1 still holds for $\beta \in (1, 2]$ with $\delta(\beta) = (10^m)^\infty$. It is worth mentioning that in [6] Kalle et al. considered a general Farey word base β, i.e., $\delta(\beta) = (s_1 \ldots s_p)^\infty$ with $s_p s_{p-1} \ldots s_2 s_1$ a non-degenerate Farey word. They showed that for a general Farey word base $\beta \in (1, 2)$, the set-valued bifurcation set $\mathscr{E}_\beta$ has no isolated points and Theorem 1.1 holds. We finish by posing the following conjecture.

Conjecture 4.1. Let $\beta \in (1,2]$. Then $\mathscr{B}_\beta = \mathscr{E}_\beta$ if and only if $\mathscr{E}_\beta$ has no isolated points.

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