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A B S T R A C T

Given two k-graphs H and F, a perfect F-packing in H is a collection of vertex-disjoint copies of F in H which together cover all the vertices in H. In the case when F is a single edge, a perfect F-packing is simply a perfect matching. For a given fixed F, it is often the case that the decision problem whether an n-vertex k-graph H contains a perfect F-packing is NP-complete. Indeed, if $k \geq 3$, the corresponding problem for perfect matchings is NP-complete [17,7] whilst if $k = 2$ the problem is NP-complete in the case when F has a component consisting of at least 3 vertices [14].

In this paper we give a general tool which can be used to determine classes of (hyper)graphs for which the corresponding decision problem for perfect F-packings is polynomial time solvable. We then give three applications of this tool: (i) Given $1 \leq \ell \leq k-1$, we give a minimum ℓ-degree condition for which it is polynomial time solvable to determine whether a k-graph satisfying this condition has a perfect matching; (ii) Given any graph F we give a minimum degree condition for which it is polynomial time solvable to determine whether a graph satisfying this condition has a perfect F-packing; (iii) We also prove a similar result for perfect K-packings in k-graphs where K is a k-partite k-graph.

For a range of values of ℓ, k (i) resolves a conjecture of Keevash, Knox and Mycroft [20]; (ii) answers a question of Yuster [47] in the negative; whilst (iii) generalises a result of

Keevash, Knox and Mycroft [20]. In many cases our results are best possible in the sense that lowering the minimum degree condition means that the corresponding decision problem becomes NP-complete.

1. Introduction

Given $k \geq 2$, a *k-uniform hypergraph* (or *k-graph*) consists of a vertex set $V(H)$ and an edge set $E(H) \subseteq \binom{V(H)}{k}$, where every edge is a k-element subset of $V(H)$. A *matching* in H is a collection of vertex-disjoint edges of H. A *perfect matching M* in H is a matching that covers all vertices of H.

The question of whether a given k-graph H contains a perfect matching is one of the most fundamental problems in combinatorics. In the graph case $k = 2$, Tutte's Theorem [46] gives necessary and sufficient conditions for H to contain a perfect matching, and Edmonds' Algorithm [5] finds such a matching in polynomial time. On the other hand, the decision problem whether a k-graph contains a perfect matching is famously NP-complete for $k \geq 3$ (see [17,7]).

An important generalisation of the notion of a perfect matching is that of a *perfect packing*: Given two k-graphs H and F, an *F-packing* in H is a collection of vertex-disjoint copies of F in H. An F-packing is called *perfect* if it covers all the vertices of H. Perfect F-packings are also referred to as *F-factors* or *perfect F-tilings*. Note that perfect matchings correspond to the case when F is a single edge. Hell and Kirkpatrick [14] showed that the decision problem whether a graph G has a perfect F-packing is NP-complete precisely when F has a component consisting of at least 3 vertices.

In light of the aforementioned complexity results, there has been significant attention to determine classes of (hyper)graphs for which the respective decision problems are polynomial time solvable. A key contribution of this paper is to provide a general tool (Theorem 3.1) that can be used to obtain such results. For this result we need to introduce several concepts so we defer its statement until Section 3.4. However, roughly speaking, for any k-graph F, Theorem 3.1 yields a general class of k-graphs within which we do have a complete characterisation of those k-graphs that contain a perfect F-packing. We then give three applications of Theorem 3.1, which we describe below. In particular, each of our applications convey an underlying theme: In each case, the class of (hyper)graphs H we consider are those satisfying some minimum degree condition which ensures an *almost* perfect matching or packing M (i.e. M covers all but a constant number of the vertices of H). Thus, in each application we show that we can detect the 'last obstructions' to having a perfect matching or packing efficiently.

1.1. Perfect matchings in hypergraphs

Given a k-graph H with an ℓ-element vertex set S (where $0 \leq \ell \leq k - 1$) we define $d_H(S)$ to be the number of edges containing S. The *minimum ℓ-degree* $\delta_\ell(H)$ of H is the minimum of $d_H(S)$ over all ℓ-element sets of vertices in H. We refer to $\delta_{k-1}(H)$ as the *minimum codegree* of H. The following conjecture from [9,28] gives a minimum ℓ-degree condition that ensures a perfect matching in a k-graph.

Conjecture 1.1. *Let $\ell, k \in \mathbb{N}$ such that $\ell \leq k - 1$. Given any $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that the following holds. Suppose H is a k-graph on $n \geq n_0$ vertices where k divides n. If*

$$\delta_\ell(H) \geq \max \left\{ (1/2 + \varepsilon), \left(1 - \left(1 - \frac{1}{k} \right)^{k-\ell} + \varepsilon \right) \right\} \binom{n}{k - \ell}$$

then H contains a perfect matching.

An 'exact' version of Conjecture 1.1 (without the error terms) was stated in [45]. There are two types of extremal examples that show, if true, Conjecture 1.1 is asymptotically best possible. The first is a so-called *divisibility barrier*: Let V_1 be a set of n vertices and A, B a partition of V_1 where $|A|, |B|$ are as equal as possible whilst ensuring $|A|$ is odd. Let H_1 be the k-graph with vertex set V_1 and edge set consisting of all those k-tuples that contain an even number of vertices from A. Then $\delta_\ell(H_1) = (1/2 + o(1))\binom{n}{k-\ell}$ for all $1 \leq \ell \leq k - 1$ but H_1 does not contain a perfect matching. (Actually note that there is a *family* of divisibility barrier constructions for this problem; see e.g. [45] for more details.) The second construction is a so-called *space barrier*: Let V_2 be a vertex set of size n and fix $S \subseteq V_2$ with $|S| = n/k - 1$. Let H_2 be the k-graph whose edges are all k-sets that intersect S. Then H_2 does not contain a perfect matching and $\delta_\ell(H_2) = \left(1 - \left(1 - \frac{1}{k} \right)^{k-\ell} + o(1) \right) \binom{n}{k-\ell}$ for all $1 \leq \ell \leq k - 1$.

In recent years Conjecture 1.1 (and its exact counterpart) has received substantial attention [1,4,9,10,21,22,27,31,34,36,37,39,40,43–45]. In particular, the *exact* threshold is known for all ℓ such that $0.42k \leq \ell \leq k-1$ as well as for a handful of other values of (k, ℓ). For example, Rödl, Ruciński and Szemerédi [40] determined the codegree threshold for this problem for sufficiently large k-graphs H on n vertices. This threshold is $n/2 - k + C$ where $C \in \{3/2, 2, 5/2, 3\}$ depends on the value of n and k.

Such results give us classes of dense k-graphs for which we are certain to have a perfect matching. This raises the question of whether one can lower the minimum ℓ-degree condition in Conjecture 1.1 whilst still ensuring it is decidable in polynomial time whether such a k-graph H has a perfect matching: Let $\mathbf{PM}(k, \ell, \delta)$ denote the problem of deciding whether there is a perfect matching in a given k-graph on n vertices with minimum ℓ-degree at least $\delta\binom{n}{k-\ell}$. Write $\mathbf{PM}(k, \delta) := \mathbf{PM}(k, k - 1, \delta)$.

The above mentioned result of Rödl, Ruciński and Szemerédi [40] implies that
$\mathbf{PM}(k, 1/2)$ is in P. On the other hand, for $k \geq 3$ Szymańska [42] proved that for
$\delta < 1/k$ the problem $\mathbf{PM}(k, \delta)$ admits a polynomial-time reduction to $\mathbf{PM}(k, 0)$ and
hence $\mathbf{PM}(k, \delta)$ is also NP-complete. Karpiński, Ruciński and Szymańska [18] proved
that there exists an $\epsilon > 0$ such that $\mathbf{PM}(k, 1/2 - \epsilon)$ is in P; they also raised the question
of determining the complexity of $\mathbf{PM}(k, \delta)$ for $\delta \in [1/k, 1/2)$. For any $\delta > 1/k$, Keevash,
Knox and Mycroft [20] recently proved that $\mathbf{PM}(k, \delta)$ is in P. Then very recently this
question was completely resolved by the first author [12] who showed that $\mathbf{PM}(k, \delta)$ is
in P for any $\delta \geq 1/k$.

Note that the minimum codegree of the space barrier construction H_2 above is
$\delta_{k-1}(H_2) = n/k - 1$. So in the case of minimum codegree, the threshold at which $\mathbf{PM}(k, \delta)$
'switches' from NP-complete to P corresponds to this space barrier. This leads to the
question whether the same phenomenon occurs in the case of minimum ℓ-degree for
$\ell \leq k - 2$. In support of this, Szymańska [42] proved that $\mathbf{PM}(k, \ell, \delta)$ is NP-complete
when $\delta < 1 - (1 - 1/k)^{k-\ell}$. This led Keevash, Knox and Mycroft [20] to pose the following
conjecture.

Conjecture 1.2 *(Keevash, Knox and Mycroft [20]).* $\mathbf{PM}(k, \ell, \delta)$ *is in P for every* $\delta >$
$1 - (1 - 1/k)^{k-\ell}$.

As an application of Theorem 3.1 we verify Conjecture 1.2 in a range of cases. To
state our result, we first must introduce the notion of a perfect fractional matching: Let
H be a k-graph on n vertices. A *fractional matching* in H is a function $w : E(H) \to [0, 1]$
such that for each $v \in V(H)$ we have that $\sum_{e \ni v} w(e) \leq 1$. Then $\sum_{e \in E(H)} w(e)$ is the
size of w. If the size of the largest fractional matching w in H is n/k then we say that w
is a *perfect fractional matching*. Given $k, \ell \in \mathbb{N}$ such that $\ell \leq k - 1$, define $c_{k,\ell}^*$ to be the
smallest number c such that every k-graph H on n vertices with $\delta_\ell(H) \geq (c + o(1))\binom{n-\ell}{k-\ell}$
contains a perfect fractional matching. We can now state our complexity result for perfect
matchings.

Theorem 1.3. *Given* $k, \ell \in \mathbb{N}$ *such that* $1 \leq \ell \leq k - 1$, *define* $\delta^* := \max\{1/3, c_{k,\ell}^*\}$.
Given any $\delta \in (\delta^*, 1]$, $\mathbf{PM}(k, \ell, \delta)$ *is in P. Indeed, for every* n-*vertex* k-*graph* H *with*
minimum ℓ-*degree at least* $\delta\binom{n-\ell}{k-\ell}$, *there is an algorithm with running time* $O(n^{k^2})$ *which*
determines whether H *contains a perfect matching.*

Alon, Frankl, Huang, Rödl, Ruciński, and Sudakov [1] conjectured that $c_{k,\ell}^* = 1 - (1 -$
$1/k)^{k-\ell}$ for all $\ell, k \in \mathbb{N}$. Thus, Theorem 1.3 verifies Conjecture 1.2 in all cases where
$c_{k,\ell}^* = 1 - (1 - 1/k)^{k-\ell}$ and $c_{k,\ell}^* \geq 1/3$. In particular, Kühn, Osthus and Townsend [30,
Theorem 1.7] proved that $c_{k,\ell}^* = 1 - (1 - 1/k)^{k-\ell}$ in the case when $\ell \geq k/2$ and the
first author [10, Theorem 1.5] proved that $c_{k,\ell}^* = 1 - (1 - 1/k)^{k-\ell}$ in the case when
$\ell = (k - 1)/2$.

Note that for all $1 \leq \ell \leq k - 1$,

$$\left(\frac{k-1}{k}\right)^{k-\ell} < \left(\frac{1}{e}\right)^{1-\frac{\ell}{k}}.$$

Thus, $1 - (1 - 1/k)^{k-\ell} \geq 1/3$ if $\ell \leq (1 + \ln(2/3))k \approx 0.5945k$. (Here ln denotes the natural logarithm function.) Altogether, this implies the following.

Corollary 1.4. *Conjecture 1.2 holds for all $k, \ell \in \mathbb{N}$ such that $(k-1)/2 \leq \ell \leq (1 + \ln(2/3))k$.*

1.2. Perfect packings in graphs

Several complexity problems for perfect packings in graphs have received attention. Given a graph F, we write $|F|$ for its order and $\chi(F)$ for its chromatic number. For approximating the size of a maximal F-packing, Hurkens and Schrijver [15] gave an $(|F|/2 + \epsilon)$-approximation algorithm (where $\epsilon > 0$ is arbitrary) which runs in polynomial time. On the other hand, Kann [16] proved that the problem is APX-hard if F has a component which contains at least three vertices. (In other words, it is impossible to approximate the optimum solution within an arbitrary factor unless P=NP.) In contrast, the results in [14] imply that the remaining cases of the problem can be solved in polynomial time.

The following classical result of Hajnal and Szemerédi [8] characterises the minimum degree that ensures a graph contains a perfect K_r-packing.

Theorem 1.5 *(Hajnal and Szemerédi [8]). Every graph G whose order n is divisible by r and whose minimum degree satisfies $\delta(G) \geq (1 - 1/r)n$ contains a perfect K_r-packing.*

By considering a complete r-partite graph G with vertex classes of almost equal size, one can see that the minimum degree condition in Theorem 1.5 cannot be lowered. Kierstead, Kostochka, Mydlarz and Szemerédi [23] gave a version of Theorem 1.5 which also yields a fast (polynomial time) algorithm for producing the perfect K_r-packing.

Up to an error term, the following theorem of Alon and Yuster [2] generalises Theorem 1.5. Let $M(n)$ be the time needed to multiply two n by n matrices with $0, 1$ entries. (Here the entries are viewed as elements of $\mathbb{Z}$.) Determining $M(n)$ is a challenging problem in theoretic computer science, and the best known bound of $M(n) = O(n^{2.3728639})$ was obtained by Le Gall [32].

Theorem 1.6 *(Alon and Yuster [2]). For every $\gamma > 0$ and each graph F there exists an integer $n_0 = n_0(\gamma, F)$ such that every graph G whose order $n \geq n_0$ is divisible by $|F|$ and whose minimum degree is at least $(1 - 1/\chi(F) + \gamma)n$ contains a perfect F-packing. Moreover, there is an algorithm which finds this F-packing in time $O(M(n))$.*

In [2], they also conjectured that the error term γn in Theorem 1.6 can be replaced by a constant $C(F) > 0$ depending only on F; this has been verified by Komlós, Sárközy and Szemerédi [25].

Theorem 1.7 *(Komlós, Sárközy and Szemerédi [25]). For every graph F there exist integers $C < |F|$ and $n_0 = n_0(F)$ such that every graph G whose order $n \geq n_0$ is divisible by $|F|$ and whose minimum degree is at least $(1-1/\chi(F))n+C$ contains a perfect F-packing. Moreover, there is an algorithm which finds this F-packing in time $O(nM(n))$.*

As observed in [2], there are graphs F for which the constant $C(F)$ cannot be omitted completely. On the other hand, there are graphs F for which the minimum degree condition in Theorem 1.7 can be improved significantly [19,3], by replacing the chromatic number with the critical chromatic number. The *critical chromatic number* $\chi_{cr}(F)$ of a graph F is defined as $(\chi(F) - 1)|F|/(|F| - \sigma(F))$, where $\sigma(F)$ denotes the minimum size of the smallest colour class in a colouring of F with $\chi(F)$ colours. Note that $\chi(F) - 1 < \chi_{cr}(F) \leq \chi(F)$ and the equality holds if and only if every $\chi(F)$-colouring of F has equal colour class sizes. If $\chi_{cr}(F) = \chi(F)$, then we call F *balanced*, otherwise *unbalanced*. Komlós [24] proved that one can replace $\chi(F)$ with $\chi_{cr}(F)$ in Theorem 1.7 at the price of obtaining an F-packing covering all but ϵn vertices. He also conjectured that the error term ϵn can be replaced with a constant that only depends on F [24]; this was confirmed by Shokoufandeh and Zhao [41] (here we state their result in a slightly weaker form).

Theorem 1.8 *(Shokoufandeh and Zhao [41]). For any F there is an $n_0 = n_0(F)$ so that if G is a graph on $n \geq n_0$ vertices and minimum degree at least $(1 - 1/\chi_{cr}(F))n$, then G contains an F-packing that covers all but at most $5|F|^2$ vertices.*

Then the question is, *for which F can we replace $\chi(F)$ with $\chi_{cr}(F)$ in Theorem 1.7?* Kühn and Osthus [26,29] answered this question completely. To state their result, we need some definitions. Write $k := \chi(F)$. Given a k-colouring c, let $x_1 \leq \cdots \leq x_k$ denote the sizes of the colour classes of c and put $D(c) = \{x_{i+1} - x_i \mid i \in [k-1]\}$. Let $D(F)$ be the union of all the sets $D(c)$ taken over all k-colourings c. Denote by $\mathrm{hcf}_\chi(F)$ the highest common factor of all integers in $D(F)$. (If $D(F) = \{0\}$, then set $\mathrm{hcf}_\chi(F) := \infty$.) Write $\mathrm{hcf}_c(F)$ for the highest common factor of all the orders of components of F (for example $\mathrm{hcf}_c(F) = |F|$ if F is connected). If $\chi(F) \neq 2$, then define $\mathrm{hcf}(F) = 1$ if $\mathrm{hcf}_\chi(F) = 1$. If $\chi(F) = 2$, then define $\mathrm{hcf}(F) = 1$ if both $\mathrm{hcf}_c(F) = 1$ and $\mathrm{hcf}_\chi(F) \leq 2$. Then let

$$\chi_*(F) = \begin{cases} \chi_{cr}(F) & \text{if } \mathrm{hcf}(F) = 1, \\ \chi(F) & \text{otherwise.} \end{cases}$$

In particular we have $\chi_{cr}(F) \leq \chi_*(F)$.

Theorem 1.9 (*Kühn and Osthus [26,29]*). *There exist integers $C = C(F)$ and $n_0 = n_0(F)$ such that every graph G whose order $n \geq n_0$ is divisible by $|F|$ and whose minimum degree is at least $(1 - 1/\chi_*(F))n + C$ contains a perfect F-packing.*

Theorem 1.9 is best possible in the sense that the degree condition cannot be lowered up to the constant C (there are also graphs F such that the constant cannot be omitted entirely). Moreover, this also implies that, *one can replace $\chi(F)$ with $\chi_{cr}(F)$ in Theorem 1.7 if and only if* $\mathrm{hcf}(F) = 1$. When $\mathrm{hcf}(F) \neq 1$ certain *divisibility barrier* constructions show that the minimum degree condition in Theorem 1.9 (and thus Theorem 1.7) is best possible up to the additive constant C (see [29]). On the other hand, the following *space barrier* construction shows that one cannot replace $\chi_*(F)$ with anything smaller than $\chi_{cr}(F)$ in Theorem 1.9; that is, when $\mathrm{hcf}(F) \neq 1$, Theorem 1.9 is best possible up to the additive constant C: Let G be the complete $\chi(F)$-partite graph on n vertices with $\sigma(F)n/|F| - 1$ vertices in one vertex class, and the other vertex classes of sizes as equal as possible. Then $\delta(G) = (1 - 1/\chi_{cr}(F))n - 1$ and G does not contain a perfect F-packing.

Now let us return to the algorithmic aspect of this problem. Let $\mathbf{Pack}(F, \delta)$ be the decision problem of determining whether a graph G whose minimum degree is at least $\delta|G|$ contains a perfect F-packing. When F contains a component of size at least 3, the result of Hell and Kirkpatrick [14] shows that $\mathbf{Pack}(F, 0)$ is NP-complete. In contrast, Theorem 1.9 gives that $\mathbf{Pack}(F, \delta)$ is (trivially) in P for any $\delta \in (1 - 1/\chi_*(F), 1]$. In [26], Kühn and Osthus showed that $\mathbf{Pack}(F, \delta)$ is NP-complete for any $\delta \in [0, 1 - 1/\chi_{cr}(F))$ if F is a clique of size at least 3 or a complete k-partite graph such that $k \geq 2$ and the size of the second smallest vertex class is at least 2.

Due to lack of knowledge on the range $\delta \in [0, 1 - 1/\chi_*(F))$ for general F, we still do not understand $\mathbf{Pack}(F, \delta)$ well in general. Indeed, even for (unbalanced) complete multi-partite graphs F with $\mathrm{hcf}(F) \neq 1$, there is a substantial hardness gap for $\delta \in [1 - 1/\chi_{cr}(F), 1 - 1/\chi_*(F)]$. In particular, Yuster asked the following question in his survey [47].

Problem 1.10 (*Yuster [47]*). *Is it true that $\mathbf{Pack}(F, \delta)$ is NP-complete for all $\delta \in [0, 1 - 1/\chi_*(F))$ and any F which contains a component of size at least 3?*

Our next result provides an algorithm showing that $\mathbf{Pack}(F, \delta)$ is in P when $\delta \in (1 - 1/\chi_{cr}(F), 1]$, which gives a negative answer to Problem 1.10 (as seen for any F such that $\chi_{cr}(F) < \chi_*(F)$). In fact, this gives the first *nontrivial* polynomial-time algorithm for the decision problem $\mathbf{Pack}(F, \delta)$. In particular, it eliminates the aforementioned hardness gap for unbalanced complete multi-partite graphs F with $\mathrm{hcf}(F) \neq 1$ almost entirely.

Theorem 1.11. *For any m-vertex k-chromatic graph F and $\delta \in (1 - 1/\chi_{cr}(F), 1]$, $\mathbf{Pack}(F, \delta)$ is in P. That is, for every n-vertex graph G with minimum degree at least*

δn, there is an algorithm with running time $O(n^{\max\{2^{m^{k-1}-1}m+1,\, m(2m-1)^m\}})$, which determines whether G contains a perfect F-packing.

In view of the aforementioned result of [26], Theorem 1.11 is asymptotically best possible if F is a complete k-partite graph such that $k \geq 2$ and the size of the second smallest cluster is at least 2 (note that when F is balanced, the result is included in Theorem 1.6). On the other hand, Theorem 1.11 complements Theorem 1.8 in the sense that when the minimum degree condition guarantees an F-packing that covers all but constant number of vertices, we can detect the 'last obstructions' efficiently.

We remark that Theorem 1.11 also appears in a conference paper of the first author [13].

1.3. Perfect packings in hypergraphs

Over the last few years there has been an interest in obtaining degree conditions that force a perfect F-packing in k-graphs where $k \geq 3$. In general though, this appears to be a harder problem than the graph version. Indeed, far less is known in the hypergraph case. See a survey of Zhao [48] for an overview of the known results in the area. Our final application of Theorem 3.1 is related to a recent general result of Mycroft [35].

Given a k-graph F and an integer n divisible by $|F|$, we define the threshold $\delta(n, F)$ as the smallest integer t such that every n-vertex k-graph H with $\delta_{k-1}(H) \geq t$ contains a perfect F-packing. Let F be a k-partite k-graph on vertex set U with at least one edge. Then a *k-partite realisation* of F is a partition of U into vertex classes $U_1, \ldots, U_k$ so that for any $e \in E(F)$ and $1 \leq j \leq k$ we have $|e \cap U_j| = 1$. Define

$$\mathcal{S}(F) := \bigcup_{\chi}\{|U_1|, \ldots, |U_k|\} \text{ and } \mathcal{D}(F) := \bigcup_{\chi}\{||U_i| - |U_j|| : i, j \in [k]\},$$

where in each case the union is taken over all k-partite realisations χ of F into vertex classes $U_1, \ldots, U_k$ of F. Then $\gcd(F)$ is defined to be the greatest common divisor of the set $\mathcal{D}(F)$ (if $\mathcal{D}(F) = \{0\}$ then $\gcd(F)$ is undefined). We also define

$$\sigma(F) := \frac{\min_{S \in \mathcal{S}(F)} S}{|V(F)|},$$

and thus in particular, $\sigma(F) \leq 1/k$. Mycroft [35] proved the following:

$$\delta(n, F) \leq \begin{cases} n/2 + o(n) & \text{if } \mathcal{S}(F) = \{1\} \text{ or } \gcd(\mathcal{S}(F)) > 1; \\ \sigma(F)n + o(n) & \text{if } \gcd(F) = 1; \\ \max\{\sigma(F)n, n/p\} + o(n) & \text{if } \gcd(\mathcal{S}(F)) = 1 \text{ and } \gcd(F) = d > 1, \end{cases} \quad (1.1)$$

where p is the smallest prime factor of d. Moreover, equality holds in (1.1) for all complete k-partite k-graphs F, as well as a wide class of other k-partite k-graphs.

Mycroft [35] also showed that minimum codegree of at least $\sigma(F)n+o(n)$ in an n-vertex k-graph H ensures an F-packing covering all but a constant number of vertices. The next two results show that above this degree threshold, one can determine in polynomial time whether H contains a perfect F-packing, whilst below the threshold the problem is NP-complete (for complete k-partite k-graphs F). Given $\delta > 0$ and a k-graph F, let **Pack**(F, δ) be the decision problem of determining whether a k-graph H whose minimum codegree is at least $\delta|H|$ contains a perfect F-packing.

Theorem 1.12. *Let $k \geq 3$ be an integer and let F be a complete k-partite k-graph. Then* **Pack**(F, δ) *is NP-complete for any $\delta \in [0, \sigma(F))$.*

Theorem 1.13. *Let $k \geq 3$ be an integer and let F be an m-vertex k-partite k-graph. For any $\delta \in (\sigma(F), 1]$,* **Pack**$(F, \delta)$ *is in P. That is, for every n-vertex k-graph H with $\delta_{k-1}(H) \geq \delta n$, there is an algorithm with running time $O(n^{m(2m-1)^m})$, which determines whether H contains a perfect F-packing.*

Note that when F is just an edge, a perfect F-packing is simply a perfect matching. Further, in this case $\sigma(F) = 1/k$. Thus, Theorem 1.13 is a generalisation of the perfect matching result of Keevash, Knox and Mycroft [20].

1.4. A general tool for complexity results

To prove the results mentioned above, we introduce a general structural theorem, Theorem 3.1. Given any k-graph F, Theorem 3.1 considers k-graphs H whose minimum ℓ-degree is sufficiently large so as to ensure H contains an *almost* perfect F-packing (that is an F-packing covering all but a constant number of vertices in H). To state Theorem 3.1 we introduce a coset group which, loosely speaking, is defined with respect to the 'distribution' of copies of F in H. In particular, Theorem 3.1 states that if this coset group Q has bounded size then we have a necessary and sufficient condition for H containing a perfect F-packing. This condition can be easily checked in polynomial time. This means if we have a class of k-graphs H (i) each of whose minimum ℓ-degree is sufficiently large and; (ii) each such H has a corresponding coset group Q of bounded size, then we can determine in polynomial time whether an element H in this class has a perfect F-packing.

Thus, in applications of Theorem 3.1 the key goal is to determine whether the corresponding coset groups have bounded size. In our applications to Theorems 1.11 and 1.13 all k-graphs H considered will have a corresponding coset group Q of bounded size. On the other hand, to prove Theorem 1.3 we show that a hypergraph H under consideration must have a corresponding coset group Q of bounded size, or failing that, *must* have a perfect matching.

The approach of using these auxiliary coset groups as a tool for such complexity results was also used in [20,12]; note that these applications were for perfect matchings

in hypergraphs of large minimum codegree. Theorem 3.1 provides a generalisation of this approach. Indeed, Theorem 3.1 is applicable to perfect matching *and* packing problems in (hyper)graphs of large minimum ℓ-degree for *any* ℓ. As such, we suspect Theorem 3.1 could have many more applications in the area.

The paper is organised as follows. In the next section we prove Theorem 1.12. In Section 3 we introduce the general structural theorem (Theorem 3.1) as well as some notation and definitions. We prove Theorem 3.1 in Sections 4 and 5. In Sections 6 and 7 we introduce some tools that are useful for the applications of Theorem 3.1. We then prove Theorems 1.3, 1.11 and 1.13 in Sections 8, 9 and 10 respectively.

2. Proof of the hardness result

In this section we prove Theorem 1.12.

Proof of Theorem 1.12. Our proof resembles the one of Szymańska [42, Theorem 1.7] and we also use the following result from it. Let $\mathbf{PM}_{lin}(k)$ be the subproblem of $\mathbf{PM}(k, 0)$ restricted to k-uniform hypergraphs which are *linear*, that is, any two edges share at most one vertex. Then it is shown in [42] that $\mathbf{PM}_{lin}(3)$ is NP-complete.

Let $K := K^{(k)}(a_1, \ldots, a_k)$ be the complete k-partite k-graph of order m with vertex classes of size $a_1 \leq \cdots \leq a_k$. We may assume that $a_k \geq 2$ as otherwise K is just a single edge and $\mathbf{Pack}(K, \delta)$ is NP-complete for $\delta \in [0, 1/k]$ as shown in [42]. We prove the theorem by the following reductions.

$$\mathbf{PM}_{lin}(3) \overset{(a)}{\leq} \mathbf{PM}_{lin}(m) \overset{(b)}{\leq} \mathbf{Pack}(K, 0) \overset{(c)}{\leq} \mathbf{Pack}(K, \delta).$$

Reduction (a). In fact, we will show that $\mathbf{PM}_{lin}(k) \leq \mathbf{PM}_{lin}(k+1)$ for any $k \geq 3$. Let H be a linear k-graph with n vertices and s edges. We construct a linear $(k+1)$-graph G by taking $k+1$ disjoint copies H_i of H, $i \in [k+1]$ and for every edge e in each copy H_i we add one vertex v_i^e to $V(G)$, i.e., $V(G) = \bigcup_{i \in [k+1]}(V(H_i) \cup \bigcup_{e \in E(H_i)}\{v_i^e\})$. Thus $|V(G)| = (k+1)(n+s)$. For every $e \in E(H)$ the $(k+1)$-tuple $\{v_i^e : i \in [k+1]\}$ forms an edge of G. Moreover, we add to $E(G)$ all sets of the form $e \cup \{v_i^e\}$ for all $i \in [k+1]$ and $e \in E(H_i)$. Hence, G has $(k+2)s$ edges and is linear by the definition.

Suppose H has a perfect matching M. Let M_i be the same matching in the copy H_i of H, $i \in [k+1]$. Then it is easy to see that G has a perfect matching $M' = \{e \cup \{v_i^e\}, e \in M_i, i \in [k+1]\} \cup \{f_e = \{v_1^e, \ldots, v_{k+1}^e\} : e \notin M\}$. On the other hand assume that G has a perfect matching $M' = \{f_1, \ldots, f_{n+s}\}$. For all $v \in V(H_1)$, let $f(v) \in M'$ be such that $v \in f(v)$. But the only edges of G containing the vertices of H_1 are of the form $e \cup \{v_1^e\}$, so $|\{f(v) : v \in V(H_1)\}| = n/k$ and $\{f(v) \cap V(H_1) : v \in V(H_1)\}$ is a perfect matching of H_1. Therefore H also has a perfect matching.

Reduction (b). Given a linear m-graph H we build a k-graph G by replacing each edge of H with a copy of K. If H has a perfect matching then G has a perfect K-packing. In turn, if G has a perfect K-packing, then by the linearity of H, each copy of K corresponds to

a single edge of H and therefore the K-packing corresponds to a perfect matching of H. In fact, since K is complete k-partite, there exists an ordering $e_1, \ldots, e_t$ of $E(K)$ (e.g., the lexicographic ordering) such that for any $2 \leq i \leq t$, there exists $1 \leq j \leq i - 1$ such that $|e_i \cap e_j| \geq 2$. Then by the linearity of H, each copy of K corresponds to a single edge of H.

Reduction (c). Let $\gamma := \sigma(K) - \delta = a_1/m - \delta$ and thus $\gamma > 0$. To achieve this, for each instance H of **Pack**$(K, 0)$ with n vertices such that $m \mid n$, we define a graph H' as follows. Let $H_0 = H_0(k, n, \gamma)$ be a k-graph, in which the vertex set is the union of two disjoint sets $A \cup B$, such that $|A| = a_1 \lceil n/\gamma \rceil$ and $|B| = (m - a_1) \lceil n/\gamma \rceil$. The edge set of H_0 consists of all k-vertex sets of $A \cup B$ which have a non-empty intersection with A. Observe that $\delta_{k-1}(H_0) = |A|$ and H_0 has a perfect K-packing (in which each copy of K contains a_1 vertices in A and $m - a_1$ vertices in B). Then let H' be the k-graph such that $V(H') = V(H) \cup V(H_0)$ and $E(H') = E(H) \cup E$, where E consists of all k-sets that intersect A and thus $E(H_0) \subseteq E$. Clearly $|V(H')| = n + m \lceil n/\gamma \rceil$ and

$$\delta_{k-1}(H') = |A| = a_1 \lceil n/\gamma \rceil \geq \left(\frac{a_1}{m} - \frac{a_1}{m^2} \gamma \right) |V(H')| > \delta |V(H')|.$$

If H has a perfect K-packing, so does H'. Now suppose that H does not have a perfect K-packing and H' has a perfect K-packing M. This means that there exists a copy of K in M with its vertex set denoted by K', such that $K' \cap A \neq \emptyset$ and $K' \cap V(H) \neq \emptyset$. First assume that $K' \cap B = \emptyset$. Then since $\frac{|A \setminus K'|}{|B \setminus K'|} = \frac{|A \setminus K'|}{|B|} < a_1/(m - a_1)$, the vertices of B cannot be covered completely by M, contradicting the existence of M. Otherwise $K' \cap B \neq \emptyset$. Then since $V(H) \cup B$ is an independent set in H', we get that $|K' \cap A| \geq a_1$, which implies that $1 \leq |K' \cap B| \leq m - a_1 - 1$. Again, $\frac{|A \setminus K'|}{|B \setminus K'|} \leq \frac{|A| - a_1}{|B| - (m - a_1 - 1)} < a_1/(m - a_1)$, so the rest of the vertices of B cannot be covered completely by M, a contradiction. $\square$

3. The general structural theorem

In order to state our general structural theorem, Theorem 3.1, we will now introduce some definitions and notation.

3.1. Almost perfect packings

Let $k, \ell \in \mathbb{N}$ where $\ell \leq k - 1$. Let F be an m-vertex k-graph and $D \in \mathbb{N}$. Define $\delta(F, \ell, D)$ to be the smallest number δ such that every k-graph H on n vertices with $\delta_\ell(H) \geq (\delta + o(1)) \binom{n-\ell}{k-\ell}$ contains an F-packing covering all but at most D vertices. We write $\delta(k, \ell, D)$ for $\delta(F, \ell, D)$ when F is a single edge.

3.2. Lattices and solubility

One concept needed to understand the statement and proof of Theorem 3.1 is that of *lattices and solubility* introduced by Keevash, Knox and Mycroft [20]. Let H be an

n-vertex k-graph. We will work with a vertex partition $\mathcal{P} = \{V_1, \ldots, V_d\}$ of $V(H)$ for some integer $d \geq 1$. In this paper, every partition has an implicit ordering of its parts. The *index vector* $\mathbf{i}_{\mathcal{P}}(S) \in \mathbb{Z}^d$ of a subset $S \subseteq V(H)$ with respect to $\mathcal{P}$ is the vector whose coordinates are the sizes of the intersections of S with each part of $\mathcal{P}$, namely, $\mathbf{i}_{\mathcal{P}}(S)|_i = |S \cap V_i|$ for $i \in [d]$, where $\mathbf{v}|_i$ is defined as the ith digit of $\mathbf{v}$. For any $\mathbf{v} = \{v_1, \ldots, v_d\} \in \mathbb{Z}^d$, let $|\mathbf{v}| := \sum_{i=1}^d v_i$. We say that $\mathbf{v} \in \mathbb{Z}^d$ is an *r-vector* if it has non-negative coordinates and $|\mathbf{v}| = r$.

Let F be an m-vertex k-graph and let $\mu > 0$. Define $I_{\mathcal{P},F}^\mu(H)$ to be the set of all $\mathbf{i} \in \mathbb{Z}^d$ such that H contains at least μn^m copies of F with index vector $\mathbf{i}$ and let $L_{\mathcal{P},F}^\mu(H)$ denote the lattice in $\mathbb{Z}^d$ generated by $I_{\mathcal{P},F}^\mu(H)$.

Let $q \in \mathbb{N}$. A (possibly empty) F-packing M in H of size at most q is a *q-solution* for $(\mathcal{P}, L_{\mathcal{P},F}^\mu(H))$ (in H) if $\mathbf{i}_{\mathcal{P}}(V(H) \setminus V(M)) \in L_{\mathcal{P},F}^\mu(H)$; we say that $(\mathcal{P}, L_{\mathcal{P},F}^\mu(H))$ is *q-soluble* if it has a q-solution.

Given a partition $\mathcal{P}$ of d parts, we write $L_{\max}^d$ for the lattice generated by all m-vectors. So $L_{\max}^d := \{\mathbf{v} \in \mathbb{Z}^d : m \text{ divides } |\mathbf{v}|\}$.

Suppose $L \subset L_{\max}^{|\mathcal{P}|}$ is a lattice in $\mathbb{Z}^{|\mathcal{P}|}$, where $\mathcal{P}$ is a partition of a set V. The *coset group* of $(\mathcal{P}, L)$ is $Q = Q(\mathcal{P}, L) := L_{\max}^{|\mathcal{P}|}/L$. For any $\mathbf{i} \in L_{\max}^{|\mathcal{P}|}$, the *residue* of $\mathbf{i}$ in Q is $R_Q(\mathbf{i}) := \mathbf{i} + L$. For any $A \subseteq V$ of size divisible by m, the *residue* of A in Q is $R_Q(A) := R_Q(\mathbf{i}_{\mathcal{P}}(A))$.

3.3. Reachability and good partitions

Let F be an m-vertex k-graph and let H be an n-vertex k-graph. We say that two vertices u and v in $V(H)$ are *(F, β, i)-reachable in H* if there are at least βn^{im-1} $(im - 1)$-sets S such that both $H[S \cup \{u\}]$ and $H[S \cup \{v\}]$ have perfect F-packings. We refer to such a set S as a *reachable $(im - 1)$-set for u and v*. We say a vertex set $U \subseteq V(H)$ is *(F, β, i)-closed in H* if any two vertices $u, v \in U$ are (F, β, i)-reachable in H. Given any $v \in V(H)$, define $\tilde{N}_{F,\beta,i}(v, H)$ to be the set of vertices in $V(H)$ that are (F, β, i)-reachable to v in H.

Let $\beta, c > 0$ and $t \in \mathbb{N}$. A partition $\mathcal{P} = \{V_1, \ldots, V_d\}$ of $V(H)$ is *(F, β, t, c)-good* if the following properties hold:

- V_i is (F, β, t)-closed in H for all $i \in [d]$;
- $|V_i| \geq cn$ for all $i \in [d]$.

3.4. Statement of the general structural theorem

With these definitions in hand, we are now able to state the general structural theorem. Throughout the paper, we write $0 < \alpha \ll \beta \ll \gamma$ to mean that we can choose the constants α, β, γ from right to left. More precisely, there are increasing functions f and g such that, given γ, whenever we choose some $\beta \leq f(\gamma)$ and $\alpha \leq g(\beta)$, all calculations needed in our proof are valid. Hierarchies of other lengths are defined in the obvious way.

Theorem 3.1 *(Structural theorem).* *Let $k, \ell \in \mathbb{N}$ where $\ell \leq k-1$ and let F be an m-vertex k-graph. Define $D, q, t, n_0 \in \mathbb{N}$ and $\beta, \mu, \gamma, c > 0$ where*

$$1/n_0 \ll \beta, \mu \ll \gamma, c, 1/m, 1/D, 1/q, 1/t.$$

Let H be a k-graph on $n \geq n_0$ vertices where m divides n. Suppose that

  (i) $\delta_\ell(H) \geq (\delta(F, \ell, D) + \gamma)\binom{n-\ell}{k-\ell}$;
  (ii) $\mathcal{P} = \{V_1, \ldots, V_d\}$ *is an (F, β, t, c)-good partition of $V(H)$;*
  (iii) $|Q(\mathcal{P}, L^\mu_{\mathcal{P},F}(H))| \leq q$.

Then H contains a perfect F-packing if and only if $(\mathcal{P}, L^\mu_{\mathcal{P},F}(H))$ is q-soluble.

At first sight Theorem 3.1 may seem somewhat technical. In particular, it may not be clear which roles conditions (i)–(iii) play. We will explain this in more detail now.

In the proof of (the backward implication of) Theorem 3.1 we will utilise the *absorbing method*. This technique was initiated by Rödl, Ruciński and Szemerédi [38] and has proven to be a powerful tool for finding spanning structures in graphs and hypergraphs. Fix an integer $i > 0$ and a k-graph F. Let H be a k-graph. For a set $S \subseteq V(H)$, we say a set $T \subseteq V(H)$ is an *absorbing (F, i)-set for S* if $|T| = i$ and both $H[T]$ and $H[T \cup S]$ contain perfect F-packings. Informally, we will refer to T as an *absorbing set for S* and say T *absorbs S*.

Often in proofs employing the absorbing method the goal is to find some small set A such that for *any* very small set of vertices S in H, A absorbs S. In particular, if one could guarantee such a set A in Theorem 3.1 then we would ensure a perfect F-packing: By (i), $H \setminus A$ would have an almost perfect F-packing. Then A can be used to absorb the uncovered vertices to obtain a perfect F-packing.

Not all k-graphs satisfying the hypothesis of Theorem 3.1 will have a perfect F-packing; so one cannot obtain such a set A in general. Instead, in the proof of Theorem 3.1 we will apply the *lattice-based absorbing method* developed recently by the first author [12]: What one can *always* guarantee in our case is a small family of absorbing sets $\mathcal{F}_{abs}$ with the property that for every m-vertex set $S \subseteq V(H)$ such that $\mathbf{i}_{\mathcal{P}}(S) \in I^\mu_{\mathcal{P},F}(H)$, there are many sets in $\mathcal{F}_{abs}$ that do absorb S. This is made precise in Lemma 4.1 in Section 4. We remark that to obtain $\mathcal{F}_{abs}$ it was crucial that condition (ii) in Theorem 3.1 holds.

Now suppose M is an almost perfect F-packing in $H \setminus V(\mathcal{F}_{abs})$. Let U denote the vertices in $H \setminus V(\mathcal{F}_{abs})$ not covered by M. If there is a partition $S_1, \ldots, S_s$ of U such that $\mathbf{i}_{\mathcal{P}}(S_i) \in I^\mu_{\mathcal{P},F}(H)$ for each i, then by definition of $\mathcal{F}_{abs}$ we can absorb the vertices in U to obtain a perfect F-packing in H. To find such a partition of U we certainly would need that $\mathbf{i}_{\mathcal{P}}(U) \in L^\mu_{\mathcal{P},F}(H)$. This is where the property that $(\mathcal{P}, L^\mu_{\mathcal{P},F}(H))$ is q-soluble is vital: by definition this allows us to find an F-packing M_1 of size at most q such that $\mathbf{i}_{\mathcal{P}}(V(H) \setminus V(M_1)) \in L^\mu_{\mathcal{P},F}(H)$. Roughly speaking, the idea is that by removing the

vertices of M_1 from H we now have a k-graph where (by following the steps outlined above) we do obtain a set of uncovered vertices U that can be fully absorbed using the family $\mathcal{F}_{abs}$. This step is a little involved; that is, some careful refinement of the almost perfect F-packing is still needed to ensure there is a partition $S_1, \ldots, S_s$ of U such that $\mathbf{i}_{\mathcal{P}}(S_i) \in I^{\mu}_{\mathcal{P}, F}(H)$ for each i.

Condition (iii) is applied in both the forward and backward implication of Theorem 3.1. In particular, this is precisely the condition required to show that if H has a perfect F-packing then $(\mathcal{P}, L^{\mu}_{\mathcal{P}, F}(H))$ is q-soluble.

In the next section we prove the absorbing lemma and in Section 5 we prove Theorem 3.1.

4. Absorbing lemma

The following result guarantees our collection $\mathcal{F}_{abs}$ of absorbing sets in the proof of Theorem 3.1.

Lemma 4.1 *(Absorbing lemma). Suppose F is an m-vertex k-graph and*

$$1/n \ll 1/c \ll \beta, \mu \ll 1/m, 1/t,$$

and H is a k-graph on n vertices. Suppose $\mathcal{P} = \{V_1, \ldots, V_d\}$ is a partition of $V(H)$ such that for each $i \in [d]$, V_i is (F, β, t)-closed. Then there is a family $\mathcal{F}_{abs}$ consisting of at most $c \log n$ disjoint tm^2-sets such that for each $A \in \mathcal{F}_{abs}$, $H[A]$ contains a perfect F-packing and every m-set $S \subseteq V(H)$ with $\mathbf{i}_{\mathcal{P}}(S) \in I^{\mu}_{\mathcal{P}, F}(H)$ has at least $\sqrt{\log n}$ absorbing (F, tm^2)-sets in $\mathcal{F}_{abs}$.

Proof. Our first task is to prove the following claim.

Claim 4.2. *Any m-set S with $\mathbf{i}_{\mathcal{P}}(S) \in I^{\mu}_{\mathcal{P}, F}(H)$ has at least $\mu \beta^{m+1} n^{tm^2}$ absorbing (F, tm^2)-sets.*

Proof. For an m-set $S = \{y_1, \ldots, y_m\}$ with $\mathbf{i}_{\mathcal{P}}(S) \in I^{\mu}_{\mathcal{P}, F}(H)$, we construct absorbing (F, tm^2)-sets for S as follows. We first fix a copy F' of F with vertex set $W = \{x_1, \ldots, x_m\}$ in H such that $\mathbf{i}_{\mathcal{P}}(W) = \mathbf{i}_{\mathcal{P}}(S) \in I^{\mu}_{\mathcal{P}, F}(H)$ and $W \cap S = \emptyset$. Note that we have at least $\mu n^m - mn^{m-1} > \frac{\mu}{2} n^m$ choices for such F'. Without loss of generality, we may assume that for all $i \in [m]$, x_i, y_i are in the same part of $\mathcal{P}$. Since x_i is (F, β, t)-reachable to y_i, there are at least βn^{tm-1} $(tm-1)$-sets T_i such that both $H[T_i \cup \{x_i\}]$ and $H[T_i \cup \{y_i\}]$ have perfect F-packings. We pick disjoint reachable $(tm-1)$-sets for each x_i, y_i, $i \in [m]$ greedily, while avoiding the existing vertices. Since the number of existing vertices is at most $tm^2 + m$, we have at least $\frac{\beta}{2} n^{tm-1}$ choices for such $(tm-1)$-sets in each step. Note that $W \cup T_1 \cup \cdots \cup T_m$ is an absorbing set for S. First, it contains a perfect F-packing because each $T_i \cup \{x_i\}$ for $i \in [m]$ spans t vertex-disjoint copies of F. Second, $H[W \cup T_1 \cup \cdots \cup T_m \cup S]$ also contains a perfect

F-packing because F' is a copy of F and each $T_i \cup \{y_i\}$ for $i \in [m]$ spans t vertex-disjoint copies of F. There were at least $\frac{\mu}{2}n^m$ choices for W and at least $\frac{\beta}{2}n^{tm-1}$ choices for each T_i. Thus we find at least

$$\frac{\mu}{2}n^m \times \frac{\beta^m}{2^m}n^{tm^2-m} \times \frac{1}{(tm^2)!} \geq \mu\beta^{m+1}n^{tm^2}$$

absorbing (F, tm^2)-sets for S. □

We pick a family $\mathcal{F}$ of tm^2-sets by including every tm^2-subset of $V(H)$ with probability $p = cn^{-tm^2}\log n$ independently, uniformly at random. Then the expected number of elements in $\mathcal{F}$ is $p\binom{n}{tm^2} \leq \frac{c}{tm^2}\log n$ and the expected number of intersecting pairs of tm^2-sets is at most

$$p^2\binom{n}{tm^2} \cdot tm^2 \cdot \binom{n}{tm^2-1} \leq \frac{c^2(\log n)^2}{n} = o(1).$$

Then by Markov's inequality, with probability at least $1 - 1/(tm^2) - o(1)$, $\mathcal{F}$ contains at most $c\log n$ sets and they are pairwise vertex disjoint.

For every m-set S with $\mathbf{i}_\mathcal{P}(S) \in I^\mu_{\mathcal{P},F}(H)$, let X_S be the number of absorbing sets for S in $\mathcal{F}$. Then by Claim 4.2,

$$\mathbb{E}(X_S) \geq p\mu\beta^{m+1}n^{tm^2} = \mu\beta^{m+1}c\log n.$$

By Chernoff's bound,

$$\mathbb{P}\left(X_S \leq \frac{1}{2}\mathbb{E}(X_S)\right) \leq \exp\left\{-\frac{1}{8}\mathbb{E}(X_S)\right\} \leq \exp\left\{-\frac{\mu\beta^{m+1}c\log n}{8}\right\} = o(n^{-m}),$$

since $1/c \ll \beta, \mu \ll 1/m$. Thus, with probability $1 - o(1)$, for each m-set S with $\mathbf{i}_\mathcal{P}(S) \in I^\mu_{\mathcal{P},F}(H)$, there are at least

$$\frac{1}{2}\mathbb{E}(X_S) \geq \frac{\mu\beta^{m+1}c\log n}{2} > \sqrt{\log n}$$

absorbing sets for S in $\mathcal{F}$. We obtain $\mathcal{F}_{abs}$ by deleting the elements of $\mathcal{F}$ that are not absorbing sets for any m-set S and thus $|\mathcal{F}_{abs}| \leq |\mathcal{F}| \leq c\log n$. □

5. Proof of Theorem 3.1

5.1. Proof of the forward implication of Theorem 3.1

If H contains a perfect F-packing M, then $\mathbf{i}_\mathcal{P}(V(H) \setminus V(M)) = \mathbf{0} \in L^\mu_{\mathcal{P},F}(H)$. We will show that there exists an F-packing $M' \subset M$ such that $|M'| \leq q$ and $\mathbf{i}_\mathcal{P}(V(H) \setminus$

$V(M')) \in L^\mu_{\mathcal{P},F}(H)$ and thus $(\mathcal{P}, L^\mu_{\mathcal{P},F}(H))$ is q-soluble. Indeed, suppose $M' \subset M$ is a minimum F-packing such that $\mathbf{i}_{\mathcal{P}}(V(H) \setminus V(M')) \in L^\mu_{\mathcal{P},F}(H)$ and $|M'| = m' \geq q$. Let $M' = \{e_1, \ldots, e_{m'}\}$ and consider the $m' + 1$ partial sums

$$\sum_{i=1}^{j} \mathbf{i}_{\mathcal{P}}(e_i) + L^\mu_{\mathcal{P},F}(H) = \sum_{i=1}^{j} R_{Q(\mathcal{P}, L^\mu_{\mathcal{P},F}(H))}(e_i),$$

for $j = 0, 1, \ldots, m'$. Since $|Q(\mathcal{P}, L^\mu_{\mathcal{P},F}(H))| \leq q \leq m'$, two of the sums must be equal. That is, there exists $0 \leq j_1 < j_2 \leq m'$ such that

$$\sum_{i=j_1+1}^{j_2} \mathbf{i}_{\mathcal{P}}(e_i) \in L^\mu_{\mathcal{P},F}(H).$$

So the F-packing $M'' := M' \setminus \{e_{j_1+1}, \ldots, e_{j_2}\}$ satisfies that $\mathbf{i}_{\mathcal{P}}(V(H) \setminus V(M'')) \in L^\mu_{\mathcal{P},F}(H)$ and $|M''| < |M'|$, a contradiction.

5.2. Proof of the backward implication of Theorem 3.1

Suppose I is a set of m-vectors of $\mathbb{Z}^d$ and J is a (finite) set of vectors such that any $\mathbf{i} \in J$ can be written as a linear combination of vectors in I, namely, there exist $a_{\mathbf{v}}(\mathbf{i}) \in \mathbb{Z}$ for all $\mathbf{v} \in I$, such that

$$\mathbf{i} = \sum_{\mathbf{v} \in I} a_{\mathbf{v}}(\mathbf{i})\mathbf{v}.$$

We denote by $C(d, m, I, J)$ as the maximum of $|a_{\mathbf{v}}(\mathbf{i})|, \mathbf{v} \in I$ over all $\mathbf{i} \in J$.

The proof of the backward implication of Theorem 3.1 consists of a few steps. We first fix an F-packing M_1, a q-solution of $(\mathcal{P}, L^\mu_{\mathcal{P},F}(H))$. We apply Lemma 4.1 to H and get a family $\mathcal{F}_{abs}$ consisting of at most $c \log n$ disjoint tm^2-sets. Let $\mathcal{F}_0$ be the subfamily of $\mathcal{F}_{abs}$ that do not intersect $V(M_1)$. Next we find a set M_2 of disjoint copies of F, which includes (constantly) many copies of F for each m-vector in $I^\mu_{\mathcal{P},F}(H)$. Now by definition of $\delta(F, \ell, D)$, in $H[V \setminus (V(\mathcal{F}_0) \cup V(M_1 \cup M_2))]$ we find an F-packing M_3 covering all but a set U of at most D vertices. The remaining job is to 'absorb' the vertices in U. Roughly speaking, by the solubility condition, we can release some copies of F in some members of $\mathcal{F}_0$ and M_3 and add their vertices to U, such that the resulting set $Y \supseteq U$ of uncovered vertices satisfies that $\mathbf{i}_{\mathcal{P}}(Y) \in L^\mu_{\mathcal{P},F}(H)$. Furthermore, by releasing some copies of F in M_2 and add their vertices to U, we can partition the new set of uncovered vertices as a collection of m-sets S such that $\mathbf{i}_{\mathcal{P}}(S) \in I^\mu_{\mathcal{P},F}(H)$ for each S. Then we can finish the absorption by the property of $\mathcal{F}_0$.

Proof of the backward implication of Theorem 3.1. Define an additional constant $C > 0$ so that

$$1/n_0 \ll 1/C \ll \beta, \mu.$$

Let H be as in the statement of the theorem. Moreover, assume that $(\mathcal{P}, L^{\mu}_{\mathcal{P},F}(H))$ is q-soluble. We first apply Lemma 4.1 to H and get a family $\mathcal{F}_{abs}$ consisting of at most $c \log n$ disjoint tm^2-sets such that every m-set S of vertices with $\mathbf{i}_{\mathcal{P}}(S) \in I^{\mu}_{\mathcal{P},F}(H)$ has at least $\sqrt{\log n}$ absorbing (F, tm^2)-sets in $\mathcal{F}_{abs}$.

Since $(\mathcal{P}, L^{\mu}_{\mathcal{P},F}(H))$ is q-soluble, there exists an F-packing M_1 of size at most q such that $\mathbf{i}_{\mathcal{P}}(V(H) \setminus V(M_1)) \in L^{\mu}_{\mathcal{P},F}(H)$. Note that $V(M_1)$ may intersect $V(\mathcal{F}_{abs})$ in at most qm absorbing sets of $\mathcal{F}_{abs}$. Let $\mathcal{F}_0$ be the subfamily of $\mathcal{F}_{abs}$ obtained from removing the tm^2-sets that intersect $V(M_1)$. Let M_0 be the perfect F-packing on $V(\mathcal{F}_0)$ that is the union of the perfect F-packings on each member of $\mathcal{F}_0$. Note that every m-set S of vertices with $\mathbf{i}_{\mathcal{P}}(S) \in I^{\mu}_{\mathcal{P},F}(H)$ has at least $\sqrt{\log n} - qm$ absorbing sets in $\mathcal{F}_0$.

Next we want to 'store' some copies of F for each m-vector in $I^{\mu}_{\mathcal{P},F}(H)$ for future use. More precisely, let J be the set of all m'-vectors in $L^{\mu}_{\mathcal{P},F}(H)$ such that $0 \le m' \le qm + D$ and set $C' := C(d, m, I^{\mu}_{\mathcal{P},F}(H), J)$. We find an F-packing M_2 in $H \setminus V(M_0 \cup M_1)$ which consists of C' copies F' of F with $\mathbf{i}_{\mathcal{P}}(F') = \mathbf{i}$ for every $\mathbf{i} \in I^{\mu}_{\mathcal{P},F}(H)$. So $|M_2| \le \binom{m+d-1}{m} C'$ and the process is possible because H contains at least μn^m copies of F for each $\mathbf{i} \in I^{\mu}_{\mathcal{P},F}(H)$ and $|V(M_0 \cup M_1 \cup M_2)| \le tm^2 C \log n + qm + \binom{m+d-1}{m} C'm < \mu n$.

Let $H' := H \setminus V(M_0 \cup M_1 \cup M_2)$ and $n' := |H'|$. So $n' \ge n - \mu n$ and

$$\delta_\ell(H') \ge \delta_\ell(H) - \mu n^{k-\ell} \ge (\delta(F, \ell, D) + \gamma/2) \binom{n' - \ell}{k - \ell}.$$

By the definition of $\delta(F, \ell, D)$ we have an F-packing M_3 in H' covering all but at most D vertices. Let U be the set of vertices in H' uncovered by M_3.

Let $Q := Q(\mathcal{P}, L^{\mu}_{\mathcal{P},F}(H))$. Recall that $\mathbf{i}_{\mathcal{P}}(V(H) \setminus V(M_1)) \in L^{\mu}_{\mathcal{P},F}(H)$. Note that by definition, the index vectors of all copies of F in M_2 are in $I^{\mu}_{\mathcal{P},F}(H)$. So we have $\mathbf{i}_{\mathcal{P}}(V(H) \setminus V(M_1 \cup M_2)) \in L^{\mu}_{\mathcal{P},F}(H)$, namely, $R_Q(V(H) \setminus V(M_1 \cup M_2)) = \mathbf{0} + L^{\mu}_{\mathcal{P},F}(H)$. Thus,

$$\sum_{F' \in M_0 \cup M_3} R_Q(V(F')) + R_Q(U) = \mathbf{0} + L^{\mu}_{\mathcal{P},F}(H).$$

Suppose $R_Q(U) = \mathbf{v}_0 + L^{\mu}_{\mathcal{P},F}(H)$ for some $\mathbf{v}_0 \in L^d_{\max}$; so

$$\sum_{F' \in M_0 \cup M_3} R_Q(V(F')) = -\mathbf{v}_0 + L^{\mu}_{\mathcal{P},F}(H).$$

Claim 5.1. *There exist* $F_1, \ldots, F_p \in M_0 \cup M_3$ *for some* $p \le q - 1$ *such that*

$$\sum_{i \in [p]} R_Q(V(F_i)) = -\mathbf{v}_0 + L^{\mu}_{\mathcal{P},F}(H). \tag{5.1}$$

Proof. Assume to the contrary that $F_1, \ldots, F_p \in M_0 \cup M_3$ is a minimum set of copies of F such that (5.1) holds and $p \geq q$. Consider the $p + 1$ partial sums $\sum_{i \in [j]} R_Q(V(F_i))$ for $j = 0, 1, \ldots, p$, where the sum equals $\mathbf{0} + L^\mu_{\mathcal{P},F}(H)$ when $j = 0$. Since $|Q| \leq q$, two of the partial sums must be equal, that is, there exist $0 \leq p_1 < p_2 \leq p$ such that $\sum_{p_1 < i \leq p_2} R_Q(V(F_i)) = \mathbf{0} + L^\mu_{\mathcal{P},F}(H)$. So we get a smaller collection of copies of F in $M_0 \cup M_3$ such that (5.1) holds, a contradiction. □

So we have $\sum_{i \in [p]} \mathbf{i}_{\mathcal{P}}(V(F_i)) + \mathbf{i}_{\mathcal{P}}(U) \in L^\mu_{\mathcal{P},F}(H)$. Let $Y := \bigcup_{i \in [p]} V(F_i) \cup U$ and thus $|Y| \leq mp + D \leq mq + D$. We now complete the perfect F-packing by absorption. Since $\mathbf{i}_{\mathcal{P}}(Y) \in L^\mu_{\mathcal{P},F}(H)$, we have the following equation

$$\mathbf{i}_{\mathcal{P}}(Y) = \sum_{\mathbf{v} \in I^\mu_{\mathcal{P},F}(H)} a_{\mathbf{v}} \mathbf{v},$$

where $a_{\mathbf{v}} \in \mathbb{Z}$ for all $\mathbf{v} \in I^\mu_{\mathcal{P},F}(H)$. Since $|Y| \leq qm + D$, by the definition of C', we have $|a_{\mathbf{v}}| \leq C'$ for all $\mathbf{v} \in I^\mu_{\mathcal{P},F}(H)$. Noticing that $a_{\mathbf{v}}$ may be negative, we can assume $a_{\mathbf{v}} = b_{\mathbf{v}} - c_{\mathbf{v}}$ such that one of $b_{\mathbf{v}}, c_{\mathbf{v}}$ is $|a_{\mathbf{v}}|$ and the other is zero for all $\mathbf{v} \in I^\mu_{\mathcal{P},F}(H)$. So we have

$$\sum_{\mathbf{v} \in I^\mu_{\mathcal{P},F}(H)} c_{\mathbf{v}} \mathbf{v} + \mathbf{i}_{\mathcal{P}}(Y) = \sum_{\mathbf{v} \in I^\mu_{\mathcal{P},F}(H)} b_{\mathbf{v}} \mathbf{v}.$$

This equation means that given a family $\mathcal{F} = \{W^{\mathbf{v}}_1, \ldots, W^{\mathbf{v}}_{c_{\mathbf{v}}} : \mathbf{v} \in I^\mu_{\mathcal{P},F}(H)\}$ of disjoint m-subsets of $V(H) \setminus Y$ such that $\mathbf{i}_{\mathcal{P}}(W^{\mathbf{v}}_i) = \mathbf{v}$ for all $i \in [c_{\mathbf{v}}]$, we can regard $V(\mathcal{F}) \cup Y$ as the union of disjoint m-sets $\{S^{\mathbf{v}}_1, \ldots, S^{\mathbf{v}}_{b_{\mathbf{v}}} : \mathbf{v} \in I^\mu_{\mathcal{P},F}(H)\}$ such that $\mathbf{i}_{\mathcal{P}}(S^{\mathbf{v}}_j) = \mathbf{v}$, $j \in [b_{\mathbf{v}}]$ for all $\mathbf{v} \in I^\mu_{\mathcal{P},F}(H)$. Since $c_{\mathbf{v}} \leq C'$ for all $\mathbf{v}$ and $V(M_2) \cap Y = \emptyset$, we can choose the family $\mathcal{F}$ as a subset of M_2. In summary, starting with the F-packing $M_0 \cup M_1 \cup M_2 \cup M_3$ leaving U uncovered, we delete the copies $F_1, \ldots, F_\ell$ of F from $M_0 \cup M_3$ given by Claim 5.1 and then leave $Y = \bigcup_{i \in [p]} V(F_i) \cup U$ uncovered. Then we delete the family $\mathcal{F}$ of copies of F from M_2 and leave $V(\mathcal{F}) \cup Y$ uncovered. Finally, we regard $V(\mathcal{F}) \cup Y$ as the union of at most $\binom{m+d-1}{d} C' + qm + D \leq \sqrt{\log n}/2$ m-sets S with $\mathbf{i}_{\mathcal{P}}(S) \in I^\mu_{\mathcal{P},F}(H)$.

Note that by definition, Y may intersect at most $qm + D$ absorbing sets in $\mathcal{F}_0$, which cannot be used to absorb those sets we obtained above. Since each m-set S has at least $\sqrt{\log n} - qm > \sqrt{\log n}/2 + qm + D$ absorbing (F, tm^2)-sets in $\mathcal{F}_0$, we can greedily match each S with a distinct absorbing (F, tm^2)-set $F_S \in \mathcal{F}_0$ for S. Replacing the F-packing on $V(F_S)$ in M_0 by the perfect F-packing on $H[F_S \cup S]$ for each S gives a perfect F-packing in H. □

6. Useful tools

In this section we collect together some results that will be used in our applications of Theorem 3.1. When considering ℓ-degree together with ℓ'-degree for some $\ell' \neq \ell$, the

following proposition is very useful (the proof is a standard counting argument, which we omit).

Proposition 6.1. *Let $0 \leq \ell \leq \ell' < k$ and H be a k-graph. If $\delta_{\ell'}(H) \geq x\binom{n-\ell'}{k-\ell'}$ for some $0 \leq x \leq 1$, then $\delta_\ell(H) \geq x\binom{n-\ell}{k-\ell}$.*

For the statements of the next three results, recall the definitions introduced in Section 3.3. Moreover, for any $S \subseteq V(H)$, let $N(S) := \{T \subseteq V(H) \setminus S : T \cup S \in E(H)\}$, and for simplicity, we write $N(x)$ for $N(\{x\})$.

Lemma 6.2. *[33, Lemma 4.2] Let $k, m \geq 2$ be integers and $\gamma > 0$. Let K be a k-partite k-graph of order m. There exists $0 < \alpha \ll \gamma$ such that the following holds for sufficiently large n. For any k-graph H of order n, two vertices $x, y \in V(H)$ are $(K, \alpha, 1)$-reachable to each other if the number of $(k-1)$-sets $S \in N(x) \cap N(y)$ with $|N(S)| \geq \gamma n$ is at least $\gamma^2\binom{n}{k-1}$.*

The following lemma gives us a sufficient condition for ensuring a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of a k-graph H such that for any $i \in [r]$, V_i is $(F, \beta, 2^{c-1})$-closed in H.

Lemma 6.3. *Given $\delta' > 0$, integers $c, k, m \geq 2$ and $0 < \alpha \ll 1/c, \delta', 1/m$, there exists a constant $\beta > 0$ such that the following holds for all sufficiently large n. Let F be an m-vertex k-graph. Assume H is an n-vertex k-graph and $S \subseteq V(H)$ is such that $|\tilde{N}_{F,\alpha,1}(v, H) \cap S| \geq \delta'n$ for any $v \in S$. Further, suppose every set of $c+1$ vertices in S contains two vertices that are $(F, \alpha, 1)$-reachable in H. Then in time $O(n^{2^{c-1}m+1})$ we can find a partition $\mathcal{P}$ of S into $V_1, \ldots, V_r$ with $r \leq \min\{c, 1/\delta'\}$ such that for any $i \in [r]$, $|V_i| \geq (\delta' - \alpha)n$ and V_i is $(F, \beta, 2^{c-1})$-closed in H.*

We will use the following simple result in the proof of Lemma 6.3.

Proposition 6.4. *[33, Proposition 2.1] Let F be a fixed k-graph on m vertices. For $\epsilon, \beta > 0$ and an integer $i \geq 1$, there exists a $\beta_0 = \beta_0(\epsilon, \beta, m, i) > 0$ and an integer $n_0 = n_0(\epsilon, \beta, m, i)$ satisfying the following. Suppose H is a k-graph of order $n \geq n_0$ and there exists a vertex $x \in V(H)$ with $|\tilde{N}_{F,\beta,i}(x, H)| \geq \epsilon n$. Then for all $0 < \beta' \leq \beta_0$, $\tilde{N}_{F,\beta,i}(x, H) \subseteq \tilde{N}_{F,\beta',i+1}(x, H)$.*

Next we prove Lemma 6.3, whose proof is almost identical to the proof of [12, Lemma 3.8].

Proof of Lemma 6.3. Let $\epsilon := \alpha/c$. We choose constants satisfying the following hierarchy

$$1/n \ll \beta = \beta_{c-1} \ll \beta_{c-2} \ll \cdots \ll \beta_1 \ll \beta_0 \ll \epsilon \ll 1/c, \delta', 1/m.$$

Let F and H be as in the statement of the lemma. Throughout this proof, given $v \in V(H)$ and $i \in [c-1]$, we write $\tilde{N}_{F,\beta_i,2^i}(v,H)$ as $\tilde{N}_i(v)$ for short. Note that for any $v \in V(H)$, $|\tilde{N}_0(v)| = |\tilde{N}_{F,\beta_0,1}(v,H)| \geq |\tilde{N}_{F,\alpha,1}(v,H)| \geq \delta'n$ because $\beta_0 < \alpha$. We also write 2^i-reachable (or 2^i-closed) for $(F,\beta_i,2^i)$-reachable (or $(F,\beta_i,2^i)$-closed). By Proposition 6.4 and the choice of β_is, we may assume that $\tilde{N}_i(v) \subseteq \tilde{N}_{i+1}(v)$ for all $0 \leq i < c-1$ and all $v \in V(H)$. Hence, if $W \subseteq V(H)$ is 2^i-closed in H for some $i \leq c-1$, then W is 2^{c-1}-closed.

We may assume that there are two vertices in S that are not 2^{c-1}-reachable to each other, as otherwise S is 2^{c-1}-closed in H and we obtain the desired (trivial) partition $\mathcal{P} = \{S\}$. Let r be the largest integer such that there exist $v_1, \ldots, v_r \in S$ such that no pair of them are 2^{c+1-r}-reachable in H. Note that r exists by our assumption and $2 \leq r \leq c$. Fix such $v_1, \ldots, v_r \in S$; by Proposition 6.4, we can assume that any pair of them are not 2^{c-r}-reachable in H. Consider $\tilde{N}_{c-r}(v_i)$ for all $i \in [r]$. Then we have the following facts.

(i) Any $v \in S \setminus \{v_1, \ldots, v_r\}$ must lie in $\tilde{N}_{c-r}(v_i)$ for some $i \in [r]$, as otherwise $v, v_1, \ldots, v_r$ contradicts the definition of r.

(ii) $|\tilde{N}_{c-r}(v_i) \cap \tilde{N}_{c-r}(v_j)| < \epsilon n$ for any $i \neq j$. Indeed, otherwise there are at least

$$\frac{\epsilon n}{(2^{c+1-r}m-1)!}(\beta_{c-r}n^{2^{c-r}m-1} - n^{2^{c-r}m-2})(\beta_{c-r}n^{2^{c-r}m-1} - 2^{c-r}mn^{2^{c-r}m-2})$$

reachable $(2^{c+1-r}m-1)$-sets for v_i, v_j. This follows because there are at least ϵn vertices $w \in \tilde{N}_{c-r}(v_i) \cap \tilde{N}_{c-r}(v_j)$, at least $\beta_{c-r}n^{2^{c-r}m-1} - n^{2^{c-r}m-2}$ reachable $(2^{c-r}m-1)$-sets T for v_i and w that do not contain v_j, and at least $\beta_{c-r}n^{2^{c-r}m-1} - 2^{c-r}mn^{2^{c-r}m-2}$ reachable $(2^{c-r}m-1)$-sets for v_j and w that avoid $\{v_i\} \cup T$; finally, we divide by $(2^{c+1-r}m-1)!$ to eliminate the effect of over-counting. Since $\beta_{c+1-r} \ll \epsilon, \beta_{c-r}, 1/c, 1/m$, this gives at least $\beta_{c+1-r}n^{2^{c+1-r}m-1}$ reachable $(2^{c+1-r}m-1)$-sets for v_i, v_j, contradicting the assumption that v_i, v_j are not 2^{c+1-r}-reachable to each other.

Note that (ii) and $|\tilde{N}_{c-r}(v_i) \cap S| \geq |\tilde{N}_0(v_i) \cap S| \geq \delta'n$ for $i \in [r]$ imply that $r\delta'n - \binom{r}{2}\epsilon n \leq |S| \leq n$. So we have $r \leq (1+c^2\epsilon)/\delta'$. Since $\epsilon \leq \alpha \ll \delta', 1/c$, we have $r \leq 1/\delta'$ and thus, $r \leq \min\{c, 1/\delta'\}$.

For $i \in [r]$, let $U_i := ((\tilde{N}_{c-r}(v_i) \cup \{v_i\}) \cap S) \setminus \bigcup_{j \in [r] \setminus \{i\}} \tilde{N}_{c-r}(v_j)$. Note that for $i \in [r]$, U_i is 2^{c-r}-closed in H. Indeed, if there exist $u_1, u_2 \in U_i$ that are not 2^{c-r}-reachable to each other, then $\{u_1, u_2\} \cup (\{v_1, \ldots, v_r\} \setminus \{v_i\})$ contradicts the definition of r.

Let $U_0 := S \setminus (U_1 \cup \cdots \cup U_r)$. By (i) and (ii), we have $|U_0| \leq \binom{r}{2}\epsilon n$. We will move each vertex of U_0 greedily to U_i for some $i \in [r]$. For any $v \in U_0$, since $|(\tilde{N}_0(v) \cap S) \setminus U_0| \geq \delta'n - |U_0| \geq r\epsilon n$, there exists $i \in [r]$ such that v is 1-reachable to at least ϵn vertices in U_i. In this case we add v to U_i (we add v to an arbitrary U_i if there are more than one such i). Let the resulting partition of S be $V_1, \ldots, V_r$. Note that we have $|V_i| \geq |U_i| \geq |\tilde{N}_{c-r}(v_i) \cap S| - r\epsilon n \geq |\tilde{N}_0(v_i) \cap S| - c\epsilon n \geq (\delta' - \alpha)n$. Observe that

in each V_i, the 'farthest' possible pairs are those two vertices both from U_0, which are 2^{c-r+1}-reachable to each other. Thus, each V_i is 2^{c-r+1}-closed, so 2^{c-1}-closed because $r \geq 2$.

We estimate the running time as follows. First, for every two vertices $u, v \in S$, we determine if they are 2^i-reachable for $0 \leq i \leq c - 1$. This can be done by testing if any $(2^i m - 1)$-set $T \in \binom{V(H) \setminus \{u,v\}}{2^i m - 1}$ is a reachable set for u and v, namely, if both $H[T \cup \{u\}]$ and $H[T \cup \{v\}]$ have perfect F-packings or not, which can be checked by listing the edges on them, in constant time. If there are at least $\beta_i n^{2^i m - 1}$ reachable $(2^i m - 1)$-sets for u and v, then they are 2^i-reachable. Since we need time $O(n^{2^{c-1}m-1})$ to list all $(2^{c-1}m-1)$-sets for each pair u, v of vertices, this can be done in time $O(n^{2^{c-1}m+1})$. Second, we search the set of vertices $v_1, \ldots, v_r$ such that no pair of them are 2^{c+1-r}-reachable for all $2 \leq r \leq c$. With the reachability information at hand, this can be done in time $O(n^c)$. We then fix the largest r as in the proof. If such r does not exist, then we get $\mathcal{P} = \{S\}$ and output $\mathcal{P}$. Otherwise, we fix any r-set $v_1, \ldots, v_r$ such that no pair of them are 2^{c+1-r}-reachable. We find the partition $\{U_0, U_1, \ldots, U_r\}$ by identifying $\tilde{N}_{c-r}(v_i)$ for $i \in [r]$, in time $O(n)$. Finally we move vertices of U_0 to $U_1, \ldots, U_r$, depending on $|\tilde{N}_0(v) \cap U_i|$ for $v \in U_0$ and $i \in [r]$, which can be done in time $O(n^2)$. Thus, the running time for finding a desired partition is $O(n^{2^{c-1}m+1})$. $\quad\square$

7. Tools for Theorem 1.3

In the following section we prove Theorem 1.3. Here we collect together some useful notation and results for this proof.

Let H be a k-graph. In the case of perfect matchings (i.e. when F is an edge) we write (β, i)-reachable, (β, i)-closed and $\tilde{N}_{\beta,i}(v, H)$ for (F, β, i)-reachable (F, β, i)-closed and $\tilde{N}_{F,\beta,i}(v, H)$ respectively.

The following result is a weaker version of Lemma 5.6 in [30].

Lemma 7.1. *[30] Let $k \geq 2$ and $1 \leq \ell \leq k - 1$ be integers, and let $\varepsilon > 0$. Suppose that for some $b, c \in (0, 1)$ and some $n_0 \in \mathbb{N}$, every k-graph H on $n \geq n_0$ vertices with $\delta_\ell(H) \geq cn^{k-\ell}$ has a fractional matching of size $(b + \varepsilon)n$. Then there exists an $n_0' \in \mathbb{N}$ such that any k-graph H on $n \geq n_0'$ vertices with $\delta_\ell(H) \geq (c+\varepsilon)n^{k-\ell}$ contains a matching of size at least bn.*

The next crucial result is an immediate consequence of Lemma 7.1 together with results from [9,11,40].

Theorem 7.2. *For $1 \leq \ell \leq k - 1$, $\delta(k, \ell, k) \leq \max\{1/3, c^*_{k,\ell}\}$.*

Summary of the proof of Theorem 7.2. Suppose $\varepsilon > 0$ and $n \in \mathbb{N}$ is sufficiently large. Consider an n-vertex k-graph H such that $\delta_\ell(H) \geq (\max\{1/3, c^*_{k,\ell}\} + \varepsilon)\binom{n-\ell}{k-\ell}$. To prove the theorem we must show that H contains a matching covering all but at most k vertices.

Note that for all $\ell, k \in \mathbb{N}$, $c_{k,\ell}^* \geq 1 - (1-1/k)^{k-\ell}$ (see e.g. [1]). In particular, $c_{k,1}^* \geq 1/2$ and $c_{k,k-1}^* \geq 1/k$.

The results from [40] imply that every sufficiently large n'-vertex k-graph H' with $\delta_{k-1}(H') \geq (1/k + \varepsilon)n'$ contains a matching covering all but at most k vertices. So since $c_{k,k-1}^* \geq 1/k$, this resolves the case when $\ell = k - 1$.

Next suppose that $k \geq 4$ and $2 \leq \ell \leq k - 2$. Then [11, Theorem 1.7] and [11, Proposition 1.11] together with Lemma 7.1 imply the following: every sufficiently large n'-vertex k-graph H' with $k \nmid n'$ and $\delta_{k-1}(H') \geq (\max\{1/3, c_{k,\ell}^*\} + \varepsilon/2)\binom{n'-\ell}{k-\ell}$ contains a matching covering all but at most k vertices. So with H as above we immediately obtain our desired matching if $k \nmid n$. If $k \mid n$ then let $H' := H \setminus x$ for some $x \in V(H)$ and set $n' := |H'|$. Hence $\delta_{k-1}(H') \geq (\max\{1/3, c_{k,\ell}^*\} + \varepsilon/2)\binom{n'-\ell}{k-\ell}$ and thus H' contains a matching M covering all but at most k vertices. Therefore since $k \mid n$ this implies that M covers all but precisely $k - 1$ vertices in H', and therefore all but precisely k vertices in H, as desired.

The final case is when $\ell = 1$. This case follows by Lemma 7.1 and the Strong Absorbing Lemma in [9, Lemma 2.4]. In particular, this uses that $c_{k,1}^* \geq 1/2$. $\square$

In fact, it is possible to show that $\delta(k, \ell, k) = c_{k,\ell}^*$ for any $1 \leq \ell \leq k - 1$, but Theorem 7.2 is enough for this paper.

8. Proof of Theorem 1.3

Let $\delta \in (\delta^*, 1]$ and define

$$0 < 1/n_0 \ll 1/c \ll \mu \ll \beta \ll \alpha' \ll \eta \ll \alpha \ll \gamma \ll (\delta - \delta^*), 1/k.$$

Let H be as in the statement of Theorem 1.3. Note that we may assume $n \geq n_0$ and $k \mid n$ since else the result is trivial (recall the use of big-O notation in the statement of the theorem). So

$$\delta_\ell(H) \geq (\delta^* + \gamma)\binom{n-\ell}{k-\ell} \geq (1/3 + \gamma)\binom{n-\ell}{k-\ell} \tag{8.1}$$

and in particular, by Proposition 6.1,

$$\delta_1(H) \geq (1/3 + \gamma)\binom{n-1}{k-1}. \tag{8.2}$$

Notice that by (8.2),

($*$) Every set of three vertices of $V(H)$ contains two vertices that are $(\alpha, 1)$-reachable.

Note that when $\ell = 1$, since $\delta^* \geq c_{k,1}^* \geq 1 - (1-1/k)^{k-1} > 1/2$, by [1, Theorem 1.1], H contains a perfect matching. So we may assume that $\ell > 1$.

We now split the argument into two cases.

8.1. There exists $v \in V(H)$ such that $|\tilde{N}_{\alpha,1}(v,H)| \leq \eta n$

In this case, we will show that H must contain a perfect matching. Let $W := \{v\} \cup \tilde{N}_{\alpha,1}(v,H)$ and thus $|W| \leq \eta n + 1$. For any two vertices $u, u' \in V(H) \setminus W$, since $u, u' \notin \tilde{N}_{\alpha,1}(v,H)$, by $(*)$, u and u' are $(\alpha, 1)$-reachable, i.e., $V(H) \setminus W$ is $(\alpha, 1)$-closed in H. Let $H_1 := H \setminus W$ and $n_1 := |H_1|$. Since $\eta \ll \alpha$ we have that $V(H) \setminus W = V(H_1)$ is $(\alpha/2, 1)$-closed in H_1.

By Lemma 4.1 (with $d = 1$) there is a set $T \subseteq V(H_1)$ (take $T := V(\mathcal{F}_{abs})$) such that $|T| \leq ck^2 \log n_1$ and both $H_1[T]$ and $H_1[T \cup S]$ contain perfect matchings for *any* set $S \subseteq V(H_1)$ where $|S| \in k\mathbb{N}$ and $|S| \leq \sqrt{\log n_1}$. We greedily construct a matching M in H such that $|M| \leq \eta n + 1$; $W \subseteq V(M)$; and $V(M) \cap T = \emptyset$. This is possible because of (8.2) and $|W| n^{k-2} < \frac{1}{3} \binom{n-1}{k-1}$. Let $H_2 := H \setminus (V(M) \cup T)$ and $n_2 := |H_2|$. Note that H_2 is a subgraph of H_1. By (8.1), the definition of δ^* and Theorem 7.2,

$$\delta_\ell(H_2) \geq (\delta^* + \gamma/2)\binom{n_2 - \ell}{k - \ell} \geq (\delta(k, \ell, k) + \gamma/2)\binom{n_2 - \ell}{k - \ell}.$$

Thus, by definition of $\delta(k, \ell, k)$, H_2 contains a matching M_1 covering all but at most k vertices of H_2. Let S denote the leftover set of vertices. (So $S = \emptyset$ or $|S| = k$.) Then $H[T \cup S]$ contains a perfect matching M_2. Altogether, $M \cup M_1 \cup M_2$ is a perfect matching in H, as desired.

8.2. Every vertex $v \in V(H)$ satisfies $|\tilde{N}_{\alpha,1}(v,H)| \geq \eta n$

Thus, since $\alpha' \ll \alpha$, every vertex $v \in V(H)$ satisfies $|\tilde{N}_{\alpha',1}(v,H)| \geq \eta n$. Apply Lemma 6.3 to H (with α', 2, η playing the roles of α, c and δ' respectively) to find a partition $\mathcal{P}$ of $V(H)$ into $V_1, \ldots, V_r$ with $r \leq 2$ such that for any $i \in [r]$, $|V_i| \geq \eta n/2$ and V_i is $(\beta, 2)$-closed in H, in time $O(n^{2k+1})$.

Our aim is to apply Theorem 3.1 to H. First, by Theorem 7.2 and (8.1), we have that $\delta_\ell(H) \geq (\delta(k, \ell, k) + \gamma)\binom{n-\ell}{k-\ell}$. Second, by definition, $\mathcal{P}$ is an $(E, \beta, 2, \eta/2)$-good partition of $V(H)$, where E is a k-graph on k vertices consisting of a single edge.

Write $L := L^\mu_{\mathcal{P},E}(H)$ and $Q := Q(\mathcal{P}, L^\mu_{\mathcal{P},E}(H))$. We will show that $|Q| \leq k$. Clearly, if $r = 1$, then $|Q| = 1$. So we may assume $r = 2$. First assume that $I^\mu_{\mathcal{P},E}(H)$ contains two distinct elements, say, $(a, b), (a', b') \in I^\mu_{\mathcal{P},E}(H)$ with $a \neq a'$. Thus $(a - a', b - b') = (a - a', a' - a) \in L^\mu_{\mathcal{P},E}(H)$. Any coset $(x, y) + L$ in Q must contain some element (x', y') so that $x' + y' = k$. Consider two vectors $(n_1, n_2), (n'_1, n'_2) \in L^2_{max}$ where $n_1 + n_2 = n'_1 + n'_2 = k$. If $n_1 \equiv n'_1 \pmod{|a - a'|}$ then these two vectors lie in the same coset in Q. (Indeed, by adding a multiple of $(a - a', a' - a)$ to (n_1, n_2) one can obtain (n'_1, n'_2).) Altogether this implies there are at most $|a - a'|$ cosets, i.e., $|Q| \leq |a - a'| \leq k$.

Second, assume that $I^\mu_{\mathcal{P},E}(H)$ contains exactly one element, say $I^\mu_{\mathcal{P},E}(H) = \{(a, b)\}$, where $a + b = k$. Note that it must hold that $a \geq \ell$ and $b \geq \ell$. Indeed, if $a < \ell$, then the number of edges that contain an ℓ-set of index vector $(\ell, 0)$ is at most $k\mu n^k$. Thus,

by averaging and since $\mu \ll \eta \ll 1/k$, there exists an ℓ-set S of index vector $(\ell, 0)$ such that $d_H(S) \leq \binom{k}{\ell} k \mu n^k / \binom{|V_1|}{\ell} \leq \sqrt{\mu} n^{k-\ell} < \delta_\ell(H)$, a contradiction. The same argument shows that $b \geq \ell$. Then for $0 \leq \ell_1 \leq \ell$, let $\ell_2 = \ell - \ell_1$. By averaging, for each $0 \leq \ell_1 \leq \ell$, there exists an ℓ-set S_{ℓ_1} of index vector (ℓ_1, ℓ_2) such that

$$d_H(S_{\ell_1}) \leq \binom{|V_1| - \ell_1}{a - \ell_1}\binom{|V_2| - \ell_2}{b - \ell_2} + \frac{\binom{k}{\ell} k \mu n^k}{\binom{|V_1|}{\ell_1}\binom{|V_2|}{\ell_2}} \leq \binom{|V_1|}{a - \ell_1}\binom{|V_2|}{b - \ell_2} + \sqrt{\mu} n^{k-\ell}.$$

Recall the identity $\sum_{0 \leq i \leq t} \binom{n_1}{i}\binom{n_2}{t-i} = \binom{n_1 + n_2}{t}$, so we have

$$\sum_{0 \leq \ell_1 \leq \ell} d_H(S_{\ell_1}) \leq \binom{n}{k - \ell} + k\sqrt{\mu} n^{k-\ell} \leq \binom{n - \ell}{k - \ell} + 2k\sqrt{\mu} n^{k-\ell}.$$

Since $\ell \geq 2$ and $a, b \geq \ell$, the above sum contains at least three terms. As $\mu \ll \gamma \ll 1/k$, there exists ℓ_1 such that $d_H(S_{\ell_1}) \leq \frac{1}{3}\binom{n-\ell}{k-\ell} + 2k\sqrt{\mu} n^{k-\ell} < (\frac{1}{3} + \gamma)\binom{n-\ell}{k-\ell}$, contradicting (8.1). That is, the case when $I_{\mathcal{P},E}^\mu(H)$ contains one element does not occur.

Therefore we can apply Theorem 3.1 to H with $D = q = k$, $t = 2$ and $c = \eta/2$ and thus conclude that H contains a perfect matching if and only if $(\mathcal{P}, L_{\mathcal{P},E}^\mu(H))$ is k-soluble.

The algorithm. Now we state our algorithm. First, for every two vertices $u, v \in V(H)$, we determine if they are $(\alpha, 1)$-reachable, which can be done by testing if any $(k-1)$-set is a reachable set in time $O(n^{k-1})$. So this step can be done in time $O(n^{k+1})$. Then we check if $|\tilde{N}_{\alpha,1}(v, H)| \geq \eta n$ for every $v \in V(H)$. With the reachability information, this can be tested in time $O(n^2)$. If $|\tilde{N}_{\alpha,1}(v, H)| < \eta n$ for some $v \in V(H)$, then we output PM and halt. Otherwise we run the algorithm with running time $O(n^{2k+1})$ provided by Lemma 6.3 and get a partition $\mathcal{P}$. By Theorem 3.1, it remains to test if $(\mathcal{P}, L_{\mathcal{P},E}^\mu(H))$ is k-soluble. This can be done by testing whether any matching M of size at most k is a solution of $(\mathcal{P}, L_{\mathcal{P},E}^\mu(H))$, in time $O(n^{k^2})$. If there is a solution M for $(\mathcal{P}, L_{\mathcal{P},E}^\mu(H))$, output PM; otherwise output NO. The overall running time is $O(n^{k^2})$.

9. The perfect graph packing result

In this section we prove Theorem 1.11. Let F be an m-vertex k-chromatic graph. By the definition of $\chi_{cr}(F)$, we have

$$\frac{1}{\chi_{cr}(F)} = \frac{m - \sigma(F)}{(k-1)m} \leq \frac{m-1}{(k-1)m}. \tag{9.1}$$

We will apply the following variant of Lemma 6.2, which can be easily derived from the original version by defining a k-graph G' where each k-set forms a hyperedge if and only if it spans a copy of K_k in G. For any vertex $u \in V(G)$, let $W(u)$ denote the collection of $(k-1)$-sets S such that $S \subseteq N(u)$ and such that S spans a clique in G. For a set $T \subseteq V(G)$, let $N(T) := \bigcap_{v \in T} N(v)$.

Lemma 9.1. *[33] Let $k, m \in \mathbb{N}$ and $\gamma' > 0$. There exists $\alpha = \alpha(k, m, \gamma') > 0$ such that the following holds for sufficiently large n. Let F be a k-chromatic graph on m vertices. For any n-vertex graph G, two vertices $x, y \in V(G)$ are $(F, \alpha, 1)$-reachable if the number of $(k-1)$-sets $S \in W(x) \cap W(y)$ with $|N(S)| \geq \gamma' n$ is at least $(\gamma')^2 \binom{n}{k-1}$.*

We apply Lemma 9.1 to prove the following result.

Proposition 9.2. *Let $k, m, n \geq 2$ be integers and $\alpha, \gamma > 0$ where $0 < 1/n \ll \alpha \ll \gamma \ll 1/m, 1/k$. Let F be a k-chromatic graph on m vertices and let G be an n-vertex graph with $\delta(G) \geq (1 - 1/\chi_{cr}(F) + \gamma)n$. Then for any $v \in V(G)$, $|\tilde{N}_{F,\alpha,1}(v, G)| \geq (1/m + \gamma/2)n$.*

Proof. For each $(k-1)$-set S, since $\delta(G) \geq (1 - 1/\chi_{cr}(F) + \gamma)n$, by (9.1) we have $|N(S)| \geq (1/m + (k-1)\gamma)n$. Then by Lemma 9.1, for any distinct $u, v \in V(G)$, $u \in \tilde{N}_{F,\alpha,1}(v, G)$ if $|W(u) \cap W(v)| \geq \gamma^2 \binom{n}{k-1}$. By double counting, we have

$$\sum_{S \in W(v)} (|N(S)| - 1) \leq |\tilde{N}_{F,\alpha,1}(v, G)| \cdot |W(v)| + n \cdot \gamma^2 \binom{n}{k-1}.$$

Note that any S in the above inequality is a $(k-1)$-set, thus $|N(S)| \geq (1/m + (k-1)\gamma)n$. On the other hand, using the minimum degree condition, it is easy to see that $|W(v)| \geq \frac{1}{m^{k-1}} \binom{n}{k-1}$. Since $\gamma \ll 1/m, 1/k$, we have

$$|\tilde{N}_{F,\alpha,1}(v, G)| \geq (1/m + (k-1)\gamma)n - 1 - \frac{\gamma^2 n^k}{|W(v)|} \geq (1/m + \gamma/2)n. \quad \square$$

The following proposition shows that $|Q(\mathcal{P}, L^{\mu}_{\mathcal{P},F}(G))|$ is bounded from above.

Proposition 9.3. *Let $t, r, k, m, n_0 \in \mathbb{N}$ where $k \geq 2$ and let $\beta, \mu, \gamma > 0$ so that*

$$1/n_0 \ll \beta, \mu \ll \gamma \ll 1/m, 1/t.$$

Let F be an unbalanced m-vertex k-chromatic graph. Suppose G is a graph on $n \geq n_0$ vertices such that $\delta(G) \geq (1 - 1/\chi_{cr}(F) + \gamma)n$ with an $(F, \beta, t, 1/m)$-good partition $\mathcal{P}$ where $|\mathcal{P}| = r$. Then $|Q(\mathcal{P}, L^{\mu}_{\mathcal{P},F}(G))| \leq (2m - 1)^r$.

We need the following simple counting result, which, for example, follows from the result of Erdős [6] on supersaturation. (Note that this result of Erdős will be later applied, as Proposition 10.1, in the proof of Theorem 1.13.)

Proposition 9.4. *Given $\gamma' > 0$, $\ell_1, \ldots, \ell_k \in \mathbb{N}$, there exists $\mu > 0$ such that the following holds for sufficiently large n. Let T be an n-vertex graph with a vertex partition $V_1 \cup \cdots \cup V_d$. Suppose $i_1, \ldots, i_k \in [d]$ are not necessarily distinct and T contains at least $\gamma' n^k$ copies of K_k with vertex set $\{v_1, \ldots, v_k\}$ such that $v_1 \in V_{i_1}, \ldots, v_k \in V_{i_k}$. Then T*

contains at least $\mu n^{\ell_1 + \cdots + \ell_k}$ copies of $K^{(2)}(\ell_1, \ldots, \ell_k)$ whose jth part is contained in V_{i_j} for all $j \in [k]$.

We write $\mathbf{u}_j$ for the 'unit' 1-vector that has 1 in coordinate j and 0 in all other coordinates.

Proof of Proposition 9.3. Write $L := L^{\mu}_{\mathcal{P}, F}(G)$. It suffices to show that for any element $\mathbf{v} \in L^r_{\max}$, there exists $\mathbf{v}' = (v'_1, \ldots, v'_r) \in L^r_{\max}$ such that $-(m-1) \leq v'_i \leq m-1$ for all $i \in [r]$ and $\mathbf{v} + L = \mathbf{v}' + L$. In particular, the number of such $\mathbf{v}'$ is at most $(2m-1)^r$. Since F is unbalanced, there exists a k-colouring with colour class sizes $a_1 \leq \cdots \leq a_k$ and $a_1 < a_k$. Set $a := a_k - a_1 < m$.

Let $\mathcal{P} = \{V_1, \ldots, V_r\}$ be the partition of $V(G)$ given in the statement of the proposition. Define a graph P on the vertex set $[r]$ such that $(i, j) \in E(P)$ if and only if $e(G[V_i, V_j]) \geq \gamma n^2$. We claim that if i and j are connected by a path in P, then $a(\mathbf{u}_i - \mathbf{u}_j) \in L$. Indeed, first assume that $(i, j) \in E(P)$. For each edge uv in $G[V_i, V_j]$, since

$$\delta(G) \geq (1 - 1/\chi_{cr}(F) + \gamma)n \overset{(9.1)}{\geq} \left(1 - \frac{m-1}{(k-1)m} + \gamma\right)n,$$

it is easy to see that uv is contained in at least $\frac{1}{m^{k-2}}\binom{n}{k-2}$ copies of K_k in G. So there are at least $\gamma n^2 \cdot \frac{1}{m^{k-2}}\binom{n}{k-2}/\binom{k}{2}$ copies of K_k in G intersecting both V_i and V_j. By averaging, there exists a k-array $(i_1, \ldots, i_k)$, $i_j \in [r]$ where $i_1 = i$ and $i_k = j$ such that G contains at least

$$\frac{1}{r^{k-2}}\gamma n^2 \cdot \frac{1}{m^{k-2}}\binom{n}{k-2}/\binom{k}{2} \geq \frac{\gamma}{m^{k-2}r^{k-2}k!}n^k$$

copies of K_k with vertex set $\{v_1, \ldots, v_k\}$ such that $v_1 \in V_{i_1}, \ldots, v_k \in V_{i_k}$. By applying Proposition 9.4 with $\ell_i := a_i$ for each $i \in [k]$, we get that there are at least μn^m copies of $K^{(2)}(a_1, \ldots, a_k)$ in G whose jth part is contained in V_{i_j} for all $j \in [k]$. We apply Proposition 9.4 again, this time with $\ell_i := a_i$ for all $2 \leq i \leq k-1$ and $\ell_1 := a_k$, $\ell_k := a_1$ and thus conclude that there are at least μn^m copies of $K^{(2)}(a_k, a_2, \ldots, a_{k-1}, a_1)$ (with a_1 and a_k exchanged) in G whose jth part is contained in V_{i_j} for all $j \in [k]$. Taking subtraction of index vectors of these two types of copies gives that $a(\mathbf{u}_i - \mathbf{u}_j) \in L$. Furthermore, note that if i and j are connected by a path in P, we can apply the argument above to every edge in the path and conclude that $a(\mathbf{u}_i - \mathbf{u}_j) \in L$, so the claim is proved.

We now distinguish two cases.

Case 1: $k \geq 3$. In this case, we first show that P is connected. Indeed, we prove that for any bipartition $A \cup B$ of $[r]$, there exists $i \in A$ and $j \in B$ such that $(i, j) \in E(P)$. Let $V_A := \bigcup_{i \in A} V_i$ and $V_B := \bigcup_{j \in B} V_j$. Without loss of generality, assume that $|V_A| \leq n/2$.

Since $\delta(G) \geq (1 - \frac{m-1}{(k-1)m})n \geq (\frac{1}{2} + \frac{1}{2m})n$, the number of edges in G that are incident to V_A is at least

$$|V_A| \cdot \left(\frac{1}{2} + \frac{1}{2m}\right)n - \binom{|V_A|}{2} \geq \binom{|V_A|}{2} + \frac{n}{2m}|V_A| \geq \binom{|V_A|}{2} + \gamma n^2 |A||B|,$$

where the last inequality follows since $|B| \leq r$, $|V_A| \geq |A|n/m$ and $\gamma \ll 1/m$. By averaging, there exists $i \in A$ and $j \in B$ such that $e(G[V_i, V_j]) \geq \gamma n^2$ and thus $(i,j) \in E(P)$.

Now let $\mathbf{v} = (v_1, \ldots, v_r) \in L^r_{\max}$. We fix an arbitrary m-vector $\mathbf{w} \in L$ and let $\mathbf{v}_1 := \mathbf{v} - (|\mathbf{v}|/m)\mathbf{w}$. So $|\mathbf{v}_1| = 0$ and $\mathbf{v}_1 + L = \mathbf{v} + L$. Since P is connected, the claim above implies that for any $i, j \in [r]$, $a(\mathbf{u}_i - \mathbf{u}_j) \in L$.

We now apply the following algorithm to $\mathbf{v}_1$. Suppose v_i^1 is the coordinate of $\mathbf{v}_1$ with the largest absolute value. If $|v_i^1| \leq m - 1$ we terminate the algorithm. Otherwise, since $|v_i^1| \geq m > |\mathbf{v}_1|$, there is some coordinate v_j^1 of $\mathbf{v}_1$ which has the opposite sign of v_i^1. We now redefine $\mathbf{v}_1$ by (i) subtracting $a(\mathbf{u}_i - \mathbf{u}_j) \in L$ from $\mathbf{v}_1$ if $v_i^1 \geq m$ or (ii) adding $a(\mathbf{u}_i - \mathbf{u}_j) \in L$ to $\mathbf{v}_1$ if $v_i^1 \leq -m$. Note that still $|\mathbf{v}_1| = 0$ and $|v_i^1|$ has decreased.

We repeat this algorithm until we obtain a vector $\mathbf{v}' = (v_1', \ldots, v_r')$ so that $|\mathbf{v}'| = 0$ and $-(m-1) \leq v_i' \leq m - 1$ for all $i \in [r]$. Note that $\mathbf{v}'$ was obtained from $\mathbf{v}_1$ by repeatedly adding and subtracting elements of L to $\mathbf{v}_1$. Since initially $\mathbf{v}_1 + L = \mathbf{v} + L$ we have that $\mathbf{v}' + L = \mathbf{v} + L$, as desired.

Case 2: $k = 2$. In this case we cannot guarantee that P is connected (we may even have some isolated vertices). First let i be an isolated vertex in P. By the definition of P, we know that $e(G[V_i, V \setminus V_i]) \leq (r-1)\gamma n^2$. Since $\delta(G) \geq n/m$,

$$e(G[V_i]) \geq \frac{1}{2}(|V_i|n/m - (r-1)\gamma n^2) \geq \frac{1}{4m}|V_i|^2.$$

Applying Proposition 9.4 on V_i shows that there are at least μn^m copies of $K^{(2)}(a_1, a_2)$ in $G[V_i]$, i.e., $m\mathbf{u}_i \in L$. Second, if $(i,j) \in E(P)$, then applying Proposition 9.4 to $G[V_i, V_j]$ gives that $a_1 \mathbf{u}_i + a_2 \mathbf{u}_j \in L$. So in both cases, for any component C in P, there exists an m-vector $\mathbf{w}_C \in L$ such that $\mathbf{w}_C|_{[r] \setminus C} = \mathbf{0}$.

Now let $\mathbf{v} = (v_1, \ldots, v_r) \in L^r_{\max}$. Consider the connected components $C_1, C_2, \ldots, C_q$ of P, for some $1 \leq q \leq r$. Let $\mathbf{v}_1$ be obtained as follows: for each component C, we add to $\mathbf{v}$ a multiple of the vector $\mathbf{w}_C$ such that $0 \leq |\mathbf{v}_1|_{C}| \leq m - 1$. Note that by definition, $\mathbf{v} - \mathbf{v}_1 \in L$. Next, recall that for each component C_i, if $j, j' \in C_i$, then $a(\mathbf{u}_j - \mathbf{u}_{j'}) \in L$. Let v_j^1 be the coordinate of $\mathbf{v}_1|_{C_i}$ with the largest absolute value. If $|v_j^1| \geq m > |\mathbf{v}_1|_{C_i}|$, then there is some coordinate v_j^1 of $\mathbf{v}_1|_{C_i}$ which has the opposite sign of v_i^1. So by using an analogous algorithm to the one in Case 1, we can obtain a vector $\mathbf{v}' = (v_1', \ldots, v_r')$ such that $\mathbf{v}' - \mathbf{v}_1 \in L$, $|\mathbf{v}'| = |\mathbf{v}_1|$ and $-(m-1) \leq v_i' \leq m - 1$ for all $i \in [r]$. We are done since $\mathbf{v}' + L = \mathbf{v}_1 + L = \mathbf{v} + L$. $\square$

Now we are ready to prove Theorem 1.11.

Proof of Theorem 1.11. We first note that it suffices to prove Theorem 1.11 in the case when F is unbalanced. Indeed, if F is balanced then $\chi(F) = \chi_{cr}(F)$ and so the result follows (trivially) from Theorem 1.6.

Given any $\delta \in (1 - 1/\chi_{cr}(F), 1]$ let $\mu, \alpha, \gamma > 0$ so that $0 < \mu \ll \alpha \ll \gamma \ll (\delta - 1 + 1/\chi_{cr}(F)), 1/m, 1/k$. Apply Lemma 6.3 with $c := m^{k-1}$, $\delta' := 1/m + \gamma/2$ to obtain some $\beta > 0$. We may assume $\beta \ll \alpha$. Finally choose $n_0 \in \mathbb{N}$ such that $1/n_0 \ll \beta, \mu$. Altogether we have

$$1/n_0 \ll \beta, \mu \ll \alpha \ll \gamma \ll (\delta - 1 + 1/\chi_{cr}(F)), 1/m, 1/k.$$

Let G be an n-vertex graph as in the statement of Theorem 1.11. We may assume that $n \geq n_0$ and m divides n since else the result is trivial. Note that $\delta(G) \geq \delta n \geq (1 - 1/\chi_{cr}(F) + \gamma)n$.

By Proposition 9.2, for any $v \in V(G)$, $|\tilde{N}_{F,\alpha,1}(v, G)| \geq \delta'n$. The degree condition and Lemma 9.1 imply that, for distinct $u, v \in V(G)$, u and v are $(F, \alpha, 1)$-reachable if $|W(u) \cap W(v)| \geq \gamma^2\binom{n}{k-1}$. Further, for any $u \in V(G)$, the minimum degree condition implies that $|W(u)| \geq \frac{1}{c}\binom{n-1}{k-1}$ (recall $c := m^{k-1}$). We claim that any set of $c + 1$ vertices $u_0, \ldots, u_c$ in $V(G)$ contains two vertices that are $(F, \alpha, 1)$-reachable. Indeed, since $|W(u)| \geq \frac{1}{c}\binom{n-1}{k-1}$ for any $0 \leq i \leq c$ and $\gamma \ll 1/m$, we have

$$\sum_{0 \leq i \leq c} |W(u_i)| \geq \frac{c+1}{c}\binom{n-1}{k-1} > \left(1 + \binom{c+1}{2}\gamma^2\right)\binom{n}{k-1}.$$

Thus, there exist distinct $0 \leq i, j \leq c$ such that $|W(u_i) \cap W(u_j)| \geq \gamma^2\binom{n}{k-1}$, namely, they are $(F, \alpha, 1)$-reachable.

So we can apply Lemma 6.3 to G to obtain a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of $V(G)$ in time $O(n^{2^{c-1}m+1})$. Note that $|V_i| \geq (\delta' - \alpha)n \geq n/m$ for all $i \in [r]$. Also $r \leq 1/\delta' \leq m$ and each V_i is $(F, \beta, 2^{c-1})$-closed in H. Thus, $\mathcal{P}$ is an $(F, \beta, 2^{c-1}, 1/m)$-good partition of $V(G)$.

Note that Theorem 1.8 shows that $\delta(F, 1, 5m^2) \leq 1 - 1/\chi_{cr}(F)$ and thus $\delta(G) \geq (1 - 1/\chi_{cr}(F) + \gamma)n \geq (\delta(F, 1, 5m^2) + \gamma)n$. Moreover, Proposition 9.3 shows that $|Q(\mathcal{P}, L^\mu_{\mathcal{P},F}(G))| \leq (2m - 1)^r$. So by Theorem 3.1 with $D := 5m^2$ and $q := (2m - 1)^r$, we conclude that G contains a perfect F-packing if and only if $(\mathcal{P}, L^\mu_{\mathcal{P},F}(G))$ is $(2m - 1)^r$-soluble.

The algorithm. Now we state the algorithm and estimate the running time. We run the algorithm with running time $O(n^{2^{m^{k-1}-1}m+1})$ provided by Lemma 6.3 and obtain a partition $\mathcal{P}$ of $V(G)$. By Theorem 3.1, it remains to test if $(\mathcal{P}, L^\mu_{\mathcal{P},F}(G))$ is $(2m - 1)^r$-soluble. This can be done by testing whether any F-packing M of size at most $(2m - 1)^r$ is a q-solution of $(\mathcal{P}, L^\mu_{\mathcal{P},F}(G))$, in time $O(n^{m(2m-1)^r}) = O(n^{m(2m-1)^m})$. If there is a q-solution M for $(\mathcal{P}, L^\mu_{\mathcal{P},F}(G))$, output YES; otherwise output NO. The overall running time is $O(n^{\max\{2^{m^{k-1}-1}m+1,\, m(2m-1)^m\}})$. $\quad\square$

10. Packing k-partite k-uniform hypergraphs

In this section we prove Theorem 1.13. For this we will first collect together a few useful results. Throughout this section we consider a (not necessarily complete) k-partite k-graph F on m vertices, and let a be the minimum of the size of the smallest vertex class over all k-partite realisations of $V(F)$. Let $K(F) \supseteq F$ be a complete k-partite k-graph on m vertices such that the smallest vertex class has a vertices. We will also write $\sigma(F) := a/m$.

The next proposition is a supersaturation result of Erdős [6].

Proposition 10.1. *Let $\eta > 0$, $k, r \in \mathbb{N}$ and let $K := K^{(k)}(a_1, \ldots, a_k)$ be the complete k-partite k-graph with $a_1 \leq \cdots \leq a_k$ vertices in each class. There exists $0 < \mu \ll \eta$ such that the following holds for sufficiently large n. Let H be an k-graph on n vertices with a vertex partition $V_1 \cup \cdots \cup V_r$. Consider not necessarily distinct $i_1, \ldots, i_k \in [r]$. Suppose H contains at least ηn^k edges $e = \{v_1, \ldots, v_k\}$ such that $v_1 \in V_{i_1}, \ldots, v_k \in V_{i_k}$. Then H contains at least $\mu n^{a_1 + \cdots + a_k}$ copies of K whose jth part is contained in V_{i_j} for all $j \in [k]$.*

We also use the following result of Mycroft [35, Theorem 1.5] which forces an almost perfect F-packing.

Theorem 10.2. *[35] Let F be a k-partite k-graph. There exists a constant $D = D(F)$ such that for any $\alpha > 0$ there exists an $n_0 = n_0(F, \alpha)$ such that any k-graph H on $n \geq n_0$ vertices with $\delta_{k-1}(H) \geq \sigma(F)n + \alpha n$ admits an F-packing covering all but at most D vertices of H.*

The following proposition shows that $Q(\mathcal{P}, L^{\mu}_{\mathcal{P}, F}(H))$ has bounded size.

Proposition 10.3. *Let $t, r, k, n_0 \in \mathbb{N}$ so that $k \geq 3$ and $\beta, \mu, \gamma > 0$ so that*

$$1/n_0 \ll \beta, \mu \ll \gamma, 1/m, 1/t, 1/r.$$

Let F be a k-partite k-graph on m vertices. Suppose H is a k-graph on $n \geq n_0$ vertices such that $\delta_{k-1}(H) \geq (\sigma(F) + \gamma)n$ with an $(F, \beta, t, 1/m)$-good partition $\mathcal{P}$ where $|\mathcal{P}| = r$. Then $|Q(\mathcal{P}, L^{\mu}_{\mathcal{P}, F}(H))| \leq (2m - 1)^r$.

Proof. Write $L := L^{\mu}_{\mathcal{P}, F}(H)$. It suffices to show that for any element $\mathbf{v} \in L^r_{\max}$, there exists $\mathbf{v}' = (v'_1, \ldots, v'_r) \in L^r_{\max}$ such that $-(m-1) \leq v'_i \leq m - 1$ for all $i \in [r]$ and $\mathbf{v} - \mathbf{v}' \in L$. In particular, the number of such $\mathbf{v}'$ is at most $(2m - 1)^r$.

Let $\mathcal{P} = \{V_1, \ldots, V_r\}$ be the partition of $V(H)$ given in the statement of the proposition. Fix any $i \in [r]$ and consider all edges that contain at least $k - 1$ vertices from V_i. Since $\delta_{k-1}(H) \geq (a/m + \gamma)n$, there are at least $\frac{1}{k}\binom{|V_i|}{k-1}(a/m + \gamma)n$ such edges. By averaging, there exists $j_i \in [r]$ (it may be that $j_i = i$) such that H contains at least

$$\frac{1}{r} \cdot \frac{1}{k} \binom{|V_i|}{k-1} (a/m + \gamma)n \geq \frac{1}{m^k k! r} n^k$$

edges with vertex set $\{v_1, \ldots, v_k\}$ such that $v_1 \in V_{j_i}$ and $\{v_2, \ldots, v_k\} \subseteq V_i$. (Here we used $|V_i| \geq n/m$ and $1/n \ll \gamma$.) By applying Proposition 10.1, since $\mu \ll 1/(m^k k! r)$, we get that there are at least μn^m copies of $K(F)$ in H whose vertex class of size a is contained in V_{j_i} and other vertex classes are contained in V_i. This means that $(m-a)\mathbf{u}_i + a\mathbf{u}_{j_i} \in L$ for each $i \in [r]$.

Now let $\mathbf{v} = (v_1, \ldots, v_r) \in L_{\max}^r$ and let $l(\mathbf{v}) := \sum_{i\in[r]} |v_i|$. We do the following process iteratively. For an intermediate step, let $\mathbf{v}^* = (v_1^*, \ldots, v_r^*)$ be the current vector and suppose v_i^* is the coordinate with the largest absolute value. We thus subtract $(m-a)\mathbf{u}_i + a\mathbf{u}_{j_i}$ from $\mathbf{v}^*$ if $v_i^* \geq m-a$ or add $(m-a)\mathbf{u}_i + a\mathbf{u}_{j_i}$ to $\mathbf{v}^*$ if $v_i^* \leq a-m$. Note that this process will end because after each step $l(\mathbf{v}^*) = \sum_{i\in[r]} |v_i^*|$ decreases by at least $m - 2a > 0$. This means that we will reach a vector $\mathbf{v}' = (v_1', \ldots, v_r') \in L_{\max}^r$ such that $-(m-1) \leq v_i' \leq m-1$ for all $i \in [r]$ and $\mathbf{v} - \mathbf{v}' \in L$. So we are done. $\square$

Proof of Theorem 1.13. Let $D := D(F)$ be given by Theorem 10.2. Given any $\delta \in (\sigma(F), 1]$ let $\mu, \alpha, \gamma > 0$ so that $0 < \mu \ll \alpha \ll \gamma \ll (\delta - \sigma(F)), 1/D, 1/m$. Apply Lemma 6.3 with $c := m$, $\delta' := 1/m + \gamma/2$ to obtain some $\beta > 0$. We may assume $\beta \ll \alpha$. Finally choose $n_0 \in \mathbb{N}$ such that $1/n_0 \ll \beta, \mu$. Altogether we have

$$1/n_0 \ll \beta, \mu \ll \alpha \ll \gamma \ll (\delta - \sigma(F)), 1/D, 1/m.$$

Let H be an n-vertex k-graph as in the statement of Theorem 1.13. Note that we may assume that $n \geq n_0$ and m divides n since else the result is trivial. We have that $\delta_{k-1}(H) \geq \delta n \geq (\sigma(F) + \gamma)n$. By Proposition 6.1, we have $\delta_1(H) \geq \delta\binom{n-1}{k-1} \geq (\sigma(F) + \gamma)\binom{n-1}{k-1}$.

First, for every $v \in V(H)$, we give a lower bound on $|\tilde{N}_{F,\alpha,1}(v, H)|$. Note that for any $(k-1)$-set $S \subseteq V(H)$, we have $|N(S)| \geq (\sigma(F) + \gamma)n$. Then by Lemma 6.2, for any distinct $u, v \in V(H)$, $u \in \tilde{N}_{F,\alpha,1}(v, H)$ if $|N(u) \cap N(v)| \geq \gamma^2\binom{n}{k-1}$. By double counting, we have

$$\sum_{S \in N(v)} (|N(S)| - 1) < |\tilde{N}_{F,\alpha,1}(v, H)| \cdot |N(v)| + n \cdot \gamma^2 \binom{n}{k-1}.$$

Note that $|N(v)| \geq \delta_1(H) \geq \delta\binom{n-1}{k-1}$. Since $\gamma \ll \delta, 1/k$, we have that

$$|\tilde{N}_{F,\alpha,1}(v, H)| > (\sigma(F) + \gamma)n - 1 - \frac{\gamma^2 n^k}{|N(v)|} \geq (\sigma(F) + \gamma/2)n \geq \left(\frac{1}{m} + \frac{\gamma}{2}\right)n. \quad (10.1)$$

Next we claim that every set A of $m+1$ vertices in $V(H)$ contains two vertices that are $(F, \alpha, 1)$-reachable in H. Indeed, since $\delta_1(H) \geq \delta\binom{n-1}{k-1}$, the degree sum of any $m+1$

vertices is at least $(m+1)\delta\binom{n-1}{k-1}$. Since $\gamma \ll 1/m$, we have

$$(m+1)\delta\binom{n-1}{k-1} > \left(1 + \binom{m+1}{2}\gamma\right)\binom{n}{k-1}.$$

Thus, there exist distinct $u, v \in A$ such that $|N(u) \cap N(v)| \geq \gamma\binom{n}{k-1}$, and so they are $(F, \alpha, 1)$-reachable by Lemma 6.2.

By (10.1) and the above claim, we can apply Lemma 6.3 to H with the constants chosen at the beginning of the proof. We get a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ of $V(H)$ such that $r \leq m$ and for any $i \in [r]$, $|V_i| \geq (\sigma(F) + \gamma/2 - \alpha)n \geq n/m$ and V_i is $(F, \beta, 2^{m-1})$-closed in H. Thus, $\mathcal{P}$ is a $(F, \beta, 2^{m-1}, 1/m)$-good partition of $V(H)$.

Note that Theorem 10.2 shows that $\delta(F, k-1, D) \leq \sigma(F)$ and thus $\delta_{k-1}(H) \geq (\sigma(F) + \gamma)n \geq (\delta(F, k-1, D) + \gamma)n$. Moreover, Proposition 10.3 shows that $|Q(\mathcal{P}, L^\mu_{\mathcal{P}, F}(H))| \leq (2m-1)^r$. So by Theorem 3.1, with $q := (2m-1)^r$, we conclude that H contains a perfect F-packing if and only if $(\mathcal{P}, L^\mu_{\mathcal{P}, F}(H))$ is $(2m-1)^r$-soluble.

The algorithm. Now we state the algorithm and estimate the running time. We run the algorithm with running time $O(n^{2^{m-1}m+1})$ provided by Lemma 6.3 and obtain a partition $\mathcal{P}$ of $V(H)$. By Theorem 3.1, it remains to test if $(\mathcal{P}, L^\mu_{\mathcal{P}, F}(H))$ is $(2m-1)^r$-soluble. This can be done by testing whether any F-packing M in H of size at most $(2m-1)^r$ is a q-solution of $(\mathcal{P}, L^\mu_{\mathcal{P}, F}(H))$, in time $O(n^{m(2m-1)^r}) = O(n^{m(2m-1)^m})$. If there is a q-solution M for $(\mathcal{P}, L^\mu_{\mathcal{P}, F}(H))$, output YES; otherwise output NO. Since $m \geq 3$ and thus $2^{m-1}m + 1 < m(2m-1)^m$, the overall running time is $O(n^{m(2m-1)^m})$. □

11. Concluding remarks

In this paper we introduced a general structural theorem (Theorem 3.1) which can be used to determine classes of (hyper)graphs for which the decision problem for perfect F-packings is polynomial time solvable. We then gave three applications of this result. It would be interesting to find other applications of Theorem 3.1.

In light of Conjecture 1.2 it is likely that one can replace the condition that $\delta^* = \max\{1/3, c^*_{k,\ell}\}$ in Theorem 1.3 with $\delta^* = c^*_{k,\ell}$. Theorem 3.1 is likely to be useful for this. However, note that in the proof of Theorem 1.3, the condition $\delta^* \geq 1/3$ ensured that the partition $\mathcal{P}$ of $V(H)$ consisted of at most 2 vertex classes. We then showed that our hypergraph H contained a perfect matching or that the coset group Q had bounded size. In particular, since $|\mathcal{P}| \leq 2$ it was relatively straightforward to show that $|Q|$ was bounded. However, if we no longer have that $\delta^* \geq 1/3$ we may have that $\mathcal{P}$ consists of many classes. Thus, determining that Q has bounded size is likely to be substantially harder in this case.

In Theorems 1.3, 1.11 and 1.13 we provided algorithms for *determining* whether a hypergraph contains a perfect matching or packing. It would be interesting to obtain analogous results which *produce* a perfect matching or packing if such a structure exists.

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