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Shifts of finite type as fundamental objects in the theory of shadowing

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Abstract Shifts of finite type and the notion of shadowing, or pseudo-orbit tracing, are powerful tools in the study of dynamical systems. In this paper we prove that there is a deep and fundamental relationship between these two concepts. Let X be a compact totally disconnected space and $f : X \to X$ a continuous map. We demonstrate that f has shadowing if and only if the system (f, X) is (conjugate to) the inverse limit of a directed system satisfying the Mittag-Leffler condition and consisting of shifts of finite type. In particular, this implies that, in the case that X is the Cantor set, f has shadowing if and only if (f, X) is the inverse limit of a sequence satisfying the Mittag-Leffler condition and consisting of shifts of finite type. Moreover, in the general compact metric case, where X is not necessarily totally disconnected, we prove that f has shadowing if (f, X) is a factor of the inverse limit of a sequence satisfying the Mittag-Leffler condition and consisting of shifts of finite type by a quotient that almost lifts pseudo-orbits.

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1 Introduction

Given a finite set of symbols, a shift of finite type consists of all infinite (or bi-infinite) symbol sequences, which do not contain any of a finite list of forbidden words, under the action of the shift map. Shifts of finite type have applications across mathematics, for example in Shannon's theory of information [26] and statistical mechanics. In particular, they have proved to be a powerful and ubiquitous tool in the study of hyperbolic dynamical systems. Adler and Weiss [1] and Sinai [31], for example, obtain Markov partitions for hyperbolic automorphisms of the torus and Anosov diffeomorphisms respectively, allowing analysis via shifts of finite type. Generalising the notion of Anosov diffeomorphisms, Smale [33] isolates subsystems conjugate to shifts of finite type in certain Axiom A diffeomorphisms. His fundamental example of a horseshoe, conjugate to the full shift space on two symbols, captures the chaotic behaviour of the diffeomorphism on the nonwandering set where the map exhibits hyperbolic behaviour. Bowen [6] then shows that the nonwandering set of any Axiom A diffeomorphism is a factor of a shift of finite type. In fact, shifts of finite type appear as horseshoes in many systems both hyperbolic (for example [34,36]) and otherwise [18].

For a map f on a metric space X, a sequence $\langle x_i \rangle_{i \in \omega}$ is a δ-pseudo-orbit if $d(f(x_i), x_{i+1}) < \delta$. Pseudo-orbits arise naturally in the numerical calculation of orbits. It turns out that pseudo-orbits can often be tracked within a specified tolerance by real orbits, in which case f is said to have the shadowing, or pseudo-orbit tracing, property. Clearly this is of importance when trying to model a system numerically (for example [9,10,20,21]), especially when the system is expanding and errors might grow exponentially (indeed shadowing follows from expansivity for open maps [25], see also [23]). However, shadowing is also of theoretical importance and the notion can be traced back to the analysis of Anosov and Axiom A diffeomorphisms. Sinai [32] isolated subsystems of Anosov diffeomorphisms with shadowing and Bowen [5] proved explicitly that for the larger class of Axiom A diffeomorphisms, the shadowing property holds on the nonwandering set. However, Bowen [6] had already used shadowing implicitly as a key step in his proof that the nonwandering set of an Axiom A diffeomorphism is a factor of a shift of finite type. The notion of structural stability of a dynamical system was instrumental in the definitions of both Anosov and Axiom A diffeomorphisms [33] and shadowing plays a key role in stability theory [22,24,35]. Shadowing is also key to characterizing omega-limit sets [2,5,19]. Moreover, fundamental to the current paper is Walters' result [35] that a shift space has shadowing if and only if it is of finite type.

In this paper we prove that there is a deep and fundamental relationship between shadowing and shifts of finite type. It is known that shadowing is generic for homeomorphisms of the Cantor set [3] and that the shifts of finite type form a dense subset of the space of homeomorphisms on the Cantor set [27]. Hirsch [17] shows that expanding differentiable maps on closed manifolds are factors of the full one sided shift. In [6], Bowen considers the induced dynamics on the shift spaces associated with Markov partitions to show that the action of an Axiom A diffeomorphism on its non-wandering set is a factor of a shift of finite type. Here we expand the scope of this type of analysis by considering the actions induced by f on shift spaces associated with several arbitrary finite open covers of the state space X, rather than the much more specific Markov partitions. In doing so, we are able to extend and clarify these results significantly, proving the following.

Theorem 18 *Let X be a compact, totally disconnected Hausdorff space. The map $f : X \to X$ has shadowing if and only if (f, X) is conjugate to the inverse limit of an inverse system satisfying the Mittag-Leffler condition and consisting of shifts of finite type.*

Corollary 19 *Let X be the Cantor set, or indeed any compact, totally disconnected metric space. The map $f : X \to X$ has shadowing if and only if (f, X) is conjugate to the inverse limit of a sequence satisfying the Mittag-Leffler condition and consisting of shifts of finite type.*

Let X and Y be compact metric spaces and $\phi : X \to Y$ be a factor map between the systems $f : X \to X$ and $g : Y \to Y$ (so that $\phi\big(f(x)\big) = g\big(\phi(x)\big)$). We say that ϕ *almost lifts pseudo-orbits* (ϕ *is ALP)* if and only if for all $\epsilon > 0$ and $\eta > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $\langle y_i \rangle$ in Y, there exists an η-pseudo-orbit $\langle x_i \rangle$ in X such that $d(\phi(x_i), y_i) < \epsilon$.

This notion is also well defined in general Hausdorff spaces (Definition 24).

Theorem 26 *Let X be a compact Hausdorff space. The map $f : X \to X$ has shadowing if (f, X) lifts via a map which is ALP to the inverse limit of an inverse system satisfying the Mittag-Leffler condition and consisting of shifts of finite type.*

Corollary 28 *Let X be a compact metric space. The map $f : X \to X$ has shadowing only if (f, X) lifts via a map which is ALP to an inverse limit of a sequence of shifts of finite type. Additionally, $f : X \to X$ has shadowing if (f, X) lifts via a map which is ALP to the inverse limit of a sequence satisfying the Mittag-Leffler condition and consisting of shifts of finite type.*

The approach we take is topological rather than metric as this seems to provide the most natural proofs and allows for simple generalizations.

Although we are considering inverse limits of dynamical systems, our techniques are very similar in flavour to the inverse limit of coupled graph covers which have been used by a number of authors to study dynamics on Cantor sets, for example [3,11–13,28–30].

The paper is arranged as follows. In Sect. 2, we formally define shadowing, shift of finite type and the inverse limit of a direct set of dynamical systems. In Sect. 3, we characterize shadowing as a topological, rather than metric property, and prove that an inverse limit satisfying the Mittag-Leffler condition which consists of systems with shadowing itself has shadowing (Theorem 8). Here we also introduce the orbit and pseudo-orbit shift spaces associated with a finite open cover of a dynamical system and observe in Theorem 12 that these capture the dynamics of f. Section 4 discusses compact, totally disconnected Hausdorff, but not necessarily metric, dynamical systems, showing that such systems have shadowing if and only if they are (conjugate to) the inverse limit of a directed system satisfying the Mittag-Leffler condition and consisting of shifts of finite type (Theorem 18). In Sect. 5, we examine the case of systems on general metric spaces, establishing in Theorem 20 a partial analogue to Theorem 18 and Corollary 19. In Sect. 6, we discuss factor maps which preserve shadowing and, in light of this, we are able to partially characterize compact metric and Hausdorff systems with shadowing in Theorem 26 and Corollary 28.

2 Preliminaries and definitions

By map, we mean a continuous function. The set of natural numbers, including 0, is denoted by ω. A dynamical system, (f, X), consists of a topological space X and a map $f : X \to X$. In what follows, X need not necessarily be metric, but will typically be compact Hausdorff. Given two dynamical systems (f, X) and (g, Y), a factor map (or semiconjugacy) from (f, X) to (g, Y) is a map $\phi : X \to Y$ that commutes with the dynamics, i.e. $\phi \circ f = g \circ \phi$. In this case, we say that the system (g, Y) lifts via ϕ to the system (f, X). A factor map which is a homeomorphism is called a conjugacy.

Definition 1 Let X be a compact metric space and let $f : X \to X$ be a continuous function. Let $\langle x_i \rangle_{i \in \omega}$ be a sequence in X. Then $\langle x_i \rangle_{i \in \omega}$ is a δ-*pseudo-orbit* provided $d(x_{i+1}, f(x_i)) < \delta$ for all $i \in \omega$ and the point z ϵ-*shadows* $\langle x_i \rangle_{i \in \omega}$ provided $d(x_i, f^i(z)) < \epsilon$ for all $i \in \omega$.

The map f has *shadowing* (or the *pseudo-orbit tracing property*) provided that for all $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit is ϵ-shadowed by a point.

A particularly nice characterization of shadowing exists if we restrict our attention to *shift spaces*. For a finite set Σ, the *full one-sided shift with alphabet* Σ consists of the space of infinite sequence in Σ, i.e. Σ^ω using the product topology on the discrete space Σ and the *shift map* σ, given by

$$\sigma\langle x_i\rangle = \langle x_{i+1}\rangle.$$

A *shift space* is a compact invariant subset X of some full-shift. A shift space X is a *shift of finite type over alphabet* Σ if there is a finite collection $\mathcal{F}$ of finite words in Σ for which $\langle x_i\rangle \in \Sigma^\omega$ belongs to X if and only if for all $i \leq j$, the word $x_i x_{i+1} \cdots x_j \notin \mathcal{F}$. A shift of finite type is said to be N-*step* provided that the length of the longest word in its associated set of forbidden words $\mathcal{F}$ is $N+1$. As mentioned above, a shift space has shadowing if and only if it a shift of finite type [35].

Inverse limit constructions arise in a variety of settings. Many of the results here hold for arbitrary (non-metric) compact Hausdorff spaces and so we consider inverse limits of dynamical systems taken along an arbitrary directed set. (Recall that $(\Lambda, \leq)$ is a directed set provided $\leq$ is a transitive order for which any pair x, y has an upper bound $x, y \leq z$.) The reader, however, will not miss much by assuming that the space is compact metric in which case the inverse limit may be indexed by $\mathbb{N}$.

Definition 2 Let $(\Lambda, \leq)$ be a directed set. For each $\lambda \in \Lambda$, let X_λ be a compact Hausdorff space and, for each pair $\lambda \leq \eta$, let $g_\lambda^\eta : X_\eta \to X_\lambda$ be a continuous map. Then $(g_\lambda^\eta, X_\lambda)$ is called an *inverse system* provided that

(1) g_λ^λ is the identity map, and
(2) for $\lambda \leq \eta \leq \nu$, $g_\lambda^\nu = g_\lambda^\eta \circ g_\eta^\nu$.

The *inverse limit of* $(g_\lambda^\eta, X_\lambda)$ is the space

$$\varprojlim\{g_\lambda^\eta, X_\lambda\} = \{\langle x_\lambda\rangle \in \Pi X_\lambda : \forall \lambda \leq \eta \; x_\lambda = g_\lambda^\eta(x_\eta)\}$$

with topology inherited as a subspace of the Tychonoff product ΠX_λ.

Since the inverse limit of compact Hausdorff spaces is a closed subset of the product space, it is itself compact and Hausdorff. The following easily proved fact is often useful. If $U \subseteq \varprojlim\{g_\lambda^\eta, X_\lambda\}$ is open, and $x \in U$, then there exists λ and $U_\lambda \subseteq X_\lambda$ open with $x \in \pi_\lambda^{-1}(U_\lambda) \cap \varprojlim\{g_\lambda^\eta, X_\lambda\} \subseteq U$. That is, the collection of sets of the form $\pi_\lambda^{-1}(U_\lambda) \cap \varprojlim\{g_\lambda^\eta, X_\lambda\}$ for U_λ open in X_λ forms a basis for $\varprojlim\{g_\lambda^\eta, X_\lambda\}$. Additionally, it is also worth noting that, in this formulation, if the bonding maps g_λ^η are surjective, then the restricted projection maps $\pi_\gamma|\varprojlim\{g_\lambda^\eta, X_\lambda\} : \varprojlim\{g_\lambda^\eta, X_\lambda\} \to X_\gamma$ are also surjective. This is easily observed as follows. Fix $\gamma \in \Lambda$ and $z \in X_\gamma$. For $\mu \geq \gamma$, define $A_\mu = \{\langle x_\lambda\rangle \in \Pi X_\lambda : \forall \lambda \leq \mu \; x_\lambda = g_\lambda^\mu(x_\mu)\} \cap \{\langle x_\lambda\rangle \in \Pi X_\lambda : x_\gamma = z\}$. Since each g_λ^η is surjective, this is nonempty and compact. Furthermore, if $\nu \geq \mu$, it is clear that $A_\nu \subseteq A_\mu$. Hence, the intersection $A = \bigcap A_\mu$ is nonempty, consists only of points belonging to the inverse limit, and has $\pi_\gamma(A) = z$.

Now, suppose that for each λ in the directed set Λ, $f_\lambda : X_\lambda \to X_\lambda$ is a continuous function. If the bonding maps g_λ^η commute with the functions f_λ, then we can extend this definition to the family of dynamical systems $\{(f_\lambda, X_\lambda) : \lambda \in \Lambda\}$. Specifically we make the following definition.

Definition 3 Let $(\Lambda, \leq)$ be a directed set. For each $\lambda \in \Lambda$, let (f_λ, X_λ) be a dynamical system on a compact Hausdorff space and, for each pair $\lambda \leq \eta$, let $g_\lambda^\eta : X_\eta \to X_\lambda$ be a continuous map. Then $(g_\lambda^\eta, (f_\lambda, X_\lambda))$ is called an *inverse system* provided that

(1) g_λ^λ is the identity map, and
(2) for $\lambda \leq \eta \leq \nu$, $g_\lambda^\nu = g_\lambda^\eta \circ g_\eta^\nu$, and
(3) for $\lambda \leq \eta$, $f_\lambda \circ g_\lambda^\eta = g_\lambda^\eta \circ f_\eta$.

The *inverse limit of* $(g_\lambda^\eta, (f_\lambda, X_\lambda))$ is the dynamical system$((f_\lambda)_*, \varprojlim\{g_\lambda^\eta, X_\lambda\})$, where $(f_\lambda)_*$ is the *induced map* given by

$$(f_\lambda)_*\big(\langle x_\lambda\rangle\big) = \big(f_\lambda(x_\lambda)\big).$$

Note that $(f_\lambda)_*$ is the restriction of the product map $\prod f_\lambda$ to the inverse limit $\varprojlim\{g_\lambda^\eta, X_\lambda\}$ and is, therefore, continuous. Moreover, it is easy to check that $(f_\lambda)_*$ maps the inverse limit into itself, and thus $((f_\lambda)_*, \varprojlim\{g_\lambda^\eta, X_\lambda\})$ is indeed a continuous dynamical system.

Given a map $f : X \to X$ from a compact metric or Hausdorff space to itself, one is frequently interested in the inverse limit space $\varprojlim(X, f) = \{(x_i) : f(x_{i+1}) = x_i\}$ under the action of the shift map $\sigma\langle x_i\rangle = \langle x_{i+1}\rangle$. We note that such spaces are a special case of Definition 3 applied with $\Lambda = \omega$, $X_n = X$ and $f_n = g_n^{n+1} = f$ for all $n \in \omega$.

An argument similar to that for the surjectivity of the restricted projection maps demonstrates that if each of the bonding maps g_λ^η and each of the maps f_λ is surjective, then the induced map $(f_\lambda)_*$ is also surjective. However, although many of the inverse limits under consideration in this paper do not have surjective bonding maps, they do satisfy a less stringent condition.

Definition 4 An inverse system (of spaces or of dynamical systems) satisfies the *Mittag-Leffler condition* provided that for all $\lambda \in \Lambda$, there exists $\gamma \geq \lambda$ such that for each $\eta \geq \gamma$, we have $g_\lambda^\gamma(X_\gamma) = g_\lambda^\eta(X_\eta)$.

For simplicity of notation going forward, we will say that an inverse system is an *ML inverse system* if it satisfies the Mittag-Leffler condition. The Mittag-Leffler condition for inverse sequences was defined by Grothendieck [15, Definition 13.1.2] although it is implicit in Bourbaki [4, Chapter II, Theorem 1] (see, for example, [16] for more on the ML condition).

We note that an inverse system with surjective bonding maps automatically satisfies the Mittag-Leffler condition. Moreover, in a system satisfying the Mittag-Leffler condition, if γ witnesses the condition with respect to μ and $x \in g_\mu^\gamma(X_\gamma) \subseteq X_\mu$, then $\pi_\mu^{-1}(x) \cap \varprojlim\{g_\lambda^\eta, X_\lambda\} \neq \varnothing$.

It is well known that for any inverse system there is an inverse system satisfying the Mittag-Leffler condition (in fact, one with surjective bonding maps) which has the same inverse limit. To see this, for each factor space X_λ define $\tilde{X}_\lambda = \bigcap_{\eta>\lambda} g_\lambda^\eta(X_\eta)$ and define the bonding map $\tilde{g}_\lambda^\eta : \tilde{X}_\eta \to \tilde{X}_\lambda$ to be the restriction of g_λ^η. The inclusion maps from $\tilde{X}_\lambda \to X_\lambda$ induce a map between the inverse limits which is easily seen to be a homeomorphism. However, in making this modification, we lose information about the factor spaces. Indeed, every system on a compact, totally disconnected Hausdorff space is conjugate to an inverse limit of shifts of finite type. Thus, by the above argument, every system on a compact, totally disconnected Hausdorff space is conjugate to the inverse limit of an inverse system satisfying the Mittag-Leffler condition, but consisting of subshifts which may or may not be of finite type. As we shall see, not every such system has the shadowing property—only those systems which are conjugate to an inverse limit of an inverse system satisfying the Mittag-Leffler condition *and* consisting of shifts of finite type have the shadowing property.

3 Shadowing without metrics

Shadowing is on first inspection a metric property, and indeed the properties of metrics often play a role in its investigation and application. However, shadowing can be viewed as a strictly topological property, defined in terms of finite open covers, provided that we restrict our attention to compact metric spaces. Similar observations have been made in [8,14].

We assume in what follows that the elements of a cover $\mathcal{U}$ are non empty open sets.

Definition 5 Let X be a space, let $f : X \to X$, and let $\mathcal{U}$ be a finite open cover of X.

(1) The sequence $\langle x_i\rangle_{i\in\omega}$ is a $\mathcal{U}$*-pseudo-orbit* provided for every $i \in \omega$, there exists $U_{i+1} \in \mathcal{U}$ with $x_{i+1}, f(x_i) \in U_{i+1}$.
(2) Let $\langle U_i\rangle$ be a sequence of elements of $\mathcal{U}$. We say that $\langle U_i\rangle$ is a $\mathcal{U}$*-pseudo-orbit pattern* provided there is a sequence $\langle x_i\rangle$ of points in X such that $x_{i+1}, f(x_i) \in U_{i+1}$ for each i. We say that $\langle U_i\rangle$ is a $\mathcal{U}$*-orbit pattern* provided there is some z such that $f^i(z) \in U_i$ for all i.
(3) The point $z \in X$ $\mathcal{U}$-shadows $\langle x_i\rangle_{i\in\omega}$ provided for each $i \in \omega$ there exists $U_i \in \mathcal{U}$ with $x_i, f^i(z) \in U_i$.

Lemma 6 *Let X be a compact metric space. Then $f : X \to X$ has shadowing if and only if for every finite open cover $\mathcal{U}$, there exists a finite open cover $\mathcal{V}$, such that every $\mathcal{V}$-pseudo-orbit is $\mathcal{U}$-shadowed by some point $z \in X$.*

Proof First, suppose that f has the shadowing property and let $\mathcal{U}$ be a finite open cover of X. Fix $\epsilon > 0$ so that for each ϵ-ball B in X, there exists $U \in \mathcal{U}$ with $\overline{B} \subseteq U$. Now, let $\delta > 0$ witness ϵ-shadowing. Let $\mathcal{V}$ be a finite open cover of X refining $\mathcal{U}$ which consists of open sets of diameter less that δ.

Now, Let $\langle x_i \rangle$ be a sequence in X as in the statement of the lemma. Then $d(x_i, f(x_{i-1})) < diam(V_i) < \delta$ for all $i \in \omega \setminus 0$. In particular, $\langle x_i \rangle$ is a δ-pseudo-orbit. Let $z \in X$ be an ϵ-shadowing point for this sequence. Then $d(x_i, f^i(z)) < \epsilon$ for each $i \in \omega$, and in particular, $\{x_i, f^i(x)\} \subseteq B_\epsilon(x_i)$. By construction, there exists $U_i \in \mathcal{U}$ for which $\{x_i, f^i(x)\} \subseteq B_\epsilon(x_i) \subseteq U_i$, satisfying the conclusion of the lemma.

Conversely, let us suppose that f satisfies the open cover condition of the lemma. Let $\epsilon > 0$, and consider a finite subcover $\mathcal{U}$ of X consisting of $\epsilon/2$-balls. Let $\mathcal{V}$ be the cover that witnesses the satisfaction of the condition, and choose $\delta > 0$ such that for each δ-ball in X, there is an element of $\mathcal{V}$ which contains it.

Now, fix a δ-pseudo-orbit $\langle x_i \rangle$. Then for each $i \in \omega \setminus 0$, $d(x_i, f(x_{i-1})) < \delta$, and hence there exists $V_i \in \mathcal{V}$ such that $x_i, f(x_{i-1}) \in V_i$. Let $z \in X$ be the point guaranteed by the open cover condition. Then, for each $i \in \omega$, there exists $U_i \in \mathcal{U}$ with $x_i, f^i(z) \in U_i$. But U_i is an $\epsilon/2$-ball and hence $d(x_i, f^i(x)) < \epsilon$, i.e. z ϵ-shadows the pseudo-orbit. □

This observation allows the decoupling of shadowing from the metric, and we can then take the following definition of shadowing, which is valid for systems with compact Hausdorff (but not necessarily metric) domain, an application that has recently seen increased interest [7,8,14].

Definition 7 Let X be a (nonempty) compact Hausdorff topological space. The map $f : X \to X$ has *shadowing* provided that for every finite open cover $\mathcal{U}$, there exists a finite open cover $\mathcal{V}$ such that every $\mathcal{V}$-pseudo-orbit is $\mathcal{U}$-shadowed by a point of X.

Clearly if the cover $\mathcal{V}$ witnesses $\mathcal{U}$-shadowing, the cover $\mathcal{V}'$ refines $\mathcal{V}$ and the cover $\mathcal{U}'$ is refined by $\mathcal{U}$, then $\mathcal{V}'$ witnesses $\mathcal{U}'$-shadowing.

With this definition in mind, we can prove the following result which will be important to the characterization of shadowing in Sect. 4.

Theorem 8 *Let $f : X \to X$ be conjugate to an ML inverse system consisting of maps with shadowing on compact spaces. Then f has shadowing.*

Proof Without loss, let $(\Lambda, \leq)$ be a directed set and $(f, X) = \varprojlim\{g^\lambda_\gamma, (f_\lambda, X_\lambda)\}$ be an ML inverse system where each of (f_λ, X_λ) is a system with shadowing on a compact space.

Let $\mathcal{U}$ be a finite open cover of X. Since $X = \underleftarrow{\lim}\{g_\gamma^\lambda, X_\lambda\}$, we can find λ and a finite open cover $\mathcal{W}_\lambda$ of X_λ so that $\mathcal{W} = \{\pi_\lambda^{-1}(W) \cap X : W \in \mathcal{W}_\lambda\}$ refines $\mathcal{U}$. There is some $\gamma \geq \lambda$ such that for each $\eta \geq \gamma$, we have $g_\lambda^\gamma(X_\gamma) = g_\lambda^\eta(X_\eta)$.

Let $W_\gamma = \{(g_\lambda^\gamma)^{-1}(W) : W \in \mathcal{W}_\lambda\}$. Since f_γ has shadowing, there is a finite non-empty open cover $\mathcal{V}_\gamma$ of X_γ such that every $\mathcal{V}_\gamma$-pseudo orbit is $\mathcal{W}_\gamma$-shadowed.

Let $\mathcal{V} = \{\pi_\gamma^{-1}(V) \cap X : V \in \mathcal{V}_\gamma\}$ and let $\langle x_i \rangle$ be a $\mathcal{V}$-pseudo-orbit with pattern $\langle \pi_\gamma^{-1}(V_i) \cap X \rangle$. Then for each $i \in \omega$, we have $f(x_i) \in f(\pi_\gamma^{-1}(V_i) \cap X) \cap \pi_\gamma^{-1}(V_{i+1}) \cap X \neq \varnothing$. It follows then, that $\langle (x_i)_\gamma \rangle$ is a $\mathcal{V}_\gamma$-pseudo-orbit with pattern $\langle V_i \rangle$. Since every $\mathcal{V}_\gamma$-pseudo orbit is $\mathcal{W}_\gamma$-shadowed, there is some $z_\gamma \in X_\gamma$ and a sequence $\langle W_i \rangle$ of elements from $\mathcal{W}_\gamma$ such that $f_\gamma^i(z_\gamma), (x_i)_\gamma \in W_i$. Note that this means that $z_\gamma \in \bigcap f_\gamma^{-i}(W_i) \neq \varnothing$.

Now each W_i is the inverse image of some element of $\mathcal{W}_\lambda$, so equivalently, we have a sequence $\langle W_i' \rangle \in \mathcal{W}_\lambda$ with $(x_i)_\lambda \in W_i'$ and with $z_\lambda = g_\lambda^\gamma(z_\gamma) \in g_\lambda^\gamma(\bigcap f_\gamma^{-i}(W_i)) \subseteq \bigcap f_\lambda^{-i}(W_i') \neq \varnothing$. In particular, since this is an ML inverse system, there is some $z \in X$ with $z_\lambda \in g_\lambda^\gamma(\bigcap f_\gamma^{-i}(W_i)) \subseteq \bigcap f_\lambda^{-i}(W_i')$.

It then follows that $f^i(z), x_i \in \pi_\lambda^{-1}(W_i') \cap X$, so that $\langle x_i \rangle$ is $\mathcal{W}$-shadowed, and hence $\mathcal{U}$-shadowed as required. $\square$

We wish to capture the dynamics of the map f on X via action of the shift map induced by f on the space of orbit or pseudo-orbit patterns for certain covers of X. If $\mathcal{U}$ is an open cover then one can prove (as in Lemma 11) that the collection of all $\mathcal{U}$-pseudo orbit patterns forms a closed subspace of the product space $\mathcal{U}^\omega$ of all sequences of elements from $\mathcal{U}$ (where $\mathcal{U}$ is given the discrete topology). This implies that the collection of all pseudo-orbit patterns of a finite open cover is a subshift of $\mathcal{U}^\omega$ (and is indeed a shift of finite type). However, the space of all orbit patterns need not even be closed subset of $\mathcal{U}^\omega$. To see this consider the period doubling map $\theta \mapsto 2\theta \mod 2\pi$ on the unit circle $[0, 2\pi)$. Let $U_1 = (0, \pi)$ and $U_2 = (\pi - \epsilon, 0 + \epsilon)$ for some suitably small ϵ. For each n, choose z_n near to 0 such that $f^{n+1}(z_n) = \pi$ and $f^{n+2}(z) = 0$. Then z_n generates an orbit pattern $\langle V_{n,i} \rangle$ such that $V_{n,i}$ is U_1 for $i \leq n$ and U_2 for $i > n$. Clearly the sequence of sequences $\langle V_{n,i} \rangle$ converges to the constant sequence $\langle U_1, U_1, U_1, \dots \rangle$, which is not an orbit pattern for the cover $\{U_1, U_2\}$, so that the collection of orbit patterns is not closed in $\mathcal{U}^\omega$. With this in mind, we have the following definitions.

Definition 9 Let $f : X \to X$ be a map on the space X, let $\mathcal{U}$ be a finite open cover of X, each element of which is nonempty and let $\mathcal{U}^\omega$ be the one-sided shift space on the alphabet $\mathcal{U}$ with shift map σ.

(1) The *$\mathcal{U}$-orbit space* is the set $\mathcal{O}(\mathcal{U}) \subseteq \mathcal{U}^\omega$ which is the closure of the set consisting of all sequences $\langle U_i \rangle$ in $\mathcal{U}$ for which there exists $z \in X$ with $f^i(z) \in U_i$.

(2) The $\mathcal{U}$*-pseudo-orbit space* is the set $\mathcal{PO}(\mathcal{U}) \subseteq \mathcal{U}^\omega$ consisting of all sequences $\langle U_i \rangle$ in $\mathcal{U}$ for which there exists a sequence $\langle x_i \rangle$ with $x_{i+1}, f(x_i) \in U_{i+1}$.

Additionally, for $U \in \mathcal{U}$ and $i \in \omega$, define $\pi_i : \mathcal{U}^\omega \to \mathcal{U}$ to be projection onto the i-th coordinate.

The following lemma is immediate and provides an alternate description of $\mathcal{O}(\mathcal{U})$ and $\mathcal{PO}(\mathcal{U})$.

Lemma 10 *Let $f : X \to X$ be a map on X and let $\mathcal{U}$ be a finite open cover. Then*

$$\begin{aligned}\mathcal{O}(\mathcal{U}) &= \overline{\{\langle U_i \rangle \in \mathcal{U}^\omega : \bigcap f^{-i}(U_i) \neq \varnothing\}} \\ &= \bigcap_{n\in\omega} \{\langle U_i \rangle \in \mathcal{U}^\omega : \bigcap_{i\le n} f^{-i}(U_i) \neq \varnothing\}\end{aligned}$$

and

$$\mathcal{P}O(\mathcal{U}) = \{\langle U_i \rangle \in \mathcal{U}^\omega : f(U_i) \cap U_{i+1} \neq \varnothing\}.$$

As consequence, we have the following relations between $\mathcal{O}(\mathcal{U})$, $\mathcal{PO}(\mathcal{U})$ and $\mathcal{U}^\omega$.

Lemma 11 *Let $f : X \to X$ be a map on X and let $\mathcal{U}$ be a finite open cover. Then, $\mathcal{O}(\mathcal{U})$ is a subset of $\mathcal{PO}(\mathcal{U})$ and both spaces are subshifts of $\mathcal{U}^\omega$. In particular, $\mathcal{PO}(\mathcal{U})$ is a 1-step shift of finite type.*

Proof That $\mathcal{O}(\mathcal{U}) \subseteq \mathcal{PO}(\mathcal{C}) \subseteq \mathcal{U}^\omega$ is immediate. It is also clear that each of these spaces is shift invariant. Hence, since $\mathcal{O}(\mathcal{U})$ is closed by definition, it is a subshift.

That $\mathcal{PO}(\mathcal{U})$ is a 1-step subshift of finite type follows by observing that if $\langle U_i \rangle$ is not a pseudo-orbit pattern, then for some i, there $f(U_i) \cap U_{i+1} = \varnothing$, and so we can forbid $\langle U_i \rangle$ from $\mathcal{PO}(\mathcal{U})$ by forbidding the word $U_i U_{i+1}$. Clearly there are only finitely many such words. □

If X is compact Hausdorff, then the entire dynamics of a map f are encoded in the orbit spaces of an appropriate system of covers of X. In particular, let $\mathcal{FOC}(X)$ be the collection of all finite open covers of X. This collection is naturally partially ordered by refinement and forms a directed set.

Theorem 12 *Let $f : X \to X$ be a map on the compact Hausdorff space X. Let $\{\mathcal{U}_\lambda\}_{\lambda\in\Lambda}$ be a cofinal directed subset of $\mathcal{FOC}(X)$. Then for all $x \in X$ there exists a choice of $U_\lambda(x) \in \mathcal{U}_\lambda$ with $\{x\} = \bigcap U_\lambda(x)$ and furthermore for any such sequence, we have for all $n \in \omega$,*

$$\{f^n(x)\} = \bigcap \pi_0 \left(\sigma^n \left(\mathcal{O}(\mathcal{U}_\lambda) \cap \pi_0^{-1}(U_\lambda(x))\right)\right).$$

Proof Let f, Λ, and $\{\mathcal{U}_\lambda\}$ be as described. Fix $x \in X$. For each $\lambda \in \Lambda$, choose $U_\lambda(x) \in \mathcal{U}_\lambda$ with $x \in U_\lambda(x)$. Then $x \in \bigcap U_\lambda(x)$. Furthermore, for all $y \in X \setminus \{x\}$, there exists a cover $\mathcal{U}$ of X such that if $x \in U \in \mathcal{U}$, then $y \notin \overline{U}$. Then for any $\lambda \in \Lambda$ with $\mathcal{U}_\lambda$ refining $\mathcal{U}$, $y \notin \overline{U_\lambda(x)}$ regardless of choice of $U_\lambda(x)$, hence $y \notin \bigcap \overline{U_\lambda(x)}$ and therefore $y \notin \bigcap U_\lambda$.

Now, for each λ, it is straightforward to show that $\pi_0 \left(\sigma^n \left(\mathcal{O}(\mathcal{U}_\lambda) \cap \pi_0^{-1}(U_\lambda(x))\right)\right)$ is equal to $f^n(U_\lambda(x))$. In particular, $f^n(x) \in \bigcap \pi_0 \left(\sigma^n \left(\mathcal{O}(\mathcal{U}_\lambda) \cap \pi_0^{-1}(U_\lambda(x))\right)\right)$. Suppose now that $z \in \bigcap \pi_0 \left(\sigma^n \left(\mathcal{O}(\mathcal{U}_\lambda) \cap \pi_0^{-1}(U_\lambda(x))\right)\right)$. Then for each λ, there exists $x_\lambda \in U_\lambda(x)$ with $z = f^n(x_\lambda)$. But, by construction, x is a limit point of $\{x_\lambda\}$, and by continuity, $z = f^n(x)$. Hence $\{f^n(x)\} = \bigcap \pi_0 \left(\sigma^n \left(\mathcal{O}(\mathcal{U}_\lambda) \cap \pi_0^{-1}(U_\lambda(x))\right)\right)$ as claimed. □

It should be noted that for general Hausdorff spaces, the structure of $\{\mathcal{U}_\lambda\}$ may be quite complex, but for metric X, it is the case that a sequence of covers will always suffice and we will make use of this fact in the following sections. In the metric case, Theorem 12 is equivalent to Theorem 3.9 of [28], although that result is expressed in terms of graph covers and relations.

4 Characterizing shadowing in totally disconnected spaces

In the sense of Theorem 12, the entire dynamics are encoded by the action of f on an appropriate collection of refining covers. This is not unlike the way that the topology is completely encoded as well. In this section we explore this analogy.

In particular, it is well known that a space X is *chainable*, i.e. can be encoded with a sequence of refining *chains* (i.e. finite covers with $U_i \cap U_j \neq \varnothing$ if and only if $|i - j| \leq 1$) if and only if X can be written as an inverse limit of arcs. In a sense, the arc is the *fundamental* chainable object. In an analogous fashion we show that shifts of finite type are the fundamental objects among dynamical systems on totally disconnected spaces with shadowing.

Without loss of generality, in the case that X is totally disconnected compact Hausdorff, the cofinal directed subset of $\mathcal{FOC}(X)$ in Theorem 12 can be taken to consist of open covers which are each finite collections of pairwise disjoint open (and hence also closed) sets. For the purposes of the following, we refer to such finite pairwise disjoint open covers as *partitions* of X.

Let $\mathcal{U}$ and $\mathcal{V}$ be arbitrary covers of X with $\mathcal{V}$ refining $\mathcal{U}$. Then let $\iota : \mathcal{V} \to \mathcal{U}$ be defined so that $V \cap \iota(V) \neq \varnothing$. In the case that $\mathcal{U}$ and $\mathcal{V}$ are both partitions, this is equivalent to asking $V \subseteq \iota(V)$, so that ι is a well-defined function.

In general, if $\mathcal{U}$ and $\mathcal{V}$ are not partitions, ι is a multifunction and it is this that creates the obstacle to dealing with non-totally disconnected spaces. We address this issue in Sect. 5. When considering partitions, the map ι naturally induces a continuous map $\iota : \mathcal{V}^\omega \to \mathcal{U}^\omega$, the domain of which can then be restricted to $\mathcal{O}(\mathcal{V})$ or $\mathcal{PO}(\mathcal{V})$ as appropriate. As the intended domain is typically clear, the symbol ι will be used for all.

Note that if $\mathcal{U}$ is a partition of X, its elements are necessarily compact, and in this case, it is easy to see that the set of orbit sequences is naturally closed, so that

$$\mathcal{O}(\mathcal{U}) = \{\langle U_i \rangle \in \mathcal{U}^\omega : \bigcap f^{-i}(U_i) \neq \varnothing\}.$$

Lemma 13 *Let $f : X \to X$ be a map on the compact, totally disconnected Hausdorff space X and let $\mathcal{U}$ and $\mathcal{V}$ be partitions of X with $\mathcal{V}$ refining $\mathcal{U}$. Then σ and ι commute and the following statements hold:*

(1) $\iota(\mathcal{O}(\mathcal{V})) = \mathcal{O}(\mathcal{U})$.
(2) $\mathcal{O}(\mathcal{U}) \subseteq \iota(\mathcal{PO}(\mathcal{V})) \subseteq \mathcal{PO}(\mathcal{U})$.

Proof It is immediate from their definitions that σ and ι commute on their unrestricted domains.

Towards proving statement (1), consider $\langle V_i \rangle \in \mathcal{O}(\mathcal{V})$. Then there is x such that $f^i(x) \in V_i$ for all i. But then $f^i(x) \in \iota(V_i)$, so that $\langle \iota(V_i) \rangle \in \mathcal{O}(\mathcal{U})$. Conversely, if $\langle U_i \rangle \in \mathcal{O}(\mathcal{U})$, then we can choose $x \in \bigcap f^{-i}(U_i) \neq \varnothing$. Now choose $\langle V_i \rangle$ so that $x \in f^{-1}(V_i)$ for each i. Clearly $\langle V_i \rangle \in \mathcal{O}(\mathcal{V})$. Since $\mathcal{V}$ refines $\mathcal{U}$ and the elements of $\mathcal{U}$ are pairwise disjoint and clopen, it follows that $V_i \subseteq U_i$, i.e. $\langle U_i \rangle = \iota \langle V_i \rangle$.

Statement (2) follows similarly. □

The additional structure of totally disconnected spaces allows us to state the following immediate corollary to Theorem 12. In particular, the collection $\mathcal{P}art(X)$ of partitions of X is a cofinal directed subset of $\mathcal{FOC}(X)$. Since the elements of the covers in $\mathcal{P}art(X)$ are closed, the refinement relations of $\mathcal{P}art(X)$ are in fact *closure refinements*, so all nested intersections are nonempty.

Corollary 14 *Let $f : X \to X$ be a map on the compact Hausdorff totally disconnected space X. Let $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ be a cofinal directed suborder of $\mathcal{P}art(X)$.*

Then the system (f, X) is conjugate to $(\sigma_*, \varprojlim\{\iota, \mathcal{O}(\mathcal{U}_\lambda)\})$ by the map

$$\langle w_\lambda \rangle \mapsto \bigcap \pi_0(w_\lambda).$$

It is important to note that the maps ι in the inverse system depend very much on their domain and range. However, if $\mathcal{W}$ refines $\mathcal{V}$ which in turn refines $\mathcal{U}$, then the composition of $\iota : \mathcal{W} \to \mathcal{V}$ and $\iota : \mathcal{V} \to \mathcal{U}$ is precisely the same as $\iota : \mathcal{W} \to \mathcal{U}$, and as such the inverse system is indeed well-defined.

The existence of partitions also allows us to state the following alternative characterization of shadowing.

Lemma 15 *Let $f : X \to X$ be a map on the compact Hausdorff totally disconnected space X. Then f has shadowing if and only if for each $\mathcal{U} \in \mathcal{P}art(X)$, there exists $\mathcal{V} \in \mathcal{P}art(X)$ which refines $\mathcal{U}$ such that for all $\mathcal{W} \in \mathcal{P}art(X)$ which refine $\mathcal{V}$, $\iota(\mathcal{PO}(\mathcal{W})) = \mathcal{O}(\mathcal{U})$.*

Proof Let f have shadowing and let $\mathcal{U} \in \mathcal{P}art(X)$. Let $\mathcal{V}$ be the cover witnessing shadowing. Without loss of generality, $\mathcal{V} \in \mathcal{P}art(X)$ and $\mathcal{V}$ refines $\mathcal{U}$. Now, let $\langle x_i \rangle$ be a $\mathcal{V}$-pseudo-orbit with $\langle V_i \rangle$ its $\mathcal{V}$-pseudo-orbit pattern, and let $z \in X$ be a shadowing point with $\langle U_i \rangle$ its shadowing pattern. By definition, x_i and $f^i(z)$ belong to U_i, and hence $V_i \cap U_i \neq \varnothing$, and since $\mathcal{V}$ refines $\mathcal{U}$ and the elements of $\mathcal{U}$ are disjoint, we have $V_i \subseteq U_i$, i.e. $\iota(V_i) = U_i$ and hence $\iota\langle V_i \rangle = \langle U_i \rangle$. Thus $\iota(\mathcal{PO}(\mathcal{V})) \subseteq \mathcal{O}(\mathcal{U})$, and the reverse inclusion is given by Lemma 13, and thus the two sets are equal. Now, for any $\mathcal{W} \in \mathcal{P}art(X)$ which refines $\mathcal{V}$, observe that $\mathcal{O}(\mathcal{U}) \subseteq \iota(\mathcal{PO}(\mathcal{W})) \subseteq \iota(\mathcal{PO}(\mathcal{V})) = \mathcal{O}(\mathcal{U})$,

Conversely, suppose that f has the stated property regarding open covers. Let $\mathcal{U}$ be a finite open cover of X. Since X is totally disconnected, let $\mathcal{U}' \in \mathcal{P}art(X)$ which refines $\mathcal{U}$. Let $\mathcal{V}$ be the cover witnessing the property with respect to $\mathcal{U}'$. Now, let $\langle x_i \rangle$ be a $\mathcal{V}$-pseudo-orbit and let $\langle V_i \rangle$ be its $\mathcal{V}$-pseudo-orbit pattern. By the property, there exists $\langle U_i' \rangle \in \mathcal{O}(\mathcal{U}')$ with $\iota\langle V_i \rangle = \langle U_i' \rangle$. Now, let $z \in \bigcap f^{-i}(U_i')$. Then $x_i, f^i(z) \in U_i'$ which in turn is a subset of some $U_i \in \mathcal{U}$. In particular, z $\mathcal{U}$-shadows the $\mathcal{V}$-pseudo-orbit $\langle x_i \rangle$. $\square$

In light of this, we have the following theorem.

Theorem 16 *Let $f : X \to X$ be a map with shadowing on the compact totally disconnected Hausdorff space X. Let $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ be a cofinal directed subset of $\mathcal{P}art(X)$.*

Then the system $(\sigma_*, \varprojlim\{\iota, \mathcal{O}(\mathcal{U}_\lambda)\})$ is conjugate to $(\sigma_*, \varprojlim\{\iota, \mathcal{PO}(\mathcal{U}_\lambda)\})$ and both systems satisfy the Mittag-Leffler condition.

Proof First, observe that for each λ, $\mathcal{O}(\mathcal{U}_\lambda)$ is a subset of $\mathcal{PO}(\mathcal{U}_\lambda)$. It is a standard result in inverse limit theory that the map $j_* : \varprojlim\{\iota, \mathcal{O}(\mathcal{U}_\lambda)\} \to \varprojlim\{\iota, \mathcal{PO}(\mathcal{U}_\lambda)\}\}$ induced by inclusion is a continuous injection, and clearly commutes with σ_*. In fact, this is a surjection, and hence demonstrates the desired conjugacy. This is easily proven by considering the following.

Define a monotone function $p : \Lambda \to \Lambda$ such that for each $\lambda \in \Lambda$, we have $p(\lambda) \geq \lambda$ and $\iota(\mathcal{PO}(\mathcal{U}_{p(\lambda)})) = \mathcal{O}(\mathcal{U}_\lambda)$. Then, define the map $\phi : \varprojlim\{\iota, \mathcal{PO}(\mathcal{U}_\lambda)\} \to \varprojlim\{\iota, \mathcal{O}(\mathcal{U}_\lambda)\}$ as follows.

$$\phi(\langle w_\gamma \rangle)_\lambda = \iota(w_{p(\lambda)})$$

That this is well-defined and continuous is a standard result in inverse limit theory. As this is induced by the maps ι, it will commute with σ_*.

Now, consider $j_* \circ \phi : \varprojlim\{\iota, \mathcal{PO}(\mathcal{U}_\lambda)\} \to \varprojlim\{\iota, \mathcal{PO}(\mathcal{U}_\lambda)\}$. Consider $\langle w_\gamma \rangle \in \varprojlim\{\iota, \mathcal{PO}(\mathcal{U}_\lambda)\}$. We see that

$$\left(j_* \circ \phi(\langle w_\gamma \rangle)\right)_\lambda = j_\lambda(\iota(w_{p(\lambda)})) = j_\lambda(w_\lambda) = w_\lambda$$

In particular, $j_* \circ \phi$ is the identity on $\varprojlim\{\iota, \mathcal{PO}(\mathcal{U}_\lambda)\}$, and since j_* is injective, it follows that both j_* and ϕ are conjugacies.

It remains to be shown that these systems satisfy the Mittag-Leffler condition. For the system $\mathcal{O}(\mathcal{U}_\lambda)$, note that the inclusion maps are surjective by Lemma 13, and hence the system has the condition. For the system $\mathcal{PO}(\mathcal{U}_\lambda)$, we proceed as follows. Let $\lambda \in \Lambda$ and choose $\gamma \geq \lambda$ so that $\mathcal{U}_\gamma$ witnesses $\mathcal{U}_\lambda$ shadowing. Then for all $\eta \geq \gamma$, by Lemma 15, we have $\iota_\lambda^\eta(\mathcal{PO}(\mathcal{U}_\eta)) = \mathcal{O}(\mathcal{U}_\lambda) = \iota_\lambda^\gamma(\mathcal{PO}(\mathcal{U}_\gamma))$. □

This theorem complements Corollary 14, and by applying Lemma 11, and the well-known fact that shifts of finite type have shadowing [35], we have the following result.

Corollary 17 *Let $f : X \to X$ be a map with shadowing on the compact totally disconnected Hausdorff space X. Then (f, X) is conjugate to an inverse limit of an ML inverse system of shifts of finite type.*

In fact, this is a complete characterization of totally disconnected systems with shadowing; the following is an immediate consequence of Corollary 17 and Theorem 8.

Theorem 18 *Let X be a compact, totally disconnected Hausdorff space. The map $f : X \to X$ has shadowing if and only if (f, X) is conjugate to the inverse limit of an ML inverse system of shifts of finite typ.*

Of course, Theorem 18 includes metric systems. However, if X is metric, we may easily find sequences $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of partitions which are cofinal directed suborders of $\mathcal{P}art(X)$. In particular, we can let $\mathcal{U}_0 = \{X\}$, and for each $\mathcal{U}_i$, let $\mathcal{U}_{i+1}$ be a partition of X with mesh less than 2^{-1} which refines $\mathcal{U}_i$ and which witnesses shadowing for $\mathcal{U}_i$. Then the function p from the proof of Theorem 16 simply increments its input. The conjugacy then follows from the induced diagonal map ι_* on the inverse systems as seen in Fig. 1.

This observation immediately implies the following.

Corollary 19 *Let X be the Cantor set, or indeed any compact, totally disconnected metric space. The map $f : X \to X$ has shadowing if and only if (f, X) is conjugate to the inverse limit of an ML sequence of shifts of finite type.*

$$
\begin{tikzcd}
\varprojlim\{\iota, \mathcal{PO}(\mathcal{U}_i)\} \arrow[d, "\iota^*"'] & & \mathcal{PO}(\mathcal{U}_0) & \mathcal{PO}(\mathcal{U}_1) \arrow[l, "\iota"'] \arrow[dl, "\iota"] & \mathcal{PO}(\mathcal{U}_2) \arrow[l, "\iota"'] \arrow[dl, "\iota"] & \mathcal{PO}(\mathcal{U}_3) \arrow[l, "\iota"'] \arrow[dl, "\iota"] & \cdots \arrow[l, "\iota"'] \arrow[dl, "\iota"] \\
\varprojlim\{\iota, \mathcal{O}(\mathcal{U}_i)\} & \simeq & \mathcal{O}(\mathcal{U}_0) \arrow[u, "j"] & \mathcal{O}(\mathcal{U}_1) \arrow[l, "\iota"'] \arrow[u, "j"] & \mathcal{O}(\mathcal{U}_2) \arrow[l, "\iota"'] \arrow[u, "j"] & \mathcal{O}(\mathcal{U}_3) \arrow[l, "\iota"'] \arrow[u, "j"] & \cdots \arrow[l, "\iota"']
\end{tikzcd}
$$

Fig. 1 Diagram for the metric case of Theorem 18

This ad hoc construction of an appropriate sequence of covers can be modified into a technique that will apply to general compact metric spaces in Sect. 5.

5 Shadowing in general metric systems

Theorem 12 applies equally well to systems in which there are non-trivial connected components, and as such, one might hope for analogue to Corollary 14.

However, as mentioned, the principal obstruction to a direct application of the methods of Sect. 4 is that the intersection relation ι is no longer necessarily single-valued, so that the induced map on the pseudo-orbit space is not only set-valued, but also not finitely determined. However, by modifiying the approach illustrated in Fig. 1, we are obtain the following. Recall that for a cover $\mathcal{C}$ of X and $A \subseteq X$, the *star of* A *in* $\mathcal{C}$ is the set $st(A, \mathcal{C})$ which is the union of all elements of $\mathcal{C}$ which meet A.

Theorem 20 *Let X be a compact metric space and $f : X \to X$ be a continuous map with shadowing. Then there is an inverse sequence (g_n^{n+1}, X_n) of shifts of finite type such that (f, X) is a factor of $(\sigma_*, \varprojlim\{g_n^{n+1}, X_n\})$.*

Proof First, observe that since X is compact metric and f has shadowing, we can easily find a sequence $\langle \mathcal{U}_i \rangle$ of finite open covers satisfying the following properties:

(1) $\mathcal{U}_{n+1}$ witnesses $\mathcal{U}_n$ shadowing,
(2) $\{\mathcal{U}_i\}$ is cofinal in $\mathcal{FOC}(X)$, and
(3) for all $U \in \mathcal{U}_{n+2}$, there exists $W \in \mathcal{U}_n$ such that $st(U, \mathcal{U}_{n+1}) \subseteq W$.

This is easily accomplished by taking $\mathcal{U}_0 = \{X\}$, and inductively letting $\mathcal{U}_{n+1}$ be a cover witnessing $\mathcal{U}_n$-shadowing with mesh less than one third the Lebesgue number of the cover $\mathcal{U}_n$. Conditions (1) and (2) are immediately met. To verify that condition (3) is satisfied, fix $n \in \mathbb{N}$ and $U \in \mathcal{U}_{n+2}$. Then U is a subset of V for some $V \in \mathcal{U}_{n+1}$, and so $st(U, \mathcal{U}_{n+1})$ is a subset of $st(V, \mathcal{U}_{n+1})$. But the diameter of $st(V, \mathcal{U}_{n+1})$ is at most three times the mesh of $\mathcal{U}_{n+1}$, and hence has diameter less than the Lebesgue number for $\mathcal{U}_n$. Hence there is some $W \in \mathcal{U}_n$ for which $W \supseteq st(V, \mathcal{U}_{n+1}) \supseteq st(V, \mathcal{U}_{n+1})$ as required.

$$\begin{tikzcd}
\mathcal{PO}(\mathcal{U}_0) & \mathcal{PO}(\mathcal{U}_1) & \mathcal{PO}(\mathcal{U}_2) \arrow[dll, "w"'] & \mathcal{PO}(\mathcal{U}_3) & \mathcal{PO}(\mathcal{U}_4) \arrow[dll, "w"'] & {} \arrow[dl, dashed] \\
\mathcal{O}(\mathcal{U}_0) \arrow[u, hook] & \mathcal{O}(\mathcal{U}_1) & \mathcal{O}(\mathcal{U}_2) \arrow[u, hook] & \mathcal{O}(\mathcal{U}_3) & \mathcal{O}(\mathcal{U}_4) \arrow[u, hook] &
\end{tikzcd}$$

Fig. 2 Diagram for the proof of Theorem 20

$$\begin{tikzcd}
\varprojlim\{\iota', \mathcal{PO}(\mathcal{U}_i)\} \arrow[d, "w^*"'] & & \mathcal{PO}(\mathcal{U}_0) & \mathcal{PO}(\mathcal{U}_2) \arrow[l, "\iota'"'] \arrow[dl, "w"'] & \mathcal{PO}(\mathcal{U}_4) \arrow[l, "\iota'"'] \arrow[dl, "w"'] & \mathcal{PO}(\mathcal{U}_6) \arrow[l, "\iota'"'] \arrow[dl, "w"'] & \cdots \arrow[l, "\iota'"'] \arrow[dl, "w"'] \\
\varprojlim\{\iota', \mathcal{O}(\mathcal{U}_{2i})\} & \simeq & \mathcal{O}(\mathcal{U}_0) \arrow[u, hook] & \mathcal{O}(\mathcal{U}_2) \arrow[l, "\iota'"'] \arrow[u, hook] & \mathcal{O}(\mathcal{U}_4) \arrow[l, "\iota'"'] \arrow[u, hook] & \mathcal{O}(\mathcal{U}_6) \arrow[l, "\iota'"'] \arrow[u, hook] & \cdots \arrow[l, "\iota'"']
\end{tikzcd}$$

Fig. 3 Diagram for the proof of Theorem 20

Let $f : X \to X$ and covers $\langle \mathcal{U}_i \rangle$ be as stated. For each $U \in \mathcal{U}_{n+2}$, fix $W(U) \in \mathcal{U}_n$ with $st(U, \mathcal{U}_{n+1}) \subseteq W(U)$, and define $w : \mathcal{PO}(\mathcal{U}_{n+2}) \to \prod \mathcal{U}_n$ by $w(\langle U_j \rangle) = \langle W(U_j) \rangle$. Note that, as this is a single letter substitution map on a shift space, it is a continuous map and commutes with the shift map by definition.

We claim that $w(\mathcal{PO}(\mathcal{U}_{n+2}))$ is a subset of $\mathcal{O}(\mathcal{U}_n)$. Indeed, let $\langle U_j \rangle \in \mathcal{PO}(\mathcal{U}_{n+2})$ and $\langle x_j \rangle$ a pseudo-orbit with this pattern. Since $\mathcal{U}_{n+2}$ witnesses $\mathcal{U}_{n+1}$-shadowing, there exists $z \in X$ and sequence $\langle V_j \rangle \in \mathcal{O}(\mathcal{U}_{n+1})$ with $f^j(z), x_j \in V_j$. In particular, for any such z and choices of $\langle V_j \rangle$, $V_j \subseteq st(U_j, \mathcal{U}_{n+1}) \subseteq W(U_j)$. Indeed, this establishes that $\langle W(U_j) \rangle \in \mathcal{O}(\mathcal{U}_n)$. It should be noted that while w is not necessarily surjective, for every $x \in X$, there is some $\langle U_j \rangle$ in $w(\mathcal{PO}(\mathcal{U}_{n+2}))$ with $f^j(x) \in U_j$ for all j. We can observe this by noting that $\langle f^j(x) \rangle$ is itself a $\mathcal{U}_{n+2}$-pseudo-orbit, and in particular, we have $f^j(x) \in W(U_j)$, and so $\langle W(U_j) \rangle$ is a $\mathcal{U}_n$ orbit pattern for x.

Since $\mathcal{O}(\mathcal{U}_n) \subseteq \mathcal{PO}(\mathcal{U}_n)$, we have the following diagram (Fig. 2).

So, while the 'natural' map from $\mathcal{PO}(\mathcal{U}_{n+2})$ is set-valued, the composition of inclusion and w gives a single-valued continuous map from $\mathcal{O}(\mathcal{U}_{n+2})$ to $\mathcal{O}(\mathcal{U}_n)$, and by reversing the order of composition, from $\mathcal{PO}(\mathcal{U}_{n+2})$ to $\mathcal{PO}(\mathcal{U}_n)$. We will denote these maps by ι'. Figure 3 then establishes the existence of a map w_* from the inverse limit of the pseudo-orbit spaces to the inverse limit of the orbit spaces which commutes with the induced maps σ_*.

All that remains is to establish that the inverse limit of orbit space is a factor of the system (f, X) and that the composition of this factor map with the map w_* is a surjection. For the former, Let $\phi : \varprojlim\{\iota', \mathcal{O}(\mathcal{U}_{2i})\} \to X$ be given by $\phi\langle u_i \rangle = \bigcap \overline{\pi_0(u_i)}$. Note that, by construction, $\pi_k(u_{i+1}) \subseteq \pi_k(u_i)$ for all k and

i, so in particular $\phi(\langle u_i \rangle)$ is a nested intersection of the closures of elements of the open covers, and hence is well-defined. That ϕ is continuous and commutes with σ_* follows from similar reasoning as Theorem 12 and Corollary 14. That ϕ is surjective follows from the same logic as Theorem 12–observe that that for all $x \in X$, and all k, there exists a nonempty subset $O_{2k}(x)$ of $\mathcal{U}_{2k}$-orbit patterns for x in $\mathcal{O}(\mathcal{U}_{2k})$, and that

$$O(x) = \bigcap \pi_{2k}^{-1}(\overline{O_{2k}}) \bigcap \underleftarrow{\lim}\{\iota', \mathcal{O}(\mathcal{U}_{2i})\}$$

is nonempty and satisfies $\phi(O(x)) = \{x\}$. This same observation coupled with the fact that $\mathcal{O}(\mathcal{U}_{2k})$ is mapped into $\mathcal{PO}(\mathcal{U}_{2k})$ by inclusion demonstrates that $\phi \circ w_*$ is indeed surjective, and is thus a factor map. $\square$

Clearly, the existence of the sequence of covers satisfying conditions (2) and (3) in this proof is a strong condition. In particular, this implies that there is such a cofinal sequence in $\mathcal{FOC}(X)$, which in turn implies that the space X is metrizable.

While it seems that this might be straightforward to generalize, a direct generalization of this argument fails due to the complexity of the partial order on the class of open covers of a general compact Hausdorff space. In particular, given any pair $\mathcal{U}$ and $\mathcal{V}$ of finite open covers of a compact Hausdorff space X, there is some cover $\mathcal{W}$ which mutually star-refines $\mathcal{U}$ and $\mathcal{V}$ and also a cover $\mathcal{T}$ which is star-refined by both. To follow the argument from before, we would need to be able to establish maps from $\mathcal{W}$ to $\mathcal{U}$ and to $\mathcal{V}$ as well as maps from $\mathcal{U}$ and $\mathcal{V}$ into $\mathcal{T}$ all of which respect star-refinement and such that the compositions agree. This is generally not possible due to the inherent 'drift' of stars of sets in covers.

6 Factor maps which preserve shadowing

We have now established that for a metric system to exhibit shadowing, it is necessary for there to exist an inverse limit of an inverse sequence of shifts of finite type of which the original system is a factor. However, it is worth noting that this is by no means sufficient, even with the added hypothesis that the inverse sequence is an ML inverse sequence. In particular, every sofic shift is a factor of such an inverse limit, but only those that are shifts of finite type exhibit shadowing.

Example 21 Let X be the subshift of $\{0, 1\}^{\mathbb{Z}}$ consisting of those bi-infinite words containing at most one 1. The system (σ, X) fails to have shadowing, but is a factor of the inverse limit of an inverse sequence of shifts of finite type.

Proof (σ, X) is a standard example of a system which does not have shadowing; it has only one non-constant full orbit, namely the orbit passing through $\cdots 0001000 \cdots$, but uncountably many distinct pseudo-orbits containing the fixed point $\langle 0 \rangle$. Now let Y be the subshift of $\{0, 1, 2\}^{\mathbb{Z}}$ consisting of those sequences in which the words 02, 10, 21 and 20 do not appear, i.e. Y is the subshift of all sequences of the form $\cdots 0000001222222 \cdots$ along with the constant sequences $\langle 0 \rangle$ and $\langle 2 \rangle$. Y is a shift of finite type. Then Y is (trivially) an inverse limit of an ML system of shifts of finite type. However, the map from Y to X induced by substituting the symbol 0 for 2 witnesses that X is a factor of Y. □

It is then natural to ask which factors of inverse limits of ML inverse systems consisting of shifts of finite type exhibit shadowing, i.e. is there a class of maps $\mathcal{P}$ such that if (f, X) is a factor by a map in $\mathcal{P}$ of an inverse limit of an ML inverse system of shifts of finite type, then (f, X) has shadowing? Of course, it is clear that there is such a class, and in fact the class of homeomorphisms have this property, but we wish to find, if possible, the maximal such class. Towards this end, we define the following.

Definition 22 Let (f, X) and (g, Y) be dynamical systems with X and Y compact Hausdorff spaces. A factor map $\phi : (f, X) \to (g, Y)$ *lifts pseudo-orbits* provided that for every $\mathcal{V}_X \in \mathcal{FOC}(X)$, there exists $\mathcal{V}_Y \in \mathcal{FOC}(Y)$ such that if $\langle y_i \rangle$ is a $\mathcal{V}_Y$-pseudo-orbit in Y, then there is a $\mathcal{V}_X$-pseudo-orbit $\langle x_i \rangle$ in X with $\langle y_i \rangle = \langle \phi(x_i) \rangle$.

Theorem 23 *Let (f, X) and (g, Y) be dynamical systems with X and Y compact Hausdorff. If (f, X) has shadowing and $\phi : (f, X) \to (g, Y)$ is a factor map that lifts pseudo-orbits, then (g, Y) has shadowing.*

Proof Fix an open cover $\mathcal{U}_Y \in \mathcal{FOC}(Y)$, and let $\mathcal{U}_X \in \mathcal{FOC}(X)$ such that $\phi(U_X)$ refines $\mathcal{U}_Y$. Since (f, X) has shadowing, let $\mathcal{V}_X \in \mathcal{FOC}(X)$ witness shadowing with respect to $\mathcal{U}_X$.

Since ϕ lifts pseudo-orbits, let $\mathcal{V}_Y$ witness this with respect to $\mathcal{V}_X$. Finally, let $\langle y_i \rangle$ be a $\mathcal{V}_Y$-pseudo-orbit.

Pick $\langle x_i \rangle$ to be a $\mathcal{V}_X$-pseudo-orbit with $\langle \phi(x_i) \rangle = \langle y_i \rangle$. As every $\mathcal{V}_X$-pseudo-orbit is $\mathcal{U}_X$-shadowed, fix $z_X \in X$ to witness this and let $z_Y = \phi(z_X)$. It then follows that for each i, we have $\phi^i(z_X), \phi(x_i) \in \phi(U_{X,i})$ for some $U_i \in \mathcal{U}_X$. As $\phi(\mathcal{U}_X)$ refines $\mathcal{U}_Y$, it follows that there exists $U_{Y,i} \in \mathcal{U}_Y$ with $\phi^i(z_Y), y_i \in \phi(U_{Y,i})$, i.e. $z_Y = \phi(z_X)$ $\mathcal{U}_Y$-shadows $\langle y_i \rangle$. □

Notwithstanding Theorem 23, a more general concept of lifting pseudo-orbits provides a much sharper insight into the relation between shadowing and shifts of finite type in compact metric systems.

Definition 24 Let (f, X) and (g, Y) be dynamical systems with X and Y compact Hausdorff spaces. A factor map $\phi : (f, X) \to (g, Y)$ *almost lifts pseudo-orbits* (or f *is an ALP map*) provided that for every $\mathcal{V}_X \in \mathcal{FOC}(X)$ and $\mathcal{W}_Y \in \mathcal{FOC}(Y)$, there exists $\mathcal{V}_Y \in \mathcal{FOC}(Y)$ such that if $\langle y_i \rangle$ is a $\mathcal{V}_Y$-pseudo-orbit in Y, then there is a $\mathcal{V}_X$-pseudo-orbit $\langle x_i \rangle$ in X such that for each $i \in \mathbb{N}$ there exists $W_i \in \mathcal{W}_Y$ with $\phi(x_i), y_i \in W_i$.

Theorem 25 *Let (f, X) and (g, Y) be dynamical systems with X and Y compact Hausdorff and let $\phi : (f, X) \to (g, Y)$ be a factor map. Then the following statements hold:*

(1) if (f, X) has shadowing and ϕ is an ALP map, then (g, Y) has shadowing, and
(2) if (g, Y) has shadowing then ϕ is an ALP map.

Proof First, we prove statement (1). Let (f, X) have shadowing and let ϕ be an ALP map. Fix an open cover $\mathcal{U}_Y \in \mathcal{FOC}(Y)$. Let $\mathcal{W}_Y \in \mathcal{FOC}(Y)$ such that if $W, W' \in \mathcal{W}_Y$ with $W \cap W' \neq \varnothing$, then there exists $U \in \mathcal{U}_Y$ with $W \cup W' \subseteq U$, and let $\mathcal{U}_X \in \mathcal{FOC}(X)$ such that $\phi(U_X)$ refines $\mathcal{W}_Y$. Since (f, X) has shadowing, let $\mathcal{V}_X \in \mathcal{FOC}(X)$ witness shadowing with respect to $\mathcal{U}_X$.

Since ϕ is ALP, let $\mathcal{V}_Y$ witness this with respect to $\mathcal{W}_Y$ and $\mathcal{V}_X$. Finally, let $\langle y_i \rangle$ be a $\mathcal{V}_Y$-pseudo-orbit.

Pick $\langle x_i \rangle$ to be a $\mathcal{V}_X$-pseudo-orbit so that $\langle \phi(x_i) \rangle$ $\mathcal{W}_Y$-shadows $\langle y_i \rangle$. As every $\mathcal{V}_X$-pseudo-orbit is $\mathcal{U}_X$-shadowed, fix $z_X \in X$ to witness this and let $z_Y = \phi(z_X)$. It then follows that for each i, we have $\phi^i(z_X), \phi(x_i) \in \phi(U_{X,i})$ for some $U_i \in \mathcal{U}_X$. As $\phi(\mathcal{U}_X)$ refines $\mathcal{W}_Y$, it follows that there exists $W_i \in \mathcal{W}_Y$ with $\phi^i(z_Y), \phi(x_i) \in W_i$. Additionally, as pseudo-orbits are almost lifted, there exists $W'_i \in \mathcal{W}_Y$ with $\phi(x_i), y_i \in W'_i$. In particular, $\phi(x_i) \in W_i \cap W'_i$, so we have that there exists $U_{Y,i} \in \mathcal{U}_Y$ with $\phi^i(z_Y), \phi(x_i), y_i \in W_i \cup W'_i \subseteq U_{Y,i} \in \mathcal{U}_Y$, i.e z_Y $\mathcal{U}_Y$-shadows $\langle y_i \rangle$.

Now, to prove statement (2), assume that (g, Y) has shadowing. let $\mathcal{V}_X \in \mathcal{FOC}(X)$ and $\mathcal{W}_Y \in \mathcal{FOC}(Y)$. Let $\mathcal{V}_Y \in \mathcal{FOC}(Y)$ witness shadowing with respect to $\mathcal{W}_Y$. Now, let $\langle y_i \rangle$ be a $\mathcal{V}_Y$ pseudo-orbit in Y. Then there is some $z \in Y$ with z $\mathcal{W}_Y$-shadowing $\langle y_i \rangle$. Now, choose $x \in \phi^{-1}(z)$ and observe that $\langle f^i(x) \rangle$ is a $\mathcal{V}_X$-pseudo-orbit (as it is in fact an orbit), and $\phi(f^i(x)) = g^i(z)$, so there exists $W_i \in \mathcal{W}_Y$ with $\phi(f^i(z)), y_i \in W_i$ (as given by the shadowing pattern of z and $\langle y_i \rangle$). Thus ϕ is an ALP map. □

One consequence of the above is that the conjugacies (factor maps) in Theorems 18 and 20 are ALP maps. This is not terribly surprising in the case of Theorem 18, as the map in question is a homeomorphism, but in the case of Theorem 20, this is interesting, and allows us to refine the characterization. It should be also be noted that, as a result of this theorem, we see that the factor maps constructed by Bowen in [6] are ALP.

Theorem 26 *Let X be a compact Hausdorff space. The map $f : X \to X$ has shadowing if (f, X) lifts via a map which is ALP to the inverse limit of an ML inverse system of shifts of finite type.*

Proof This follows immediately from Theorems 8 and 25. □

In the metric case, we can say a bit more, but first we first translate the notion of almost lifting pseudo-orbits from the language of covers into the language of metric spaces. This is not completely necessary, but allows for a different perspective on the property. As this is a direct translation and application of Theorems 20 and 26, we state the following results without proof.

Lemma 27 *Let (f, X) and (g, Y) be dynamical systems with X and Y compact metric spaces. A factor map $\phi : (f, X) \to (g, Y)$ is an ALP map if and only if for all $\epsilon > 0$ and $\eta > 0$, there exists $\delta > 0$ such that if $\langle y_i \rangle$ is a δ-pseudo-orbit in Y, there exists an η-pseudo-orbit $\langle x_i \rangle$ in X with $d(\phi(x_i), y_i) < \epsilon$.*

Corollary 28 *Let X be a compact metric space. The map $f : X \to X$ has shadowing only if (f, X) is a factor of an inverse limit of a sequence of shifts of finite type by a map which is ALP. Additionally, $f : X \to X$ has shadowing if (f, X) lifts via a map which is ALP to the inverse limit of an ML sequence of shifts of finite type.*

Of course, it would be of significant benefit if ALP maps had an alternate characterization. In particular, it is clear that homeomorphisms and covering maps lift pseudo-orbits. However, there are maps which are neither covering maps nor homemorphisms which almost lift pseudo-orbits, in particular, the factor maps given in Theorem 20 are not typically open, much less covering maps.

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