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Nemeth, Sandor; Xie, Jiaxin; Zhang, Guohan

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Sándor Zoltán Németh · Jiaxin Xie · Guohan Zhang

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Abstract A positive operator on a cone is a linear operator that maps the cone to a subcone of itself. The extended second order cones were introduced by Németh and Zhang (J Glob Optim 62(3):443–457, 2015) as a working tool to solve mixed complementarity problems. Sznajder (J Global Optim 66(3): 585–593, 2016) determined the automorphism group and the Lyapunov (or bilinearity) ranks of these cones. Ferreira and Németh (J Global Optim 70(4): 707–718, 2018) reduced the problem of projecting onto the second order cone to a piecewise linear equation. Németh and Xiao (J Optim Theory Appl 176(2): 269–288, 2018) solved linear complementarity problems on the extended second order cone (motivated by portfolio optimization models) by reducing them to mixed complementarity problems with respect to the nonnegative orthant. As an extension of Sznajder's results this paper aims to be a first work about finding necessary conditions and sufficient conditions for a linear operator to be a positive operator (which extends the notion of an automorphism) on an extended second order cone. Although, in the particular case of second order

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All authors contributed equally to this work.

S. Z. Németh
School of Mathematics, University of Birmingham, Watson Building, Edgbaston, Birmingham B15 2TT, United Kingdom.
E-mail: s.nemeth@bham.ac.uk

Jiaxin Xie,
School of Mathematical Sciences, Beihang University, Beijing, 100191, China.
E-mail: xiejx@buaa.edu.cn

Guohan Zhang, Corresponding author
School of Sciences, Beijing University of Posts and Telecommunications, Beijing, 100876, China.
E-mail: zhanggh@bupt.edu.cn

cones a necessary and sufficient condition is known, for extended second order cone such a condition is very difficult to find without restricting the structure of the linear operator. If the matrix of the linear operator is block-diagonal, we give such a necessary and sufficient condition.

Keywords Extended second order cone · Positive operator

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1 Introduction

The second order cone (also called ice-cream or Lorentz cone) plays an important role in optimization problems. It has a variety of applications in financial management [15,26], signal processing [12,29] and machine learning [11,20,27]. We refer to the survey [1] and the references therein. In recent papers, the authors shows the characteristic conditions the spherical convexity of quadratic functions defined on Lorentz spherically convex sets [10]. In [17] and [18], we generalized the n-dimensional second order cone to mutually dual $(p+q)$-dimensional cones. In these papers we used the extended second order cones as a tool to solve mixed complementarity problems defined on general cones and variational inequalities defined on cylinders. Sznajder [23] determined the automorphism group and the Lyapunov (or bilinearity) ranks of these cones. He also proved that the extended second order cones are irreducible. In [9], the authors gave explicit formulas for projecting onto the extended second order cones. In the most recent research about extended second order cones, Németh and Xiao [16] discussed the properties and solutions of linear complementarity problems on extended second order cones. They also formulated a class of mean-variance portfolio optimization problems via extended second order cones. It is expected that the theory of extended second order cones will be further developed and applied in various areas.

The study of positive operators (mappings) on a cone is an attractive topic in the theory of fixed points and Hilbert spaces. Birkhoff [6] defined the contraction ratio of a positive operator and proved its connection with the projective diameter. Later Bushell [7] found a simpler proof. One may refer to [28] for a simple review. Loewy and Schneider [14] discussed the necessary and sufficient conditions of positive operators on an n-dimensional second order cone. They showed that a matrix is a positive operator if and only if it satisfies a positive semidefinite condition (we will show the details of this result in the next section). Tam [24] showed properties of positive operators on polyhedral and simplicial cones and studied the structure and properties of the cone of positive operators [25]. A recent work [2] showed how the (semi)positivity influences the solutions of semidefinite linear complementarity problems.

Considering the positive operators on extended second order cones is a very natural and interesting idea, which is motivated by the theoretical and practical aspects described above. What properties should these positive operators satisfy? What is the difference between the positive operators on second order cones and the positive operators on extended second order cones? Can the elegant necessary and sufficient conditions of second order cones in [14] be directly extended to extended second order cones? These questions are the key motivators of this article. As it is shown in previous papers, in general the extended second order cones do not have good properties of self-duality or polyhedrality, though they are a useful tool for solving complementarity problems and variational inequalities [17, 18].

This paper is the first work which aims to answer the above questions. Our investigations are also related to the famous $\mathcal{S}$-Lemma which is first illustrated in control theory by Yakubovič [31, 32]. Yuan [33] showed an equivalent form of this lemma and an elementary proof. The study of the $\mathcal{S}$-Lemma is still an essential topic in optimization theory [30]. Its development and applications are reviewed in [4, 19].

The paper is organized as follows. In Section 2, we list the definitions and terminology used throughout this paper. In this section we also present the necessary and sufficient conditions for a linear operator to be a positive operator on an n-dimensional second order cone (Theorem 2.3 [14]). In Section 3, we first shows some algebraic properties of the positive operators on extended second order cones. This section contains various necessary and sufficient conditions for these operators. Section 4 presents a necessary and sufficient condition for the block diagoal case. In Section 5 we show some numerical examples. In Section 6 we give some conclusions and outline the proposed future work.

2 Preliminaries

Let ℓ, m, p, q be positive integers such that $m = p+q$. Define the direct product space $\mathbb{R}^p \times \mathbb{R}^q$ as the set of all pairs of vectors (x, u), where $x \in \mathbb{R}^p$ and $u \in \mathbb{R}^q$. We identify the vectors of $\mathbb{R}^m$ by $m \times 1$ real matrices and define the canonical inner product on $\mathbb{R}^m$ by the mapping

$$\mathbb{R}^m \times \mathbb{R}^m \ni (x, y) \mapsto \langle x, y \rangle := x^\top y \in \mathbb{R}$$

with the induced norm

$$\mathbb{R}^m \ni x \mapsto \|x\| = \sqrt{\langle x, x \rangle} \in \mathbb{R}.$$

The inner product in $\mathbb{R}^p \times \mathbb{R}^q \equiv \mathbb{R}^{p+q}$ is given by

$$\langle (x, u), (y, v) \rangle := \langle x, y \rangle + \langle u, v \rangle.$$

A set $\mathcal{K} \subseteq \mathbb{R}^\ell$ is called *cone* if $\lambda\mathcal{K} \subseteq \mathcal{K}$ for any $\lambda \geq 0$. For convenience we identify a linear operator A of $\mathbb{R}^\ell$ with its matrix. A cone $\mathcal{K}$ is called a *closed convex cone* if it is a closed and convex set. A closed convex cone is called *proper cone* if it has nonempty interior and $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$. For any cone $\mathcal{K} \subseteq \mathbb{R}^\ell$, its *dual cone* is defined by

$$\mathcal{K}^* := \{y \in \mathbb{R}^\ell : \langle x, y \rangle \geq 0, \quad \forall x \in \mathcal{K}\}.$$

A cone $\mathcal{K} \subseteq \mathbb{R}^\ell$ is called subdual if $\mathcal{K} \subseteq \mathcal{K}^*$, superdual if $\mathcal{K}^* \subseteq \mathcal{K}$ and self-dual if $\mathcal{K}^* = \mathcal{K}$. If $\mathcal{K}, \mathcal{D} \subseteq \mathbb{R}^\ell$ are closed convex cones cones such that $\mathcal{D} = \mathcal{K}^*$, then $\mathcal{D}^* = \mathcal{K}$ and the cones $\mathcal{K}, \mathcal{D}$ are called *mutually dual*. The set $\Gamma(\mathcal{K})$ of *positive operators* on a cone $\mathcal{K} \subseteq \mathbb{R}^\ell$ (see [14]) is defined by

$$\Gamma(\mathcal{K}) = \{B \in \mathbb{R}^{\ell\times\ell} : Bx \in \mathcal{K} \ \text{ for any } \ x \in \mathcal{K}\}. \tag{1}$$

The set of positive operators is a cone in $\mathbb{R}^{\ell\times\ell}$ [22]. It can be easily checked that B is a positive operator on $\mathcal{K}$ if and only if $B^\top$ is a positive operator on $\mathcal{K}^*$.

The *complementarity set* of a closed convex cone $\mathcal{K} \subseteq \mathbb{R}^\ell$ is defined by

$$\mathcal{C}(\mathcal{K}) = \{(x, y) : x \in \mathcal{K}, y \in \mathcal{K}^*, \langle x, y\rangle = 0\}.$$

A matrix $B \in \mathbb{R}^{\ell\times\ell}$ is said to be *Lyapunov-like* on $\mathcal{K}$ if $\langle Bx, y\rangle = 0$ for all $(x, y) \in \mathcal{C}(\mathcal{K})$ (see [21]). We call a matrix B an *automorphism* on the cone $\mathcal{K}$ if it is invertible and $B\mathcal{K} = \mathcal{K}$, where $B\mathcal{K} = \{Bx : x \in \mathcal{K}\}$. The group of such matrices on $\mathcal{K}$ is denoted by $Aut(\mathcal{K})$. A Lyapunov-like matrix B can also be characterized by the condition (see [8], [22])

$$e^{tB} \in Aut(\mathcal{K}) \ \text{ for all } t \in \mathbb{R}.$$

Note that any automorphism is a positive operator.

For $z \in \mathbb{R}^\ell$, we denote $z = (z_1, \dots, z_\ell)^\top$. Let $\geq$ denote the *component-wise order* in $\mathbb{R}^\ell$, that is, the order defined by $\mathbb{R}^\ell \ni x \geq y \in \mathbb{R}^\ell$ if and only if $x_i \geq y_i$ for any $i = 1, \dots, \ell$. Depending on the context, denote by 0 either the zero vector in $\mathbb{R}^\ell$ or the scalar zero; by e the vector of ones in $\mathbb{R}^\ell$, i.e., $e = (1, \dots, 1)^\top$; by e_i the ith vector of the standard canonical basis of $\mathbb{R}^\ell$; and by $\mathbb{R}^\ell_+ = \{x \in \mathbb{R}^\ell : x \geq 0\}$ the nonnegative orthant. We call a matrix $B \in \mathbb{R}^{\ell\times\ell}$ positive semidefinite if $x^\top Bx \geq 0$ holds for any $x \in \mathbb{R}^\ell$. For any two matrices B and D, we write $B \succeq D$ to represent $B - D$ is positive semidefinite.

We recall from [17] the following definition of a pair of mutually dual *extended second order cones* $\mathcal{L}(p, q)$ and $\mathcal{M}(p, q)$.

Definition 1 The extended second order cones $\mathcal{L}(p, q)$ and $\mathcal{M}(p, q)$ are defined by

$$\mathcal{L}(p, q) := \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^q : x \geq \|u\|e\} \tag{2}$$

and

$$\mathcal{M}(p, q) := \{(x, u) \in \mathbb{R}^p \times \mathbb{R}^q : \langle x, e\rangle \geq \|u\|, \ x \geq 0\}, \tag{3}$$

respectively. Here $\geq$ denotes the component-wise order.

It can be easily checked that both $\mathcal{L}(p, q)$ and $\mathcal{M}(p, q)$ are proper cones. Actually, they are natural extensions of the second order cones, as both $\mathcal{L}(p, q)$ and $\mathcal{M}(p, q)$ will become second order cones exactly in the special case $p = 1$. $\mathcal{M}(p, q)$ is superdual and $\mathcal{L}(p, q) = \mathcal{M}(p, q)$ becomes selfdual exactly when $p = 1$. It is easy to show that $\mathcal{L}(p, q)$ is polyhedral if and only if $q = 1$.

We now present the famous $\mathcal{S}$-Lemma, which will be used in the next section.

Lemma 1 **($\mathcal{S}$-Lemma, [4])** *Let B, D be two symmetric real $n \times n$ matrices, and assume that the quadratic inequality*

$$x^\top B x \geq 0 \tag{4}$$

is strictly feasible, i.e., there exists $u \in \mathbb{R}^n$ such that $u^\top B u > 0$. Then the quadratic inequality

$$x^\top D x \geq 0$$

is a consequence of (4) if and only if there exists a nonnegative λ such that

$$D \succeq \lambda B.$$

3 Conditions for positive operators on extended second order cones

In this section, we will show some algebraic properties of the positive operators on $\mathcal{L}(p, q)$ and $\mathcal{M}(p, q)$. In addition, we also present a kind of extension of [22, Theorem 2.3], which shows that one can use a positive semidefinite condition to characterize a positive operator on a second order cone.

The following theorem states necessary conditions for a linear operator to be a positive operator on the extended second order cone $\mathcal{M}(p, q) = (\mathcal{L}(p, q))^*$, where $\mathcal{L}(p, q)$ is defined by (2) and $\mathcal{M}(p, q)$ is defined by (3).

Theorem 1 **(Necessary conditions for positive operators on $\mathcal{M}(p, q)$)** *Let $p \geq 1$ and $q \geq 1$ be two integers. If $A \in \mathbb{R}^{(p+q)\times(p+q)}$ is a positive operator on $\mathcal{M}(p, q)$, then the following hold:*

(*i*) *The transpose of the first p rows of A are in $\mathcal{L}(p, q)$.*
(*ii*) *The first p columns of A are in $\mathcal{M}(p, q)$.*
(*iii*) *By adding any i-th column $i = 1, \ldots, p$ to the linear combination of the columns $p+1, \ldots, p+q$ with coefficients $u_1, \ldots, u_q$ such that the Euclidean norm of $u = (u_1, \ldots, u_q)^\top$ is one, we obtain an element in $\mathcal{M}(p, q)$.*
(*iv*) *The sum of any i-th column $i = 1, \ldots, p$ with any $(p + j)$-th column $j = 1, \ldots, q$ is in $\mathcal{M}(p, q)$.*

Proof (i) Since A is a positive operator on $\mathcal{M}(p, q)$, the first p entries of Az are nonnegative for any $z \in \mathcal{M}(p, q)$. Hence, the inner product of z and the transpose of any row vector of the first p rows of A is nonnegative. Therefore, the transpose of these row vectors must be in the dual cone of $\mathcal{M}(p, q)$, i.e., $\mathcal{L}(p, q)$.

(ii) Since $A^\top$ is a positive operator on $\mathcal{L}(p, q) = (\mathcal{M}(p, q))^*$, (ii) follows similarly to (i).

(iii) To state conveniently, let the columns of A be the vectors $a_1, a_2, \ldots, a_{p+q} \in \mathbb{R}^{p+q}$. For $i = 1, \ldots, p$, we set

$$b_i := a_i + \sum_{j=1}^{q} u_j a_{p+j}.$$

It suffices to prove that for any $z \in \mathcal{L}(p,q)$ and $i \in \{1,\ldots,p\}$,

$$\langle z, b_i\rangle = \langle z, a_i\rangle + \sum_{j=1}^{q} u_j \langle z, a_{p+j}\rangle \geq 0. \tag{5}$$

Note that

$$\sum_{j=1}^{q} u_j \langle z, a_{p+j}\rangle \geq -\sqrt{\sum_{j=1}^{q}\langle z, a_{p+j}\rangle^2}\sqrt{\sum_{j=1}^{q} u_i^2} = -\sqrt{\sum_{j=1}^{q}\langle z, a_{p+j}\rangle^2}$$

where the first inequality follows by the Cauchy-Schwarz inequality and the equality follows from $\|u\|_2 = 1$. So for any $z \in \mathcal{L}(p,q)$, we have

$$\langle z, b_i\rangle \geq \langle z, a_i\rangle - \sqrt{\sum_{j=1}^{q}\langle z, a_{p+j}\rangle^2}.$$

Recall that A is a positive operator on $\mathcal{M}(p,q)$, which implies that $A^\top$ is a positive operator on $\mathcal{L}(p,q)$. Thus $A^\top z \in \mathcal{L}(p,q)$. Observe that $\langle z, a_k\rangle$ is the k-th entry of $A^\top z$, and by the definition of $\mathcal{L}(p,q)$ we know that

$$\langle z, a_i\rangle \geq \sqrt{\sum_{j=1}^{q}\langle z, a_{p+j}\rangle^2}, \quad i = 1,\ldots,p.$$

Thus we arrive at (5), which implies that $b_i \in \mathcal{L}^*(p,q) = \mathcal{M}(p,q)$.

(iv) Obviously, it is a special case of the above assertion. □

Next, we will present the sufficient conditions for positive operators on $\mathcal{M}(p,q)$. Let us begin with the following lemma, which tells us that $\mathcal{M}(p,q)$ can be formulated by using a quadratic form with respect to the matrix $J_{p,q}$ defined below.

Lemma 2 *Let*

$$J_{p,q} = \begin{bmatrix} ee^\top & 0 \\ 0 & -I_q \end{bmatrix}, \tag{6}$$

where $e \in \mathbb{R}^p$ is the vector with all entries 1 and I_q denotes the identity mapping in $\mathbb{R}^q$. Then

$$\mathcal{M}(p,q) = \{z = (x,u) \in \mathbb{R}^p \times \mathbb{R}^q : z^\top J_{p,q} z \geq 0,\ x \in \mathbb{R}^p_+\}. \tag{7}$$

Proof Set

$$\Omega := \{z = (x,u) \in \mathbb{R}^p \times \mathbb{R}^q : z^\top J_{p,q} z \geq 0, x \in \mathbb{R}^p_+\}.$$

We have $z = (x,u) \in \Omega$ if and only if $z^\top J_{p,q} z \geq 0$ and $x \geq 0$, or equivalently,

$$z^\top J_{p,q} z = x^\top ee^\top x - u^\top u = \langle x, e\rangle^2 - \|u\|^2 \geq 0$$

and $x \geq 0$. Recall that $z = (x,u) \in \mathcal{M}(p,q)$ if and only if $\langle x, e\rangle \geq \|u\|$ and $x \geq 0$. Hence $\Omega = \mathcal{M}(p,q)$. □

We have the following result show the sufficient conditions for positive operators on $\mathcal{M}(p,q)$.

Theorem 2 (Sufficient conditions for positive operators on $\mathcal{M}(p,q)$) *Suppose that $A \in \mathbb{R}^{(p+q)\times(p+q)}$, where $p \geq 1$ and $q \geq 1$ are two integers. Then the following hold:*

(i) If A is $\mathcal{M}(p,q)$-Lyapunov like, then for any $t \in \mathbb{R}$, $e^{tA} \in Aut(\mathcal{M}(p,q))$.
(ii) Let $J_{p,q}$ be defined by (6). If there exists a $\lambda \geq 0$ such that $A^\top J_{p,q} A - \lambda J_{p,q}$ is positive semidefinite and the first p rows of A (more precisely, their transposes) are in $\mathcal{L}(p,q)$, then A is a positive operator on $\mathcal{M}(p,q)$.

Proof (i) See [8], [22].

(ii) To this end, we need prove that for any $z \in \mathcal{M}(p,q)$, $Az \in \mathcal{M}(p,q)$. Note that $A^\top J_{p,q} A - \lambda J_{p,q}$ is positive semidefinite, we obtain that

$$z^\top (A^\top J_{p,q} A - \lambda J_{p,q}) z = (Az)^\top J_{p,q} (Az) - \lambda z^\top J_{p,q} z \geq 0.$$

Let $z \in \mathcal{M}(p,q)$. From Lemma 2, we know that $z^\top J_{p,q} z \geq 0$. As $\lambda \geq 0$, we have $(Az)^\top J_{p,q}(Az) \geq 0$. Recall that the first p rows of A are in $\mathcal{L}(p,q)$, then the first p entries of Az are nonnegative for any $z \in \mathcal{M}(p,q)$. Hence by Lemma 2, we get $Az \in \mathcal{M}(p,q)$ for any $z \in \mathcal{M}(p,q)$. We conclude that A is a positive operator on $\mathcal{M}(p,q)$. □

The rest of this section aims to show that one can use a *positive semidefinite condition* to characterize a positive operator on the extended second order cone. This work is inspired by the results in [22], where they show that a positive operator on a second order cone can be characterized by a positive semidefinite condition. Recall that $\Gamma(\cdot)$ given in (1) is the set of all the positive operators on a certain cone. Specifically, they prove the following result.

Lemma 3 ([22], Theorem 2.3) *Let $B \in \mathbb{R}^{(1+q)\times(1+q)}$. If $B \in \Gamma(\mathcal{L}(1,q)) \cup \Gamma(-\mathcal{L}(1,q))$, then there exists a $\mu \geq 0$ such that*

$$B^\top J_{1,q} B \succeq \mu J_{1,q}. \tag{8}$$

Conversely, if $rank(B) \neq 1$ and (8) holds for some $\mu \geq 0$, then

$$B \in \Gamma(\mathcal{L}(1,q)) \cup \Gamma(-\mathcal{L}(1,q)).$$

Before inducing the main result, we first prove the following.

Lemma 4 *For $i = 1, \ldots, p$, let*

$$H_i = \begin{bmatrix} diag(e_i) & 0 \\ 0 & -I_q \end{bmatrix}, \tag{9}$$

where $e_i \in \mathbb{R}^p$ is the ith vector of the standard basis of $\mathbb{R}^p$, and I_q denotes the identity mapping in $\mathbb{R}^q$. Then

$$\mathcal{L}(p,q) = \{z \in \mathbb{R}^p \times \mathbb{R}^q : z^\top H_i z \geq 0,\ z_i \geq 0, \quad i = 1, \ldots, p\}. \tag{10}$$

Proof Note that the definition of $\mathcal{L}(p,q)$ means the first p entries of the vectors in $\mathcal{L}(p,q)$ are nonnegative. So we can conclude the above directly. □

For $i = 1, \ldots, p$, we also introduce the following cones

$$\mathcal{N}_i(p,q) = \{(x,u) \in \mathbb{R}^p \times \mathbb{R}^q : |x_i| \geq \|u\|\}, \tag{11}$$

and

$$\mathcal{N}(p,q) = \{(x,u) \in \mathbb{R}^p \times \mathbb{R}^q : |x_i| \geq \|u\|, i = 1, \ldots, p\}. \tag{12}$$

It is obvious that $\cap_{i=1}^p \Gamma(\mathcal{N}_i) \subseteq \Gamma(\cap_{i=1}^p \mathcal{N}_i) = \Gamma(\mathcal{N})$. Our theorem is based on this simple observation.

Theorem 3 *Suppose that $A \in \mathbb{R}^{(p+q)\times(p+q)}$. Then $A \in \cap_{i=1}^p \Gamma(\mathcal{N}_i)$ if and only if there exist p nonnegative numbers $\lambda_1, \ldots, \lambda_p$ such that for any $(\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p_+$,*

$$A^\top \left(\sum_{i=1}^p \alpha_i H_i \right) A \succeq \sum_{i=1}^p \alpha_i \lambda_i H_i. \tag{13}$$

Moreover, if the first p rows of A are in $\mathcal{M}(p,q)$, then A is a positive operator on $\mathcal{L}(p,q)$.

Proof If $z \in \mathcal{N}_i(p,q)$, then $z^\top H_i z \geq 0$. If $A \in \Gamma(\mathcal{N}_i(p,q))$, then we have

$$(Az)^\top H_i (Az) = z^\top A^\top H_i A z \geq 0, \quad i = 1, \ldots, p.$$

We can easily find $v \in \mathcal{N}_i$ such that $v^\top H_i v > 0$ for $i = 1, \ldots, p$. Hence, by the $\mathcal{S}$-Lemma, there exists a $\lambda_i \geq 0$ such that

$$A^\top H_i A \succeq \lambda_i H_i. \tag{14}$$

Then, just multiply both sides of (14) by α_i and sum up with respect to i, we get (13).

Conversely, given the inequality (13), let $\alpha_i = 1$ when $j = i$ and $\alpha_i = 0$ when $j \neq i$, we will get (14). Then, by the converse of $\mathcal{S}$-lemma, we get $A \in \Gamma(\mathcal{N}_i)$ for any i. Moreover, if the first p rows of A are in the dual cone of $\mathcal{L}(p,q)$, the first p entries of Az are positive for any $z \in \mathcal{L}(p,q)$. Together with the definition of $\mathcal{N}(p,q)$, we have A is a positive operator on $\mathcal{L}(p,q)$. □

Remark 1 If $p = 1$, then $\mathcal{L}(1,q)$ is a second order cone and $\cap_{i=1}^p \Gamma(\mathcal{N}_i) = \Gamma(\cap_{i=1}^p \mathcal{N}_i) = \Gamma(\mathcal{N})$. In this case Theorem 3 reduces to Lemma 3.

4 A necessary and sufficient condition for block diagonal cases

In this section we consider the special case when A is a block-diagonal matrix. We can apply our previous result in [18, Proposition 6.1].

Theorem 4 *Let $A \in \mathbb{R}^{(p+q)\times(p+q)}$ be given by*

$$A = \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix},$$

where $S \in \mathbb{R}^{p\times p}$, $T \in \mathbb{R}^{q\times q}$ and $\|T\| = \alpha$ for some $\alpha \geq 0$. Then,

(i) A is a positive operator on $\mathcal{L}(p,q)$ if and only if each entry of S is nonnegative and the sum of each row of S is at least α.
(ii) There exist $\lambda \geq 0$ such that $AJ_{p,q}A^\top - \lambda J_{p,q}$ is positive semidefinite if and only if the sum of each row of S are the same and at least α.
(iii) If there exist $\lambda \geq 0$ such that $AJ_{p,q}A^\top - \lambda J_{p,q}$ is positive semidefinite and each entry of S is nonnegative, then A is a positive operator on $\mathcal{L}(p,q)$.

Proof (i) "$\Leftarrow$": Suppose $(x,u) \in \mathcal{L}(p,q)$. Thus, $x_j \geq \|u\|$ where x_j is the jth entry of x. Let $(y,v) = A(x,u) = (Sx, Tu) \in \mathbb{R}^p \times \mathbb{R}^q$. It follows that

$$y_i = \sum_{j=1}^{p} s_{ij}x_j \geq \|u\| \sum_{j=1}^{p} s_{ij} \geq \|T\|\|u\| \geq \|Tu\| = \|v\|,$$

for any ith entry of y. Then $A(x,u) \in \mathcal{L}(p,q)$ and therefore A is a positive operator on $\mathcal{L}(p,q)$.

"$\Rightarrow$": Suppose A is a positive operator on $\mathcal{L}(p,q)$. By Theorem 1, it is obvious that each entry of S is nonnegative. Suppose that the sum of the kth row of S is strictly less than α. Let u_0 be the vector with $\|Tu_0\| = \alpha\|u_0\|$ (such an u_0 always exists). Then $(\|u_0\|e, u_0) \in \mathcal{L}(p,q)$. If the sum of the jth row of S is $s_j < \alpha$, then $A(\|u_0\|e, u_0) = (\|u_0\|Se, Tu_0)$ and the jth entry of this vector will be $s_j\|u_0\| < \alpha\|u_0\| = \|Tu_0\|$. That means $A(\|u_0\|e, u_0) \notin \mathcal{L}(p,q)$, which is a contradiction.

(ii) "$\Leftarrow$": By the assumptions, if the sum of each row of S is s, we get that $AJ_{p,q}A^\top - \lambda J_{p,q}$ is

$$\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} ee^\top & 0 \\ 0 & -I_q \end{bmatrix} \begin{bmatrix} S^\top & 0 \\ 0 & T^\top \end{bmatrix} - \lambda \begin{bmatrix} ee^\top & 0 \\ 0 & -I_q \end{bmatrix} = \begin{bmatrix} (s^2-\lambda)E & 0 \\ 0 & \lambda I_q - TT^\top \end{bmatrix}.$$

Since E is positive semidefinite, there exists a $\lambda \in (\alpha^2, s^2)$ such that $AJ_{p,q}A^\top - \lambda J_{p,q}$ is positive semidefinite.

"$\Rightarrow$": Suppose $s_i \neq s_j$ where s_i and s_j are the sums of the ith and jth row, respectively. The lower right block of $AJ_{p,q}A^\top - \lambda J_{p,q}$ will be $\lambda I_q - TT^\top$. So if

$AJ_{p,q}A^\top - \lambda J_{p,q}$ is positive semidefinite, we have $\lambda \geq \alpha^2 > 0$. Then the upper left block of $AJ_{p,q}A^\top - \lambda J_{p,q}$ will be of the following form

$$\begin{bmatrix} \ddots & & & & \\ & s_i^2 - \lambda & \dots & s_i s_j - \lambda & \\ & \vdots & \ddots & \vdots & \\ & s_i s_j - \lambda & \dots & s_j^2 - \lambda & \\ & & & & \ddots \end{bmatrix}.$$

If $AJ_{p,q}A^\top - \lambda J_{p,q}$ is positive semidefinite, then

$$(s_i^2 - \lambda)(s_j^2 - \lambda) - (s_i s_j - \lambda)^2 \geq 0$$

which implies $\lambda(s_i - s_j)^2 \leq 0$. As $s_i \neq s_j$, we have $\lambda \leq 0$ which is contradictory to $\lambda \geq \alpha^2 > 0$. Now we can conclude that the sums of each row are the same. Then it follows from equation (15) that s must be at least α.

(iii) Since there exist $\lambda \geq 0$ such that $AJ_{p,q}A^\top - \lambda J_{p,q}$ is positive semidefinite, we have that the sum of each row of S are the same and at least α by (ii). Together with the condition that each entry of S are nonnegative, by (i), we can conclude that A is a positive operator on $\mathcal{L}(p, q)$.

□

Remark 2 Regarding Theorem 4, the first p entries of $A(x^\top, u^\top)^\top$ are only determined by S and the last q entries are determined by T. We just need to control the sum of each row of S and the norm of T to keep $A(x^\top, u^\top)^\top$ in $\mathcal{L}(p, q)$. Note that to have $AJ_{p,q}A^\top - \lambda J_{p,q}$ positive semidefinite, it is not enough for A to be a positive operator. The extra condition that the sums of each column of S are the same is also required. We will emphasize these ideas by explicit examples in the next section.

5 Numerical examples

In this section, we will show some numerical examples. Let us begin with the following example which shows that one can find a positive operator on $\mathcal{L}(2, 2)$, although it does not satisfy the conditions described in Theorem 2. That is Theorem 2 describes only a sufficient condition for positivity.

Example 1 Now, consider the following matrix

$$A_2 = \begin{bmatrix} S_2 & 0 \\ 0 & T_2 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.45 & 0 & 0 \\ 0.3 & 0.4 & 0 & 0 \\ 0 & 0 & 0.2 & 0.3 \\ 0 & 0 & 0.3 & 0.2 \end{bmatrix}.$$

By using Theorem 4 (i), it is easy to verify that A_2 is a positive operator on $\mathcal{L}(2,2)$. However,

$$A_2^\top J_{2,2}A_2 - \lambda J_{2,2} = \begin{bmatrix} 0.3025-\lambda & 0.4675-\lambda & 0 & 0 \\ 0.4675-\lambda & 0.7225-\lambda & 0 & 0 \\ 0 & 0 & \lambda-0.13 & -0.12 \\ 0 & 0 & -0.12 & \lambda-0.13 \end{bmatrix}.$$

If $A_2^\top J_{2,2}A_2 - \lambda J_{2,2}$ is positive semidefinite, then $0.25 \le \lambda \le 0.3025$ and $\det(S_2^\top ee^\top S_2 - \lambda ee^\top) \ge 0$. So,

$$(0.3025-\lambda)(0.7225-\lambda) - (0.4675-\lambda)(0.4675-\lambda) \ge 0.$$

Then $\lambda \le 0$, which is a contradiction. That means there is no λ such that $A_2^\top J_{2,2}A_2 - \lambda J_{2,2}$ is positive semidefinite. Hence the conditions of Theorem 2 are not satisfied which shows that the theorem describes only a sufficient condition for positivity.

The following example is to show the necessity and sufficiency of Theorem 4.

Example 2 Let $A_1 \in \mathbb{R}^{4\times 4}$ provided by

$$A_1 = \begin{bmatrix} S_1 & 0 \\ 0 & T_1 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.45 & 0 & 0 \\ 0.45 & 0.25 & 0 & 0 \\ 0 & 0 & 0.2 & 0.3 \\ 0 & 0 & 0.3 & 0.2 \end{bmatrix}.$$

Here S_1 is a 2×2 matrix with the sums of each row at least 0.55 and the sum of each column equal to 0.7. Because the eigenvalues of T_1 are 0.5 and -0.1, so $\|T_1\| \le 0.5 < 0.55$. Therefore, by the result of Theorem 4 (i), we have $A_1\mathcal{L}(2,2) \subseteq \mathcal{L}(2,2)$. Since $A_1 = A_1^\top$ and $A_1^\top \in \Gamma(\mathcal{M}(2,2))$, A_1 is a positive operator on $\mathcal{M}(2,2)$.

On the other hand, note that

$$A_1^\top J_{2,2}A_1 - \lambda J_{2,2} = \begin{bmatrix} 0.49-\lambda & 0.49-\lambda & 0 & 0 \\ 0.49-\lambda & 0.49-\lambda & 0 & 0 \\ 0 & 0 & \lambda-0.13 & -0.12 \\ 0 & 0 & -0.12 & \lambda-0.13 \end{bmatrix},$$

we have $A_1^\top J_{2,2}A_1 - \lambda J_{2,2}$ is positive semidefinite for every $\lambda \in [0.25, 0.49]$. Since every entry of S is positive and the first 2 lines of A_1 are in $\mathcal{L}(2,2)$, it follows from Theorem 2 that A_1 is a positive operator on $\mathcal{M}(2,2)$.

Finally, let us propose two examples in support of Theorem 3.

Example 3 Let

$$A_3 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, we can choose $(\alpha_1, \lambda_1) = (5, 6)$ and $(\alpha_2, \lambda_2) = (5, 3)$ such that

$$A_3^\top \left(\sum_{i=1}^{2} \alpha_i H_i \right) A_3 - \sum_{i=1}^{2} \alpha_i \lambda_i H_i$$

is positive definite. It is easy to see that A_3 is a positive operator on $\mathcal{L}(2, 2)$.

Example 4 This example shows a matrix B that is a positive operator on $\mathcal{L}(2, 2)$ while $B \notin \cap_{i=1}^{2} \Gamma(\mathcal{N}_i)$. Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

It is trivial to show that $B \in \Gamma(\mathcal{L}(2, 2))$. Let $z = (2, -2, 1, -1)^\top \in \mathcal{N}_2$, $Bz = (0, 0, 2, 2)^\top \notin \mathcal{N}_2$.

6 Conclusions and future works

In this note, we showed some conditions for the positivity of operators on extended second order cones. Theorem 2.3 in [14] showed a necessary and sufficient condition for a linear operator to be a positive operator on a second order cone. But when $p > 1$, the study of positive operators on an extended second order cone $\mathcal{L}(p, q)$ (or $\mathcal{M}(p, q)$) becomes much more complicated. In Theorem 4, we showed a necessary and sufficient condition for a block diagonal linear operator A to be a positive operator on $\mathcal{L}(p, q)$. In general, if

$$A = \begin{bmatrix} S & R \\ W & T \end{bmatrix},$$

where $R \in \mathbb{R}^{p \times q}$ and $W \in \mathbb{R}^{q \times p}$, then $A(x^\top, u^\top)^\top = (Sx + Ru, Wx + Tu)^\top$. Hence, each entry is determined by both x and u. It is difficult to ensure the first p entries of $A(x^\top, u^\top)^\top$, that is, $Sx + Ru$, to be all positive or all negative simultaneously. Similar difficulties occur in Theorem 3. Though the $\mathcal{S}$-lemma provides a necessary and sufficient condition, the matrix B in Example 4 shows that $\cap_{i=1}^{p} \Gamma(\mathcal{N}_i) \neq \Gamma(\mathcal{N})$. This is because the extended second order cones are irreducible [23].

Since the extended second order cones are not self-dual when $p > 1$ and are not polyhedral when $q > 1$ (see [17]), it seems hard to find a unified feature of such a positive operator A. To improve our results, we may consider some other direction to investigate the necessary conditions and sufficient conditions for a linear operator to be a positive operator. The $\mathcal{S}$-Lemma (Yuan's Lemma) was extended to be defined in a first order cone (direct sum of a subspace and a ray) rather than the whole space in the following way (see [3, Corollary 3.2], [13, Lemma 2.1]):

Lemma 5 *Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices and $\mathcal{K} \subseteq \mathbb{R}^n$ be a first-order cone. Then, the following conditions are equivalent:*

1. $\max\{x^\top Ax, x^\top Bx\} \geq 0$, $x \in \mathcal{K}$.
2. *There exists $t_1 \geq 0$, $t_2 \geq 0$ with $t_1 + t_2 = 1$ such that $t_1A + t_2B$ is positive semidefinite on $\mathcal{K}$.*

It is a natural but challenging question to find a version of the $\mathcal{S}$-lemma defined on second order cones or extended second order cones which could be used to characterise the positive operators on the latter cones. This direction seems very interesting and challenging and if successful it would lead to many applications.

References

1. Alizadeh, F., Goldfarb, D. Second-order cone programming. Math. Program., Ser. B, 95(1): 3–51, 2003.
2. Arumugasamy, C., Jayaraman, S., Mer, V.N. Semipositivity of linear maps relative to proper cones in finite dimensional real Hilbert spaces. Electron. J. Linear Algebra., 34: 304–319, 2018.
3. Baccari, A., Trad, A. On the classical necessary second-order optimality conditions in the presence of equality and inequality constraints. SIAM J. Optim. 15(2) :394-408, 2004/05
4. Ben-Tal, A., Nemirovski, A. Lectures on modern convex optimization: analysis, algorithms, and engineering applications. Vol. 2, SIAM, 2001.
5. Berman, A., Gaiha, P.. A generalization of irreducible monotonicity. Linear Algebra and Appl., 5: 29–38, 1972.
6. Birkhoff, G. Extensions of Jentzsch's theorem. Trans. Amer. Math. Soc., 85: 219–227, 1957.
7. Bushell, P. J. On the projective contraction ratio for positive linear mappings. J. London Math. Soc., 2(2): 256–258, 1973.
8. Faraut, J., Korányi, A. Analysis on symmetric cones. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1994.
9. Ferreira, O. P., Németh, S. Z. How to project onto extended second order cones. J. Global Optim., 70(4): 707–718, 2018.
10. Ferreira, O. P., Németh, S. Z., On the spherical convexity of quadratic functions. J. Global Optim., 73(3): 537–545, 2019.
11. Gu, B, Sun, X.M., Sheng, V.S. Structural minimax probability machine. IEEE Trans. Neural Netw. Learn. Syst. , 28(7): 1646-1656, 2017.
12. Jiang, X., Chen, J.Y.,So, H.C., Liu, X.Z. Large-Scale Robust Beamforming via l(infinity)-Minimization. IEEE Trans. Signal Process., 66(14): 3824-3837, 2018.
13. Haeser, G. An extension of Yuan's lemma and its applications in optimization. J. Optim. Theory Appl., 174(3): 641–649, 2017.
14. Loewy, R., Schneider, H. Positive operators on the n-dimensional ice cream cone. J. Math. Anal. Appl., 49: 375–392, 1975.
15. Mencarelli, L. ; D'Ambrosio, C. Complex portfolio selection via convex mixed-integer quadratic programming: a survey. Intl. Trans. Op. Res., 26(2): 389–414, 2019.
16. Németh,S. Z., Xiao, L. Linear complementarity problems on extended second order cones. J. Optim. Theory Appl., 176(2): 269–288, 2018.
17. Németh,S. Z., Zhang, G. Extended Lorentz cones and mixed complementarity problems. J. Global Optim., 62(3): 443–457, 2015.
18. Németh,S. Z., Zhang, G. Extended Lorentz cones and variational inequalities on cylinders. J. Optim. Theory Appl., 168(3): 756–768, 2016.
19. Pólik, I., Terlaky, T. A survey of the S-lemma. SIAM Rev., 49(3): 371–418, 2007.

20. Qi, Z. Q., Tian, Y. J., Shi, Y. Robust twin support vector machine for pattern classification. Pattern Recognit., 46(1): 305-316, 2013.
21. Rudolf, G., Noyan, Dávid Papp, D., Alizadeh, F.. Bilinear optimality constraints for the cone of positive polynomials. Math. Program., Ser. B 129(1): 5–31, 2011.
22. Schneider, H., Vidyasagar, M. Cross-positive matrices. SIAM J. Numer. Anal., 7: 508–519, 1970.
23. Sznajder, R. The Lyapunov rank of extended second order cones. J. Global Optim., 66(3): 585–593, 2016.
24. Tam, B. S. Some results of polyhedral cones and simplicial cones. Linear Multilinear Alg., 4(4): 281–284, 1977.
25. Tam, B. S. On the structure of the cone of positive operators. Linear Algebra Appl., 167: 65–85, 1992.
26. Tanaka, K. Forecasting scenarios from the perspective of a reverse stress test using second-order cone programming. Journal of Risk Model Validation, 11(2): 1–21, 2017.
27. Tzelepis, C; Mezaris, V; Patras, I. Linear Maximum Margin Classifier for Learning from Uncertain Data. IEEE Trans. Neural Netw. Learn. Syst. , 40(12): 2948-2962, 2018.
28. Istrăţescu, V. I. Fixed point theory. Mathematics and its Applications, D. Reidel Publishing Co., Dordrecht-Boston, Mass. 1981
29. Wang, G., Zamzam, A.S., Giannakis, G.B., Sidiropoulos, N.D. Power System State Estimation via Feasible Point Pursuit: Algorithms and Cramer-Rao Bound. IEEE Trans. Singal Process., 66(6): 1649–1658, 2018.
30. Xia, Y., Wang, S., Sheu, R.L. S-lemma with equality and its applications. Math. Program., Ser. A 156(1-2): 513–547, 2016.
31. Yakubovič, V. A. The S-procedure in nonlinear control theory. Vestnik Leningrad. Univ., 1: 62–77, 1971.
32. Yakubovič, V. A. Minimization of quadratic functionals under quadratic constraints, and the necessity of a frequency condition in the quadratic criterion for the absolute stability of nonlinear control systems. Dokl. Akad. Nauk SSSR, 209: 1039–1042, 1973.
33. Yuan,Y.X. On a subproblem of trust region algorithms for constrained optimization. Math. Program., Ser. A 47(1): 53–63, 1990.