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# Set families with a forbidden pattern

Ilan Karpas \*

Eoin Long †

## Abstract

A *balanced pattern* of order  $2d$  is an element  $P \in \{+, -\}^{2d}$ , where both signs appear  $d$  times. Two sets  $A, B \subset [n]$  form a  $P$ -pattern, which we denote by  $\text{pat}(A, B) = P$ , if  $A \Delta B = \{j_1, \dots, j_{2d}\}$  with  $1 \leq j_1 < \dots < j_{2d} \leq n$  and  $\{i \in [2d] : P_i = +\} = \{i \in [2d] : j_i \in A \setminus B\}$ . We say  $\mathcal{A} \subset \mathcal{P}[n]$  is  $P$ -free if  $\text{pat}(A, B) \neq P$  for all  $A, B \in \mathcal{A}$ . We consider the following extremal question: how large can a family  $\mathcal{A} \subset \mathcal{P}[n]$  be if  $\mathcal{A}$  is  $P$ -free?

We prove a number of results on the sizes of such families. In particular, we show that for some fixed  $c > 0$ , if  $P$  is a  $d$ -balanced pattern with  $d < c \log \log n$  then  $|\mathcal{A}| = o(2^n)$ . We then give stronger bounds in the cases when (i)  $P$  consists of  $d +$  signs, followed by  $d -$  signs and (ii)  $P$  consists of alternating signs. In both cases, if  $d = o(\sqrt{n})$  then  $|\mathcal{A}| = o(2^n)$ . In the case of (i), this is tight.

## 1 Introduction

A central goal in extremal set theory is to understand how large a set family can be subject to some restriction on the intersections of its elements. Given  $\mathcal{L} \subset \mathbb{N} \cup \{0\}$ , we say that a set family  $\mathcal{A}$  is  $\mathcal{L}$ -intersecting if  $|A \cap B| \in \mathcal{L}$  for all distinct  $A, B \in \mathcal{A}$ . Taking  $\mathcal{L}_t = \{s \in \mathbb{N} : s \geq t\}$ , a fundamental theorem of Erdős, Ko and Rado [6] shows that  $\mathcal{L}_t$ -intersecting families  $\mathcal{A} \subset \binom{[n]}{k}$  satisfy  $|\mathcal{A}| \leq \binom{n-t}{k-t}$ , provided  $n \geq n_0(k, t)$ . Another important theorem due to Frankl and Füredi [8] shows that if  $\mathcal{L}_{\ell, \ell'} := \{s < \ell \text{ or } s \geq k - \ell'\}$ , then any  $\mathcal{L}_{\ell, \ell'}$ -intersecting family  $\mathcal{A} \subset \binom{[n]}{k}$  satisfies  $|\mathcal{A}| \leq cn^{\max(\ell, \ell')}$ , for some constant  $c$  depending on  $k, \ell$  and  $\ell'$ . See [2], [3], [7], [9] for an overview of this extensive topic.

Here we are concerned with understanding the effect of restricting the *pattern* formed between elements of a set family. A *difference pattern* or *pattern* of order  $t$  is an element  $P \in \{+, -\}^t$ . Given such a pattern  $P$ , let  $S_+(P) = \{i \in [t] : P_i = +\} \subset [t]$  and  $s_+(P) = |S_+(P)|$ . Define  $S_-(P)$  and  $s_-(P)$  analogously. Two sets  $A, B \subset [n]$  form a *difference pattern*  $P$  if:

- (i)  $A \Delta B = \{j_1, \dots, j_t\}$  with  $j_1 < \dots < j_t$ , and
- (ii)  $\{i \in [t] : P_i = +\} = \{i \in [t] : j_i \in A \setminus B\}$ .

We denote this by writing  $\text{pat}(A, B) = P$ . A family of subsets  $\mathcal{A} \subset \mathcal{P}[n]$  is  $P$ -free if  $\text{pat}(A, B) \neq P$  for all distinct  $A, B \in \mathcal{A}$ . In this paper we consider the following natural question: given a pattern  $P$ , how large can a family  $\mathcal{A} \subset \mathcal{P}[n]$  be if it is  $P$ -free?

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First note the following simple observation. If  $s_+(P) \neq s_-(P)$  then large  $P$ -free families exist. Indeed, if  $|s_+(P) - s_-(P)| = m > 0$  then the following families are  $P$ -free:

$$\mathcal{B}_1 = \{A \subset [n] : |A| \in [0, m-1] \pmod{2m}\}; \quad \mathcal{B}_2 = \{A \subset [n] : |A| \in [m, 2m-1] \pmod{2m}\}.$$

Clearly either  $|\mathcal{B}_1| \geq 2^{n-1}$  or  $|\mathcal{B}_2| \geq 2^{n-1}$ . We will therefore focus on the case when  $s_+(P) = s_-(P) = d$ . We say that such patterns are  $d$ -balanced. For a balanced pattern  $P$  it is only possible that  $\text{pat}(A, B) = P$  if  $|A| = |B|$ . Thus, our question on balanced patterns essentially reduces to a question for uniform families. Given  $0 \leq k \leq n$ , define

$$f(n, k, P) := \max \left\{ |A| : P\text{-free families } \mathcal{A} \subset \binom{[n]}{k} \right\}.$$

Let  $f(n, k, d) = \max\{f(n, k, P) : P \text{ is } d\text{-balanced}\}$ . We will also write  $\delta(n, k, P)$  and  $\delta(n, k, d)$  for the corresponding extremal densities, i.e.  $\delta(n, k, P) := f(n, k, P) / \binom{n}{k}$ , and  $\delta(n, k, d) := f(n, k, d) / \binom{n}{k}$ . Note also that if  $\mathcal{A} \subset \binom{[n]}{k}$  is  $P$ -free then the family  $\mathcal{A}^c = \{[n] \setminus A : A \in \mathcal{A}\} \subset \binom{[n]}{n-k}$  is also  $P$ -free. Therefore  $f(n, k, P) = f(n, n-k, P)$  and it suffices to bound  $f(n, k, P)$  for  $k \leq n/2$ .

Our first aim is to prove a density result for  $d$ -balanced patterns of small order. That is, we will show that for fixed  $d$ , any sequence of integers  $\{k_n\}_{n=1}^\infty$  tending to infinity with  $n$  with  $k_n \leq n/2$  satisfies  $\lim_{n \rightarrow \infty} \delta(n, k_n, d) = 0$ . The condition that  $k$  is not fixed and tends to infinity with  $n$  will be crucial. This is different from the case in the Frankl-Füredi Theorem, which tells us that we can take some fixed  $k \geq 2d-1$ ,  $\ell = k-d$  and  $\ell' = d-1$ , and if  $\mathcal{A} \subset \binom{[n]}{k}$  with  $|\mathcal{A}| = \omega(n^{k-d})$  then there are  $A, B \in \mathcal{A}$  with  $|A \triangle B| = 2d$ , i.e.  $A$  and  $B$  form a  $P$ -pattern for *some*  $d$ -balanced pattern  $P$ . Indeed, take any fixed  $k := k(d)$ , and consider the family  $\mathcal{A}_0 \subset \binom{[n]}{k}$  given by

$$\mathcal{A}_0 = \left\{ A \subset [n] : \left| A \cap \left( \frac{(i-1)n}{k}, \frac{in}{k} \right] \right| = 1 \text{ for all } i \in [k] \right\}.$$

Then  $|\mathcal{A}_0| \geq c_k n^k$  for some absolute constant  $c_k > 0$ , but it is easily seen that  $\mathcal{A}_0$  does not contain the pattern  $++--$ . Therefore, there does not exist a density theorem for  $d$ -balanced patterns in subsets of  $\binom{[n]}{k}$  with fixed  $k$ , as in the Frankl-Füredi theorem.

Our first result shows that such a density theorem does hold for  $k$  growing with  $n$ .

**Theorem 1.** *Given  $d, k, n \in \mathbb{N}$  with  $2k \leq n$  and taking  $a_d = (8d)^{5d}$  and  $c_d = 6d8^{-d}$  we have*

$$\delta(n, k, d) \leq a_d k^{-c_d}.$$

By our discussion above for fixed  $k$  we see that Theorem 1 is in a sense a ‘high-dimensional’ result. Also note that Theorem 1 shows there is a constant  $c > 0$  with the property that if  $P$  is a  $d$ -balanced pattern with  $d \leq c \log \log n$  and  $\mathcal{A} \subset \mathcal{P}[n]$  which is  $P$ -free, then  $|\mathcal{A}| = o(2^n)$ .

Let  $\text{IP}(d)$  denote the  $d$ -balanced pattern consisting of  $d$  plus signs, followed by  $d$  minus signs. We refer to these as *interval patterns*. Given the obstruction of  $\text{IP}(2)$  above, it is natural to ask for bounds on  $f(n, k, \text{IP}(d))$ .

**Theorem 2.** *Given  $d, k, n \in \mathbb{N}$  with  $2k \leq n$  we have*

$$\delta(n, k, \text{IP}(d)) = O(d^2 k^{-1}).$$

In particular, families  $\mathcal{A} \subset \mathcal{P}[n]$  which are  $\text{IP}(d)$ -free for all  $d = o(\sqrt{n})$  satisfy  $|\mathcal{A}| = o(2^n)$ . Furthermore, this turns out to be tight – if  $d \geq c\sqrt{n}$  then there are  $\text{IP}(d)$ -free families  $\mathcal{A} \subset \mathcal{P}[n]$  with  $|\mathcal{A}| = \Omega_c(2^n)$ .

Lastly, we consider the  $d$ -balanced pattern  $\text{AP}(d)$  consisting of alternating plus and minus signs, e.g.  $\text{AP}(2) = + - + -$ . We refer to these as *alternating patterns*. Our next result proves a density result for such patterns.

**Theorem 3.** *Given  $d, k, n \in \mathbb{N}$  with  $2k \leq n$  we have*

$$\delta(n, k, \text{AP}(d)) = O\left(\log^{-1}\left(\frac{k}{d^2}\right)\right).$$

Thus again, all families  $\mathcal{A} \subset \mathcal{P}[n]$  which are  $\text{AP}(d)$ -free for  $d = o(\sqrt{n})$  satisfy  $|\mathcal{A}| = o(2^n)$ . Unlike in the case of the interval patterns, we do not know if this is tight.

Before closing the introduction, we mention some further results related to this topic. A family  $\mathcal{A} \subset \mathcal{P}[n]$  is said to be a *tilted Sperner family* if for all distinct  $A, B \in \mathcal{A}$  we have  $|B \setminus A| \neq 2|A \setminus B|$ . Equivalently,  $\mathcal{A}$  is  $P$ -free for all patterns  $P$  with  $|S_-(P)| = 2|S_+(P)|$ . Kalai raised the question of how large a tilted Sperner family  $\mathcal{A} \subset \mathcal{P}[n]$  can be. In [13], Leader and the second author proved that such families satisfy  $|\mathcal{A}| \leq (1 + o(1))\binom{n}{n/2}$ , which is asymptotically optimal. For sufficiently large  $n$ , the extremal families were also determined. In [14], the second author proved that this bound almost still applies if we only forbid ‘tilted pairs’  $A, B$  with a single pattern. It was shown that if  $\mathcal{A} \subset \mathcal{P}[n]$  does not contain  $A, B \in \mathcal{P}[n]$  with  $|B \setminus A| \neq 2|A \setminus B|$  for all distinct  $A, B \in \mathcal{A}$  and satisfying  $a < b$  for all  $a \in A \setminus B$  and  $b \in B \setminus A$  then  $|\mathcal{A}| \leq C\sqrt{\log n} \binom{n}{n/2}$ , for some constant  $C > 0$ . This condition is equivalent to  $\mathcal{A}$  being  $P(d)$ -free for all patterns  $P(d)$  consisting of  $d +$  signs followed by  $2d -$  signs. This bound was recently improved by Gerbner and Vizer in [11]. They proved that such families satisfy  $|\mathcal{A}| \leq C\sqrt{\log n} \binom{n}{n/2}$ . No family is known for this problem with order more than  $C\binom{n}{n/2}$ .

Lastly, we mention a fascinating question raised by Johnson and Talbot [12] related to Theorem 1 (similar conjectures have been raised by Bollobás, Leader and Malvenuto [4], and Bukh [5]). Our phrasing slightly differs from that in [12].

**Question** (Johnson–Talbot). *Is it true that for any  $k \in \mathbb{N}$  and  $\alpha > 0$  there is  $n_0(k, \alpha) \in \mathbb{N}$  with the following property. Suppose that  $n \geq n_0(k, \alpha)$  and that  $\mathcal{A} \subset \binom{[n]}{n/2}$  with  $|\mathcal{A}| \geq \alpha \binom{n}{n/2}$ . Then there are disjoint sets  $S \in \binom{[n]}{n/2 - \lfloor k/2 \rfloor}$  and  $T \in \binom{[n] \setminus S}{k}$  such that the family  $\mathcal{C}_{T,S} := \{S \cup U : U \in \binom{T}{\lfloor k/2 \rfloor}\}$  is contained in  $\mathcal{A}$ .*

This is true for  $k = 3$ , but is already open for  $k = 4$ . In this case it is possible to guarantee that  $|\mathcal{C}_{T,S} \cap \mathcal{A}| \geq 5$  for some  $T, S$  (note  $|\mathcal{C}_{T,S}| = 6$  for  $k = 4$ ). More generally, Johnson and Talbot [12] proved that under the hypothesis above,  $|\mathcal{C}_{T,S} \cap \mathcal{A}| \geq 4 \cdot 3^{(k-4)/3}$  for some  $T, S$ . We note the conclusion that dense subsets of  $\mathcal{P}[n]$  contain all small patterns (from Theorem 1) would immediately follow from a positive answer to this question. Indeed, for  $k = 2d$  any set  $\mathcal{C}_{T,S}$  contains every  $d$ -balanced pattern. Theorem 1 may be seen as giving (weak) evidence for the question: for  $k = 2d$  and any  $d$ -balanced pattern  $P$ , there is  $T$  and  $S$  and sets  $A, B \in \mathcal{C}_{T,S} \cap \mathcal{A}$  with  $\text{pat}(A, B) = P$ .

**Notation:** Given a set  $X$ , we write  $\mathcal{P}(X)$  for the power set of  $X$  and  $\binom{X}{k} = \{A \subset X : |A| = k\}$ . Given integers  $m, n \in \mathbb{N}$  with  $m \leq n$ , we let  $[n] = \{1, \dots, n\}$  and  $[m, n] = \{m, m+1, \dots, n\}$ . We also write  $(n)_m$  for the falling factorial  $(n)_m = n(n-1) \cdots (n-m+1)$ .

## 2 Small balanced patterns

In this section we prove Theorem 1. We will find it convenient to prove many of our results restricted of the middle layer. We then simply write  $f(k, P)$  for  $f(2k, k, P)$ ,  $\delta(k, P)$  for  $\delta(2k, k, P)$ , etc.. The following simple observation is useful to move results between different layers of the cube.

**Proposition 2.1.** *Let  $n, m, k, l \in \mathbb{N}$  with  $m \leq n$ ,  $l \leq k$  and  $k + m - l \leq n$ . Let  $P$  be a pattern. Then  $\delta(n, k, P) \leq \delta(m, l, P)$ .*

*Proof.* Suppose  $\mathcal{A} \subset \binom{[n]}{k}$  is  $P$ -free with  $|\mathcal{A}| = \delta(n, k, P) \binom{n}{k}$ . Select two disjoint sets  $T$  and  $U$  of order  $m$  and  $k - l$  uniformly at random (possible as  $k + m - l \leq n$ ). Then let  $\mathcal{A}_{T,U} = \{A \in \binom{[n]}{k} : A \cup U \in \mathcal{A}\}$ . As  $\mathcal{A}$  is  $P$ -free, the set  $\mathcal{A}_{T,U}$  must also be  $P$ -free for all  $T, U$ , giving  $|\mathcal{A}_{T,U}| \leq \delta(m, l, P) \binom{m}{l}$ . However,  $\mathbb{E}_{T,U} |\mathcal{A}_{T,U}| = \delta(n, k, P) \binom{m}{l}$ . The result follows.  $\square$

Our next two lemmas are the main steps in the proof of Theorem 1. Combined they will allow a recursive bound for  $\delta(k, d)$  based on bounds on  $\delta(k', d')$  for  $k' < k$  and  $d' < d$ .

**Lemma 2.2.** *Let  $d, k \in \mathbb{N}$  with  $k^{1/2} \geq 16 \log k$  and let  $P$  be a  $d$ -balanced pattern with  $P_1 \neq P_{2d}$ . Then given any  $\gamma \in [\frac{16 \log k}{k^{1/2}}, 1]$  we have*

$$\delta(k, P) \leq \max \left( \gamma, 6\sqrt{\delta(\lceil \gamma^2 k / 64 \rceil, d - 1)} \right).$$

*Proof.* Let  $\gamma$  be chosen as above and let  $\mathcal{A} \subset \binom{[2k]}{k}$  be  $P$ -free with  $|\mathcal{A}| = \alpha \binom{2k}{k}$ . If  $\alpha \leq \gamma$  then there is nothing to prove, so we will assume that  $\alpha > \gamma \geq \frac{16 \log k}{k^{1/2}}$ . We will first show that there are many pairs  $A, B \in \mathcal{A}$  with  $|A \Delta B| = 2$ . Indeed, given  $C \in \binom{[2k]}{k+1}$  let  $y_C$  denote the number of  $A \in \mathcal{A}$  with  $A \subset C$ . Then we have

$$\sum_{C \in \binom{[2k]}{k+1}} y_C = |\{(A, C) \in \mathcal{A} \times \binom{[2k]}{k+1} : A \subset C\}| = |\mathcal{A}| k \geq \alpha k \binom{2k}{k+1}.$$

As for every pair  $A, B \in \mathcal{A}$  with  $|A \Delta B| = 2$  there is a unique set  $C \in \binom{[2k]}{k+1}$  with  $A, B \subset C$ , we obtain

$$\begin{aligned} \left| \left\{ (A, B) \in \binom{[2k]}{2} : |A \Delta B| = 2 \right\} \right| &= \sum_{C \in \binom{[2k]}{k+1}} \binom{y_C}{2} \geq \binom{2k}{k+1} \binom{\alpha k}{2} \\ &\geq \frac{\alpha^2 k^2}{4} \times \frac{2k(2k-1)}{(k+1)k} \binom{2k-2}{k-1} \geq \frac{\alpha^2 k^2}{2} \binom{2k-2}{k-1}. \end{aligned} \quad (1)$$

The first inequality holds by the convexity of  $\binom{x}{2}$  and the second since  $\alpha k - 1 \geq \alpha k / 2$  as  $\alpha \geq 2/k$ .

Now given  $1 \leq i < j \leq 2k$ , let  $\mathcal{A}_{i,j} := \{A \in \binom{[2k]}{k} \setminus \{i, j\} : A \cup \{i\}, A \cup \{j\} \in \mathcal{A}\}$ . Note that from (1) we have

$$\sum_{i < j} |\mathcal{A}_{i,j}| = \left| \{(A, B) \in \mathcal{A} \times \mathcal{A} : |A \Delta B| = 2\} \right| \geq \frac{\alpha^2 k^2}{2} \binom{2k-2}{k-1}. \quad (2)$$

Also let  $\alpha_{i,j}$  and  $\beta_{i,j}$  be defined so that  $|\mathcal{A}_{i,j}| = \alpha_{i,j} \binom{2k-2}{k-1}$  and  $\beta_{i,j} = (j-i)/2k$ . By (2) we find  $\{i, j\}$  with  $\alpha_{i,j} \geq \frac{\alpha^2}{8}$  and  $\beta_{i,j} \geq \frac{\alpha^2}{16}$ . Indeed, we have

$$\sum_{\{i,j\}:\alpha_{i,j} \geq \frac{\alpha^2}{8}} |\mathcal{A}_{i,j}| + \sum_{\{i,j\}:\beta_{i,j} \geq \frac{\alpha^2}{16}} |\mathcal{A}_{i,j}| < \binom{2k}{2} \frac{\alpha^2}{8} \binom{2k-2}{k-1} + 2k \times \frac{\alpha^2}{16} 2k \binom{2k-2}{k-1} \leq \frac{\alpha^2 k^2}{2} \binom{2k-2}{k-1}.$$

Combined with (2) we see that a claimed pair  $\{i, j\}$  exists. Fix such a pair  $\{i, j\}$  and set  $\mathcal{B} = \mathcal{A}_{i,j}$ . Now let  $X = [i+1, j-1]$  and  $Y = [n] \setminus [i, j]$  so that  $\mathcal{B} \subset \binom{X \cup Y}{k-1}$ . Partition elements from  $\binom{X \cup Y}{k-1}$  according to how they intersect  $X$ , for each  $\ell \in [0, j-i-2]$  letting

$$X_\ell = \left\{ A \in \binom{X \cup Y}{k-1} : |A \cap X| = \ell \right\}.$$

Also let  $\mathcal{B}_\ell = \mathcal{B} \cap X_\ell$  and  $L = \left\{ \ell : \left| \ell - \frac{|X|}{2} \right| \leq \sqrt{|X| \log \left( \frac{8}{\alpha} \right)} \right\}$ . By Chernoff's inequality we have

$$\sum_{\ell \notin L} |X_\ell| \leq \frac{\alpha^2}{32} \binom{|X| + |Y|}{k-1}.$$

Using that  $|\mathcal{B}| = \alpha_{i,j} \binom{|X|+|Y|}{k-1} \geq \frac{\alpha^2}{16} \binom{|X|+|Y|}{k-1}$  this shows that

$$\sum_{\ell \in L} |\mathcal{B}_\ell| \geq |\mathcal{B}| - \frac{\alpha^2}{32} \binom{|X| + |Y|}{k-1} \geq \frac{\alpha^2}{32} \binom{|X| + |Y|}{k-1} \geq \frac{\alpha^2}{32} \sum_{\ell \in L} |X_\ell|.$$

The last inequality here holds since the sets  $X_\ell$  are disjoint subsets of  $\binom{X \cup Y}{k-1}$ . Thus for some  $\ell \in L$  we have  $|\mathcal{B}_\ell| \geq \frac{\alpha^2}{32} |X_\ell|$ . By averaging, we find a set  $U \subset Y$  with  $|U| = k - \ell - 1$  such that the family  $\mathcal{C} = \{C \in \binom{X}{\ell} : C \cup U \in \mathcal{B}_\ell\}$  satisfies  $|\mathcal{C}| \geq \frac{\alpha^2}{32} \binom{|X|}{\ell}$ .

To complete the proof, let  $Q$  denote the pattern obtained from  $P$  by removing  $P_1$  and  $P_{2d}$ , i.e.  $Q = P_2 \cdots P_{2d-1}$ . Note that as  $P_1 \neq P_{2d}$  we see that  $Q$  is  $(d-1)$ -balanced. We claim that  $\mathcal{C}$  is  $Q$ -free. Indeed, suppose  $C_1, C_2 \in \mathcal{C}$  with  $\text{pat}(C_1, C_2) = Q$ . Then by definition of  $\mathcal{C}$  and  $\mathcal{B} = \mathcal{A}_{i,j}$  we have

$$\left\{ C_a \cup U \cup \{h\} : a \in \{1, 2\}, h \in \{i, j\} \right\} \subset \mathcal{A}.$$

If  $P_1 = +$  we find  $\text{pat}(C_1 \cup U \cup \{i\}, C_2 \cup U \cup \{j\}) = P$ . If  $P_1 = -$  we find  $\text{pat}(C_1 \cup U \cup \{j\}, C_2 \cup U \cup \{i\}) = P$ . Thus  $\mathcal{C}$  must be  $Q$ -free and

$$\frac{\alpha^2}{32} \binom{|X|}{\ell} \leq |\mathcal{C}| \leq \delta(|X|, \ell, Q) \binom{|X|}{\ell}.$$

Take  $k' = \lfloor \frac{|X|}{2} - \sqrt{|X| \log \left( \frac{8}{\alpha} \right)} \rfloor$ . A calculation shows that  $\frac{|X|}{4} \geq \sqrt{|X| \log \left( \frac{8}{\alpha} \right)} + 2$  since  $\alpha \geq \frac{16 \log k}{k^{1/2}}$  and  $|X| + 2 = \beta_{i,j} 2k \geq \frac{\alpha^2 k}{8}$ . This gives

$$k' \geq \frac{|X|}{2} - \sqrt{|X| \log \left( \frac{8}{\alpha} \right)} - 1 \geq \frac{|X|}{4} + 1 \geq \left\lceil \frac{\beta_{i,j} k}{2} \right\rceil \geq \left\lceil \frac{\alpha^2 k}{64} \right\rceil \geq \left\lceil \frac{\gamma^2 k}{64} \right\rceil.$$

Since  $\ell \in L$  we have  $k' \leq \ell \leq |X| - k'$ . Using Proposition 2.1 we find that  $\frac{\alpha^2}{32} \leq \delta(|X|, \ell, d-1) \leq \delta(2k', k', d-1) = \delta(k', d-1) \leq \delta(\lceil \frac{\gamma^2 k}{64} \rceil, d-1)$ . Rearranging this gives  $\alpha \leq 6 \sqrt{\delta(\lceil \frac{\gamma^2 k}{64} \rceil, d-1)}$ .  $\square$

Our second lemma deals with the case where  $P$  starts and ends with the same signs.

**Lemma 2.3.** *Let  $d \in \mathbb{N}$  and let  $P$  be a  $d$ -balanced pattern with  $P_1 = P_{2d}$ . Then there are  $d_1, d_2 \geq 1$  with  $d_1 + d_2 = d$  such that the following holds. For every  $k_1, k_2$  with  $2k_1 + k_2 = k$  we have*

$$\delta(k, P) \leq \max\left(2e^{-k_1/12}, 4\delta(k_1, d_1), 4(3k_1)^{2d_1}\delta(k_2, d_2)\right).$$

Similarly for every  $k_1, k_2$  with  $k_1 + 2k_2 = k$  we have

$$\delta(k, P) \leq \max\left(2e^{-k_2/12}, 4\delta(k_2, d_2), 4(3k_2)^{2d_2}\delta(k_1, d_1)\right).$$

*Proof.* To begin, for each  $\ell \in [0, 2d]$  let

$$c_\ell = |\{j \in [\ell] : P_j = +\}| - |\{j \in [\ell] : P_j = -\}|.$$

As  $P$  is  $d$ -balanced and  $P_1 = P_{2d}$ , we have  $c_{2d-1} = -c_1$ . Combined with the fact that  $c_0 = c_{2d} = 0$  and  $c_\ell$  changes by exactly 1 as  $\ell$  increases, we see that  $c_{2d_1} = 0$  for some  $1 \leq d_1 \leq d-1$ . Setting  $d_2 := d - d_1$  and  $Q_1 = P_1 \cdots P_{2d_1}$ ,  $Q_2 = P_{2d_1+1} \cdots P_{2d}$  it is easy to see that these patterns are  $d_1$ -balanced and  $d_2$ -balanced respectively.

Now suppose that  $\mathcal{A} \subset \binom{[2k]}{k}$  with  $|\mathcal{A}| = \alpha \binom{2k}{k}$  and that  $\mathcal{A}$  is  $P$ -free. We will prove the first bound above as the second bound is proved identically. We will assume that  $\alpha \geq 2e^{-k_1/12}$  as otherwise there is nothing to show. Partition  $[2k]$  into two consecutive intervals  $I_1 = [3k_1]$  and  $I_2 = [3k_1 + 1, 2k]$ . For each  $\ell \in I_1$  let  $Z_\ell := \binom{I_1}{\ell} \times \binom{I_2}{k-\ell}$ . Let  $L = \left\{ \ell \in I_1 : |\ell - 3k_1/2| \leq \sqrt{3k_1 \log\left(\frac{2}{\alpha}\right)} \right\}$ . Note that as  $|\bigcup_{\ell \notin L} Z_\ell| \leq \frac{\alpha}{2} \binom{2k}{k}$  by Chernoff's inequality, we have  $|\mathcal{A} \cap Z_\ell| \geq \frac{\alpha}{2} |Z_\ell|$  for some  $\ell \in L$ . Fix such a choice of  $\ell$  and set  $Z := Z_\ell$  and  $\mathcal{B} = \mathcal{A} \cap Z_\ell$  so that  $\mathcal{B} \subset Z$  with  $|\mathcal{B}| \geq \frac{\alpha}{2} |Z|$ .

We will now prove that  $\alpha$  satisfies

$$\alpha \leq \max\left(4\delta(|I_1|, \ell, Q_1), 4|I_1|^{2d_1}\delta(|I_2|, k - \ell, Q_2)\right). \quad (3)$$

To see this, we may assume that  $\alpha \geq 4\delta(|I_1|, \ell, Q_1)$  as otherwise there is nothing to show. Consider the set  $\mathcal{P}_{Q_1}$  given by

$$\mathcal{P}_{Q_1} = \left\{ (A, B) \in Z \times Z : \text{pat}(A \cap I_1, B \cap I_1) = Q_1 \text{ and } A \cap I_2 = B \cap I_2 \right\}.$$

We will first show that  $|\mathcal{B} \times \mathcal{B} \cap \mathcal{P}_{Q_1}| \geq \frac{\alpha}{4|I_1|^{2d_1}} |\mathcal{P}_{Q_1}|$ . Indeed, for each  $D \in \binom{I_2}{k-\ell}$  let

$$\mathcal{E}(D) := \left\{ C \in \binom{I_1}{\ell} : C \cup D \in \mathcal{B} \right\}; \quad \mathcal{P}_{Q_1}(D) := \left\{ C, C' \in \mathcal{E}(D) : \text{pat}(C, C') = Q_1 \right\}.$$

Noting that each  $\mathcal{C} \subset \binom{I_1}{\ell}$  with  $|\mathcal{C}| > \delta(|I_1|, \ell, Q_1) \binom{|I_1|}{\ell}$  contains  $C, C'$  with  $\text{pat}(C, C') = Q_1$ , we find  $|\mathcal{P}_{Q_1}(D)| \geq |\mathcal{E}(D)| - \delta(|I_1|, \ell, Q_1) \binom{|I_1|}{\ell}$ . Combined these give

$$|\mathcal{B} \times \mathcal{B} \cap \mathcal{P}_{Q_1}| = \sum_{D \in \binom{I_2}{k-\ell}} |\mathcal{P}_{Q_1}(D)| \geq \sum_{D \in \binom{I_2}{k-\ell}} \left( |\mathcal{E}(D)| - \delta(|I_1|, \ell, Q_1) \binom{|I_1|}{\ell} \right) \geq \frac{\alpha}{4} |Z|, \quad (4)$$

The final inequality here holds since  $\sum_{D \in \binom{I_2}{k-\ell}} |\mathcal{E}(D)| = |\mathcal{B}| \geq \frac{\alpha}{2}|Z|$  and  $\alpha \geq 4\delta(|I_1|, \ell, Q_1)$ . Lastly, using that  $|\mathcal{P}_{Q_1}| \leq |I_1|^{2d_1}|Z|$  together with (4), we obtain  $|(\mathcal{B} \times \mathcal{B}) \cap \mathcal{P}_{Q_1}| \geq \frac{\alpha}{4|I_1|^{2d_1}}|\mathcal{P}_{Q_1}|$ .

Now, from this bound we find a choice of  $C, C' \in \binom{I_1}{\ell}$  with  $\text{pat}(C, C') = Q_1$  such that the set

$$\mathcal{F}_{C, C'} = \left\{ D \in \binom{I_2}{k-\ell} : C \cup D, C' \cup D \in \mathcal{B} \right\}$$

satisfies  $|\mathcal{F}_{C, C'}| \geq \frac{\alpha}{4|I_1|^{2d_1}} \binom{n_2}{k-\ell}$ . However, if  $D, D' \in \mathcal{F}$  with  $\text{pat}(D, D') = Q_2$  then  $C \cup D, C' \cup D' \in \mathcal{A}$  and  $\text{pat}(C \cup D, C' \cup D') = Q_1 Q_2 = P$ . As  $\mathcal{A}$  is  $P$ -free we see  $\mathcal{F}_{C, C'} \subset \binom{I_2}{k-\ell}$  is  $Q_2$ -free. This gives  $\frac{\alpha}{4|I_1|^{d_1}} \leq \delta(|I_2|, k-\ell, Q_2)$  and proves (3).

To complete the proof, note that as  $\alpha \geq 2e^{-k_1/12}$ , by definition of  $L$  we have  $\ell \in L \subset [k_1, 2k_1]$  and  $k-\ell \in [k_2, k_2+k_1]$ . As  $|I_1| = 3k_1$  and  $|I_2| = 2k - 3k_1 = 2k_2 + k_1$ , by Proposition 2.1 we find

$$\delta(|I_1|, \ell, Q_1) \leq \delta(2k_1, k_1, Q_1) = \delta(k_1, d_1) \quad \text{and} \quad \delta(|I_2|, k-\ell, Q_1) \leq \delta(2k_2, k_2, Q_2) = \delta(k_2, d_2).$$

Combined with (3) this completes the proof.  $\square$

*Proof of Theorem 1.* We prove by induction on  $d$  that with  $a_d = (8d)^{5d}$  and  $c_d = 6d8^{-d}$  we have

$$\delta(k, d) \leq a_d k^{-c_d}. \quad (5)$$

For  $d=1$  we have  $P = +-$  or  $P = -+$  and  $\mathcal{A} \subset \binom{[2k]}{k}$  is  $P$ -free simply means that  $|A \triangle B| \neq 2$  for all distinct  $A, B \in \mathcal{A}$ . It is well known that such families satisfy  $|\mathcal{A}| \leq \frac{1}{k} \binom{2k}{k}$ . Indeed, for each  $C \in \binom{[2k]}{k+1}$  let  $y_C$  denote the number of  $A \in \mathcal{A}$  with  $A \subset C$ . Then

$$\sum_{C \in \binom{[2k]}{k+1}} y_C = |\{(A, C) \in \mathcal{A} \times \binom{[2k]}{k+1} : A \subset C\}| = |\mathcal{A}| \times k.$$

However, if  $|A \triangle B| \neq 2$  for all distinct  $A, B \in \mathcal{A}$  we must have  $y_C \leq 1$  for all  $C$ . Rearranging, we obtain the claimed upper bound on  $|\mathcal{A}|$ . This easily gives that (5) holds for  $d=1$ .

We now prove the result for a  $d$ -balanced pattern  $P$ , assuming by induction that the theorem holds for all  $d'$ -balanced patterns with  $d' < d$ . We can assume that  $k \geq a_d^{1/c_d} \geq 16^8$  as otherwise the statement is trivial. We will first prove this when  $P$  begins and ends with different signs, using Lemma 2.2, noting that in this range  $k^{1/2} \geq 16 \log k$ . To apply this, let  $\gamma = 8(a_{d-1})^{1/2} k^{-\frac{c_{d-1}}{4}}$  and note that  $\gamma \geq 8(a_{d-1})^{1/2} k^{-\frac{1}{4}} \geq 16(\log k) k^{-1/2}$  since  $k^{1/4}/\log k \geq 1/32 \geq 2(a_{d-1})^{-1/2}$ . Therefore we can apply Lemma 2.2 to find

$$\begin{aligned} \delta(k, P) &\leq \max\left(\gamma, 6\sqrt{\delta\left(\left\lceil \frac{\gamma^2 k}{64} \right\rceil, d-1\right)}\right) \leq \max\left(8(a_{d-1})^{1/2} k^{-\frac{c_{d-1}}{4}}, 6\sqrt{a_{d-1}(a_{d-1} k^{1-\frac{c_{d-1}}{2}})^{-c_{d-1}}}\right) \\ &\leq 8(a_{d-1})^{1/2} k^{-\frac{c_{d-1}}{4}} \leq a_d k^{-c_d}. \end{aligned}$$

The second inequality here uses that Lemma 2.2 holds for  $d-1$  by induction, the third that  $(a_{d-1})^{-c_{d-1}} \leq 1$  and  $1 - \frac{c_{d-1}}{2} \geq \frac{1}{2}$  and the last inequality uses that  $c_d \leq \frac{c_{d-1}}{4}$ .

We now move to the case where  $P$  starts and ends with the same signs. Given  $P$  let  $d_1$  and  $d_2$  be as in Lemma 2.3 so that  $d_1 + d_2 = d$  with  $d_i \geq 1$ . We will assume that  $d_1 \leq d_2$  as the other case follows



similarly. Let us set  $k_1 = \lceil k^\beta \rceil$  where  $\beta = \frac{c_{d_2}}{2d_1+c_{d_1}}$ . Set  $k_2 = k - 2k_1 \geq k - 4k^\beta \geq k - 4k^{1/2} \geq \frac{k}{2}$  for  $k \geq 2^6$ . Then by Lemma 2.3 we have

$$\begin{aligned} \delta(k, P) &\leq \max\left(2e^{-k_1/12}, 4\delta(k_1, d_1), 4(3k_1)^{2d_1}\delta(k_2, d_2)\right) \\ &\leq \max\left(2e^{-k^\beta/12}, 4a_{d_1}r^{-\beta c_{d_1}}, 4(6k^\beta)^{2d_1}a_{d_2}\left(\frac{k}{2}\right)^{-c_{d_2}}\right) \\ &\leq \max\left(2e^{-k^\beta/12}, 4a_{d_1}k^{-\beta c_{d_1}}, 8^{2d_1+3}a_{d_2}k^{2d_1\beta-c_{d_2}}\right) \\ &\leq \max\left(2e^{-k^\beta/12}, 8^{2d_1+3}a_{d_2}k^{-\frac{c_{d_1}c_{d_2}}{2d_1+c_{d_1}}}\right) \leq a_d k^{-c_d}. \end{aligned}$$

The first part of the final inequality here uses  $a_d \geq 2k^{c_d}$  for  $k \leq (a_d/2)^{1/c_d}$  and that  $e^{-k^\beta/12} \leq k^{-c_d}$  for  $k \geq (a_d/2)^{1/c_d}$ . The second part uses that  $8^{2d_1+3}a_{d_2} \leq a_d$  and that since  $d = d_1 + d_2$  and  $d \leq 2d_2$  we have  $c_d \leq 12d_28^{-d} \leq \frac{36d_1d_28^{-(d_1+d_2)}}{2d_1+1} \leq \frac{c_{d_1}c_{d_2}}{2d_1+1} \leq \frac{c_{d_1}c_{d_2}}{2d_1+c_{d_1}}$ . This completes this case and the proof of the theorem.  $\square$

### 3 Interval patterns

In this section, we first prove Theorem 2. We then give several lower bounds for the case  $n = 2k$  depending on value of  $d$ .

#### 3.1 Upper Bound on $\delta(n, n/2, \text{IP}(d))$

*Proof of Theorem 2.* Let  $m = \lfloor \frac{n}{8d^2} \rfloor$ . We partition  $[n]$  into  $m$  intervals,  $[n] = I_1 \cup \dots \cup I_m$  with  $|I_i| = \lfloor 8d^2 \rfloor$  or  $|I_i| = \lceil 8d^2 \rceil$  for all  $i \in [m]$ .

Consider the following way of choosing elements from  $\binom{[n]}{n/2}$ . First select a set  $T \subset \binom{[n]}{n/2-d}$  uniformly at random. Let  $J = \{i \in [m] : |I_i \setminus T| \geq d\}$ . As  $d < |I_i|/2$ , for every  $i \in [m]$  we have

$$\mathbb{P}(i \in J) = \mathbb{P}(|I_i \setminus T| \geq d) > \mathbb{P}(|I_i \cap T| \leq |I_i|/2) \geq \frac{1}{2}.$$

If  $i \in J$ , further select a set  $S_i \subset \binom{I_i \setminus T}{d}$  uniformly at random, and set  $A_i = T \cup S_i$ . If  $i \in [m] \setminus J$  simply set  $A_i = \emptyset$ .

Now for every  $i, j \in J$  with  $i < j$ , we have  $\text{pat}(A_i, A_j) = \text{IP}(d)$ . Also for  $i \notin J$  we have  $A_i \notin \mathcal{A}$ , since  $|A_i| = 0 \neq n/2$ . We conclude that there is at most one index  $i \in [m]$  with  $A_i \in \mathcal{A}$ . Equivalently,

$$\sum_{i=1}^m \mathbf{1}_{A_i \in \mathcal{A}} = \sum_{i=1}^m \sum_{A \in \mathcal{A}} \mathbf{1}_{A_i = A} \leq 1.$$

This is true for any choice of  $T$  and  $S_i$ 's, so in particular if we take the expectation on both sides, we have

$$\sum_{i=1}^m \sum_{A \in \mathcal{A}} \mathbb{P}(A_i = A) \leq 1. \tag{6}$$

But as  $A_i \notin \mathcal{A}$  for  $i \notin J$ , given any  $A \in \mathcal{A}$  we get that  $\mathbb{P}(A_i = A) = \mathbb{P}(A_i = A | i \in J) \mathbb{P}(i \in J) > \frac{1}{2} \mathbb{P}(A_i = A | i \in J)$ . Rewriting (6), this gives

$$\sum_{i=1}^m \sum_{A \in \mathcal{A}} \frac{\mathbb{P}(A_i = A | i \in J)}{2} \leq 1. \quad (7)$$

**Lemma 3.1.** *Let  $A \in \binom{[n]}{n/2}$  be a fixed set. If  $|A \cap I_i| \geq \frac{|I_i|}{2} + d$ , then  $\mathbb{P}(A_i = A | i \in J) \geq \frac{1}{\binom{n}{n/2}}$ .*

*Proof.* Indeed,  $\mathbb{P}(A_i = A | i \in J) = \frac{N_i(A)}{N_i}$  where

$$N_i(A) := \left| \left\{ (S_i, T) : S_i \in \binom{I_i}{d}, T \in \binom{[n] \setminus S_i}{n/2 - d}, S_i \cup T = A \right\} \right|;$$

$$N_i := \left| \left\{ (S_i, T) : S_i \in \binom{I_i}{d}, T \in \binom{[n] \setminus S_i}{n/2 - d} \right\} \right|.$$

However, we have

$$\frac{N_i(A)}{N_i} \geq \frac{\binom{4d^2+d}{d}}{\binom{8d^2}{d} \binom{n-d}{\frac{n}{2}-d}} = \frac{(4d^2+d)_d \left(\frac{n}{2}-d\right)! \frac{n!}{2!}}{(8d^2)_d (n-d)!} \geq \frac{\left(\frac{n}{2}-d\right)! \frac{n!}{2!}}{2^d (n-d)!} > \frac{(n/2)!(n/2)!}{(n)!} = \frac{1}{\binom{n}{n/2}}.$$

□

For a set  $A \in \binom{[n]}{n/2}$ , denote by  $G(A) = \left| \left\{ i \in [m] : |A \cap I_i| \geq \frac{|I_i|}{2} + d \right\} \right|$ . From Lemma 3.1 it follows that for any given  $A$ , we have  $\sum_{i=1}^m \mathbb{P}(A_i = A | i \in J) \geq G(A) \times \frac{1}{\binom{n}{n/2}}$ . Together with (7), we obtain

$$\sum_{A \in \mathcal{A}} G(A) \leq 2 \binom{n}{n/2}. \quad (8)$$

We call a set  $A \in \binom{[n]}{n/2}$  *bad*, if  $G(A) < m/5$ . Otherwise, we say that  $A$  is *good*. Let  $\mathcal{B}$  be the family of all bad sets.

**Lemma 3.2.**  $|\mathcal{B}| = o\left(\frac{1}{m} \binom{n}{n/2}\right)$  for sufficiently large  $n$ .

*Proof.* For a uniform random choice of a set  $A \subseteq \binom{[n]}{n/2}$ , let  $X_i$  be a random variable, with  $X_i = 1$  if  $|A \cap I_i| > \frac{|I_i|}{2} + d$ , and  $X_i = 0$  otherwise. Let  $Z = X_1 + \dots + X_m$ . To prove the lemma, we need to show that  $\mathbb{P}(Z < m/5) = o\left(\frac{1}{m}\right)$ . By linearity of expectation,  $\mathbb{E}Z = m\mathbb{E}X_i = m\mathbb{P}(X_i = 1)$ . Notice that for every  $i \neq j$ ,  $X_i$  and  $X_j$  are negatively correlated, since if  $A$  has many elements in one interval, it is less likely to have many elements on another interval.

$$\mathbb{P}(X_i = 0) = \frac{\sum_{i=0}^{4d^2+d} \binom{8d^2}{i} \binom{n-8d^2}{n/2-i}}{\binom{n}{n/2}} \leq \frac{1}{2} + \frac{\sum_{i=4d^2}^{4d^2+d} \binom{8d^2}{i} \binom{n-8d^2}{n/2-i}}{\binom{n}{n/2}} \leq \frac{1}{2} + \frac{d \binom{8d^2}{4d^2} \binom{n-8d^2}{n/2-4d^2}}{\binom{n}{n/2}} < 0.79.$$

The second inequality uses Stirling's formula. Therefore  $\mathbb{P}(X_i = 1) = \mathbb{E}X_i > 0.21$ . Using linearity of expectation gives  $\mathbb{E}Z = \sum_{i=1}^m \mathbb{E}X_i > 0.21m$ .

By a version of the Chernoff-Hoeffding bound for negatively correlated variables [15], we deduce that  $\mathbb{P}(A \in \mathcal{B}) = \mathbb{P}(Z < 0.2m) < \mathbb{P}(Z - \mathbb{E}Z > 0.01m) = o\left(\frac{1}{m}\right)$ , finishing the proof. □

Therefore, if  $|\mathcal{A}| \geq \frac{2}{m} \binom{n}{n/2}$ , then  $|\mathcal{A} \setminus \mathcal{B}| = (1 - o(1))|\mathcal{A}|$ . Using (8), we see that

$$(1 - o(1)) \frac{m|\mathcal{A}|}{10} \leq \sum_{A \in \mathcal{A} \setminus \mathcal{B}} G(A) \leq \sum_{A \in \mathcal{A}} G(A) \leq \binom{n}{n/2}. \quad (9)$$

Equivalently  $|\mathcal{A}| = O(\frac{1}{m} \binom{n}{n/2}) = O(\frac{d^2}{n} \binom{n}{n/2})$ , as required.  $\square$

### 3.2 Lower Bound on $\delta(n, n/2, \text{IP}(d))$

For the lower bounds, we provide different lower bounds, depending on the range of  $d$ .

**Theorem 4.** *The following hold:*

(i) *If  $d = o(\sqrt{n})$ , there is an  $\text{IP}(d)$ -free family  $\mathcal{A} \subseteq \binom{[n]}{n/2}$  with  $|\mathcal{A}| = \Omega(\max\{\frac{1}{nd}, \frac{d^2}{n^{3/2}}\} \cdot \binom{n}{n/2})$ .*

(ii) *If  $d = c\sqrt{n}$ , there is an  $\text{IP}(d)$ -free family  $\mathcal{A} \subseteq \binom{[n]}{n/2}$  with  $|\mathcal{A}| = \Omega_c(\binom{n}{n/2})$ .*

*Proof.* First we prove (i). For a set  $A \in \binom{[n]}{n/2}$  let  $S(A) := \sum_{i \in A} i$ , the sum of the elements in  $A$ . Observe that if  $\text{pat}(A, B) = \text{IP}(d)$  then  $0 < |S(A) - S(B)| < nd$ . Thus for any  $0 \leq i \leq nd - 1$ , the family  $\mathcal{A}_i := \{A \in \binom{[n]}{n/2} \mid S(A) \equiv i \pmod{nd}\}$  forms an  $\text{IP}(d)$ -free family. By the pigeonhole principle, we can find such  $i$  so that  $|\mathcal{A}_i| \geq \frac{1}{nd} \binom{n}{n/2}$ .

To obtain the second bound from (i), note that if we choose a set  $A \in \binom{[n]}{n/2}$  uniformly at random,

$$\mathbb{E}[S(A)] = \frac{n(n+1)}{4}. \quad (10)$$

To calculate the variance, let

$$X_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A. \end{cases}$$

Then  $S(A) = \sum_{i=1}^n iX_i$ . Now  $\mathbb{E}[X_i] = \frac{1}{2}$  every  $i \in [n]$  and  $\mathbb{E}[X_iX_j] = \frac{1}{4}(1 - \frac{1}{n-1})$ . Using this, we find

$$\begin{aligned} \text{Var}(S(A)) &= \mathbb{E}[(\sum_{i=1}^n iX_i)^2] - \mathbb{E}[\sum_{i=1}^n iX_i]^2 \leq \sum_{i \in [n]} i^2 \mathbb{E}[X_i] + \sum_{i \neq j} ij(\mathbb{E}[X_iX_j] - \mathbb{E}[X_i]\mathbb{E}[X_j]) \\ &\leq \sum_{i \in [n]} \frac{i^2}{2} \leq \frac{n^3}{2}. \end{aligned} \quad (11)$$

From (10) and (11) together, by Chebyshev's inequality we get  $\mathbb{P}(|S(A) - n(n+1)/4| \leq n^{3/2}) \geq 1/2$ . Equivalently,  $|\{A \in \binom{[n]}{n/2} : |S(A) - n(n+1)/4| \leq n^{3/2}\}| \geq \frac{1}{2} \binom{n}{n/2}$ . By an easy averaging argument, for some value  $m \in [\frac{n(n+1)}{4} - \frac{n^{3/2}}{3}, \frac{n(n+1)}{4} + \frac{n^{3/2}}{2}]$ .

$$|\{A \in \binom{[n]}{n/2} : S(A) \in [m - \frac{d^2}{2}, m + \frac{d^2}{2}]\}| \geq \frac{1}{2(2n^{3/2}/d^2 + 2)} \binom{n}{n/2} = \Omega\left(\frac{d^2}{n^{3/2}} \binom{n}{n/2}\right)$$

However, since two sets  $A, B \in \binom{[n]}{n/2}$  with  $\text{pat}(A, B) = \text{IP}(d)$  have  $|S(A) - S(B)| > d^2$ , this completes the proof of (i).

To prove (ii), let  $c > 0$  be given and let  $d = c\sqrt{n}$ . Note that if  $\text{pat}(A, B) = IP(d)$  then for some  $i \in [n]$  we have  $|A \cap [i]| \geq |B \cap [i]| + d$ . This shows that  $\mathcal{A} = \{A \in \binom{[n]}{n/2} : ||A \cap [i]| - i/2| < d/4 \text{ for all } i \in [n]\}$  is an  $IP(d)$ -free family. We will now show that  $|\mathcal{A}| = \Omega_c(\binom{n}{n/2})$ .

To see this, it is convenient to identify elements of  $\binom{[n]}{n/2}$  with certain walks. Let  $\mathcal{W}_0$  denote the set of all walks  $W = W_0 \cdots W_n$  of length  $n$  on  $\mathbb{Z}$  with  $W_0 = W_n = 0$  and which either increase or decrease by 1 in each step (i.e.  $|W_i - W_{i-1}| = 1$  for all  $i \in [n]$ ). Note that each walk  $W \in \mathcal{W}_0$  naturally corresponds to a subset of  $[n]$  of size  $n/2$  consisting of those steps in  $[n]$  where the walk increases. Under this correspondence, the set  $\mathcal{A}$  corresponds to those walks in  $\mathcal{W}_0$  which lie entirely in  $[-d/4, d/4]$ .

Now select a walk  $W \in \mathcal{W}_0$  uniformly at random. Letting  $T$  denote a value to be determined, consider the following events:

$$\begin{aligned} A &= \{W_j \in [-d/4, d/4] \text{ for all } j \in [n]\} \\ B &= \{W_{in/T} \in [-d/12, d/12] \text{ for all } i \in [T-1]\} \\ C_i &= \{W_j \in [-d/4, d/4] \text{ for all } j \in [\frac{(i-1)n}{T}, \frac{in}{T}]\}, \text{ where } i \in [T]. \end{aligned}$$

Also for  $i \in [T-1]$  and  $a_i \in [-d/12, d/12]$ , let  $B_i(a_i)$  denote the event  $B_i(a_i) = \{W_{in/T} = a_i\}$ . We will show that

$$\mathbb{P}_{W \sim \mathcal{W}_0} \left( B \wedge \bigwedge_{i \in [T]} C_i \right) \geq c' > 0, \quad (12)$$

where  $c'$  depends only on  $c$ . Since  $\bigwedge_{i \in [T]} C_i \subset A$ , this will prove the result.

To begin, note that we have

$$\begin{aligned} \mathbb{P}_{W \sim \mathcal{W}_0} \left( B \wedge \bigwedge_{i \in [T]} C_i \right) &\geq \sum_{a_1, \dots, a_{T-1} \in [-d/12, d/12]} \mathbb{P}_{W \sim \mathcal{W}_0} \left( \bigwedge_{i \in [T-1]} B_i(a_i) \wedge \bigwedge_{i \in [T]} C_i \right) \\ &= \sum_{a_1, \dots, a_{T-1} \in [-d/12, d/12]} \mathbb{P}_{W \sim \mathcal{W}_0} \left( \bigwedge_{i \in [T]} C_i \mid \bigwedge_{i \in [T-1]} B_i(a_i) \right) \\ &\quad \times \mathbb{P}_{W \sim \mathcal{W}_0} \left( \bigwedge_{i \in [T-1]} B_i(a_i) \right). \end{aligned} \quad (13)$$

Let  $\mathcal{W}(a, b)$  denote the collection of random walks of length  $n/T$  which start at  $a$  and end at  $b$ . Since  $C_i$  depends only on  $\{W_j : j \in [(i-1)n/T, in/T]\}$ , taking  $a_0 = a_T = 0$  we have

$$\begin{aligned} \mathbb{P}_{W \sim \mathcal{W}_0} \left( \bigwedge_{i \in [T]} C_i \mid \bigwedge_{i \in [T-1]} B_i(a_i) \right) &= \prod_{i \in [T]} \mathbb{P}_{W \sim \mathcal{W}_0} \left( C_i \mid B_{i-1}(a_{i-1}) \wedge B_i(a_i) \right) \\ &= \prod_{i \in [T]} \mathbb{P}_{W \sim \mathcal{W}(a_{i-1}, a_i)} \left( W \text{ lies entirely in } [-d/4, d/4] \right). \end{aligned} \quad (14)$$

**Claim:** For every  $a, b \in [-d/12, d/12]$  we have  $\mathbb{P}_{W \sim \mathcal{W}(a, b)} \left( W \text{ lies entirely in } [-d/4, d/4] \right) \geq 1/2$ .

Let  $\mathcal{W}(a)$  denote the collection of all walks of length  $n/T$  which begin at  $a$ . Let us select  $W$  from  $\mathcal{W}(a)$  uniformly at random and let  $S_{n/T}$  denote the final vertex. By the reflection principle for random

walks, we have

$$\begin{aligned}
\mathbb{P}_{W \sim \mathcal{W}(a,b)}(W \text{ exceeds } d/4) &= \mathbb{P}_{W \sim \mathcal{W}(a)}(W \text{ exceeds } d/4 | S_{n/T} = b) \\
&= \frac{\mathbb{P}_{W \sim \mathcal{W}(a)}(S_{n/T} = d/2 - b)}{\mathbb{P}_{W \sim \mathcal{W}(a)}(S_{n/T} = b)} \\
&= \frac{\binom{n/T}{n/2T + (d/2 - b) - a}}{\binom{n/T}{n/2T + b - a}} \leq \frac{\binom{n/T}{n/2T + d/3}}{\binom{n/T}{n/2T + d/6}} \\
&= \frac{(n/2T - d/6)_{d/6}}{(n/2T + d/6)_{d/6}} \leq \left(1 - \frac{dT}{3n}\right)^{d/6} \leq e^{-d^2 T/36n}.
\end{aligned}$$

Taking  $T = 72/c^2$  say, we find  $\mathbb{P}_{W \sim \mathcal{W}(a,b)}(W \text{ exceeds } d/4) \leq e^{-2} < 1/4$ . By symmetry, this gives  $\mathbb{P}_{W \sim \mathcal{W}(a,b)}(W \text{ lies entirely in } [-d/4, d/4]) \geq 1 - 2 \times (1/4) = 1/2$ , as claimed.

Now by combining (14) together with the claim in (13) we find

$$\mathbb{P}_{W \sim \mathcal{W}_0} \left( B \wedge \bigwedge_{i \in [T]} C_i \right) \geq \sum_{a_1, \dots, a_{T-1} \in [-d/12, d/12]} 2^{1-T} \times \mathbb{P}_{W \sim \mathcal{W}_0} \left( \bigwedge_{i \in [T-1]} B_i(a_i) \right). \quad (15)$$

But letting  $b_i := \frac{n}{2T} + a_i - a_{i-1}$  for all  $i \in [T]$  where  $a_0 = a_T = 0$ , we have

$$\mathbb{P}_{W \sim \mathcal{W}_0} \left( \bigwedge_{i \in [T-1]} B_i(a_i) \right) = \frac{\prod_{i \in [T]} \binom{n/T}{b_i}}{\binom{n}{n/2}} = \Omega_{c,T}(d^{1-T}).$$

The final inequality follows by Stirling's approximation, using that  $b_i \in [\frac{n}{2T} - \frac{d}{6}, \frac{n}{2T} + \frac{d}{6}]$  for all  $i \in [T]$ . Combined with (15), this gives  $\mathbb{P}_{W \sim \mathcal{W}_0} \left( B \wedge \bigwedge_{i \in [T]} C_i \right) = \Omega_{c,T}(1) = \Omega_c(1)$ , as required.  $\square$

## 4 Alternating patterns

To begin, we prove an auxiliary lemma. Given  $\mathbf{x} = (x_i)$  and  $\mathbf{y} = (y_i)$  in  $[m]^D$  we say that  $\mathbf{y}$   $d$ -dominates  $\mathbf{x}$  if  $|\{i \in [D] : x_i \neq y_i\}| = d$  and  $x_i \leq y_i$  for all  $i \in [D]$ .

**Lemma 4.1.** *Let  $d, m, D \in \mathbb{N}$  with  $2md^2 \leq D$ . Suppose that  $\mathcal{C} \subset [m]^D$  does not contain  $\mathbf{x}$  and  $\mathbf{y}$  such that  $\mathbf{y}$   $d$ -dominates  $\mathbf{x}$ . Then  $|\mathcal{C}| \leq 2m^{D-1}$ .*

*Proof.* To begin, choose a set  $S \subset [D]$  with  $|S| = d$  and a vector  $\mathbf{z} \in [m]^{[D] \setminus S}$  uniformly at random. For each  $i \in [m]$  let  $\mathbf{z}_S(i) \in [m]^D$  denote the vector which agrees with  $\mathbf{z}$  on coordinates in  $[D] \setminus S$  and equals  $i$  everywhere else. Also let  $\mathcal{B}_{S, \mathbf{z}}$  denote the combinatorial line  $\mathcal{B}_{S, \mathbf{z}} := \{\mathbf{z}_S(i) : i \in [m]\}$ .

Now as  $\mathcal{C}$  does not contain any  $d$ -dominating pairs, for any choice of  $S$  and  $\mathbf{z}$  we have  $|\mathcal{C} \cap \mathcal{B}_{S, \mathbf{z}}| \leq 1$ . Letting  $X_i$  denote the indicator random variable which is 1 if  $\mathbf{z}_S(i) \in \mathcal{C}$  and 0 otherwise, this gives

$$\sum_{i \in [m]} X_i \leq 1.$$

Taking expectations over all choice of  $S$  and  $\mathbf{z}$ , this gives

$$\sum_{C \in \mathcal{C}} \mathbb{P}(C \in \mathcal{B}_{S, \mathbf{z}}) = \sum_{i \in [m]} \sum_{C \in \mathcal{C}} \mathbb{P}(\mathbf{z}_S(i) = C) \leq 1. \quad (16)$$

However, an easy calculation gives that if  $C$  has  $k_i$  entries  $i$  for all  $i \in [m]$ , then

$$\mathbb{P}(C \in \mathcal{B}_{S,\mathbf{z}}) = \sum_{i \in [m]} \frac{\binom{k_i}{d}}{m^{D-d} \binom{D}{d}}.$$

This expression is minimized when all  $k_i$  are as equal as possible. Thus

$$\begin{aligned} \mathbb{P}(C \in \mathcal{B}_{S,\mathbf{z}}) &\geq m \frac{\binom{D/m}{d}}{m^{D-d} \binom{D}{d}} = m \frac{(D/m)_d}{m^{D-d} D_d} = \frac{m}{m^D} \prod_{l \in [0, d-1]} \left(1 - \frac{l(m-1)}{D-l}\right) \\ &\geq \frac{1}{m^{D-1}} \left(1 - \sum_{l \in [0, d-1]} \frac{l(m-1)}{D/2}\right) \\ &\geq \frac{1}{m^{D-1}} \left(1 - \frac{md^2}{D}\right) \geq \frac{1}{2m^{D-1}}. \end{aligned}$$

The final line here used  $2md^2 \leq D$ . Combined with (16) this gives  $|\mathcal{C}|/2m^{D-1} \leq 1$ , as required.  $\square$

We are now ready for the proof of Theorem 3.

*Proof of Theorem 3.* By Proposition 2.1 it suffices to prove the theorem for  $n = 2k$ . Let  $m = \lfloor \frac{\log_2(n/d^2)}{2} \rfloor$ . For convenience we assume that  $n$  is divisible by  $m$ , with  $Km = n$ . Let  $[n] = \bigcup_{i=1}^K I_i$  be a partition of  $[n]$  where  $I_i = \{(i-1)m+1, \dots, im\}$  for all  $i \in [K]$ . Given a set  $T \subset [K]$ , let  $T^c = [K] \setminus T$  and let

$$\mathcal{B}_T := \left\{ A \subset \bigcup_{i \in T^c} I_i : |A \cap I_i| \neq 1 \text{ for all } i \in T^c \right\}.$$

Given  $B \in \mathcal{B}_T$  and  $\mathbf{x} \in [m]^T$  we also let  $B(\mathbf{x}) := B \cup \{(i-1)m+j-1 : i \in T, x_i = j\}$  and

$$\mathcal{C}_B := \{B(\mathbf{x}) : \mathbf{x} \in [m]^T\}.$$

Note that for every  $A \subset [n]$  there is a unique  $T \subset [K]$ ,  $B \in \mathcal{B}_T$  and  $\mathbf{x} \in [m]^T$  such that  $A = B(\mathbf{x})$ . Thus we have the disjoint union

$$\binom{[n]}{n/2} = \bigcup_{T \subset [K]} \bigcup_{\substack{B \in \mathcal{B}_T \\ |B| = \frac{n}{2} - |T|}} \mathcal{C}_B. \quad (17)$$

We will first show that almost all sets  $A$  in  $\binom{[n]}{n/2}$  are of the form  $A = B(\mathbf{x})$  where  $T \subset [K]$  and  $B \in \mathcal{B}_T$  with  $|T| \geq mK/2^{m+1} = n/2^{m+1}$ . To see this, given a set  $A \subset [n]$ , let  $A_i = A \cap I_i$  for all  $i \in [K]$ . We will say that  $A \subset [n]$  is *bad* if  $T(A) = \{i \in [K] : |A_i| = 1\}$  satisfies  $|T(A)| \leq \frac{m}{2^{m+1}} K$ . We claim that there are at most  $O(e^{-n^{1/2}/2^{2n}})$  sets are bad. Indeed, if we select  $A \subset [n]$  uniformly at random, we have  $\mathbb{P}(|A_i| = 1) = m/2^m$ , which gives  $\mathbb{E}(|T(A)|) = \frac{mK}{2^m} = \frac{n}{2^m}$ . As  $|A_i| = 1$  for each  $i \in [K]$  independently, by Chernoff's inequality, we find that  $\mathbb{P}(|T(A)| - \frac{n}{2^m} \leq -\frac{n}{2^{m+1}}) \leq e^{-\frac{n}{2^{m+1}}}$ . As  $m \leq \log_2(n/d^2)/2 \leq \frac{\log_2 n}{2}$  we find that  $\mathbb{P}(A \text{ is bad}) \leq e^{-n^{1/2}/2}$ . Equivalently,  $|\{A \subset [n] : A \text{ is bad}\}| = O(e^{-n^{1/2}/2^n})$ .

Now suppose that  $T \subset [K]$  with  $|T| \geq n/2^{m+1}$  and  $B \in \mathcal{B}_T$ . Note that given  $\mathbf{x}, \mathbf{y} \in [m]^T$ , if  $\mathbf{y}$   $d$ -dominates  $\mathbf{x}$  then  $\text{pat}(B(\mathbf{x}), B(\mathbf{y})) = \text{AP}(d)$ . Noting that as  $m = \lfloor \log_2(n/d^2)/2 \rfloor$  we have  $|T| \geq n/2^{m+1} \geq 2^m d^2 \geq 2md^2$ . Setting  $D = |T|$ , Lemma 4.1 therefore shows that any  $\mathcal{A} \subset \binom{[n]}{n/2}$  which is  $\text{AP}(d)$ -free satisfies

$$|\mathcal{A} \cap \mathcal{C}_B| \leq 2m^{|T|-1} = \frac{2}{m} |\mathcal{C}_B|. \quad (18)$$

Summing over all  $T \subset [K]$  and  $B \in \mathcal{C}_T$ , combined with (17) and (18), this gives

$$\begin{aligned} |\mathcal{A}| &\leq \sum_{T \subset [K]} |\mathcal{A} \cap \bigcup_{\substack{B \in \mathcal{B}_T \\ |B|=n/2-|T|}} \mathcal{C}_B| \leq |\{A \subset [n] : A \text{ bad}\}| + \sum_{\substack{T \subset [K]: \\ |T| \geq 2md^2}} \sum_{B \in \mathcal{B}_T} |\mathcal{A} \cap \mathcal{C}_B| \\ &\leq O\left(\frac{2^n}{e^{\sqrt{n}/2}}\right) + \sum_{\substack{T \subset [K]: \\ |T| \geq 2md^2}} \sum_{\substack{B \in \mathcal{B}_T \\ |B|=n/2-|T|}} \frac{2}{m} |\mathcal{C}_B| \\ &\leq \frac{2 + o(1)}{m} \binom{n}{n/2}. \end{aligned}$$

This completes the proof of the theorem.  $\square$

## 5 Concluding remarks and open problems

In this paper we proved bounds on the size of families  $\mathcal{A} \subset \mathcal{P}[n]$  which avoid a  $d$ -balanced pattern  $P$ . Our proof shows that such families satisfy

$$|\mathcal{A}| = O(a_d n^{-c_d 2^n}),$$

where  $a_k = (8d)^{5d}$  and  $c_d = 6d8^{-d}$ . In particular, families  $\mathcal{A}$  which avoid a  $d$ -balanced pattern with  $d < c \log \log n$  satisfy  $|\mathcal{A}| = o(2^n)$  for some absolute constant  $c > 0$ . It would be interesting to improve the density bound here and/or extend the range of  $d$  for which this zero density property holds.

Another interesting question is the following: which balanced pattern  $P$  has the strongest effect on the density of  $P$ -free families  $\mathcal{A} \subset \mathcal{P}[n]$ ? That is, what is  $\min_P \delta(n, k, P)$ , where the minimum is taken over all balanced patterns  $P$ ? If instead of patterns we only forbid intersection sizes (as discussed in the Introduction) then there are a number of very strong density results for subsets of  $\mathcal{P}[n]$ . For example, the Frankl-Rödl [10] theorem shows that given  $\epsilon > 0$ , if  $\mathcal{A} \subset \mathcal{P}[n]$  and  $|A \cap B| \neq t$  for some  $\epsilon n \leq t \leq (1/2 - \epsilon)n$  then  $|\mathcal{A}| \leq (2 - \delta)^n$ , where  $\epsilon = \epsilon(\delta) > 0$ . It would be very interesting to know if there exists a pattern which forces a superpolynomial density in  $n$ . That is, does there an increasing sequence of naturals  $(n_k)_{k \in \mathbb{N}}$  and balanced patterns  $(P_k)_k$  with  $\delta(n_k, n_k/2, P_k) = n_k^{-\omega_k(1)}$  for some function  $\omega_k(1)$  tending to infinity with  $k$ ?

Lastly, how large can  $d$  be (as a function of  $n$ ) while still giving  $\delta(n, n/2, \text{AP}(d)) \rightarrow 0$  as  $n \rightarrow \infty$ . Theorem 3 proves that this holds for any  $d = o(\sqrt{n})$ .

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