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## On a Ramsey-type problem of Erdős and Pach<sup>\*</sup>

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July 10, 2017

#### Abstract

Affirmatively answering a question of Erdős and Pach from 1983, we show for some constant C > 0 that for any graph G on  $Ck \ln k$  vertices either G or its complement  $\overline{G}$  has an induced subgraph on k vertices with minimum degree at least  $\frac{1}{2}(k-1)$ .

Keywords: Ramsey theory, quasi-Ramsey numbers, graph discrepancy, probabilistic method. MSC: 05C55 (Primary) 05D10, 05D40 (Secondary).

#### 1 Introduction

Recall that the (diagonal, two-colour) Ramsey number is defined as the smallest integer R(k) for which any graph on R(k) vertices is guaranteed to contain a homogeneous set of order k — that is, a set of k vertices corresponding to either a complete or independent subgraph. The search for better bounds on R(k), particularly asymptotic bounds as  $k \to \infty$ , is a challenging topic that has long played a central role in combinatorial mathematics (see [4, 8]).

We are interested in a degree-based generalisation of R(k) where, rather than seeking a clique or coclique of order k, we seek an induced subgraph of order (at least) k with high minimum degree (clique-like) or symmetrically low maximum degree (coclique-like). By gradually relaxing the degree requirement, a spectrum of Ramsey-type, or *quasi-Ramsey*, problems arise. Erdős and Pach [1] introduced these problems in 1983 and showed that there is a sharp change in behaviour at a certain point along the spectrum. More precisely, they gave good estimates for the smallest integer  $R_{1/2}(k)$  such that for any graph G on  $R_{1/2}(k)$  vertices either G or its complement  $\overline{G}$  contains some subgraph on  $\ell \geq k$  vertices with

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minimum degree at least  $\frac{1}{2}(\ell - 1)$ . They showed that  $R_{1/2}(k) = O(k \ln k)$  and  $R_{1/2}(k) = \Omega(k \ln k/ \ln \ln k)$ , and moreover that for the corresponding problem where  $\frac{1}{2}$  is replaced with some strictly larger constant c the corresponding parameter  $R_c(k)$  must be at least exponential in k. (We may take c = 1 to recover the original Ramsey numbers.) Three of the authors recently revisited this topic together with Pach [5] to give a more refined understanding of the threshold around  $\frac{1}{2}$ , showing that the change from polynomial to super-polynomial growth in k occurs when one seeks a subgraph on  $\ell \ge k$  vertices with minimum degree at least  $\frac{1}{2}(\ell - 1) + \Theta(\sqrt{(\ell - 1) \ln \ell})$  (consult [5] for precise details). The problems just described relate to the so-called *variable quasi-Ramsey* numbers, whereas here we focus on the harder version, namely the *fixed quasi-Ramsey* problem where the sought subgraph is required to have *exactly k* vertices rather than at least k vertices as above.

Using a result on graph discrepancy, Erdős and Pach [1] proved that there is a constant C > 0 such that for any graph G on at least  $Ck^2$  vertices either G or its complement  $\overline{G}$  has an induced subgraph on (exactly) k vertices with minimum degree at least  $\frac{1}{2}(k-1)$ . As alluded to in the previous paragraph, they also showed (by way of an unusual random graph construction) that the previous statement does not hold with  $C'k \ln k / \ln \ln k$  in place of  $Ck^2$  for some constant C' > 0. They asked if it holds instead with  $Ck \ln k$ , as is the case for the variable quasi-Ramsey problem. Our main contribution here is to confirm this.

**Theorem 1.** There exists a constant C > 0 such that for any graph G on  $Ck \ln k$  vertices, either G or its complement  $\overline{G}$  has an induced subgraph on k vertices with minimum degree at least  $\frac{1}{2}(k-1)$ .

Although it is short, our proof of Theorem 1 has a number of different ingredients, including the use of graph discrepancy in Section 2, an application of the celebrated 'six standard deviations' result of Spencer [9] in Section 3 and a greedy algorithm in Section 4 that was inspired by similar procedures for max-cut and min-bisection. It is interesting to remark that the two discrepancy results we use are of a different nature; the one in Section 2 is an anti-concentration result while the result of Spencer is a concentration result.

### 2 An auxiliary result via graph discrepancy

Our first step in proving Theorem 1 will be to apply the following result. This is a bound on a variable quasi-Ramsey number which is similar to Theorem 3(a) in [5]. The idea of the proof of this auxiliary result is inspired by the sketch argument for Theorem 2 in [1], in spite of the error contained in that sketch (cf. [5]).

**Theorem 2.** For any constant  $v \ge 0$ , there exists a constant C = C(v) > 1 such that for any graph G on  $Ck \ln k$  vertices, G or its complement  $\overline{G}$  has an induced subgraph on  $\ell \ge k$  vertices with minimum degree at least  $\frac{1}{2}(\ell - 1) + v\sqrt{\ell - 1}$ .

Note that the  $O(k \ln k)$  quantity is tight up to an  $O(\ln \ln k)$  factor by the unusual construction in [1] (cf. also Theorem 4 in [5]). The astute reader may later notice that the second-order term  $\nu\sqrt{\ell-1}$  in the minimum degree guarantee of Theorem 2 can be straightforwardly improved to an  $\Omega(\sqrt{(\ell-1)\ln \ln \ell})$  term. Since this does not seem to help in our results, we have omitted this improvement to minimise technicalities. On the other hand, a standard random graph construction yields the following, which certifies that the second-order term cannot be improved to a  $\omega(\sqrt{(\ell-1)\ln \ln \ell})$  term.

**Proposition 3.** For any c > 0, for large enough k there is a graph G with at least  $k \ln^c k$  vertices such that the following holds. If H is any induced subgraph of G or  $\overline{G}$  on  $\ell \ge k$  vertices, then H has minimum degree less than  $\frac{1}{2}(\ell - 1) + \sqrt{3c(\ell - 1) \ln \ln \ell}$ .

*Proof.* Substitute  $\nu(\ell) = \sqrt{(2c \ln \ln \ell) / \ln \ell}$  into the proof of Theorem 3(b) in [5]. (We may not use Theorem 3(b) in [5] directly as stated as it needs  $\nu(\ell)$  to be non-decreasing in  $\ell$ .)

We use a result on graph discrepancy to prove Theorem 2. Given a graph G = (V, E), the *discrepancy* of a set  $X \subseteq V$  is defined as

$$D(X) := e(X) - \frac{1}{2} \binom{|X|}{2},$$

where e(X) denotes the number of edges in the subgraph G[X] induced by X. We use the following result of Erdős and Spencer [2, Ch. 7].

**Lemma 4** (Theorem 7.1 of [2]). Provided *n* is large enough and  $t \in \mathbb{N}$  satisfies  $\frac{1}{2}\log_2 n < t \le n$ , then any graph G = (V, E) of order *n* satisfies

$$\max_{S \subseteq V, |S| \le t} |D(S)| \ge \frac{t^{3/2}}{10^3} \sqrt{\ln(5n/t)}.$$

*Proof of Theorem 2.* Let G = (V, E) be any graph on at least  $N = \lceil Ck \ln k \rceil$  vertices for a sufficiently large choice of *C*. We may assume that  $k > \frac{1}{2} \log_2 N$  because otherwise *G* or  $\overline{G}$  contains a clique of order *k* by the Erdős-Szekeres bound [3] on ordinary Ramsey numbers.

For any  $X \subseteq V$  and  $\nu > 0$ , we define the following skew form of discrepancy:

$$D_{\nu}(X) := |D(X)| - \nu |X|^{3/2}$$

We now construct a sequence  $(H_0, H_1, \ldots, H_t)$  of graphs as follows. Let  $H_0$  be G or G. At step i + 1, we form  $H_{i+1}$  from  $H_i = (V_i, E_i)$  by letting  $X_i \subseteq V_i$  attain the maximum skew discrepancy  $D_v$  and setting  $V_{i+1} := V_i \setminus X_i$  and  $H_{i+1} := H[V_{i+1}]$ . We stop after step t + 1 if  $|V_{t+1}| < \frac{1}{2}N$ . Let  $I^+ \subseteq \{1, \ldots, t\}$  be the set of indices i for which  $D(X_i) > 0$ . By symmetry, we may assume

$$\sum_{i\in I^+} |X_i| \ge \frac{1}{4}N. \tag{1}$$

**Claim 1.** For any  $i \in I^+$  and  $x \in X_i$ ,  $\deg_{H_i}(x) \ge \frac{1}{2}(|X_i| - 1) + \nu(|X_i| - 1)^{1/2}$ .

*Proof.* Write  $|X_i| = n_i$ . We are trivially done if  $n_i = 1$ , so assume  $n_i \ge 2$ . Suppose  $x \in X_i$  has strictly smaller degree than claimed and set  $X'_i := X_i \setminus \{x\}$ . Then, since  $i \in I^+$ ,

$$D_{\nu}(X'_{i}) \ge e(X'_{i}) - \frac{1}{2} \binom{n_{i} - 1}{2} - \nu(n_{i} - 1)^{3/2}$$
  
>  $e(X_{i}) - \frac{1}{2} \binom{n_{i}}{2} - \nu\sqrt{n_{i} - 1} - \nu(n_{i} - 1)^{3/2}$ 

Note that  $n_i^{3/2} > n_i^{1/2} + (n_i - 1)^{3/2}$ , which by the above implies  $D_{\nu}(X'_i) > D_{\nu}(X_i)$ , contradicting the maximality of  $D_{\nu}(X_i)$ .

Claim 1 implies that we may assume for each  $i \in I^+$  that  $|X_i| \le k - 1$ , or else we are done. This gives for any  $i_1, \ldots, i_4 \in I^+$  that

$$\left(\sum_{s=1}^{4} |X_{i_s}|\right)^{3/2} \le 8(k-1)^{3/2}.$$
(2)

Writing  $I^+ = \{i_1, \ldots, i_m\}$ , we next show the following.

**Claim 2.** For any  $\ell \in \{1, ..., m-3\}$ ,  $D(X_{i_{\ell+3}}) \leq \frac{5}{6}D(X_{i_{\ell}})$ .

*Proof.* For  $X \subseteq V$ , let us write  $\nu(X) := \nu |X|^{3/2}$  so that  $D_{\nu}(X) = |D(X)| - \nu(X)$ . For  $i_1, \ldots, i_r \in I^+$ , we may write  $X_{i_1, \ldots, i_r} := \bigcup_{s=1}^r X_{i_s}$ . For disjoint  $X, Y \subseteq V$ , we define the *relative discrepancy* between X and Y to be

$$D(X,Y) := e(X,Y) - \frac{1}{2}|X||Y|,$$

where e(X, Y) denotes the number of edges between X and Y.

Now let  $i, j \in I^+$  with i < j. Then, by the maximality of  $D_{\nu}(X_i)$ , we have  $D_{\nu}(X_i \cup X_j) \le D_{\nu}(X_i)$ , i.e.

 $|D(X_i) + D(X_i, X_j) + D(X_j)| - \nu(X_{i,j}) \le |D(X_i)| - \nu(X_i) = D(X_i) - \nu(X_i),$ 

and hence

$$D(X_{i}) \leq -D(X_{i}, X_{i}) + \nu(X_{i,i}).$$
 (3)

Applying (3) (and the fact that  $\nu(X_{i_{\ell+r},i_{\ell+s}}) \leq \nu(\bigcup_{s=0}^{3} X_{i_{\ell+s}})$  for any  $r,s \in \{0,1,2,3\}$ ), we find that

$$D(X_{i_{\ell+1}}) + 2D(X_{i_{\ell+2}}) + 3D(X_{i_{\ell+3}}) \le -\sum_{0 \le r < s \le 3} D(X_{i_{\ell+r}}, X_{i_{\ell+s}}) + 6\nu(\bigcup_{s=0}^{3} X_{i_{\ell+s}}).$$
(4)

Using  $-D(\bigcup_{s=0}^{3} X_{i_{\ell+s}}) - \nu(\bigcup_{s=0}^{3} X_{i_{\ell+s}}) \le D_{\nu}(\bigcup_{s=0}^{3} X_{i_{\ell+s}}) \le D_{\nu}(X_{i_{\ell}})$ , we obtain

$$-\sum_{s=0}^{3} D(X_{i_{\ell+s}}) - \sum_{0 \le r < s \le 3} D(X_{i_{\ell+r}}, X_{i_{\ell+s}}) \le D(X_{i_{\ell}}) + \nu(\bigcup_{s=0}^{3} X_{i_{\ell+s}}),$$

which combined with (4) implies that  $D(X_{i_{\ell+2}}) + 2D(X_{i_{\ell+3}}) \leq 2D(X_{i_{\ell}}) + 7\nu(\bigcup_{s=0}^{3} X_{i_{\ell+s}})$ . From this, we obtain that

$$3D(X_{i_{\ell+3}}) \le 2D(X_{i_{\ell}}) + 8\nu(\bigcup_{s=0}^{3} X_{i_{\ell+s}}),$$
(5)

where we have used the fact that  $D(X_{i_{\ell+3}}) \leq D(X_{i_{\ell+2}}) + \nu(\bigcup_{s=0}^{3} X_{i_{\ell+s}})$ , which follows since  $D_{\nu}(X_{i_{\ell+3}}) \leq D_{\nu}(X_{i_{\ell+2}})$ . Using the fact that the graph  $H_{i_s}$  for any  $s \in \{1, \ldots, m\}$  has at least  $\frac{1}{2}N \geq \frac{C}{2}k \ln k$  vertices, it follows by Lemma 4 (using our assumption on k) that there exists a subset  $Y_s \subseteq V_{i_s}$  of size at most k which satisfies

$$|D(Y_s)| \ge k^{3/2} \frac{\sqrt{\ln(C \ln k)}}{10^3}.$$

However, by our choice of  $X_{i_s}$ , we have

$$D(X_{i_s}) \ge D_{\nu}(X_{i_s}) \ge D_{\nu}(Y_s) \ge |D(Y_s)| - \nu k^{3/2}$$
$$\ge k^{3/2} \left( \frac{\sqrt{\ln(C \ln k)}}{10^3} - \nu \right) \ge 2 \left( 8\nu \left( \bigcup_{s=0}^3 X_{i_{\ell+s}} \right) \right),$$

by (2), provided *C* is sufficiently large. Therefore, from (5) we find that  $3D(X_{i_{\ell+3}}) \leq 2D(X_{i_{\ell}}) + \frac{1}{2}D(X_{i_{\ell}})$ , proving the claim.

Claim 2 now implies that  $(5/6)^{(m-1)/3}D(X_{i_1}) \ge D(X_{i_m}) \ge 1$  (assuming for simplicity  $m \equiv 1 \pmod{3}$ ), which then implies

$$m-1 \leq \frac{3\ln(D(X_{i_1}))}{\ln(6/5)} \leq \frac{6}{\ln(6/5)}\ln(k-1).$$

By (1), we deduce that at least one of the *m* sets  $X_i$  with  $i \in I^+$  satisfies

$$|X_i| \geq \frac{N\ln(6/5)}{25\ln k}.$$

This last quantity is at least *k* by a choice of *C* sufficiently large, contradicting our assumption that  $|X_i| \le k - 1$  for each  $i \in I^+$ . This completes the proof.

## 3 Subgraphs of high minimum degree via set-system discrepancy

In this section we prove, based on a well known discrepancy result of Spencer [9], that from a graph on  $\ell = Ck$  vertices with minimum degree at least  $\ell/2 + C'\sqrt{\ell}$  (with C' depending on C) we can select a subgraph on k vertices that has minimum degree at least k/2.

We start by defining the various standard notions of discrepancy that we need. Suppose  $\mathcal{H} = \{A_1, \ldots, A_n\}$  where  $A_i \subseteq V = [n]$ . Let  $\chi : V \to \{-1, 1\}$  be a colouring of V with the colours -1 and 1. For any  $S \subseteq V$ , we write  $\chi(S) := \sum_{i \in S} \chi(i)$  and we define the *discrepancy* of  $\mathcal{H}$  to be

$$\operatorname{disc}(\mathcal{H}) := \min_{\chi \in \{-1,1\}^V} \max_{S \in \mathcal{H}} \chi(S).$$

The result of Spencer [9] states that for any such  $\mathcal{H}$  we have  $disc(\mathcal{H}) \leq 6\sqrt{n}$ .

For  $X \subseteq V$ , we define  $\mathcal{H}|_X := \{A_1 \cap X, \dots, A_n \cap X\}$ . Then the *hereditary discrepancy* of  $\mathcal{H}$  is defined by

herdisc(
$$\mathcal{H}$$
) :=  $\max_{X \subseteq V} \operatorname{disc}(\mathcal{H}|_X)$ .

The result of Spencer also immediately implies that  $\operatorname{herdisc}(\mathcal{H}) \leq 6\sqrt{n}$  for any  $\mathcal{H}$ .

Let *A* be the incidence matrix of  $\mathcal{H}$ , i.e. *A* is the *n* × *n* matrix given by

$$A_{ij} = egin{cases} 1 & ext{if } j \in A_i, \ 0 & ext{otherwise} \end{cases}$$

Then we clearly have

$$\operatorname{disc}(\mathcal{H}) = \min_{\mathbf{x} \in \{-1,1\}^{V}} \|\mathbf{A}\mathbf{x}\|_{\infty} = 2\min_{\mathbf{x} \in \{0,1\}^{V}} \left\| \mathbf{A}\left(\mathbf{x} - \frac{1}{2}\mathbb{1}\right) \right\|_{\infty},$$

where 1 is the all 1 vector.

Now we define the *linear discrepancy* by

$$\text{lindisc}(\mathcal{H}) := \max_{c \in [0,1]^V} \min_{x \in \{0,1\}^V} \|A(x-c)\|_{\infty}.$$
 (6)

Note that here we are using  $\{0,1\}$ -colourings again. Similarly, we define the hereditary linear discrepancy of  $\mathcal{H}$  by

$$\text{herlindisc}(\mathcal{H}) := \max_{X \subseteq V} \text{lindisc}(\mathcal{H}|_X).$$

A result of Lovász, Spencer, and Vestergombi [7] states that  $herlindisc(\mathcal{H}) \leq herdisc(\mathcal{H})$ . (Note that the factor of 2 from [7] is missing to adjust for the slightly different definition we are using.) Combining with Spencer's result, we have

$$\operatorname{lindisc}(\mathcal{H}) \leq \operatorname{herlindisc}(\mathcal{H}) \leq \operatorname{herdisc}(\mathcal{H}) \leq 6\sqrt{n}.$$

If we set *c* to be the all *p* vector (for some  $p \in [0, 1]$ ) in (6), we obtain the following result.

**Lemma 5.** Let  $A_1, \ldots, A_n \subseteq V = [n]$  and  $p \in [0, 1]$ . Then there exists  $Y \subseteq V$  such that, for all  $i \in [n]$ ,

$$||A_i \cap Y| - p|A_i|| \le 6\sqrt{n}$$

We use the previous lemma to prove the following result.

**Lemma 6.** Suppose G = (V, E) is a graph with  $\ell = Pk$  vertices for some P > 1 and k a positive integer, and suppose

$$\delta(G) \ge \frac{1}{2}\ell + \eta\sqrt{\ell}$$

for some  $\eta > 0$ . Then G has an induced subgraph H on k vertices with minimum degree

$$\delta(H) \ge \frac{1}{2}k + \left(\frac{\eta}{\sqrt{P}} - 19\sqrt{P}\right)\sqrt{k}.$$

*Proof.* Write  $V = \{v_1, \ldots, v_\ell\}$ , let  $A_0 = V$  and for each  $i \in [\ell]$  let  $A_i \subseteq V$  be the neighbourhood of  $v_i$  in *G*. We apply Lemma 5 to the sets  $A_0, \ldots, A_{\ell-1}$  with  $p = (k+1+6\sqrt{\ell})/\ell$ . (Note that if p > 1 then with a simple calculation it is easy to see we can obtain the desired graph *H* simply by deleting any  $\ell - k$  vertices from *G*.) Thus there exists  $Y \subseteq V$  satisfying

$$||A_i \cap Y| - p|A_i|| \le 6\sqrt{\ell}$$

for all  $i \in \{0, ..., \ell - 1\}$ . Applying this for i = 0 and noting  $A_0 \cap Y = Y$  gives

$$k+1 = p|A_0| - 6\sqrt{\ell} \le |Y| \le p|A_0| + 6\sqrt{\ell} = k + 1 + 12\sqrt{Pk}$$

and applying it for  $i \in [\ell - 1]$  gives

$$\begin{aligned} |A_i \cap Y| \ge p|A_i| - 6\sqrt{\ell} \ge \frac{k}{\ell} \left(\frac{1}{2}\ell + \eta\sqrt{\ell}\right) - 6\sqrt{\ell} &= \frac{1}{2}k + \eta\frac{k}{\sqrt{\ell}} - 6\sqrt{\ell} \\ &= \frac{1}{2}k + \left(\frac{\eta}{\sqrt{P}} - 6\sqrt{P}\right)\sqrt{k}. \end{aligned}$$

Thus *Y* has between k + 1 and  $k + 1 + 12\sqrt{P}\sqrt{k}$  vertices. Let *Z* be an arbitrary subset of  $Y \setminus \{v_{\ell}\}$  of size *k* and let H = G[Z]. Then since we have removed at most  $12\sqrt{Pk} + 1 \le 13\sqrt{Pk}$  vertices from *Y* to obtain *Z*, we have for each  $i \in [\ell - 1]$  that

$$|A_i \cap Z| \ge \frac{1}{2}k + \left(\frac{\eta}{\sqrt{P}} - 19\sqrt{P}\right)\sqrt{k}.$$

In particular this means

$$\delta(H) \ge \frac{1}{2}k + \left(\frac{\eta}{\sqrt{P}} - 19\sqrt{P}\right)\sqrt{k},$$

as desired.

#### 4 **Proof of Theorem 1**

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To prove the theorem, we use as a subroutine the following algorithm, which is inspired by the greedy algorithm for max-cut or min-bisection.

**Lemma 7.** Let G = (V, E) be a graph of order n with  $\delta(G) \ge \frac{1}{2}(n-1) + t$  for some number t. Let  $\alpha \in [0,1]$  and let  $a, b \in \mathbb{N}$  such that a + b = n. Then either there exists  $A \subseteq V$  of size a such that  $\delta(G[A]) \ge \frac{1}{2}a - 1 + \alpha t$ , or there exists  $B \subseteq V$  of size b such that  $\delta(G[B]) \ge \frac{1}{2}b - 1 + (1 - \alpha)t$ .

*Proof.* Take any  $A \subseteq V$  of size a and let  $B := V \setminus A$ . If there exists  $x \in A$  with  $\deg_A(x) < \frac{1}{2}a - 1 + \alpha t$  and  $y \in B$  with  $\deg_B(y) < \frac{1}{2}b - 1 + (1 - \alpha)t$ , then move x to B and y to A, i.e. *swap* x and y. Note that when there is no such pair of vertices x, y we are done. We just need to prove that, if we keep iterating, then this procedure must stop at some point.

Consider the number of edges in G[A] before and after we swap x and y. The number of edges in G[A] increases by at least

$$\deg_A(y) - \deg_A(x) - 1 \ge \delta(G) - \deg_B(y) - \deg_A(x) - 1 \ge 1/2,$$

(where we subtracted 1 in case *x* and *y* are adjacent). This shows that we cannot continue to swap pairs indefinitely.  $\Box$ 

At last we are ready to prove the main result. In fact, we prove something stronger.

**Theorem 8.** There exist constants D, D' > 0 such that for  $k \ge 2$  and any graph G on  $Dk \ln k$  vertices, G or its complement  $\overline{G}$  has an induced subgraph on k vertices with minimum degree at least  $\frac{1}{2}(k-1) + D'\sqrt{(k-1)/\ln k}$ .

*Proof.* Set v = 160, C = C(v) as defined according to Theorem 2, and D := 4C. Also set  $D' := 1/\sqrt{D}$ .

By Theorem 2, since  $C \cdot 2k \ln(2k) \le 4Ck \ln k = Dk \ln k \le |V(G)|$ , we find G or  $\overline{G}$  has an induced subgraph H on  $\ell \ge 2k$  vertices with  $\delta(H) \ge \frac{1}{2}(\ell - 1) + \nu\sqrt{\ell - 1}$ . If  $\ell \equiv 0 \pmod{k}$  then we can and will repeatedly apply Lemma 7 to split the graph into parts whose sizes are multiples of k, eventually finding the desired subgraph. Otherwise we must take an extra application of Lemma 7 at the beginning to break off the residual vertices mod k and treat these separately, which we now do.

Let  $x = \ell \mod k$  (so  $x \in \{0, ..., k-1\}$ ). We can now apply Lemma 7 to H with a = k + x,  $b = \ell - k - x$ ,  $t = \nu \sqrt{\ell - 1}$  and  $\alpha = 1/2$ . Suppose this gives us a subset  $A \subseteq V(H)$  of size a such that

$$\delta(H[A]) \ge \frac{1}{2}a - 1 + \frac{1}{2}\nu\sqrt{\ell - 1} \ge \frac{1}{2}a + \frac{1}{4}\nu\sqrt{\ell} \ge \frac{1}{2}a + \frac{1}{4}\nu\sqrt{a}.$$

Then  $k \le a < 2k$  and, so applying Lemma 6 (with  $P = a/k \in [1, 2]$  and  $\eta = \nu/4 = 40$ ) yields a subset  $A' \subseteq A$  of size k such that

$$\delta(H[A']) \ge \frac{1}{2}k + \left(\frac{40}{\sqrt{P}} - 19\sqrt{P}\right)\sqrt{k} \ge \frac{1}{2}k + \left(\frac{40}{\sqrt{2}} - 19\sqrt{2}\right)\sqrt{k} \ge \frac{1}{2}k + \sqrt{2k},$$

which is more than required. In case Lemma 7 does not produce such a set *A*, it gives instead a subset *B* of size  $b = \ell - k - x \equiv 0 \pmod{k}$  such that  $\delta(H[B]) \geq \frac{1}{2}(b-1) + \frac{1}{2}\nu\sqrt{\ell-1} - \frac{1}{2}$ . We iteratively apply Lemma 7 to H[B] in a binary search to find a desired induced subgraph as follows.

Set  $G_0 = H[B]$ . Let  $\ell_0 := |V(G_0)| = b$  (so that  $k \le \ell_0 \le Dk \ln 2k$  and  $\ell_0 \equiv 0 \pmod{k}$ ) and set  $t_0 := \frac{1}{2}\nu\sqrt{\ell-1} - \frac{1}{2} \ge \frac{1}{2}\nu\sqrt{\ell_0 - 1} - \frac{1}{2}$  (so that  $\delta(G_0) \ge \frac{1}{2}(\ell_0 - 1) + t_0$ ). Suppose that  $G_i$  is given, where  $G_i$  has  $\ell_i$  vertices with  $\ell_i \equiv 0 \pmod{k}$  and  $\delta(G_i) \ge \frac{1}{2}(\ell_i - 1) + t_i$  for some number  $t_i$ . Set  $a_i = \lfloor \ell_i / 2k \rfloor k$  and  $b_i = \lceil \ell_i / 2k \rceil k$  so that  $a_i + b_i = \ell_i$  and  $a_i \equiv b_i \equiv 0$ (mod k). Apply Lemma 7 with  $G = G_i$ ,  $a = a_i$ ,  $b = b_i$ ,  $t = t_i$ , and  $\alpha = \frac{1}{2}$ . Then we either obtain a set of vertices  $A_i$  of size  $a_i$  such that  $\delta(G_i[A_i]) \ge \frac{1}{2}a_i - 1 + \frac{1}{2}t_i$ , in which case we set  $G_{i+1} := G_i[A_i] = H[A_i]$ , or we obtain a set of vertices  $B_i$  of size  $b_i$  such that  $\delta(G_i[B_i]) \ge \frac{1}{2}b_i - 1 + \frac{1}{2}t_i$ , in which case we set  $G_{i+1} := G_i[B_i] = H[B_i]$ . Now set  $\ell_{i+1} = |V(G_{i+1})|$  and note that  $\ell_{i+1} \equiv 0 \pmod{k}$  and  $\delta(G_{i+1}) \ge \frac{1}{2}(\ell_{i+1} - 1) + t_{i+1}$ , where  $t_{i+1} = \frac{1}{2}(t_i - 1)$ . Note also that  $\ell_{i+1}/k \le \lfloor \ell_i/2k \rfloor$ .

In this way we obtain subgraphs  $G_0, G_1, \ldots$  of  $G_0 = H[B]$  and we see from the recursion for  $\ell_i$  above that if  $\ell_i > k$  then  $\ell_{i+1} < \ell_i$ . Thus there exists some j such that  $\ell_j = k$  (since  $\ell_i \equiv 0 \pmod{k}$  for all i) and an easy computation shows we can assume that  $j \le \log_2(\ell_0/k) + 1$ . The recursion for  $t_i$  implies that  $t_i \ge t_0 2^{-i} - 1$  so that

$$t_j \ge \frac{t_0 k}{2\ell_0} - 1 \ge \frac{\nu(\sqrt{\ell_0 - 1} - 1)k}{4\ell_0} \ge \frac{k}{\sqrt{\ell_0}} \ge \frac{\sqrt{k}}{\sqrt{D\ln k}} = D'\sqrt{\frac{k}{\ln k}}$$

(where we used that  $t_0 \ge \frac{1}{2}\nu\sqrt{\ell_0 - 1} - \frac{1}{2}$ , that  $\ell_0 \ge k \ge 2$  with  $\nu = 160$ , and that  $\ell_0 \le Dk \ln k$ ). Thus  $G_j$  has k vertices and minimum degree at least  $\frac{1}{2}(k-1) + D'\sqrt{(k-1)/\ln k}$  and is an induced subgraph of H[B] and hence of G or  $\overline{G}$ .

#### 5 Concluding remarks

It is tempting to try using the greedy subroutine (Lemma 7) in a binary search on the output of Theorem 3(a) of [5], but since we cannot control the order of this output graph, the search might require  $O(\ln k)$  steps, which would destroy the minimum degree bounds.

Determination of the second-order term in the minimum degree threshold for polynomial to super-polynomial growth of the fixed quasi-Ramsey numbers is an open problem. (The corresponding term for the variable quasi-Ramsey numbers was determined in [5].) We define the fixed quasi-Ramsey number as the least integer  $R_c^*(k)$  such that for any graph G on  $R_c^*(k)$  vertices either G or its complement  $\overline{G}$  contains some subgraph on (exactly) k vertices with minimum degree at least c(k-1). By Theorem 8 if  $c - \frac{1}{2} = O(\sqrt{1/(k-1) \ln k})$  then  $R_c^*(k)$  is polynomial in k, and by Proposition 3 if  $c - \frac{1}{2} = \omega(\sqrt{\ln \ln k/(k-1)})$  then  $R_c^*(k)$  is superpolynomial in k. Hence the choice of  $c - \frac{1}{2}$  for which we find a transition between polynomial and super-polynomial growth in k of  $R_c^*(k)$  is determined to within a  $O(\sqrt{\ln k \ln \ln k})$  factor of  $\sqrt{1/(k-1)}$ . What is it precisely?

Last, we remark that, in the above notation, our main result is that  $R_{1/2}^*(k) = O(k \ln k)$ , while Erdős and Pach showed that  $R_{1/2}^*(k) = \Omega(k \ln k / \ln \ln k)$ . They also asked if  $R_{1/2}^*(k) = \Omega(k \ln k)$ . This question remains open.

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