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Nemeth, Sandor; Gowda, Muddappa Seetharama

DOI:
10.13001/1081-3810.4083

Citation for published version (Harvard):
Nemeth, S & Gowda, MS 2019, 'The cone of Z-transformations on the Lorentz cone', *Electronic Journal of Linear Algebra*, vol. 35, no. 1, 25, pp. 387-393. https://doi.org/10.13001/1081-3810.4083

THE CONE OF $\mathcal{Z}$-TRANSFORMATIONS ON THE LORENTZ CONE*

SÁNDOR Z. NÉMETH† AND M. SEETHARAMA GOWDA‡

Abstract. In this paper, we describe the structural properties of the cone of $\mathcal{Z}$-transformations on the Lorentz cone in terms of the semidefinite cone and copositive/completely positive cones induced by the Lorentz cone and its boundary. In particular, we describe its dual as a slice of the semidefinite cone as well as a slice of the completely positive cone of the Lorentz cone. This provides an example of an instance where a conic linear program on a completely positive cone is reduced to a problem on the semidefinite cone.

Key words. $\mathcal{Z}$-transformation, Lorentz cone, semidefinite cone, copositive cone, completely positive cone

AMS subject classifications. 90C33, 15A48

1. Introduction. Given a proper cone $\mathcal{K}$ in a finite dimensional real Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, a linear transformation $A:\mathcal{H}\to\mathcal{H}$ is said to be a *$\mathcal{Z}$-transformation on* $\mathcal{K}$ if

$$\Big[x\in\mathcal{K}, y\in\mathcal{K}^*, \text{ and } \langle x,y\rangle=0\Big]\Rightarrow\langle Ax,y\rangle\le 0,$$

where $\mathcal{K}^*$ denotes the dual of $\mathcal{K}$ in $\mathcal{H}$. Such transformations appear in various areas including economics, dynamical systems, optimization, see e.g., [3, 2, 12, 9] and the references therein. When $\mathcal{H}$ is $\mathbb{R}^n$ and $\mathcal{K}$ is the nonnegative orthant, $\mathcal{Z}$-transformations become $\mathcal{Z}$-matrices, which are square matrices with nonpositive off-diagonal entries.

The set $\mathcal{Z}(\mathcal{K})$ of all $\mathcal{Z}$-transformations on $\mathcal{K}$ is a closed convex cone in the space of all (bounded) linear transformations on $\mathcal{H}$. Given their appearance and importance in various areas, describing/characterizing elements of $\mathcal{Z}(\mathcal{K})$ and its interior, boundary, dual, etc., is of interest. An early result of Schneider and Vidyasagar [16] asserts that A is a $\mathcal{Z}$-transformation on $\mathcal{K}$ if and only if $e^{-tA}(\mathcal{K})\subseteq\mathcal{K}$ for all $t\ge 0$; consequently,

$$\mathcal{Z}(\mathcal{K})=\overline{\mathbb{R}\,I-\pi(\mathcal{K})}, \tag{1.1}$$

where $\pi(\mathcal{K})$ denotes the set of all linear transformations that leave $\mathcal{K}$ invariant, I denotes the identity transformation, and overline denotes the closure. To see another description of $\mathcal{Z}(\mathcal{K})$, let $\mathrm{LL}(\mathcal{K}) := \mathcal{Z}(\mathcal{K})\cap-\mathcal{Z}(\mathcal{K})$ denote the lineality space of $\mathcal{Z}(\mathcal{K})$, the elements of which are called Lyapunov-like transformations. Then the inclusions

$$\mathbb{R}\,I-\pi(\mathcal{K})\ \subseteq\ \mathrm{LL}(\mathcal{K})-\pi(\mathcal{K})\ \subseteq\ \mathcal{Z}(\mathcal{K})=\overline{\mathbb{R}\,I-\pi(\mathcal{K})}$$

imply that

$$\mathcal{Z}(\mathcal{K})=\overline{\mathrm{LL}(\mathcal{K})-\pi(\mathcal{K})}.$$

As the cones $\mathcal{Z}(\mathcal{K})$, $\pi(\mathcal{K})$, and $\mathrm{LL}(\mathcal{K})$ are generally difficult to describe for an arbitrary proper cone $\mathcal{K}$, we consider special cases. When $\mathcal{K}$ is the nonnegative orthant, $\mathcal{Z}(\mathcal{K})$ consists of square matrices with nonpositive off-diagonal entries, $\pi(\mathcal{K})$ consists of nonnegative matrices, and $\mathrm{LL}(\mathcal{K})$ consists of diagonal matrices. Consequently, proper polyhedral cones can be handled via isomorphism arguments. Moving away from

*Submitted to the editors July 15, 2019.

†School of Mathematics, University of Birmingham, Watson Building, Edgbaston, Birmingham B15 2TT, United Kingdom (s.nemeth@bham.ac.uk)

‡Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, Maryland 21250, USA (gowda@umbc.edu)

proper polyhedral cones, in this paper, we focus on the Lorentz cone (also called the ice-cream cone or the second-order cone as it is induced by the 2-norm) in the Euclidean space $\mathbb{R}^n$, $n > 1$, defined by:

$$\mathcal{L} := \Big\{(t,u)^\top : t \in \mathbb{R},\, u \in \mathbb{R}^{n-1},\, t \geq ||u||\Big\}. \tag{1.2}$$

This, being an example of a symmetric cone, appears prominently in conic optimization [1]. For this cone, Stern and Wolkowicz [17] have shown that $A \in \mathcal{Z}(\mathcal{L})$ *if and only if for some real number* γ, *the matrix* $\gamma J - (JA + A^\top J)$ *is positive semidefinite,* where J is the diagonal matrix $\mathrm{diag}(1, -1, -1, \ldots, -1)$. Another result of Stern and Wolkowicz ([18], Theorem 4.2) asserts that

$$\mathcal{Z}(\mathcal{L}) = \mathrm{LL}(\mathcal{L}) - \pi(\mathcal{L}). \tag{1.3}$$

(Going in the reverse direction, in a recent paper, Kuzma et al., [13] have shown that for an irreducible symmetric cone $\mathcal{K}$, the equality $\mathcal{Z}(\mathcal{K}) = \mathrm{LL}(\mathcal{K}) - \pi(\mathcal{K})$ holds only when $\mathcal{K}$ is isomorphic to $\mathcal{L}$.) Characterizations of $\pi(\mathcal{L})$ and $\mathrm{LL}(\mathcal{L})$ appear, respectively, in [14] and [20].

In this paper, we describe $\mathcal{Z}(\mathcal{L})$ and its interior, boundary, and dual in terms of the semidefinite cone and the so-called copositive and completely positive cones induced by $\mathcal{L}$ (or its boundary $\partial(\mathcal{L})$), see below for the definitions. In particular, we describe the dual of $\mathcal{Z}(\mathcal{L})$ as a slice of the semidefinite cone and also of the completely positive cone of $\mathcal{L}$. This provides an example of an instance where a conic linear optimization problem over a completely positive cone is reduced to a semidefinite problem. To elaborate, consider $\mathbb{R}^n$, the Euclidean n-space of (column) vectors with the usual inner product, $\mathbb{R}^{n\times n}$, the space of all real $n \times n$ matrices with the inner product $\langle X, Y\rangle = \mathrm{tr}(X^\top Y)$, and $\mathcal{S}^n$, the subspace of all real $n \times n$ symmetric matrices in $\mathbb{R}^{n\times n}$. Corresponding to a *closed cone* $\mathcal{C}$ *(which is not necessarily convex)* in $\mathbb{R}^n$, let

$$\mathcal{E}_\mathcal{C} := \mathrm{copos}(\mathcal{C}) := \Big\{A \in \mathcal{S}^n : x^\top Ax \geq 0 \text{ for all } x \text{ in } \mathcal{C}\Big\}$$

denote the *copositive cone of* $\mathcal{C}$ and

$$\mathcal{K}_\mathcal{C} := \mathrm{compos}(\mathcal{C}) := \left\{\sum_{u\in U} uu^\top : U \text{ is a finite subset of } \mathcal{C}\right\}$$

denote the *completely positive cone of* $\mathcal{C}$. When $\mathcal{C} = \mathbb{R}^n$, these two cones coincide with the *semidefinite cone* $\mathcal{S}^n_+$ (consisting of all real $n\times n$ symmetric positive semidefinite matrices); when $\mathcal{C} = \mathbb{R}^n_+$, these reduce, respectively, to the (standard) copositive cone and completely positive cone. All these cones appear prominently in conic optimization. A result of Burer [5] (see also, [4, 7]) says that any nonconvex quadratic programming problem over a closed cone with additional linear and binary constraints can be reformulated as a linear program over a suitable completely positive cone. For this and other reasons, there is a strong interest in understanding copositive and completely positive cones. For the closed convex cones $\mathcal{E}_\mathcal{C}$ and $\mathcal{K}_\mathcal{C}$, various structural properties (such as the interior, boundary) as well as duality, irreducibility, and homogeneity properties, have been investigated in the literature, see for example, [19, 6, 8, 11]. Taking $\mathcal{C}$ to be one of $\mathbb{R}^n$, $\mathcal{L}$, or $\partial(\mathcal{L})$, we show that

$$\mathcal{Z}(\mathcal{L})^* = \Big\{B \in \mathbb{R}^{n\times n} : \langle B, I\rangle = 0, -JB \in \mathcal{K}_\mathcal{C}\Big\} \tag{1.4}$$

and deduce the equality of slices

$$(1.5)\qquad \Big\{X \in \mathbb{R}^{n\times n} : \langle J, X\rangle = 0,\ X \in \mathcal{S}^n_+\Big\} = \Big\{X \in \mathbb{R}^{n\times n} : \langle J, X\rangle = 0,\ X \in \mathcal{K}_{\mathcal{C}}\Big\}.$$

2. Preliminaries.

2. Preliminaries. In a (finite dimensional real) Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, a nonempty set $\mathcal{C}$ is said to be a *closed cone* if it closed and $tx \in \mathcal{C}$ whenever $x \in \mathcal{C}$ and $t \geq 0$ in $\mathbb{R}$. *Throughout this paper, $\mathcal{C}$ denotes a closed cone.*
A nonempty set $\mathcal{K}$ is said to be a *closed convex cone* if it is a closed cone which is also convex. Such a cone is said to be *proper* if $\mathcal{K} \cap -\mathcal{K} = \{0\}$ and has nonempty interior. Corresponding to a closed convex cone $\mathcal{K}$, we define its dual in $\mathcal{H}$ as the set

$$\mathcal{K}^* = \Big\{x \in \mathcal{H} : \langle x, y\rangle \geq 0,\ \forall y \in \mathcal{K}\Big\}.$$

We say that a linear transformation $A : \mathcal{H} \to \mathcal{H}$ is *copositive* on $\mathcal{K}$ if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{K}$. We also let $\pi(\mathcal{K}) = \{A : A(\mathcal{K}) \subseteq \mathcal{K}\}$, where A denotes a linear transformation on $\mathcal{H}$. For a set S in $\mathcal{H}$, we denote the closure, interior, and the boundary by $\overline{S}$, S°, and $\partial(S)$ respectively.

We will be considering closed convex cones in the space $\mathcal{H} = \mathbb{R}^n$ which carries the usual inner product and in the space $\mathbb{R}^{n\times n}$ which carries the inner product $\langle X, Y\rangle := \mathrm{tr}(X^\top Y)$, where the trace of a square matrix is the sum of its diagonal entries. In $\mathbb{R}^{n\times n}$, $\mathcal{S}^n$ denotes the subspace of all symmetric matrices and $\mathcal{A}^n$ denotes the subspace of all skew-symmetric matrices. We note that $\mathbb{R}^{n\times n}$ is the orthogonal direct sum of $\mathcal{S}^n$ and $\mathcal{A}^n$.

We recall some (easily verifiable) properties of the Lorentz cone $\mathcal{L}$ given by (1.2). $\mathcal{L}$ is a self-dual cone in $\mathbb{R}^n$, that is, $\mathcal{L}^* = \mathcal{L}$; its interior and boundary are given, respectively, by

$$\mathcal{L}^\circ = \Big\{(t, u)^\top : t > ||u||\Big\},$$

$$\partial(\mathcal{L}) = \Big\{(t, u)^\top : t = ||u||\Big\} = \Big\{\alpha\,(1, u)^\top : \alpha \geq 0,\ ||u|| = 1\Big\}.$$

We also have

$$(2.1)\qquad \Big[0 \neq x, y \in \mathcal{L},\ \langle x, y\rangle = 0\Big] \Rightarrow x = \alpha\,(1, u)^\top \text{ and } y = \beta\,(1, -u)^\top, \text{ for some } \alpha, \beta > 0 \text{ and } ||u|| = 1.$$

For a closed cone $\mathcal{C}$ in $\mathbb{R}^n$, we consider the copositive cone $\mathcal{E}_{\mathcal{C}}$ and the completely positive cone $\mathcal{K}_{\mathcal{C}}$ (defined in the Introduction). Note that these are cones of symmetric matrices.

In the Hilbert space $\mathcal{S}^n$ (which carries the inner product from $\mathbb{R}^{n\times n}$), the following hold.

1. *$\mathcal{K}_{\mathcal{C}}$ is the dual cone of $\mathcal{E}_{\mathcal{C}}$ [19].*
2. *When $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$, both $\mathcal{E}_{\mathcal{C}}$ and $\mathcal{K}_{\mathcal{C}}$ are proper cones ([10], Proposition 2.2). In particular, this holds when $\mathcal{C}$ is one of $\mathbb{R}^n$, $\mathcal{L}$, or $\partial(\mathcal{L})$.*
3. *We have $\mathcal{S}^n_+ = \mathcal{E}_{\mathbb{R}^n} \subset \mathcal{E}_{\mathcal{L}} \subset \mathcal{E}_{\partial(\mathcal{L})}$, or equivalently, $\mathcal{K}_{\partial(\mathcal{L})} \subset \mathcal{K}_{\mathcal{L}} \subset \mathcal{K}_{\mathbb{R}^n} = \mathcal{S}^n_+$.*

3. Main results. In this section, we provide a closure-free description of $\mathcal{Z}(\mathcal{L})$ and, additionally, describe the dual, interior, and the boundary of $\mathcal{Z}(\mathcal{L})$. We recall that $J = \mathrm{diag}(1, -1, -1, \ldots, -1)$ and $\mathcal{A}^n$ denotes the set of all skew-symmetric matrices in $\mathbb{R}^{n\times n}$.

THEOREM 3.1. *Let $\mathcal{C}$ denote one of $\mathbb{R}^n$, $\mathcal{L}$, or $\partial(\mathcal{L})$. Then,*

$$\mathcal{Z}(\mathcal{L}) = \mathbb{R}\, I - J(\mathcal{E}_{\mathcal{C}} + \mathcal{A}^n). \tag{3.1}$$

Proof. Let $A \in \mathcal{Z}(\mathcal{L})$. From the result of Stern and Wolkowicz [17] mentioned in the Introduction, we have

$$2\gamma J - (JA + A^\top J) = 2P$$

for some $\gamma \in \mathbb{R}$ and $P \in \mathcal{S}^n_+$. Hence, $JA + (JA)^\top = 2(\gamma J - P)$, which implies

$$2JA = JA + (JA)^\top - \big[(JA)^\top - JA\big] = 2(\gamma J - P) - 2Q, \tag{3.2}$$

where $2Q = (JA)^\top - JA$ is skew-symmetric. Since $J^2 = I$, this leads to

$$A = \gamma\, I - J(P + Q),$$

where $P \in \mathcal{S}^n_+$ and $Q \in \mathcal{A}^n$. As $\mathcal{S}^n_+ \subset \mathcal{E}_{\mathcal{L}} \subset \mathcal{E}_{\partial(\mathcal{L})}$, this proves that

$$\mathcal{Z}(\mathcal{L}) \subseteq \mathbb{R}\, I - J(\mathcal{S}^n_+ + \mathcal{A}^n) \subseteq \mathbb{R}\, I - J(\mathcal{E}_{\mathcal{L}} + \mathcal{A}^n) \subseteq \mathbb{R}\, I - J(\mathcal{E}_{\partial(\mathcal{L})} + \mathcal{A}^n). \tag{3.3}$$

Now, to see the reverse inclusions, suppose $A = \gamma\, I - J(P + Q)$ for some $\gamma \in \mathbb{R}$, $P \in \mathcal{E}_{\partial(\mathcal{L})}$, and Q skew-symmetric. Let $0 \neq x, y \in \mathcal{L}$ with $\langle x, y\rangle = 0$. By (2.1), x and y are in $\partial(\mathcal{L})$, and Jy is a positive multiple of x. Hence, $\langle Px, Jy\rangle \geq 0$ as $P \in \mathcal{E}_{\partial(\mathcal{L})}$ and $\langle Qx, Jy\rangle = 0$ as Q is skew-symmetric. Thus,

$$\langle Ax, y\rangle = \gamma\langle x, y\rangle - \langle JPx, y\rangle + \langle JQx, y\rangle = -\langle Px, Jy\rangle + \langle Qx, Jy\rangle \leq 0.$$

This shows that $A \in \mathcal{Z}(\mathcal{L})$ and so, inclusions in (3.3) turn into equalities. Thus we have (3.1). ∎

Remarks. From the above theorem, we have

$$\mathbb{R}\, I - J(\mathcal{S}^n_+ + \mathcal{A}^n) = \mathbb{R}\, I - J(\mathcal{E}_{\mathcal{L}} + \mathcal{A}^n) = \mathbb{R}\, I - J(\mathcal{E}_{\partial(\mathcal{L})} + \mathcal{A}^n).$$

Multiplying throughout by J and noting $-\mathcal{A}^n = \mathcal{A}^n$, we get the equality of sets

$$(\mathbb{R}\, J - \mathcal{S}^n_+) + \mathcal{A}^n = (\mathbb{R}\, J - \mathcal{E}_{\mathcal{L}}) + \mathcal{A}^n = (\mathbb{R}\, J - \mathcal{E}_{\partial(\mathcal{L})}) + \mathcal{A}^n,$$

where each set is a sum of $\mathcal{A}^n$ and a subset of $\mathcal{S}^n$. Since $\mathbb{R}^{n\times n} = \mathcal{S}^n + \mathcal{A}^n$ is an (orthogonal) direct sum decomposition, we see that

$$\mathbb{R}\, J - \mathcal{S}^n_+ = \mathbb{R}\, J - \mathcal{E}_{\mathcal{L}} = \mathbb{R}\, J - \mathcal{E}_{\partial(\mathcal{L})}. \tag{3.4}$$

These equalities can also be established via different arguments. A result of Loewy and Schneider [14] asserts that *A symmetric matrix X is copositive on $\mathcal{L}$ if and only if there exists $\mu \geq 0$ such that $X - \mu\, J \in \mathcal{S}^n_+$.* (This is essentially a consequence of the so-called S-Lemma [15]: If A and B are two symmetric matrices with $\langle Ax_0, x_0\rangle > 0$ for some x_0 and $\langle Ax, x\rangle \geq 0 \Rightarrow \langle Bx, x\rangle \geq 0$, then there exists $\mu \geq 0$ such that $B - \mu A$ is positive semidefinite.) This result gives the equality

$$\mathcal{E}_{\mathcal{L}} = \mathcal{S}^n_+ + \mathbb{R}_+\, J$$

and consequently $\mathbb{R}\,J - \mathcal{S}^n_+ = \mathbb{R}\,J - \mathcal{E}_{\mathcal{L}}$. The equality

$$\mathcal{E}_{\partial(\mathcal{L})} = \mathcal{S}^n_+ + \mathbb{R}\,J$$

can be seen via an application of Finsler' theorem [15] that says that if A and B are two symmetric matrices with $[x \neq 0, \langle Ax, x\rangle = 0] \Rightarrow \langle Bx, x\rangle > 0$, then there exists $\mu \in \mathbb{R}$ such that $B + \mu A$ is positive semidefinite. (For $M \in \mathcal{E}_{\partial(\mathcal{L})}$ and vectors $u, v \in \mathcal{L}^\circ$, one has $\langle Jx, x\rangle = 0 \Rightarrow \langle M_k x, x\rangle > 0$, where k is a natural number and $M_k := M + \frac{1}{k} uv^\top$. When $M_k + \mu_k\,J$ is positive semidefinite for all k, it follows that the sequence μ_k is bounded. One can then use a limiting argument.) From this equality, one gets $\mathbb{R}\,J - \mathcal{S}^n_+ = \mathbb{R}\,J - \mathcal{E}_{\partial(\mathcal{L})}$.

Our next result deals with the dual of $\mathcal{Z}(\mathcal{L})$.

THEOREM 3.2. *Let $\mathcal{C}$ denote one of $\mathbb{R}^n$, $\mathcal{L}$, or $\partial(\mathcal{L})$. Then,*

$$\mathcal{Z}(\mathcal{L})^* = \Big\{B \in \mathbb{R}^{n\times n} : \langle B, I\rangle = 0, -JB \in \mathcal{K}_{\mathcal{C}}\Big\}.$$

In particular, (1.5) holds.

Proof. We fix $\mathcal{C}$. From (3.1), we see that $B \in \mathcal{Z}(\mathcal{L})^*$ if and only if

$$0 \leq \langle B, \gamma\, I - J(P+Q)\rangle$$

for all γ real, P in $\mathcal{E}_{\mathcal{C}}$, and Q in $\mathcal{A}^n$. Clearly, this holds if and only if

$$\langle B, I\rangle = 0,\ \langle -JB, P\rangle \geq 0, \text{ and } \langle -JB, Q\rangle = 0$$

for all γ, P, and Q specified above. Now, with the observation that a (real) matrix is orthogonal to all skew-symmetric matrices in $\mathbb{R}^{n\times n}$ if and only if it is symmetric, this further simplifies to

$$\langle B, I\rangle = 0 \text{ and } -JB \in \mathcal{E}^*_{\mathcal{C}},$$

where $\mathcal{E}^*_{\mathcal{C}}$ is the dual of $\mathcal{E}_{\mathcal{C}}$ computed in $\mathcal{S}^n$. Since $\mathcal{K}_{\mathcal{C}} = \mathcal{E}^*_{\mathcal{C}}$ in $\mathcal{S}^n$, we see that $B \in \mathcal{Z}(\mathcal{L})^*$ if and only if $\langle B, I\rangle = 0$ and $-JB \in \mathcal{K}_{\mathcal{C}}$. This completes the proof. ∎

We remark that (1.5) can be deduced directly from (3.4) by taking the duals in $\mathcal{S}^n$.

In our final result, we describe the interior and boundary of $\mathcal{Z}(\mathcal{L})$. First, we recall some definitions from [9]. Let

$$\Omega := \Big\{(x, y) \in \mathcal{L} \times \mathcal{L} : ||x|| = 1 = ||y|| \text{ and } \langle x, y\rangle = 0\Big\}.$$

It is easy to see that Ω is compact and, from (2.1),

$$\Omega = \Big\{(x, Jx) : x \in \partial(\mathcal{L}),\ ||x|| = 1\Big\}. \tag{3.5}$$

For any $A \in \mathbb{R}^{n\times n}$, let

$$\gamma(A) := \max\Big\{\langle Ax, y\rangle : (x, y) \in \Omega\Big\}.$$

Note that $A \in \mathcal{Z}(\mathcal{L})$ if and only if $\gamma(A) \leq 0$. We say that $A \in \mathbb{R}^{n\times n}$ is a *strict-$\mathcal{Z}$-transformation on $\mathcal{L}$* if

$$\Big[0 \neq x, y \in \mathcal{L},\, \langle x, y\rangle = 0\Big] \Rightarrow \langle Ax, y\rangle < 0.$$

The set of all such transformations is denoted by $str(\mathcal{Z}(\mathcal{L}))$. For $A \in \mathbb{R}^{n\times n}$, the following statements are shown in [9], Theorem 3.1:

$$\gamma(A) < 0 \iff A \in \mathcal{Z}(\mathcal{L})^\circ \iff A \in str(\mathcal{Z}(\mathcal{L}))$$

and

$$\gamma(A) = 0 \iff A \in \partial(\mathcal{Z}(\mathcal{L})).$$

Recall that $\mathcal{E}_{\mathcal{L}}$ consists of all symmetric matrices that are copositive on $\mathcal{L}$. We say that a symmetric matrix P is *strictly copositive on* $\mathcal{L}$ if $0 \neq x \in \mathcal{L} \Rightarrow \langle Px, x\rangle > 0$; the set of all such matrices is denoted by $str(\mathcal{E}_{\mathcal{L}})$. Similarly, one defines $str(\mathcal{E}_{\partial(\mathcal{L})})$.

Corollary 3.3. *The following statements hold:*

$$\mathcal{Z}(\mathcal{L})^\circ = str(\mathcal{Z}(\mathcal{L})) = \mathbb{R}\, I - J\left(str(\mathcal{E}_{\partial(\mathcal{L})}) + \mathcal{A}^n\right)$$

and

$$\partial(\mathcal{Z}(\mathcal{L})) = \mathbb{R}\, I - J\left(\partial_*(\mathcal{E}_{\partial(\mathcal{L})}) + \mathcal{A}^n\right),$$

where $\partial_*(\mathcal{E}_{\partial(\mathcal{L})})$ denotes the boundary of $\mathcal{E}_{\partial(\mathcal{L})}$ in $\mathcal{S}^n$.

Proof. We first deal with the interior of $\mathcal{Z}(\mathcal{L})$. The equality

$$\Big\{A \in \mathbb{R}^{n\times n} : \gamma(A) < 0\Big\} = \mathcal{Z}(\mathcal{L})^\circ = str(\mathcal{Z}(\mathcal{L}))$$

has already been observed in [9], Theorem 3.1. To see the first assertion, we show that $\gamma(A) < 0$ if and only if $A = \theta\, I - J(P + Q)$ for some $\theta \in \mathbb{R}$, P (symmetric) strictly copositive on $\partial(\mathcal{L})$, and Q skew-symmetric. Suppose $\gamma(A) < 0$. Then, for any $\theta \in \mathbb{R}$,

$$\max\Big\{\big\langle (A - \theta\, I)x, y\big\rangle : (x, y) \in \Omega\Big\} < 0,$$

which, from (3.5) becomes

$$\min\Big\{\big\langle J(\theta\, I - A)x, x\big\rangle :\ x \in \partial(\mathcal{L}), ||x|| = 1\Big\} > 0.$$

Now, fix θ and let $J(\theta\, I - A) = P + Q$, where $P \in \mathcal{S}^n$ and $Q \in \mathcal{A}^n$. As $\langle Qx, x\rangle = 0$ for any x, the above inequality implies that $\min\Big\{\big\langle Px, x\big\rangle :\ x \in \partial(\mathcal{L}), ||x|| = 1\Big\} > 0$. This proves that P is strictly copositive on $\partial(\mathcal{L})$. Rewriting $J(\theta\, I - A) = P + Q$, we see that $A = \theta\, I - J(P + Q)$ which is of the required form.
To see the converse, suppose $A = \theta\, I - J(P+Q)$, where $\theta \in \mathbb{R}$, P (symmetric) strictly copositive on $\partial(\mathcal{L})$, and Q skew-symmetric. Using (3.5), we can easily verify that $\gamma(A) < 0$. Thus, $A \in str(\mathcal{Z}(\mathcal{L}))$.
An argument similar to the above will show that $\gamma(A) = 0$ if and only if $A = \theta\, I - J(P + Q)$ for some $\theta \in \mathbb{R}$, $P \in \partial_*(\mathcal{E}_{\partial(\mathcal{L})})$, and Q skew-symmetric. This gives the statement regarding the boundary of $\mathcal{Z}(\mathcal{L})$. ∎

We end the paper with a remark dealing with conic linear programs. Motivated by the result of Burer (mentioned in the Introduction), we consider a conic linear program on a completely positive cone $\mathcal{K}_{\mathcal{C}}$ (where $\mathcal{C}$ is a closed cone):

$$\min \Big\{ \langle c, x \rangle : Ax = b, x \in \mathcal{K}_{\mathcal{C}} \Big\}.$$

While such a problem is generally hard to solve, we ask: (When) can we replace $\mathcal{K}_{\mathcal{C}}$ by $\mathcal{S}^n_+$ and thus reduce the above problem to the semidefinite programming problem $\min \Big\{ \langle c, x \rangle : Ax = b, x \in \mathcal{S}^n_+ \Big\}$? Just replacing $\mathcal{K}_{\mathcal{C}}$ by $\mathcal{S}^n_+$ without handling the constraint $Ax = b$ is not viable as $\mathcal{K}_{\mathcal{C}} = \mathcal{S}^n_+$ if and only if $\mathcal{C} \cup -\mathcal{C} = \mathbb{R}^n$ (which fails to hold when $n > 1$ and $\mathcal{C}$ is pointed), see [11]. While we do not answer this broad question, we point out, as a consequence of (1.5) that for any $C \in \mathcal{S}^n$,

$$\min \Big\{ \langle C, X \rangle : \langle X, J \rangle = 0, X \in \mathcal{K}_{\mathcal{L}} \Big\} = \min \Big\{ \langle C, X \rangle : \langle X, J \rangle = 0, X \in \mathcal{S}^n_+ \Big\}.$$

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