Challenges associated with finite element methods for forward modelling of unbounded gravity fields

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Abstract

Gravity forward modelling is the calculation of gravity field from a specific density distribution, and is essential for reconstructing ground density in an inversion process. Finite element (FE) methods have been effectively used for forward modelling of gravity data. In contrast to the closed-form and analytical methods, FEM can model complicated geometries and density distributions. Since the gravity field is an unbounded domain, numerical modelling of the boundary condition is the main challenge associated with the FE formulation of the gravity data. In the majority of numerical cases, the domain of the gravity is truncated at a relatively far distance from the zone of density contrast in order to reduce the effect of the boundary conditions on the results. Some researchers have applied a zero gravitational potential value to the boundary while others have applied an estimated gravitational potential value to the truncated edges. In both cases, a large zone of zero-density contrast has to be added around the zone of density contrast which considerably increases the computational time. Another type of the boundary condition, which is less developed in the field of gravity modelling, is the use of a single layer of infinite elements around the zone of density contrast to model the boundary. The scope of this paper is the discussion and comparison of the aforementioned types of modelling boundary conditions in FE analyses with respect to gravity field modelling. The advantages and disadvantages of each method, especially the infinite element boundary over the truncation methods, are presented. The results show that a trade-off between the size of the additional zero density zone around the zone of density contrast, and the meshing element sizes is essential for the truncation methods. Furthermore, the infinite element boundary is shown to have great potential to overcome the computational issues related to the truncation methods. A high accuracy in the results with less computational time can be obtained using infinite elements.
Keywords: gravity forward modelling, Finite element analysis, Boundary conditions, Infinite elements

1. Introduction

Modelling of gravity plays an important role to solve different issues, such as identifying and locating buried objects. Gravity values can be predicted from a specific distribution of ground density in a process called forward modelling of gravity data. Forward modelling is an essential stage of the inversion process in which the subsurface density is reconstructed using a non-destructive approach. Different methods can be used for gravity forward modelling, including closed-form solutions and numerical methods.

Numerical methods contain different techniques one of which is the finite element (FE) method, which is a powerful technique used for forward modelling of gravity. The importance of FE analysis becomes vital when the geometry of the problem and the density distribution are complicated. In such cases, the complexity of the problem makes it impossible to use simplified methods or closed-form solutions [2].

This paper is focused on the use of FE analysis for forward modelling of gravity, considering three different techniques within the FE method. The difference between the three techniques is related to the way the boundary conditions of the gravity field is modelled, since gravity is an unbounded problem. Background to the use of FE analysis, including the gaps in knowledge, is presented in Section 2. Utilising a set of rigorous numerical simulations, this work aims to evaluate the performance of FE methods to model gravity, and to introduce robust indicators to improve the numerical outcomes when using them for gravity forward modelling.

2. Background

FE modelling of gravity, as opposed to closed-form solutions, initially solves the gravitational potential, and then calculates the gravity values from the gradient of the potential field. This needs a solution for

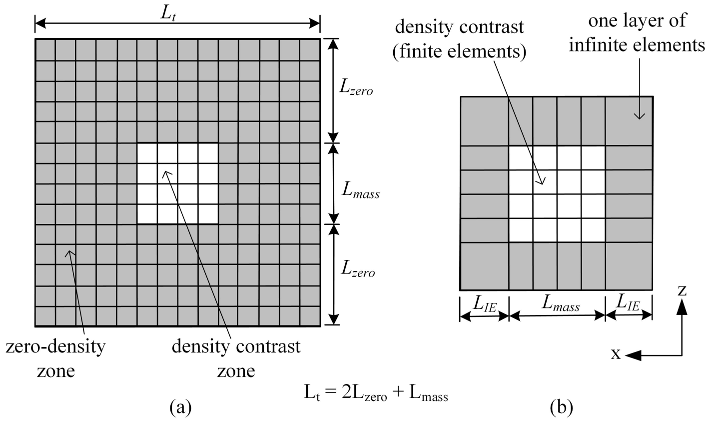

Figure 1: A typical numerical domain for (a) the truncation boundary and (b) an infinite element boundary (L_{mass} and L_{zero} are the lengths of the density contrast and the zero-density zones, respectively, and L_{IE} is the length of infinite elements)

an unbounded problem. The main concern associated with the FE modelling of such problems is the simulation of the boundary conditions since theoretically, it extends to infinity.

The truncation of the boundary at a far, but finite, distance from the zone of density contrast is the most popular technique used for the forward modelling of gravity. The space from the edge of the density contrast zone to the truncated boundary of the whole domain needs to be modelled with zero-density elements (zero-density here means no contrast within the density distribution). At the truncated boundary, either a zero gravitational potential is applied, as used by Jahandari and Farquharson [4] and Zhang et al. [8], or the gravitational potential is estimated and applied to the boundary, as proposed by Cai and Wang [2]. In the rest of this paper, the former method is called the zero-potential boundary, and the latter is called the Cai and Wang method. A typical numerical model of the truncation boundary is shown in Figure 1a. It is worth noting that a comparison of these two boundary modelling methods was made by May and Knepley [7] in their paper about forward models for the calculation of gravity anomalies. Their focus was mainly on the effect of the domain size on the results. A trade-off between the domain size and the mesh fineness, and the influence of the element size outside the zone of density contrast on the results were not discussed.

Another type of boundary condition, simulated with one layer of infinite elements around the zone of density contrast, was used by Gharti and Tromp [3] to model gravity in a spherical domain. It should be mentioned that this method is less developed in the field of gravity modelling. Furthermore, no comparison of this method with the truncation techniques is available in the literature.

This paper presents the results from a series of analyses investigating the infinite element boundary condition. A brief methodology of the numerical simulation of this boundary is presented in Section 3. Figure 1b shows a typical numerical domain with an infinite element boundary.

3. Methodology and Model Description

For details of the simulations by Cai and Wang, the reader is referred to reference [2] who originally proposed the method. This section presents some insight into the formulation of 2D infinite elements, which is then followed by a description of the model used in this paper.

The aim behind using infinite elements to model boundaries is that the infinity nature of the unbounded domains can be reproduced using only one layer of elements that have special types of shape functions. The coordinate ascent approach, in which an infinite element is compressed to become a conventional finite element [5], was used to formulate infinite elements in this paper.

The difference between a conventional finite element analysis and an infinite element formulation is the type of shape functions used to form the Jacobian matrix for the conversion between the global and local coordinate systems. In the local coordinate system, finite element shape functions are used to interpolate the field variables (in the case of gravity modelling, the gravitational potential, $\mathbf{\Phi}$). It is worth noting that when simulating infinite elements, an arbitrary position called the pole has to be specified. This position is used to find the length of the infinite elements, however it is not explicitly modelled. There is not a specific criterion to accurately determine the pole position. The factors that help with specifying this position include engineering knowledge, the problem characteristics (geometrical and physical), and fit to the decay pattern. More details about the pole position can be found in [6] and [9]. In this work, the

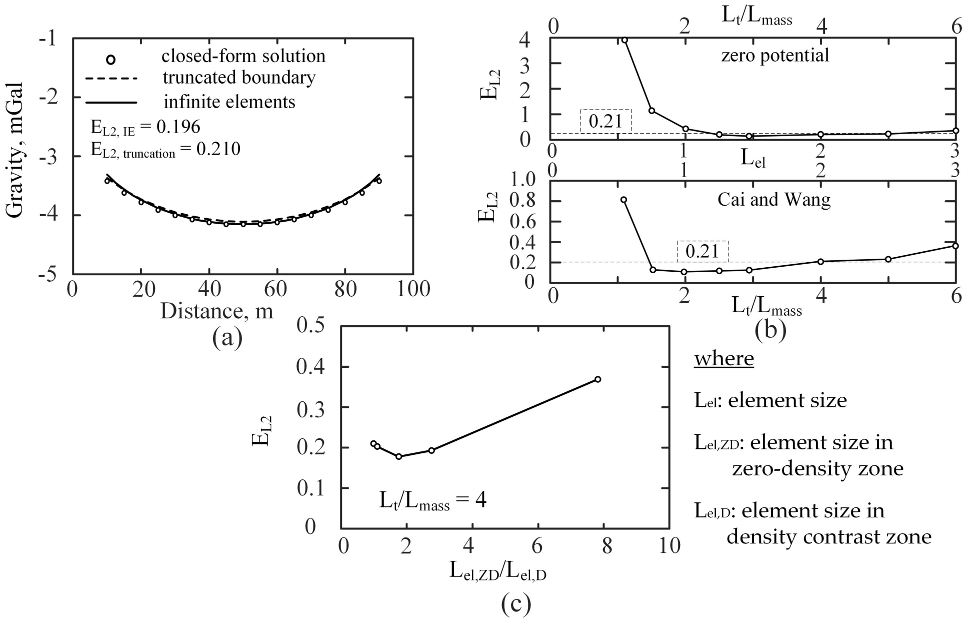

Figure 2: (a) Gravity values predicted by the infinite element and truncated boundary methods, (b) the combined effect of the mesh fineness and zero-density size, and (c) the effect of mesh non-uniformity on the truncated boundary results

infinite element formulation proposed by Marques and Owen [6] was used to model the boundary of the problem.

To simplify the problem and to focus on the intended aim of the paper, a simple domain of density contrast was simulated that had dimensions of 100 m ×100 m (i.e. $L_{mass} = 100$ m in Figure 1), and a uniform density contrast of 1800 kg/m^3. In all cases, the domain was discretised using 2D square elements with 4 nodes at their corners. All the numerical simulations were performed using MATLAB. Observation points were located at the top of the density contrast zone, starting from $x_{mass} = 10$ m to $x_{mass} = 90$ m where x_{mass} is the horizontal distance along the zone of the density contrast. The element and domain sizes were variable. Extra information is presented in Section 4 where the results are discussed. It should be mentioned that for the infinite element model, one layer of infinite elements were used to model the boundary of the problem. The length of these elements was 22.5 m (obtained from comparing the numerical results to the exact solution of the problem).

In order to validate the results, an exact solution of the problem was calculated using Equation 1 [1]. Furthermore, the error between the numerical and synthetic data was calculated using the L_2 norm, Equation 2.

$$g_{synth} = 2G\rho \iint \frac{z}{r^2} dxdz \tag{1}$$

$$E_{L_2} = \sqrt{\sum_{i=1}^{n_m} \left| g_{num,i} - g_{synth,i} \right|^2} \tag{2}$$

where $g_{synth,i}$ is the synthetic gravity at i^{th} observation point, G is Newton's gravitational constant, ρ is the density, z is the vertical distance, r is the distance between the mass and the observation point, n_m is the number of measurement points, and g_{num} is the gravity values computed numerically (i.e. FEM).

4. Results & discussion

Figure 2a shows the gravity prediction and L_2 error for the different numerical techniques. The zero-density size in the truncation methods was $L_t/L_{mass} = 4$ (L_t is the length of the whole numerical domain, as shown in Figure 1). The numerical domain was meshed uniformly with elements of 2 m side length. The total number of elements was 2,704 and 40,000 in the infinite element boundary and truncation methods, respectively. Figure 2a also states that E_{L2} =0.196 and 0.210, respectively for the infinite element and the truncation boundaries. For the same meshing resolution, the infinite element boundary gives slightly better results than the truncation method, while using considerably fewer elements.

The results of Figure 2a indicate that substantially less computational cost is involved when modelling gravity using an infinite element boundary. It should be noted that when using infinite elements, the choice of a suitable pole position has to be made carefully due to its impact on the results. In addition, for more complicated gravity problems, non-linear shape functions for the infinite elements could improve the results significantly.

Figure 2b presents the combined effect of the element and zero-density domain sizes on the results of the truncation boundaries. The total number of elements for all the simulations in Figure 2b was 40,000. The meshing was gradually made coarser as the zero-density domain was added to the model. The magnitude of E_{L2} is large when there is no zero-density domain, despite having a very fine mesh in the zone of the density contrast. Adding the zero-density domain reduces the error up to a specific point (up to $L_t/L_{mass} = 2$ for the Cai and Wang method, and 3 for the zero-potential boundary). After this point, the error starts to bounce which shows the role of the element size in the analysis. It should be noted that after $L_t/L_{mass} = 4$, both the truncation methods give similar results.

The simulations linked with Figures 2a and 2b had uniform meshes throughout the domain whilst Figure 2c shows the results with non-uniform meshing. Figure 2c displays the effect of the element size in the zero-density domain on the results for the truncation methods with $L_t/L_{mass} = 4$. All the simulations are for a constant element number of 40,000. Figure 2c shows that the results improve as the ratio of the element size of the zero-density to that of density contrast zone, $L_{el,ZD}/L_{el,D}$, increases up to 1.75. After this point, an increase in the $L_{el,ZD}/L_{el,D}$ ratio results in a larger error. This effect is the same for both truncation methods, and shows that non-uniformity meshing could be helpful to improve the results.

5. Conclusions

This paper has presented insights into the forward modelling of gravity data using finite element methods with three types of boundary formulations: infinite elements, zero-potential and the Cai and Wang methods. The results showed that the infinite element boundary presented accurate predictions of gravity data with fewer elements compared to the zero-potential and Cai and Wang methods. This could lead to a considerable reduction in computational cost if proper infinite element shape functions and accurate pole positions are used.

Regarding the zero-potential and Cai and Wang methods, it was shown that a trade-off between the element size and the size of the zero-density domain is necessary to obtain optimal results. Furthermore, the results showed that the meshing does not need to be uniform throughout the domain. A finer mesh within the zone of the density contrast improves the gravity prediction of the truncation methods up to a specific limit. This limit can be determined by finding an optimal ratio between the element sizes within and outside the zone of the density contrast.

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