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Bicategories in Univalent Foundations

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Abstract

We develop bicategory theory in univalent foundations. Guided by the notion of univalence for (1-)categories studied by Ahrens, Kapulkin, and Shulman, we define and study univalent bicategories. To construct examples of those, we develop the notion of “displayed bicategories”, an analog of displayed 1-categories introduced by Ahrens and Lumsdaine. Displayed bicategories allow us to construct univalent bicategories in a modular fashion. To demonstrate the applicability of this notion, we prove several bicategories are univalent. Among these are the bicategory of univalent categories with families and the bicategory of pseudofunctors between univalent bicategories. Our work is formalized in the **UniMath** library of univalent mathematics.

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1 Introduction

Category theory (by which we mean 1-category theory) is established as a convenient language to structure, and discuss, mathematical objects and maps between them. To axiomatize the fundamental objects of category theory itself – categories, functors, and natural transformations – the theory of 1-categories is not enough. Instead, category-like structures allowing for “morphisms between morphisms” were developed to account for the natural transformations. Among those are *bicat​​egories*.

Bicat​​egory theory was originally developed by Bénabou [5] in set-theoretic foundations. The goal of our work is to develop bicat​​egory theory in univalent foundations. Specifically, we give here a notion of *univalent bicat​​egory* and show that some bicat​​egories of interest are univalent. To this end, we generalize *displayed categories* [2] to the bicat​​egorical setting, and prove that the total bicat​​egory spanned by a displayed bicat​​egory is univalent, if the constituent pieces are.

Univalent foundations and categories therein. According to Voevodsky, a foundation of mathematics consists of three things:

1. a language for mathematical objects;
2. a notion of proposition and proof; and
3. an interpretation of those into a world of mathematical objects.

By “univalent foundations”, we mean the foundation given by univalent type theory as described, *e.g.*, in the HoTT book [21], with its notion of “univalent logic”, and the interpretation of univalent type theory in simplicial sets expected to arise from Voevodsky’s simplicial set model [14].

In the simplicial set model, univalent categories (just called “categories” in [1]) correspond to truncated complete Segal spaces, which in turn are equivalent to ordinary (set-theoretic) categories. This means that univalent categories are “the right” notion of categories in univalent foundations: they correspond exactly to the traditional set-theoretic notion of category. Similarly, the notion of *univalent bicat​​egory*, studied in this paper, provides the correct notion of bicat​​egory in univalent foundations.

Throughout this article, we work in type theory with function extensionality. We explicitly mention any use of the univalence axiom. We use the notation standardized in [21]; a significantly shorter overview of the setting we work in is given in [1]. As a reference for 1-category theory in univalent foundations, we refer to [1], which follows a path suggested by Hofmann and Streicher [13, Section 5.5].

Bicat​​egories for Type Theory. Our motivation for this work stems from several particular (classes of) bicat​​egories, that come up in our work on the semantics of type theories and Higher Inductive Types (HITs).

Firstly, we are interested in the “categories with structure” that have been used in the model theory of type theories. The purpose of the various categorical structures is to model context extension and substitution. Prominent such notions are categories with families (see, *e.g.*, [8, 9]), categories with attributes (see, *e.g.*, [19]), and categories with display maps (see, *e.g.*, [20]). Each notion of “categorical structure” gives rise to a bicat​​egory whose objects are categories equipped with such a structure. Secondly, in the study of HITs, bicat​​egories of algebras feature prominently, see, *e.g.*, work by Dybjer and Moenclaey [10]. Our long term goal is to show that these bicat​​egories are univalent.

Displayed bicategories. In this work, we develop the notion of *displayed bicategory* analogous to the 1-categorical notion of displayed category introduced in [2]. Intuitively, a displayed bicategory D over a bicategory B represents data and properties to be added to B to form a new bicategory: D gives rise to the *total bicategory* $\int D$. Its cells are pairs (b, d) where d in D is a “displayed cell” over b in B .

When a bicategory is built as the total bicategory $\int D$ of a displayed bicategory D over base B , univalence of $\int D$ can be shown from univalence of B and “displayed univalence” of D . The latter two conditions are easier to show, sometimes significantly easier.

Two features make the displayed point of view particularly useful: firstly, displayed structures can be *iterated*, making it possible to build bicategories of very complicated objects layerwise. Secondly, displayed “building blocks” can be provided, for which univalence is proved once and for all. These building blocks, *e.g.*, cartesian product, can be used like LEGOTM pieces to *modularly* build complicated bicategories that are automatically accompanied by a proof of univalence.

We demonstrate these features in examples, proving univalence of three complicated (classes of) bicategories: first, the bicategory of pseudofunctors between two univalent bicategories; second, bicategories of algebraic structures; and third, the bicategory of categories with families.

Formalization. The results presented here are mechanized in the **UniMath** library [22], which is based on the Coq proof assistant [17]. The **UniMath** library is under constant development; in this paper, we refer to the version with `git` hash `ab97d96`. Throughout the paper, definitions and statements are accompanied by a link to the online documentation of that version. For instance, the link `bicat` points to the definition of a bicategory.

Related work. Our work extends the notion of univalence from 1-categories [1] to bicategories. Similarly, we extend the notion of displayed 1-category [2] to the bicategorical setting.

Capriotti and Kraus [7] study univalent $(n, 1)$ -categories for $n \in \{0, 1, 2\}$. They only consider bicategories where the 2-cells are equalities between 1-cells; in particular, all 2-cells in [7] are invertible, and their $(2, 1)$ -categories are by definition locally univalent. Consequently, the condition called *univalence* by Capriotti and Kraus is what we call *global univalence*, cf. Definition 3.1, Item 2. In this work, we study bicategories, a.k.a. $(2, 2)$ -categories, that is, we allow for non-invertible 2-cells. The examples we study in Section 6 are proper $(2, 2)$ -categories and are not covered by [7].

Lafont, Hirschowitz, and Tabareau [15] are working on formalizing ω -categories in type theory. Their work is guided by work by Finster and Mimram [11], who develop a type theory for which the models (in set theory) are precisely weak ω -categories.

2 Bicategories and Some Examples

Bicategories were introduced by Bénabou [5] in 1967, encompassing monoidal categories, 2-categories (in particular, the 2-category of categories), and other examples. He (and later many other authors) defines bicategories in the style of “categories weakly enriched in categories”. That is, the homs $B_1(a, b)$ of a bicategory B are taken to be (1-)categories, and composition is given by a functor $B_1(a, b) \times B_1(b, c) \rightarrow B_1(a, c)$. This presentation of bicategories is concise and convenient for communication between mathematicians.

In this article, we use a different, more unfolded definition of bicategories, which is inspired by Bénabou [5, Section 1.3] and [18, Section “Details”]. It is more verbose than the definition via weak enrichment. However, it is better suited for our purposes, in particular, it is suitable for defining *displayed bicategories*, cf. Section 4.

► **Definition 2.1** (`bicat`). A **bicategory** \mathbf{B} consists of

1. of a type \mathbf{B}_0 of **objects**;
2. a type $\mathbf{B}_1(a, b)$ of **1-cells** for all $a, b : \mathbf{B}_0$;
3. a *set* $\mathbf{B}_2(f, g)$ of **2-cells** for all $a, b : \mathbf{B}_0$ and $f, g : \mathbf{B}_1(a, b)$;
4. an **identity 1-cell** $\text{id}_1(a) : \mathbf{B}_1(a, a)$;
5. a **composition** $\mathbf{B}_1(a, b) \times \mathbf{B}_1(b, c) \rightarrow \mathbf{B}_1(a, c)$, written $f \cdot g$;
6. an **identity 2-cell** $\text{id}_2(f) : \mathbf{B}_2(f, f)$;
7. a **vertical composition** $\theta \bullet \gamma : \mathbf{B}_2(f, h)$ for all 1-cells $f, g, h : \mathbf{B}_1(a, b)$ and 2-cells $\theta : \mathbf{B}_2(f, g)$ and $\gamma : \mathbf{B}_2(g, h)$;
8. a **left whiskering** $f \triangleleft \theta : \mathbf{B}_2(f \cdot g, f \cdot h)$ for all 1-cells $f : \mathbf{B}_1(a, b)$ and $g, h : \mathbf{B}_1(b, c)$ and 2-cells $\theta : \mathbf{B}_2(g, h)$;
9. a **right whiskering** $\theta \triangleright h : \mathbf{B}_2(f \cdot h, g \cdot h)$ for all 1-cells $f, g : \mathbf{B}_1(a, b)$ and $h : \mathbf{B}_1(b, c)$ and 2-cells $\theta : \mathbf{B}_2(f, g)$;
10. a **left unitor** $\lambda(f) : \mathbf{B}_2(\text{id}_1(a) \cdot f, f)$ and its inverse $\lambda(f)^{-1} : \mathbf{B}_2(f, \text{id}_1(a) \cdot f)$;
11. a **right unitor** $\rho(f) : \mathbf{B}_2(f \cdot \text{id}_1(b), f)$ and its inverse $\rho(f)^{-1} : \mathbf{B}_2(f, f \cdot \text{id}_1(b))$;
12. a **left associator** $\alpha(f, g, h) : \mathbf{B}_2(f \cdot (g \cdot h), (f \cdot g) \cdot h)$ and a **right associator** $\alpha(f, g, h)^{-1} : \mathbf{B}_2((f \cdot g) \cdot h, f \cdot (g \cdot h))$ for $f : \mathbf{B}_1(a, b)$, $g : \mathbf{B}_1(b, c)$, and $h : \mathbf{B}_1(c, d)$ such that, for all suitable objects, 1-cells, and 2-cells,
13. $\text{id}_2(f) \bullet \theta = \theta$, $\theta \bullet \text{id}_2(g) = \theta$, $\theta \bullet (\gamma \bullet \tau) = (\theta \bullet \gamma) \bullet \tau$;
14. $f \triangleleft (\text{id}_2 g) = \text{id}_2(f \cdot g)$, $f \triangleleft (\theta \bullet \gamma) = (f \triangleleft \theta) \bullet (f \triangleleft \gamma)$;
15. $(\text{id}_2 f) \triangleright g = \text{id}_2(f \cdot g)$, $(\theta \bullet \gamma) \triangleright g = (\theta \triangleright g) \bullet (\gamma \triangleright g)$;
16. $(\text{id}_1(a) \triangleleft \theta) \bullet \lambda(g) = \lambda(f) \bullet \theta$;
17. $(\theta \triangleright \text{id}_1(b)) \bullet \rho(g) = \rho(f) \bullet \theta$;
18. $(f \triangleleft (g \triangleleft \theta)) \bullet \alpha(f, g, i) = \alpha(f, g, h) \bullet ((f \cdot g) \triangleleft \theta)$;
19. $(f \triangleleft (\theta \triangleright i)) \bullet \alpha(f, h, i) = \alpha(f, g, i) \bullet ((f \triangleleft \theta) \triangleright i)$;
20. $(\theta \triangleright (h \cdot i)) \bullet \alpha(g, h, i) = \alpha(f, h, i) \bullet ((\theta \triangleright h) \triangleright i)$;
21. $\lambda(f) \bullet \lambda(f)^{-1} = \text{id}_2(\text{id}_1(a) \cdot f)$, $\lambda(f)^{-1} \bullet \lambda(f) = \text{id}_2(f)$;
22. $\rho(f) \bullet \rho(f)^{-1} = \text{id}_2(f \cdot \text{id}_1(b))$, $\rho(f)^{-1} \bullet \rho(f) = \text{id}_2(f)$;
23. $\alpha(f, g, h) \bullet \alpha(f, g, h)^{-1} = \text{id}_2(f \cdot (g \cdot h))$, $\alpha(f, g, h)^{-1} \bullet \alpha(f, g, h) = \text{id}_2((f \cdot g) \cdot h)$;
24. $\alpha(f, \text{id}_1(b), g) \bullet (\rho(f) \triangleright g) = f \triangleleft \lambda(f)$;
25. $\alpha(f, g, h \cdot i) \bullet \alpha(f \cdot g, h, i) = (f \triangleleft \alpha(g, h, i)) \bullet \alpha(f, g \cdot h, i) \bullet (\alpha(f, g, h) \triangleright i)$.

We write $a \rightarrow b$ for $\mathbf{B}_1(a, b)$ and $f \Rightarrow g$ for $\mathbf{B}_2(f, g)$. Riley formalized a definition of bicategories via weak enrichment in `UniMath`, based on work by Lumsdaine. These two definitions are equivalent.

► **Proposition 2.2.** *The definition of bicategories given in Definition 2.1 is equivalent to the formalized definition in terms of weak enrichment.*

This result is not formalized in our computer-checked library. However, as a sanity check for our definition of bicategory, we constructed maps between the two variations of bicategories, see `BicategoryOfBicat.v` and `BicatOfBicategory.v`.

Recall that our goal is to study univalence of bicategories, which is a property that relates equivalence and equality. For this reason, we study the two analogs of the 1-categorical notion of isomorphism. The first one is the notion of *invertible 2-cells*.

► **Definition 2.3** (`is_invertible_2cell`). A 2-cell $\theta : f \Rightarrow g$ is called **invertible** if we have $\gamma : g \Rightarrow f$ such that $\theta \bullet \gamma = \text{id}_2(f)$ and $\gamma \bullet \theta = \text{id}_2(g)$. An **invertible 2-cell** consists of a 2-cell and a proof that it is invertible, and `inv2cell(f, g)` is the type of invertible 2-cells from f to g .

Since inverses are unique, being an invertible 2-cell is a proposition. In addition, $\text{id}_2(f)$ is invertible, and we write $\text{id}_2(f) : \text{inv2cell}(f, f)$ for this invertible 2-cell. The second analog of isomorphisms is the notion of *adjoint equivalences*.

► **Definition 2.4** (`adjoint_equivalence`). An **adjoint equivalence structure** on a 1-cell $f : a \rightarrow b$ consists of a 1-cell $g : b \rightarrow a$ and invertible 2-cells $\eta : \text{id}_2(f) \Rightarrow f \cdot g$ and $\varepsilon : g \cdot f \Rightarrow \text{id}_2(g)$ together with paths

$$\begin{aligned} \rho(g)^{-1} \bullet (g \triangleleft \eta) \bullet \alpha(g, f, g)^{-1} \bullet (\varepsilon \triangleright g) \bullet \lambda(g) &= \text{id}_2(g), \\ \lambda(g)^{-1} \bullet (\eta \triangleright f) \bullet \alpha(f, g, f) \bullet (f \triangleleft \varepsilon) \bullet \rho(f) &= \text{id}_2(f). \end{aligned}$$

An **adjoint equivalence** consists of a map f together with an adjoint equivalence structure on f . The type $\text{AdjEquiv}(a, b)$ consists of all adjoint equivalences from a to b .

We call η and ε the unit and counit of the adjunction, and we call g the right adjoint. The prime example of an adjoint equivalence is the identity 1-cell $\text{id}_1(a)$ and we denote it by $\text{id}_1(a) : \text{AdjEquiv}(a, a)$. Sometimes, we write $a \simeq b$ for $\text{AdjEquiv}(a, b)$.

Before we continue our study of univalence, we present some examples of bicategories.

► **Example 2.5** (`fundamental_bigroupoid`). Let X be a 2-type. Then we define a bicategory whose 0-cells are inhabitants of X , 1-cells from x to y are paths $x = y$, and 2-cells from p to q are higher paths $p = q$. The operations are defined with path induction. Every 1-cell is an adjoint equivalence and every 2-cell is invertible.

► **Example 2.6** (`one_types`). Let U be a universe. The objects of the bicategory 1-Type_U of 1-types from U are 1-truncated types of the universe U , the 1-cells are functions between the underlying types, and the 2-cells are homotopies between functions. The 1-cells $\text{id}_1(X)$ and $f \cdot g$ are defined as the identity and composition of functions. The 2-cell $\text{id}_2(f)$ is refl , the 2-cell $p \bullet q$ is the concatenation of paths. The unitors and associators are defined as the identity path. Every 2-cell is invertible and adjoint equivalences between X and Y are the same as equivalences from X to Y .

► **Example 2.7** (`bicat_of_cats`). We define the bicategory Cat of univalent categories as the bicategory whose 0-cells are univalent categories, 1-cells are functors, and 2-cells are natural transformations. For the operations, we use the identity and composition of functors, and whiskering of functors and transformations. The internal invertible 2-cells are the natural isomorphisms of functors, and the internal adjoint equivalences correspond to external adjoint equivalences of categories.

3 Univalent Bicategories

Recall that a (1-)category C (called “precategory” in [1]) is called *univalent* if, for every two objects $a, b : C_0$, the canonical map $\text{idtoiso}_{a,b} : (a = b) \rightarrow \text{Iso}(a, b)$ from identities between a and b to isomorphisms between them is an equivalence. For bicategories, where we have one more layer of structure, univalence can be imposed both *locally* and *globally*.

► **Definition 3.1** (`Univalence.v`). Univalence for bicategories is defined as follows:

1. Let $a, b : B_0$ and $f, g : B_1(a, b)$ be objects and morphisms of B ; by path induction we define a map $\text{idtoiso}_{f,g}^{2,1} : f = g \rightarrow \text{inv2cell}(f, g)$ which sends $\text{refl}(f)$ to $\text{id}_2(f)$. A bicategory B is **locally univalent** if, for every two objects $a, b : B_0$ and two 1-cells $f, g : B_1(a, b)$, the map $\text{idtoiso}_{f,g}^{2,1}$ is an equivalence.

2. Let $a, b : B_0$ be objects of B ; using path induction we define $\text{idtoiso}_{a,b}^{2,0} : a = b \rightarrow \text{AdjEquiv}(a, b)$ sending $\text{refl}(a)$ to $\text{id}_1(a)$. A bicategory B is **globally univalent** if, for every two objects $a, b : B_0$, the canonical map $\text{idtoiso}_{a,b}^{2,0}$ is an equivalence.
3. (`is_univalent_2`) We say that B is **univalent** if B is both locally and globally univalent.

While right adjoints are only unique up to equivalence in general, they are unique up to identity when the bicategory is locally univalent:

► **Proposition 3.2** (`isaprop_left_adjoint_equivalence`). *Let B be locally univalent. Then having an adjoint equivalence structure on a 1-cell in B is a proposition.*

As a corollary of this proposition we get the following:

► **Theorem 3.3.** *In a univalent bicategory B , the type B_0 of 0-cells is a 2-type, and for any two objects $a, b : B_0$, the type $a \rightarrow b$ of 1-cells from a to b is a 1-type.*

To prove global univalence of a bicategory, we need to show that $\text{idtoiso}_{a,b}^{2,0}$ is an equivalence. Often we do that by constructing a map in the other direction and showing these two are inverses. This requires comparing adjoint equivalences, which is done with the help of Proposition 3.2.

Now let us prove the examples from Section 2 are univalent.

► **Example 3.4.** The following bicategories are univalent:

1. (`TwoType.v`, Example 2.5 cont'd) The fundamental bigroupoid of each 2-type is univalent.
2. (`OneTypes.v`, Example 2.6 cont'd) The bicategory of 1-types of a universe U is locally univalent; this is a consequence of function extensionality. If we assume the univalence axiom for U , then 1-types form a univalent bicategory.

It is more difficult to prove the bicategory of univalent categories is univalent, and we only give a brief sketch of this proof.

► **Proposition 3.5** (`BicatOfCats.v`, Example 2.7 cont'd). *The bicategory Cat is univalent.*

Local univalence follows from the fact that the functor category $[C, D]$ is univalent if D is. For global univalence, we use that the type of identities on categories is equivalent to the type of adjoint equivalences between categories [1, Theorem 6.17]. The proof proceeds by factoring $\text{idtoiso}^{2,0}$ as a chain of equivalences $(C = D) \xrightarrow{\sim} \text{CatIso}(C, D) \xrightarrow{\sim} \text{AdjEquiv}(C, D)$. To our knowledge, a proof of global univalence was first computer-formalized by Rafaël Bocquet¹.

In the previous examples, we proved univalence directly. However, in many complicated bicategories such proofs are not feasible. An example of such a bicategory is the bicategory $\text{Pseudo}(B, C)$ of pseudofunctors from B to C , pseudotransformations, and modifications [16] (for a univalent bicategory C). Even in the 1-categorical case, proving the univalence of the category $[C, D]$ of functors from C to D , and natural transformations between them, is tedious. In Section 5, we develop some machinery to prove the following theorem.

► **Theorem 3.6** (`psfunctor_bicat_is_univalent_2`). *If B is a (not necessarily univalent) bicategory and C is a univalent bicategory, then the bicategory $\text{Pseudo}(B, C)$ of pseudofunctors from B to C is univalent.*

¹ <https://github.com/mortberg/cubicaltt/blob/master/examples/category.ctt>

4 Displayed Bicategories

In this section, we introduce *displayed bicategories*, the bicategorical analog to the notion of displayed category developed in [2]. A displayed (1-)category D over a given (base) category C consists of a family of objects over objects in C and a family of morphisms over morphisms in C together with suitable displayed operations of composition and identity. A category $\int D$ is then constructed, the objects and morphisms of which are pairs of objects and morphisms from C and D , respectively. Properties of $\int D$, in particular univalence, can be shown from analogous, but simpler, conditions on C and D .

A prototypical example is the following displayed category over $C := \mathbf{Set}$: an object over a set X is a group structure on X , and a morphism over a function $f : X \rightarrow X'$ from group structure G (on X) to group structure G' (on X') is a proof of the fact that f is compatible with G and G' . The total category is the category of groups, and its univalence follows from univalence of \mathbf{Set} and a univalence property of the displayed data.

Just like in 1-category theory, many examples of bicategories are obtained by endowing previously considered bicategories with additional structure. An example is the bicategory of pointed 1-types in \mathbf{U} . The objects in this bicategory are pairs of a 1-type A and an inhabitant $a : A$. The morphisms are pairs of a morphism f of 1-types and a path witnessing that f preserves the selected points. Similarly, the 2-cells are pairs of a homotopy p and a proof that this p commutes with the point preservation proofs. Thus, this bicategory is obtained from $\mathbf{1-Type_U}$ by endowing the cells on each level with additional structure.

Of course, the structure should be added in such a way that we are guaranteed to obtain a bicategory at the end. Now let us give the formal definition of displayed bicategories.

► **Definition 4.1** (`disp_bicat`). Given a bicategory B , a **displayed bicategory D over B** is given by data analogous to that of a bicategory, to which the numbering refers:

1. for each $a : B_0$ a type D_a of displayed 0-cells over a ;
2. for each $f : a \rightarrow b$ in B and $\bar{a} : D_a, \bar{b} : D_b$ a type $\bar{a} \xrightarrow{f} \bar{b}$ of displayed 1-cells over f ;
3. for each $\theta : f \Rightarrow g$ in B , $\bar{f} : \bar{a} \xrightarrow{f} \bar{b}$ and $\bar{g} : \bar{a} \xrightarrow{g} \bar{b}$ a set $\bar{f} \xRightarrow{\theta} \bar{g}$ of displayed 2-cells over θ and dependent versions of operations and laws from Definition 2.1, which are
4. for each $a : B_0$ and $\bar{a} : D_0(a)$, we have $\text{id}_1(\bar{a}) : \bar{a} \xrightarrow{\text{id}_1(a)} \bar{a}$;
5. for all 1-cells $f : a \rightarrow b$, $g : b \rightarrow c$, and displayed 1-cells $\bar{f} : \bar{a} \xrightarrow{f} \bar{b}$ and $\bar{g} : \bar{b} \xrightarrow{g} \bar{c}$, we have a displayed 1-cell $\bar{f} \cdot \bar{g} : \bar{a} \xrightarrow{f \cdot g} \bar{c}$;
6. for all $f : B_1(a, b)$, $\bar{a} : D_0(a)$, $\bar{b} : D_0(b)$, and $\bar{f} : \bar{a} \xrightarrow{f} \bar{b}$, we have $\text{id}_2(\bar{f}) : \bar{f} \xRightarrow{\text{id}_2(f)} \bar{f}$;
7. for 2-cells $\theta : f \Rightarrow g$ and $\gamma : g \Rightarrow h$, and displayed 2-cells $\bar{\theta} : \bar{f} \xRightarrow{\theta} \bar{g}$ and $\bar{\gamma} : \bar{g} \xRightarrow{\gamma} \bar{h}$, we have a displayed 2-cell $\bar{\theta} \bullet \bar{\gamma} : \bar{f} \xRightarrow{\theta \bullet \gamma} \bar{h}$.
8. for each displayed 1-cell $\bar{f} : \bar{a} \xrightarrow{f} \bar{b}$ and each displayed 2-cell $\bar{g} \xRightarrow{\theta} \bar{h}$, we have a displayed 2-cell $\bar{f} \triangleleft \bar{\theta} : \bar{f} \cdot \bar{g} \xRightarrow{f \triangleleft \theta} \bar{f} \cdot \bar{h}$;
9. for each displayed 1-cell $\bar{h} : \bar{b} \xrightarrow{h} \bar{c}$ and each displayed 2-cell $\bar{\theta} : \bar{f} \xRightarrow{\theta} \bar{g}$, we have a displayed 2-cell $\bar{\theta} \triangleright \bar{h} : \bar{f} \cdot \bar{h} \xRightarrow{\theta \triangleright h} \bar{g} \cdot \bar{h}$;
10. for each $\bar{f} : \bar{a} \xrightarrow{f} \bar{b}$, we have displayed 2-cells $\lambda(\bar{f}) : \text{id}_1(\bar{a}) \cdot \bar{f} \xRightarrow{\lambda f} \bar{f}$ and $\lambda(\bar{f})^{-1} : \bar{f} \xRightarrow{\lambda f^{-1}} \text{id}_1(\bar{a}) \cdot \bar{f}$;
11. for each $\bar{f} : \bar{a} \xrightarrow{f} \bar{b}$, displayed 2-cells $\rho(\bar{f}) : \bar{f} \cdot \text{id}_1(\bar{b}) \xRightarrow{\rho f} \bar{f}$ and $\rho(\bar{f})^{-1} : \bar{f} \xRightarrow{\rho f^{-1}} \bar{f} \cdot \text{id}_1(\bar{b})$;
12. for each $\bar{f} : \bar{a} \xrightarrow{f} \bar{b}$, $\bar{g} : \bar{b} \xrightarrow{g} \bar{c}$, and $\bar{h} : \bar{c} \xrightarrow{h} \bar{d}$, we have displayed 2-cells $\alpha(\bar{f}, \bar{g}, \bar{h}) : \bar{f} \cdot (\bar{g} \cdot \bar{h}) \xRightarrow{\alpha(f, g, h)} (\bar{f} \cdot \bar{g}) \cdot \bar{h}$ and $\alpha(\bar{f}, \bar{g}, \bar{h})^{-1} : (\bar{f} \cdot \bar{g}) \cdot \bar{h} \xRightarrow{\alpha(f, g, h)} \bar{f} \cdot (\bar{g} \cdot \bar{h})$.

Note that we use the same notation for the displayed and the non-displayed operations.

These operations are subject to laws, which are derived systematically from the non-displayed version. Just as for displayed 1-categories, the laws of displayed bicategories are heterogeneous, because they are transported along the analogous law in the base bicategory. For instance, the displayed left-unitary law for identity reads as $\text{id}_2(\bar{f}) \bullet \bar{\theta} =_e \bar{\theta}$, where e is the corresponding identity of Item 13 in Definition 2.1.

13. $\text{id}_2(f) \bullet \theta =_* \theta, \quad \theta \bullet \text{id}_2(g) =_* \theta, \quad \theta \bullet (\gamma \bullet \tau) =_* (\theta \bullet \gamma) \bullet \tau;$
14. $f \triangleleft (\text{id}_2 g) =_* \text{id}_2(f \cdot g), \quad f \triangleleft (\theta \bullet \gamma) =_* (f \triangleleft \theta) \bullet (f \triangleleft \gamma);$
15. $(\text{id}_2 f) \triangleright g =_* \text{id}_2(f \cdot g), \quad (\theta \bullet \gamma) \triangleright g =_* (\theta \triangleright g) \bullet (\gamma \triangleright g);$
16. $(\text{id}_1(a) \triangleleft \theta) \bullet \lambda(g) =_* \lambda(f) \bullet \theta;$
17. $(\theta \triangleright \text{id}_1(b)) \bullet \rho(g) =_* \rho(f) \bullet \theta;$
18. $(f \triangleleft (g \triangleleft \theta)) \bullet \alpha(f, g, i) =_* \alpha(f, g, h) \bullet ((f \cdot g) \triangleleft \theta);$
19. $(f \triangleleft (\theta \triangleright i)) \bullet \alpha(f, h, i) =_* \alpha(f, g, i) \bullet ((f \triangleleft \theta) \triangleright i);$
20. $(\theta \triangleright (h \cdot i)) \bullet \alpha(g, h, i) =_* \alpha(f, h, i) \bullet ((\theta \triangleright h) \triangleright i);$
21. $\lambda(f) \bullet \lambda(f)^{-1} =_* \text{id}_2(\text{id}_1(a) \cdot f), \quad \lambda(f)^{-1} \bullet \lambda(f) =_* \text{id}_2(f);$
22. $\rho(f) \bullet \rho(f)^{-1} =_* \text{id}_2(f \cdot \text{id}_1(b)), \quad \rho(f)^{-1} \bullet \rho(f) =_* \text{id}_2(f);$
23. $\alpha(f, g, h) \bullet \alpha(f, g, h)^{-1} =_* \text{id}_2(f \cdot (g \cdot h)), \quad \alpha(f, g, h)^{-1} \bullet \alpha(f, g, h) =_* \text{id}_2((f \cdot g) \cdot h);$
24. $\alpha(f, \text{id}_1(b), g) \bullet (\rho(f) \triangleright g) =_* f \triangleleft \lambda(f);$
25. $\alpha(f, g, h \cdot i) \bullet \alpha(f \cdot g, h, i) =_* (f \triangleleft \alpha(g, h, i)) \bullet \alpha(f, g \cdot h, i) \bullet (\alpha(f, g, h) \triangleright i).$

The purpose of displayed bicategories is to give rise to a total bicategory together with a projection pseudofunctor. They are defined as follows:

► **Definition 4.2** (`total_bicat`). Given a displayed bicategory D over a bicategory B , we can form a **total bicategory** $\int D$ (or $\int_B D$) which has:

1. as 0-cells tuples (a, \bar{a}) , where $a : B$ and $\bar{a} : D_a$;
2. as 1-cells tuples $(f, \bar{f}) : (a, \bar{a}) \rightarrow (b, \bar{b})$, where $f : a \rightarrow b$ and $\bar{f} : \bar{a} \xrightarrow{f} \bar{b}$;
3. as 2-cells tuples $(\theta, \bar{\theta}) : (f, \bar{f}) \Rightarrow (g, \bar{g})$, where $\theta : f \Rightarrow g$ and $\bar{\theta} : \bar{f} \xRightarrow{\theta} \bar{g}$.

We also have a **projection pseudofunctor** $\pi_D : \text{Pseudo}(\int D, B)$.

As mentioned before, the bicategory of pointed 1-types is the total bicategory of the following displayed bicategory.

► **Example 4.3** (`p1types_disp`, Example 3.4, Item 2 cont'd). Given a universe U , we build a displayed bicategory of pointed 1-types over the base bicategory of 1-types in U (Example 2.6).

- For 1-type A in U , the objects over A are inhabitants of A .
- For $f : A \rightarrow B$ with A, B 1-types in U , the maps over f from a to b are paths $f(a) = b$.
- Given two maps $f, g : A \rightarrow B$, a path $p : f = g$, two points $a : A$ and $b : B$, and paths $q_f : f(a) = b$ and $q_g : g(a) = b$, the 2-cells over p are paths $\text{transport}^{x \mapsto x=b}(p, q_f) = q_g$.

The bicategory of pointed 1-types is the total bicategory of this displayed bicategory.

► **Example 4.4** (`disp_fullsubbicat`). We can select the 0-cells of a bicategory B by attaching a property $P : B \rightarrow \mathbf{hProp}$. Define a displayed bicategory D over B such that $D_x := P(x)$, and the types of displayed 1-cells and 2-cells are the unit type. Now we define the full subbicategory of B with cells satisfying P to be the total bicategory of D .

We end this section with several general constructions of displayed bicategories.

► **Definition 4.5** (Various constructions of displayed bicategories).

1. (`disp_dirprod_bicat`) Given displayed bicategories D_1 and D_2 over a bicategory B , we construct the product $D_1 \times D_2$ over B . The 0-cells, 1-cells, and 2-cells are pairs of 0-cells, 1-cells, and 2-cells respectively.

2. (**sigma_bicat**) Given a displayed bicategory D over a base B and a displayed bicategory E over $\int D$, we construct a displayed bicategory $\sum_D E$ over B as follows. The objects over $a : B$ are pairs (\bar{a}, e) , where $\bar{a} : D_a$ and $e : E_{(a, \bar{a})}$, the morphisms over $f : a \rightarrow b$ from (\bar{a}, e) to (\bar{b}, e') are pairs (\bar{f}, φ) , where $\bar{f} : \bar{a} \xrightarrow{f} \bar{b}$ and $\varphi : e \xrightarrow{(f, \bar{f})} e'$, and similarly for 2-cells.
3. (**trivial_displayed_bicat**) Every bicategory D is, in a trivial way, a displayed bicategory over any other bicategory B . Its total bicategory is the direct product $B \times D$.
4. (**disp_cell_unit_bicat**) We say a displayed bicategory D over B is **chaotic** if, for each $\alpha : f \Rightarrow g$ and $\bar{f} : \bar{a} \xrightarrow{f} \bar{b}$ and $\bar{g} : \bar{a} \xrightarrow{g} \bar{b}$, the type $\bar{f} \xRightarrow{\alpha} \bar{g}$ is contractible. Let B be a bicategory and suppose we have
 - a type D_0 and a type family D_1 on B as in Definition 4.1;
 - displayed 1-identities id_1 and compositions (\cdot) of displayed 1-cells as Definition 4.1.
 Then we have an associated **chaotic displayed bicategory** $\hat{D}(D_0, D_1, \text{id}_1, (\cdot))$ over B by stipulating that the types of 2-cells are the unit type.

5 Displayed univalence

Given a bicategory B and a displayed bicategory D on B , our goal is to prove the univalence of $\int D$ from conditions on B and D . For that, we develop the notion of *univalent displayed bicategories*. We start by defining displayed versions of invertible 2-cells.

► **Definition 5.1** (**is_disp_invertible_2cell**). Given are a bicategory B and a displayed bicategory D over B . Suppose we have objects $a, b : B_0$, two 1-cells $f, g : B_1(a, b)$, and an invertible 2-cell $\theta : B_2(f, g)$. Suppose that we also have $\bar{a} : D_0(a)$, $\bar{b} : D_0(b)$, $\bar{f} : \bar{a} \xrightarrow{f} \bar{b}$, $\bar{g} : \bar{a} \xrightarrow{g} \bar{b}$, and $\bar{\theta} : \bar{f} \xRightarrow{\theta} \bar{g}$. Then we say $\bar{\theta}$ is **invertible** if we have $\bar{\gamma} : \bar{g} \xRightarrow{\theta^{-1}} \bar{f}$ such that $\bar{\theta} \bullet \bar{\gamma}$ and $\bar{\gamma} \bullet \bar{\theta}$ are identities modulo transport over the corresponding identity laws of θ .

A **displayed invertible 2-cell over θ** , where θ is an invertible 2-cell, is a pair of a displayed 2-cell $\bar{\theta}$ over θ and a proof that $\bar{\theta}$ is invertible. The type of displayed invertible 2-cells from \bar{f} to \bar{g} over θ is denoted by $\bar{f} \cong_{\theta} \bar{g}$.

Being a displayed invertible 2-cell is a proposition and the displayed 2-cell $\text{id}_2(\bar{f})$ over $\text{id}_2(f)$ is invertible. Next we define displayed adjoint equivalences.

► **Definition 5.2** (**disp_left_adjoint_equivalence**). Given are a bicategory B and a displayed bicategory D over B . Suppose we have objects $a, b : B_0$ and a 1-cell $f : B_1(a, b)$ together with an adjoint equivalence structure A on f . We write r, η, ε for the right adjoint, unit, and counit of f respectively. Furthermore, suppose that we have $\bar{a} : D_0(a)$, $\bar{b} : D_0(b)$, and $\bar{f} : \bar{a} \xrightarrow{f} \bar{b}$. A **displayed adjoint equivalence structure** on \bar{f} consists of

- A displayed 1-cell $\bar{r} : \bar{b} \xrightarrow{r} \bar{a}$;
- An invertible displayed 2-cell $\text{id}_1(\bar{a}) \xRightarrow{\eta} \bar{f} \cdot \bar{r}$;
- An invertible displayed 2-cell $\bar{r} \cdot \bar{f} \xRightarrow{\varepsilon} \text{id}_1(\bar{b})$.

In addition, two laws reminiscent of those in Definition 2.4 need to be satisfied.

A **displayed adjoint equivalence** over the adjoint equivalence A is a pair of a displayed 1-cell \bar{f} over f together with a displayed adjoint equivalence structure on \bar{f} . The type of displayed adjoint equivalences from \bar{a} to \bar{b} over f is denoted by $\bar{a} \simeq_f \bar{b}$.

The displayed 1-cell $\text{id}_1(\bar{a})$ is a displayed adjoint equivalence over $\text{id}_1(a)$.

Using these definitions, we define univalence of displayed bicategories similarly to univalence for ordinary bicategories. Again we separate it in a local and global condition.

► **Definition 5.3** (`DispUnivalence.v`). Let D be a displayed bicategory over B .

1. Let $a, b : B$, and $\bar{a} : D_a, \bar{b} : D_b$. Let $f, g : a \rightarrow b$, let $p : f = g$, and let \bar{f} and \bar{g} be displayed morphisms over f and g respectively. Then we define a function

$$\text{disp_idtoiso}_{p, \bar{f}, \bar{g}}^{2,1} : \bar{f} =_p \bar{g} \rightarrow \bar{f} \simeq_{\text{idtoiso}_{f,g}^{2,1}(p)} \bar{g}$$

sending `refl` to the identity displayed isomorphism. We say that D is **locally univalent** if the map $\text{disp_idtoiso}_{p, \bar{f}, \bar{g}}^{2,1}$ is an equivalence for each p , \bar{f} , and \bar{g} .

2. Let $a, b : B$, and $\bar{a} : D_a, \bar{b} : D_b$. Given $p : a = b$, we define a function

$$\text{disp_idtoiso}_{p, \bar{a}, \bar{b}}^{2,0} : \bar{a} =_p \bar{b} \rightarrow \bar{a} \simeq_{\text{idtoiso}_{a,b}^{2,0}(p)} \bar{b}$$

sending `refl` to the identity displayed adjoint equivalence. We say that D is **globally univalent** if the map $\text{disp_idtoiso}_{p, \bar{a}, \bar{b}}^{2,0}$ is an equivalence for each p , \bar{a} , and \bar{b} .

3. (`disp_univalent_2`) We call D **univalent** if it is both locally and globally univalent.

Now we give the main theorem of this paper. It says that the total bicategory $\int_B D$ is univalent if B and D are.

► **Theorem 5.4** (`total_is_univalent_2`). *Let B be a bicategory and let D be a displayed bicategory over B . Then*

1. $\int D$ is locally univalent if B is locally univalent and D is locally univalent;
2. $\int D$ is globally univalent if B is globally univalent and D is globally univalent.

Proof. The main idea behind the proof is to characterize invertible 2-cells in the total bicategory as pairs of an invertible 2-cell p in the base bicategory, and a displayed invertible 2-cell over p . Concretely, for the local univalence, we factor $\text{idtoiso}^{2,1}$ as a composition of the following equivalences:

$$\begin{array}{ccc} (f, \bar{f}) = (g, \bar{g}) & \xrightarrow{\text{idtoiso}^{2,1}} & \text{inv2cell}((f, \bar{f}), (g, \bar{g})) \\ w_1 \downarrow \sim & & \sim \uparrow w_3 \\ \sum_{(p:f=g)} \bar{f} =_p \bar{g} & \xrightarrow{\sim_{w_2}} & \sum_{(p:\text{inv2cell}(f,g))} \bar{f} \simeq_p \bar{g} \end{array}$$

The map w_1 is just a characterization of paths in a sigma type. The map w_2 turns equalities into (displayed) invertible 2-cells, and it is an equivalence by local univalence of B and displayed local univalence of D . Finally, the map w_3 characterizes invertible 2-cells in the total bicategory.

The proof is similar in the case of global univalence. The most important step is the characterization of adjoint equivalences in the total bicategory.

$$(a, \bar{a}) \simeq (b, \bar{b}) \xrightarrow{\sim} \sum_{(p:a \simeq b)} \bar{a} \simeq_p \bar{b}. \quad \blacktriangleleft$$

To check displayed univalence, it suffices to prove the condition in the case where p is reflexivity. This step, done by path induction, simplifies some proofs of displayed univalence.

► **Proposition 5.5.** *Given a displayed bicategory D over B , then D is univalent if the following maps are equivalences:*

■ (`fiberwise_local_univalent_is_univalent_2_1`)

$$\text{disp_idtoiso}_{\text{refl}(f), \bar{f}, \bar{f}'}^{2,1} : \bar{f} = \bar{f}' \rightarrow \bar{f} \simeq_{\text{id}_2(f)} \bar{f}'$$

■ $(\text{fiberwise_univalent_2_0_to_disp_univalent_2_0})$

$$\text{disp_idtoiso}_{\text{refl}(a), \bar{a}, \bar{a}'}^{2,0} : \bar{a} = \bar{a}' \rightarrow \bar{a} \simeq_{\text{id}_1(a)} \bar{a}'$$

Now we establish the univalence of several examples.

► **Example 5.6.** The following bicategories and displayed bicategories are univalent:

1. The category of pointed 1-types (see Example 4.3) is univalent (`p1types_univalent_2`).
2. The full subcategory (see Example 4.4) of a univalent bicategory is univalent (`is_univalent_2_fullsubbicat`).
3. The product of univalent displayed bicategories (Definition 4.5, Item 1) is univalent (`is_univalent_2_dirprod_bicat`).
4. Given univalent displayed bicategories D_1 and D_2 on B and $\int D_1$ respectively, the displayed bicategory $\sum_{D_1} D_2$ (Definition 4.5, Item 2) is univalent (`sigma_is_univalent_2`).

Lastly, we give a condition for when the chaotic displayed bicategory is univalent.

► **Proposition 5.7** (`disp_cell_unit_bicat_univalent_2`). *Let B be a univalent bicategory. Given $D = (D_0, D_1, \text{id}_1, (\cdot))$ as in Definition 4.5, Item 4, such that D_0 is a set and D_1 is a family of propositions. Then the chaotic displayed bicategory on D is univalent if we have a map in the opposite direction of $\text{disp_idtoiso}^{2,0}$.*

6 Univalence of Complicated Bicategories

In this section, we demonstrate the power of displayed bicategories on a number of complicated examples. We show the univalence of the bicategory of pseudofunctors between univalent bicategories and of univalent categories with families. In addition, we give two constructions to define univalent bicategories of algebras.

6.1 Pseudofunctors

As promised, we use displayed bicategories to prove Theorem 3.6. For the remainder, fix bicategories B and C such that C is univalent. Recall that a pseudofunctor consists of an action on 0-cells, 1-cells, 2-cells, a family of 2-cells witnessing the preservation of composition and identity 1-cells, such that a number of laws are satisfied. We call the 2-cells witnessing the preservation of composition and identity the *compositor* and *identitor* respectively.

To construct $\text{Pseudo}(B, C)$, we add structure to a base bicategory in several layers. This base bicategory consists of functions from B_0 to C_0 . Each layer is given by a displayed bicategory on the total bicategory of the preceding layer. We start by defining a displayed bicategory of actions on 1-cells. On its total bicategory, we define three displayed bicategories: one for the preservation of composition, one for the preservation of identities, and one for the action on 2-cells. We take the product of these three and we finish by taking a full subcategory with the required laws. To show the resulting bicategory is univalent, we show the base and each layer is univalent.

Now let us look at the formal definitions.

► **Definition 6.1** (`ps_base`). The bicategory $\text{Base}(B, C)$ is defined as follows.

- The objects are maps $B_0 \rightarrow C_0$;
- The 1-cells from F_0 to G_0 are maps $\eta_0, \beta_0 : \prod_{(x:B_0)} F_0(x) \rightarrow G_0(x)$;
- The 2-cells from η_0 to β_0 are maps $\Gamma : \prod_{(x:B_0)} \eta_0(x) \Rightarrow \beta_0(x)$.

The operations are defined pointwise.

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Next we define a displayed bicategory on $\mathbf{Base}(\mathbf{B}, \mathbf{C})$. The displayed 0-cells are actions of pseudofunctors on 1-cells. The displayed 1-cells over η_0 are 2-cells witnessing the naturality of η_0 . The displayed 2-cells over Γ are equalities which show that Γ is a modification.

► **Definition 6.2** (`map1cells_disp_bicat`). We define a displayed bicategory $\mathbf{Map1D}(\mathbf{B}, \mathbf{C})$ on $\mathbf{Base}(\mathbf{B}, \mathbf{C})$ such that

- the objects over $F_0 : \mathbf{B}_0 \rightarrow \mathbf{C}_0$ are maps

$$F_1 : \prod_{(X, Y : \mathbf{B}_0)} \mathbf{B}_1(X, Y) \rightarrow \mathbf{C}_1(F_0(X), F_0(Y));$$

- the 1-cells over $\eta_0 : F_0(x) \rightarrow G_0(x)$ from F_1 to G_1 are invertible 2-cells

$$\eta_1 : \prod_{(X, Y : \mathbf{B}_0)(f : X \rightarrow Y)} \eta_0(X) \cdot G_1(f) \Rightarrow F_1(f) \cdot \eta_0(Y);$$

- the 2-cells over $\Gamma : \eta_0(x) \Rightarrow \beta_0(x)$ from η_1 to β_1 are equalities

$$\prod_{(X, Y : \mathbf{B}_0)(f : X \rightarrow Y)} \eta_1(f) \bullet (F_1(f) \triangleleft \Gamma(Y)) = (\Gamma(X) \triangleright G_1(f)) \bullet \beta_1(f).$$

We denote the total bicategory of $\mathbf{Map1D}(\mathbf{B}, \mathbf{C})$ by $\mathbf{Map1}(\mathbf{B}, \mathbf{C})$. Now we define three displayed bicategories over $\mathbf{Map1}(\mathbf{B}, \mathbf{C})$. Each of them is defined as a chaotic displayed bicategory (Item 4 in Definition 4.5).

► **Definition 6.3** (`identitor_disp_cat`). We define a displayed bicategory $\mathbf{MapId}(\mathbf{B}, \mathbf{C})$ over $\mathbf{Map1}(\mathbf{B}, \mathbf{C})$ as follows:

- The objects over (F_0, F_1) are identitors

$$F_i : \prod_{(X : \mathbf{B}_0)} \text{id}_1(F_0(X)) \Rightarrow F_1(\text{id}_1(X));$$

- The morphisms over (η_0, η_1) from F_i to G_i are equalities

$$\rho(\eta_0(X)) \bullet \lambda(\eta_0(X))^{-1} \bullet (F_i(X) \triangleright \eta_0(X)) = (\eta_0(X) \triangleleft G_i(X)) \bullet \eta_1(\text{id}_1(X)).$$

► **Definition 6.4** (`compositor_disp_cat`). We define a displayed bicategory $\mathbf{MapC}(\mathbf{B}, \mathbf{C})$ over $\mathbf{Map1}(\mathbf{B}, \mathbf{C})$ as follows:

- The objects over (F_0, F_1) are compositors

$$F_c : \prod_{(X, Y, Z : \mathbf{B}_0)(f : \mathbf{B}_1(X, Y))(g : \mathbf{B}_1(Y, Z))} F_1(f) \cdot F_1(g) \Rightarrow F_1(f \cdot g);$$

- The morphisms over (η_0, η_1) from F_c to G_c consists of equalities

$$\alpha \bullet (\eta_1(f) \triangleright G_1(g)) \bullet \alpha^{-1} \bullet (F_1(f) \triangleleft \eta_1(g)) \bullet \alpha \bullet (F_c \triangleright \eta_0(Z)) = (\eta_0(X) \triangleleft G_c) \bullet \eta_1(f \cdot g)$$

for all $X, Y, Z : \mathbf{B}_0$, $f : \mathbf{B}_1(X, Y)$ and $g : \mathbf{B}_1(Y, Z)$.

► **Definition 6.5** (`map2cells_disp_cat`). We define a displayed bicategory $\mathbf{Map2D}(\mathbf{B}, \mathbf{C})$ over $\mathbf{Map1}(\mathbf{B}, \mathbf{C})$ as follows:

- The objects over (F_0, F_1) are

$$F_2 : \prod_{(a, b : \mathbf{B}_0)(f, g : a \rightarrow b)} \mathbf{B}_2(f, g) \rightarrow F_1(f) \Rightarrow F_1(g);$$

- The morphisms over (η_0, η_1) from F_2 to G_2 consist of equalities

$$\prod_{(\theta: f \Rightarrow g)} (\eta_0(X) \triangleleft G_2(\theta)) \bullet \eta_1(g) = \eta_1(f) \bullet (F_2(\theta) \triangleright \eta_0(Y)).$$

We denote the total category of the product of $\text{Map2D}(\mathbf{B}, \mathbf{C})$, $\text{MapId}(\mathbf{B}, \mathbf{C})$, and $\text{MapC}(\mathbf{B}, \mathbf{C})$ by $\text{RawPseudo}(\mathbf{B}, \mathbf{C})$. Note that its objects are of the form $((F_0, F_1), (F_2, F_i, F_c))$, its 1-cells are pseudotransformations, and its 2-cells are modifications. However, its objects are not yet pseudofunctors, because they also need to satisfy several laws.

► **Definition 6.6** (`psfunctor_bicat`). We define the bicategory $\text{Pseudo}(\mathbf{B}, \mathbf{C})$ as the full subcategory of $\text{RawPseudo}(\mathbf{B}, \mathbf{C})$ where the objects satisfy the following laws

- $F_2(\text{id}_2(f)) = \text{id}_2(F_1(f))$ and $F_2(f \bullet g) = F_2(f) \bullet F_2(g)$;
- $\lambda(F_1(f)) = (F_i(a) \triangleright F_1(f)) \bullet F_c(\text{id}_1(a), f) \bullet F_2(\lambda(f))$;
- $\rho(F_1(f)) = (F_1(f) \triangleleft F_i(b)) \bullet F_c(f, \text{id}_1(b)) \bullet F_2(\rho(f))$;
- $(F_1(f) \bullet F_c(g, h)) \bullet F_c(f, g \cdot h) \bullet F_2(\alpha) = \alpha \bullet (F_c(f, g) \triangleright F_1(h)) \bullet F_c(f \cdot g, h)$;
- $F_c(f, g_1) \bullet F_2(f \triangleleft \theta) = (F_1(f) \triangleleft F_2(\theta)) \bullet F_c(f, g_2)$;
- $F_c(f_1, g) \bullet F_2(\theta \triangleright g) = (F_2(\theta) \triangleright F_1(g)) \bullet F_c(f_2, g)$;
- $F_i(X)$ and $F_c(f, g)$ are invertible 2-cells.

Each displayed layer in this construction is univalent. In addition, if \mathbf{C} is univalent, then so is $\text{Base}(\mathbf{B}, \mathbf{C})$. Hence, Theorem 3.6 follows from repeated application of Theorem 5.4.

6.2 Algebraic Examples

Next, we consider two constructions to build bicategories of algebras. To illustrate their usage, we show how to define the bicategory of monads internal to a bicategory. Note that each monad has a 0-cell X and a 1-cell $X \rightarrow X$. This structure is encapsulated by *algebras of a pseudofunctor* [6].

► **Definition 6.7** (`disp_alg_bicat`). Let \mathbf{B} be a bicategory and let $F : \text{Pseudo}(\mathbf{B}, \mathbf{B})$ be a pseudofunctor. We define a displayed bicategory $\text{Alg}_D(F)$.

- The objects over $a : \mathbf{B}$ are 1-cells $F(a) \rightarrow a$.
- The 1-cells over $f : \mathbf{B}_1(a, b)$ from $h_a : F(a) \rightarrow a$ to $h_b : F(b) \rightarrow b$ are invertible 2-cells $h_a \cdot f \Rightarrow F_1(f) \cdot h_b$.
- Given $f, g : \mathbf{B}_1(a, b)$, algebras $h_a : F(a) \rightarrow a$ and $h_b : F(b) \rightarrow b$, and h_f and h_g over f and g respectively, a 2-cell over $\theta : f \Rightarrow g$ is an equality

$$(h_a \triangleleft \theta) \bullet h_g = h_f \bullet (F_2(\theta) \triangleright h_b).$$

We write $\text{Alg}(F)$ for the total category of $\text{Alg}_D(F)$.

► **Theorem 6.8** (`bicat_algebra_is_univalent_2`). Let \mathbf{B} be a bicategory and let $F : \text{Pseudo}(\mathbf{B}, \mathbf{B})$ be a pseudofunctor. If \mathbf{B} is univalent, then so is $\text{Alg}(F)$.

► **Example 6.9** (Example 4.3 cont'd). The bicategory of pointed 1-types is the bicategory of algebras for the constant pseudofunctor $F(a) = 1$.

Define \mathbf{M}_1 to be $\text{Alg}(\text{id}_0(\mathbf{B}))$. Objects of \mathbf{M}_1 consist of an $X : \mathbf{B}_0$ and a 1-cell $X \rightarrow X$. These are not monads yet, because those are supposed to also have two 2-cells: the unit and multiplication. To add this structure, we define two displayed bicategories on \mathbf{M}_1 . Both are defined via a more general construction, for which we use that there is an identity pseudofunctor $\text{id}_0(\mathbf{B}) : \text{Pseudo}(\mathbf{B}, \mathbf{B})$ and that for all $F_1 : \text{Pseudo}(\mathbf{B}_1, \mathbf{B}_2)$ and $F_2 : \text{Pseudo}(\mathbf{B}_2, \mathbf{B}_3)$, we have a composition $F_1 \cdot F_2 : \text{Pseudo}(\mathbf{B}_1, \mathbf{B}_3)$.

Before giving this construction, let us describe the setting. Suppose that we have a displayed bicategory D over some B . Our goal is to define a displayed bicategory over $\int D$ where the displayed 0-cells are certain 2-cells in B . We define the endpoints of these as natural 1-cells, so we use pseudotransformations. The source of these is $\pi_D \cdot S$ for some $S : \text{Pseudo}(B, B)$, and the target is defined to be $\pi_D \cdot \text{id}_0(B)$ where π_D is the projection from $\int D$ to B . Note that instead of π_D , we use $\pi_D \cdot \text{id}_0(B)$, which is symmetric to the source $\pi_D \cdot S$. This allows us to construct such transformations by composing them. In addition, pseudotransformations $l, r : \pi_D \cdot S \rightarrow \pi_D \cdot \text{id}_0(B)$ give, for each $(a, h_a) : \int D$, a 1-cell $l(a) : B_1(S(a), a)$. The construction adds 2-cells from $l(a)$ to $r(a)$ and formally, we define the following displayed bicategory.

► **Definition 6.10** (`add_cell_disp_cat`). Suppose that D is a displayed bicategory over B and given are $S : \text{Pseudo}(B, B)$ and $l, r : \pi_D \cdot S \rightarrow \pi_D \cdot \text{id}_0(B)$. We use Item 4 in Definition 4.5 to define a displayed bicategory $\text{Add2Cell}(D, l, r)$ over $\int D$.

- Its objects over a are 2-cells $l(a) \Rightarrow r(a)$.
- The morphisms over $f : B_1(a, b)$ from η_1 to η_2 are equalities

$$(\eta_1 \triangleright \pi_D(f)) \bullet r(f) = l(f) \bullet (S_1(\pi_D(f)) \triangleleft \eta_2).$$

► **Theorem 6.11.** *The displayed bicategory $\text{Add2Cell}(D, l, r)$ is locally univalent (`add_cell_disp_cat_univalent_2_1`). Moreover, if C is locally univalent and D is locally univalent, then $\text{Add2Cell}(D, l, r)$ is globally univalent (`add_cell_disp_cat_univalent_2_0`).*

Let us show how to add the unit and multiplication to the structure. For that, we first need the following pseudotransformation.

► **Definition 6.12** (`alg_map`). Let B be a bicategory, and let $F : \text{Pseudo}(B, B)$ be pseudo-functors. We define a pseudotransformation $h : \pi_{\text{Alg}_D F} \cdot F \rightarrow \pi_{\text{Alg}_D F} \cdot \text{id}_0(B)$. On objects (a, h_a) , we define $h(a, h_a) = h_a$ and on 1-cells (f, h_f) , we define $h(f, h_f) = h_f$.

To add the unit to the structure, we use Definition 6.10. For S , we take id_0 , for l we take the identity transformation on $\pi_{\text{id}_0} \cdot \text{id}_0$ and for r we take h . The multiplication is done similarly, but instead we take $h \cdot h$ for l .

Let M_2 be the total bicategory of the product of these two displayed bicategories. To obtain actual monads, the structure needs to satisfy three laws, namely

- $\lambda(f)^{-1} \bullet (\eta \triangleright f) \bullet \mu = \text{id}_2(f)$;
- $\rho(f)^{-1} \bullet (f \triangleleft \eta) \bullet \mu = \text{id}_2(f)$;
- $(f \triangleleft \mu) \bullet \mu = \alpha(f, f, f) \bullet (\mu \triangleright f) \bullet \mu$.

We define $M(B)$ to be the full subcategory of M_2 with respect to these laws. From Theorems 6.8 and 6.11 and Example 5.6 we conclude:

► **Theorem 6.13** (`monad_is_univalent_2`). *If B is univalent, then so is $M(B)$.*

6.3 Categories with Families

Finally, we discuss the last example: the bicategory of (univalent) categories with families (CwFs) [9]. We follow the formulation by Fiore [12] and Awodey [4], which is already formalized in `UniMath` [3]: a CwF consists of a category C , two presheaves Ty and Tm on C , a morphism $p : \text{Tm} \rightarrow \text{Ty}$, and a representation structure for p .

However, rather than defining CwFs in one step, we use a stratified construction yielding the sought bicategory as the total bicategory of iterated displayed layers. The base bicategory is Cat (cf. Example 2.7). The second layer of data consists of two presheaves, each described by the following construction.

► **Definition 6.14** (`disp_presheaf_bicat`). Define the displayed bicategory \mathbf{PShD} on \mathbf{Cat} :

- The objects over C are functors from C^{op} to the univalent category \mathbf{Set} ;
- The 1-cells from $T : C \rightarrow \mathbf{Set}$ to $T' : D \rightarrow \mathbf{Set}$ over $F : C \rightarrow D$ are natural transformations from T to $F^{\text{op}} \cdot T'$;
- The 2-cells from $\beta : T \Rightarrow F^{\text{op}} \cdot T'$ to $\beta' : T \Rightarrow G^{\text{op}} \cdot T'$ over $\gamma : F \Rightarrow G$ are equalities

$$\beta = \beta' \bullet (\gamma^{\text{op}} \triangleright T').$$

Denote by \mathbf{CwF}_1 the total category of the product of \mathbf{PShD} with itself. An object in \mathbf{CwF}_1 consists of a category C and two presheaves $\mathbf{Ty}, \mathbf{Tm} : C^{\text{op}} \rightarrow \mathbf{Set}$. The third piece of data is a natural transformation between them.

► **Definition 6.15** (`morphisms_of_presheaves_display`). We define a displayed bicategory \mathbf{dCwF}_2 on \mathbf{CwF}_1 as the chaotic displayed bicategory (Item 4 in Definition 4.5) such that

- The objects over $(C, (\mathbf{Ty}, \mathbf{Tm}))$ are natural transformations from \mathbf{Ty} to \mathbf{Tm} .
- Suppose we have two objects $(C, (\mathbf{Ty}, \mathbf{Tm}))$ and $(C', (\mathbf{Ty}', \mathbf{Tm}'))$, two natural transformations $p : \mathbf{Tm} \Rightarrow \mathbf{Ty}$ and $p' : \mathbf{Tm}' \Rightarrow \mathbf{Ty}'$, and suppose we have a 1-cell f from $(C, (\mathbf{Ty}, \mathbf{Tm}))$ to $(C', (\mathbf{Ty}', \mathbf{Tm}'))$. Note that f consists of a functor $F : C \rightarrow C'$ and two transformations $\beta : \mathbf{Ty} \Rightarrow F^{\text{op}} \circ \mathbf{Ty}'$ and $\beta' : \mathbf{Tm} \Rightarrow F^{\text{op}} \circ \mathbf{Tm}'$. Then a 1-cell over f is an equality

$$p \bullet \beta = \beta' \bullet (F^{\text{op}} \triangleleft p').$$

With \mathbf{dCwF}_2 and the sigma construction from Item 2 in Definition 4.5, we get a displayed bicategory over \mathbf{Cat} and we denote its total bicategory by \mathbf{CwF}_2 . As the last piece of data, we add the representation structure for the morphism p of presheaves.

► **Definition 6.16** (`cwf_representation`). Given a category C together with functors $\mathbf{Ty}, \mathbf{Tm} : C^{\text{op}} \rightarrow \mathbf{Set}$ and a natural transformation $p : \mathbf{Tm} \Rightarrow \mathbf{Ty}$, we say $\text{isCwF}(C, \mathbf{Ty}, \mathbf{Tm}, p)$ if for each $\Gamma : C$ and $A : \mathbf{Ty}(\Gamma)$, we have a representation of the fiber of p over A .

A detailed definition can be found in [3, Definition 3.1]. Since C is univalent, the type $\text{isCwF}(C, \mathbf{Ty}, \mathbf{Tm}, p)$ is a proposition, and thus we define \mathbf{CwF} as a full subcategory of \mathbf{CwF}_2 .

► **Proposition 6.17** ([3, Lemma 4.3], `isaprop_cwf_representation`). $\text{isCwF}(C, \mathbf{Ty}, \mathbf{Tm}, p)$ is a proposition.

► **Definition 6.18** (`cwf`). We define \mathbf{CwF} as the full subcategory of \mathbf{dCwF}_2 with isCwF .

► **Theorem 6.19** (`cwf_is_univalent_2`). \mathbf{CwF} is univalent.

7 Conclusions and Further Work

In the present work, we studied univalent bicategories. Showing that a bicategory is univalent can be challenging; to simplify this task, we introduced displayed bicategories, which provide a way to modularly reason about complicated bicategories. We then demonstrated the usefulness of displayed bicategories by showing, using the displayed technology, that several complicated bicategories are univalent.

For the practical mechanization of mathematics in a computer proof assistant, two issues may arise when building complicated bicategories as the total bicategory of iterated displayed bicategories. Firstly, the structures may not be parenthesized as desired. This problem can be avoided or at least alleviated through a suitable use of the sigma construction of displayed bicategories (Item 2 in Definition 4.5). Secondly, “meaningless” terms of unit type may occur

in the cells of this bicategory. We are not aware of a way of avoiding these occurrences while still using displayed bicategories. However, both issues can be addressed through the definition of a suitable “interface” to the structures, in form of “builder” and projection functions, which build, or project a component out of, an instance of the structure. The interface hides the implementation details of the structure, and thus provides a welcome separation of concerns between mathematical and foundational aspects.

We have only started, in the present work, the development of bicategory theory in univalent foundations and its formalization. Our main goals for the future are

A bicategorical Rezk completion: to construct the free univalent bicategory associated to a bicategory. It will fundamentally use Definition 6.6 and Theorem 3.6.

Equivalence Principle: to show that identity is biequivalence for univalent bicategories.

More displayed machinery: to define and study displayed notions of pseudofunctors, biequivalences, etc over the respective notions in the base. In particular, the extra displayed machinery will allow us to build not just (univalent) bicategories layerwise, but also maps and equivalences between them.

The envisioned displayed machinery can also be used to study the semantics of higher inductive types (HITs). Using Definitions 6.7 and 6.10, we can define bicategories of algebras on a signature; its initial object is the HIT specified by the signature. The Rezk completion $\eta : \mathbf{Grpd}_U \rightarrow \mathbf{1-Type}_U$ from groupoids to 1-types can then be used to construct a biadjunction – obtained as the total biadjunction of a suitable displayed biadjunction – between algebras of 1-types and algebras of groupoids. To construct higher inductive 1-types, we just need to show that the groupoid model has HITs, which was proved by Dybjer and Moenclaey [10].

Displayed notions naturally appear in Clairambault and Dybjer’s [8] pair of biequivalences $\mathbf{FL} \xrightarrow{\sim} \mathbf{CwF}_{\text{dem}}^{\text{I}_{\text{ext}}, \Sigma}$ and $\mathbf{LCC} \xrightarrow{\sim} \mathbf{CwF}_{\text{dem}}^{\text{I}_{\text{ext}}, \Sigma, \Pi}$ relating categories with families equipped with structure modelling type and term formers to finite limit categories and locally cartesian closed categories, respectively. Here, the latter biequivalence is an “extension” of the former; this can be made formal by a displayed biequivalence relating the Π -structure with the locally cartesian closed structure.

More generally, we aim to use the displayed machinery when extending to the bicategorical setting the comparison of different categorical structures for type theories started in [3].

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