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Bart De Bruyn · Sergey Shpectorov

Abstract We prove that the near hexagon associated with the extended ternary Golay code has, up to isomorphism, 25 hyperplanes, and give an explicit construction for each of them. As a main tool in the proof, we show that the classification of these hyperplanes is equivalent to the determination of the orbits on vectors of certain modules for the group $2 \cdot M_{12}$.

Keywords extended ternary Golay code · near hexagon · hyperplane · Mathieu group M_{12}

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1 Introduction

The extended ternary Golay code is an important object in several areas of mathematics like coding theory, group theory and combinatorics. The ternary Golay code which is obtained from the extended ternary Golay code by deleting one coordinate position is an example of a so-called perfect code (for which not many examples and families are known to exist). The importance of this code is also shown by the several finite simple groups and combinatorial objects that are somehow related to it. This paper regards one such combinatorial

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object, namely one of the "exceptional" near hexagons. This particular near hexagon is very regular in the sense that its collinearity graph is a so-called distance-regular graph.

Our aim is to classify all hyperplanes of this near hexagon. Hyperplanes are very important objects in Incidence Geometry and are often studied in connection with projective representations (embeddings) of the geometry under consideration. They play for instance a crucial role in Tits' classification of polar spaces. The problem of classifying hyperplanes of geometries is often equivalent with classifying orbits of certain group modules. Although the hyperplanes of the near hexagon related to the extended ternary Golay code can be determined computationally, we also present an entirely theoretical treatment and this for two reasons: (1) This treatment offers extra insight into the structure of certain modules for the group $2 \cdot M_{12}$; (2) The treatment also allows to give computer free descriptions of the hyperplanes.

We now proceed to giving the definitions of the basic objects occurring in this paper, and to describing our main results. We denote by $W = \mathbb{F}_3^{12}$ the 12-dimensional vector space over the field $\mathbb{F}_3 = \{0, 1, -1\}$ whose vectors are row matrices of length 12 with entries in $\mathbb{F}_3$. The six rows of the matrix

$$M := \begin{bmatrix} 1\,0\,0\,0\,0\,0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0\,1\,0\,0\,0\,0 & 0 & 1 & -1 & -1 & 1 & -1 \\ 0\,0\,1\,0\,0\,0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0\,0\,0\,1\,0\,0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 0\,0\,0\,0\,1\,0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0\,0\,0\,0\,0\,1 & 1 & -1 & -1 & 1 & 0 & -1 \end{bmatrix}$$

span a 6-dimensional subspace C of $\mathbb{F}_3^{12}$, called the *extended ternary Golay code*. By deleting one coordinate position, the 11-dimensional perfect linear code is obtained which was discovered by Golay in [14].

In [19], Shult and Yanushka constructed a near hexagon from the code C. By a *near hexagon*, we mean a point-line geometry of diameter 3 such that for every point x and every line L there exists a unique point on L nearest to x, i.e., there exists a unique point y on L for which the distance $\mathrm{d}(x, y)$ between x and y in the collinearity graph is minimal.

Let $\mathbb{E}_1$ be the point-line geometry whose points are all the cosets of C (in W) and whose lines are all the triples of the form $\{\bar{w} + C, \bar{w} + \bar{e}_i + C, \bar{w} - \bar{e}_i + C\}$, with incidence being containment. Here, $\bar{w} \in W$ and $\bar{e}_i$ with $i \in \{1, 2, \ldots, 12\}$ denotes the row matrix all whose entries are 0 except for the ith one which is equal to 1. Shult and Yanushka [19, pp. 30–33] proved that $\mathbb{E}_1$ is a near hexagon. The near hexagon $\mathbb{E}_1$ is *dense* which means that every line contains at least three points and every two points at distance 2 have at least two common neighbours. In fact, every line of $\mathbb{E}_1$ is incident with precisely three points, every point is incident with exactly 12 lines and every two points at distance 2 have exactly two common neighbours. By the result of Brouwer [1], we know that $\mathbb{E}_1$ is, up to isomorphism, the unique near hexagon having these three properties. The full automorphism group G of $\mathbb{E}_1$ is a group of

type	index	# hyperplanes	# points	Line distribution	D$\textsc{e}$	S$\textsc{i}$	O$\textsc{v}$
1	1	36	243	0^{243}	0	0	5346
2	2	594	405	$6^{324}12^{81}$	810	2916	1620
3	3	5940	351	$0^{27}6^{324}$	216	2916	2214
4	4	40095	369	$4^{72}6^{216}8^{72}12^{9}$	354	3096	1896
5	5	192456	363	$0^{3}4^{120}6^{120}8^{120}$	320	3000	2026
6	6	32076	429	$2^{6}6^{90}8^{270}10^{60}12^{3}$	946	3300	1100
7	6	64152	333	$2^{45}4^{180}6^{45}8^{45}10^{18}$	150	2520	2676
8	6	577368	365	$4^{120}6^{124}8^{120}12^{1}$	330	3036	1980
9	7	577368	343	$0^{1}2^{20}4^{130}6^{150}8^{40}10^{2}$	160	2820	2366
10	7	1154736	375	$2^{6}4^{75}6^{159}8^{120}10^{15}$	420	3096	1830
11	8	360855	393	$2^{8}4^{16}6^{200}8^{136}10^{32}12^{1}$	578	3216	1552
12	8	2886840	361	$2^{15}4^{84}6^{181}8^{69}10^{12}$	302	2988	2056
13	9	160380	355	$0^{1}2^{24}4^{108}6^{144}8^{54}10^{24}$	288	2832	2226
14	9	2886840	355	$2^{20}4^{94}6^{172}8^{61}10^{8}$	256	2928	2162
15	9	1283040	387	$4^{45}6^{189}8^{126}10^{27}$	516	3204	1626
16	10	48114	325	$2^{60}4^{180}6^{24}8^{60}12^{1}$	90	2436	2820
17	10	962280	389	$4^{57}6^{158}8^{144}10^{30}$	546	3180	1620
18	10	2886840	357	$2^{15}4^{102}6^{159}8^{75}10^{6}$	270	2952	2124
19	11	8748	463	$0^{1}8^{330}10^{132}$	1320	3300	726
20	11	192456	335	$2^{36}4^{150}6^{104}8^{45}$	120	2676	2550
21	11	1924560	367	$2^{6}4^{93}6^{157}8^{102}10^{9}$	348	3048	1950
22	12	729	289	$2^{264}12^{25}$	66	1320	3960
23	12	17496	353	$0^{12}4^{165}6^{110}10^{66}$	330	2640	2376
24	12	192456	385	$2^{15}4^{30}6^{169}8^{165}12^{6}$	510	3156	1680
25	12	320760	353	$0^{4}4^{126}6^{162}8^{54}10^{6}12^{1}$	234	2928	2184

Table 1 The hyperplanes of $\mathbb{E}_1$

type $3^6 : 2 \cdot M_{12}$, i.e., a semi-direct product of the elementary abelian 3-group $(C_3)^6$ with the non-split double cover $2 \cdot M_{12}$ of the Mathieu group M_{12}.

A *hyperplane* of a point-line geometry $\mathcal{S}$ is a set H of points, distinct from the whole point set, such that every line of $\mathcal{S}$ has either one or all its points in H. Two hyperplanes of $\mathcal{S}$ are called *isomorphic* if there exists an automorphism of $\mathcal{S}$ mapping one hyperplane to the other. This paper is part of a project to classify the hyperplanes of all dense near hexagons with three points on each line. By [2], there are 11 such near hexagons. For some of them, a complete classification of the hyperplanes is already available, see e.g. [3] for the M_{24} near hexagon $\mathbb{E}_2$. Also classification results have been obtained for hyperplanes of non-dense near hexagons with three points per line, see e.g. [12] for the generalized hexagons of order $(2, 2)$. The following is the main result of this paper.

Theorem 1 *Up to isomorphism, the near hexagon* $\mathbb{E}_1$ *has 25 hyperplanes.*

The 25 hyperplanes together with some of their basic combinatorial properties are listed in Table 1. For a given hyperplane H of $\mathbb{E}_1$, we mention the total number of hyperplanes isomorphic to H (column 3) and the total number of points H has (column 4). A hyperplane H is said to have *line distribution* $n_1^{e_1} n_2^{e_2} \dots n_k^{e_k}$, where $n_1, n_2, \dots, n_k$ are nonnegative integers satisfying

$n_1 < n_2 < \cdots < n_k$, if $|H| = e_1 + e_2 + \cdots + e_k$ and if for every $i \in \{1, 2, \ldots, k\}$, H contains exactly $e_i \geq 1$ points which are incident with precisely n_i lines that are completely contained in H. The line distribution is mentioned in the fifth column of the table.

The near hexagon $\mathbb{E}_1$ has 5346 subgeometries that are (3×3)-grids, the so-called *quads*. If H is a hyperplane of $\mathbb{E}_1$ and Q is a quad, then either $Q \subseteq H$, $Q \cap H$ is a union of two intersecting lines or $Q \cap H$ is a set of three pairwise noncollinear points. We say that Q is *deep*, *singular* or *ovoidal* depending on whether the first, second or third possibility occurs. The table also mentions the number De of deep quads, the number Si of singular quads and the number Ov of ovoidal quads for each of the 25 hyperplanes.

In a sense, two of the 25 hyperplanes of $\mathbb{E}_1$ are special. If x is a point of $\mathbb{E}_1$, then the set of points of $\mathbb{E}_1$ at distance at most 2 from x is a hyperplane of $\mathbb{E}_1$, the so-called *singular hyperplane with center* x. The singular hyperplanes are precisely the hyperplanes of type 22 in the table. Another special hyperplane is the so-called *ovoid*, a set of points meeting each line in a singleton. The ovoids of $\mathbb{E}_1$ have been classified by De Bruyn [6, Theorem 4.2]. There are 36 of them and they occur as hyperplanes of type 1 in the table. The set of 36 ovoids of $\mathbb{E}_1$ can be divided into 12 equivalence classes of size 3, by calling two ovoids *equivalent* whenever they are equal or disjoint.

If X_1 and X_2 are two sets of points of $\mathbb{E}_1$, then we denote by $X_1 + X_2$ the complement of the symmetric difference of X_1 and X_2 (with respect to the whole point set $\mathcal{P}$ of $\mathbb{E}_1$). The "addition" turns the set of all subsets of $\mathcal{P}$ into an elementary abelian 2-group with neutral element $\mathcal{P}$. If H_1 and H_2 are two distinct hyperplanes of $\mathbb{E}_1$, then $H_1 + H_2$ is again a hyperplane of $\mathbb{E}_1$. In Section 6, we indicate how each hyperplane of $\mathbb{E}_1$ can be written as a sum of ovoids and/or singular hyperplanes.

We will prove in Section 3 that for every hyperplane H of $\mathbb{E}_1$, there exists a unique set $\{O_1, O_2, \ldots, O_k\}$ of mutually non-equivalent ovoids such that $H = O_1 + O_2 + \cdots + O_k$. The number k of ovoids in this set is called the *ovoid index* of the hyperplane H and is mentioned in the second column of the table.

The full automorphism group K of the extended ternary Golay code C is a non-split double cover $2 \cdot M_{12}$ of M_{12}. For every subset $S \subseteq I$, we denote by W_S the subspace $\langle \bar{e}_i \mid i \in S \rangle$ of W, by W^S the subspace $\langle \bar{e}_i \mid i \in I \setminus S \rangle$ of W, and by K_S the stabiliser of W_S inside K. In case S is a singleton $\{i\}$, we denote W_S and W^S also by W_i and W^i. The group K_S stabilises the subspace $W^S + C$ of W and so naturally acts on the quotient vector space $W/(W^S + C)$. In Section 4, we show that enumerating the isomorphism classes of hyperplanes of $\mathbb{E}_1$ is equivalent with enumerating the orbits (on vectors) of the action of K_S on $W/(W^S + C)$, for nonempty subsets $S \subseteq I$.

The latter fact is then used in Section 5 to classify all hyperplanes of $\mathbb{E}_1$. We show that there are 25 of them (up to isomorphism) and determine all orbit sizes. These goals will be achieved without the aid of a computer.

In Section 6, we show that every hyperplane of $\mathbb{E}_1$ can be obtained as a sum of singular hyperplanes. As singular hyperplanes are easily implemented, this fact then allows us to enumerate all hyperplanes of $\mathbb{E}_1$ by means of a

computer. Our computer computations confirm the classification of the hyperplanes obtained in Section 5. The combinatorial information mentioned in Table 1 (columns 4–8) has been obtained from the computer models of the hyperplanes.

2 Preliminaries

In Section 1, we described a model for the near hexagon $\mathbb{E}_1$ using the cosets of the extended ternary Golay code C. It turns out that some of the results of this paper will be more easily proved if we use an equivalent model for $\mathbb{E}_1$, discussed in De Bruyn and De Clerck [8].

Let PG$(5,3)$ be a hyperplane of the projective space PG$(6,3)$. After fixing some reference system in PG$(5,3)$, the twelve columns of the matrix M considered in Section 1 define a set $\mathcal{K}$ of twelve points of PG$(5,3)$. This set $\mathcal{K}$ of twelve points satisfies several nice properties, see e.g. Coxeter [5]. The stabiliser J of $\mathcal{K}$ inside the automorphism group $PGL(6,3)$ of PG$(5,3)$ is isomorphic to M_{12} and acts sharply 5-transitive on $\mathcal{K}$. Every $k \in \{1,2,3,4,5\}$ points of $\mathcal{K}$ generate a $(k-1)$-dimensional subspace of PG$(5,3)$ which contains precisely k points of $\mathcal{K}$ if $k \leq 4$ and precisely 6 points of $\mathcal{K}$ if $k = 5$. The sets of six points that arise by intersecting $\mathcal{K}$ with hyperplanes of PG$(5,3)$ define a Steiner system $S(5,6,12)$ on the set $\mathcal{K}$. This Steiner system, which is uniquely determined by its parameters, is one of the (small) Witt designs. It has the property that the complement of any block is again a block.

We denote by $\widetilde{\mathbb{E}_1}$ the point-line geometry whose points are the points of the affine space AG$(6,3) :=$ PG$(6,3) \setminus$ PG$(5,3)$ and whose lines are the lines of PG$(6,3)$ not contained in PG$(5,3)$ that intersect PG$(5,3)$ in a point of $\mathcal{K}$ (natural incidence). By [8], $\widetilde{\mathbb{E}_1}$ is a near hexagon isomorphic to $\mathbb{E}_1$.

For every point p of PG$(5,3)$, let $i_{\mathcal{K}}(p)$ denote the smallest number of points of $\mathcal{K}$ that generate a subspace containing p. We call $i_{\mathcal{K}}(p)$ the $\mathcal{K}$-index of p. By [8], we have $i_{\mathcal{K}}(p) \in \{1,2,3\}$. Clearly, $i_{\mathcal{K}}(p) = 1$ if and only if $p \in \mathcal{K}$. If x and y are two distinct points of AG$(6,3)$ and p is the unique point of PG$(5,3)$ on the line xy, then the distance d(x,y) between x and y in $\widetilde{\mathbb{E}_1}$ is equal to $i_{\mathcal{K}}(p)$ by [8, Lemma 4.2].

Suppose α is a plane of PG$(6,3)$ that intersects PG$(5,3)$ in a line L such that $|L \cap \mathcal{K}| = 2$. Then $\alpha \setminus L$ is a subspace of $\widetilde{\mathbb{E}_1}$ on which the induced subgeometry is a (3×3)-grid, i.e., $\alpha \setminus L$ is a quad. By [6, Section 4.2], every quad of $\widetilde{\mathbb{E}_1}$ can be obtained in this way. If $\{x_1, x_2, x_3\}$ is an ovoid of a quad, then $\{x_1, x_2, x_3\}$ is a line of AG$(6,3)$ by [6, p. 28]. We call any such line a *quad line* of AG$(6,3)$.

Suppose β is a 3-dimensional subspace of PG$(6,3)$ that intersects PG$(5,3)$ in a plane α such that $|\alpha \cap \mathcal{K}| = 3$. Then $\beta \setminus \alpha$ is a subspace of $\widetilde{\mathbb{E}_1}$ on which the induced subgeometry is a $(3 \times 3 \times 3)$-cube (i.e., a direct product of three lines of size 3). If $\{x_1, x_2, x_3\}$ is a set of mutually opposite points in such a

$(3 \times 3 \times 3)$-cube, then $\{x_1, x_2, x_3\}$ is a line of $\mathrm{AG}(6,3)$ by [6, p. 28]. We call any such line a *cube line* of $\mathrm{AG}(6,3)$.

The lines of $\mathrm{AG}(6,3)$ that are contained in lines of $\widetilde{\mathbb{E}_1}$ will be called *hexagon lines*. Any line of $\mathrm{AG}(6,3)$ with corresponding point $p \in \mathrm{PG}(5,3)$ at infinity is either a hexagon line (if $p \in \mathcal{K}$), a quad line (if $i_{\mathcal{K}}(p) = 2$) or a cube line (if $i_{\mathcal{K}}(p) = 3$). Every automorphism θ of $\widetilde{\mathbb{E}_1}$ should map hexagon lines to hexagon lines, quad lines to quad lines, and cube lines to cube lines, implying that θ is induced by a (unique) automorphism of $\mathrm{PG}(6,3)$ that stabilises the hyperplane $\mathrm{PG}(5,3)$. In the sequel, we regard $\widetilde{G} := \mathrm{Aut}(\widetilde{\mathbb{E}_1})$ as a subgroup of the group of automorphisms of $\mathrm{PG}(6,3)$ that stabilise $\mathrm{PG}(5,3)$. $\widetilde{G}$ is then precisely the group of automorphisms of $\mathrm{PG}(6,3)$ that stabilise the set $\mathcal{K}$ (and hence also $\mathrm{PG}(5,3)$). From this fact, it can be seen that $\widetilde{G}$ is the semidirect product $\widetilde{T} : \widetilde{K}$, where $\widetilde{T} \cong (C_3)^6$ is the subgroup of $\widetilde{G}$ induced by the translations of $\mathrm{AG}(6,3)$ and $\widetilde{K} \cong 2 \cdot M_{12}$ is the subgroup of $\widetilde{G}$ that fixes a distinguished point o of $\mathrm{AG}(6,3)$, called the *origin* of $\mathrm{AG}(6,3)$. Note also that the group of dilations of $\mathrm{AG}(6,3)$ determines a subgroup $\widetilde{T} : \langle \sigma \rangle$ of type $3^6 : 2$ of $\widetilde{G}$, with σ being the unique nontrivial central collineation with center o and axis $\mathrm{PG}(5,3)$.

The automorphism group $3^6 : 2 \cdot M_{12}$ can also be recognised inside the original model $\mathbb{E}_1$ of the near hexagon. The subgroup K of $GL(W)$ that stabilises the set $\{W_1, W_2, \ldots, W_{12}\}$ and also the code C is called the *automorphism group* of C. It is known that K is a group of type $2 \cdot M_{12}$, see e.g. Wilson [20, §5.3.5] or MacWilliams & Sloane [15, p. 647]. Each element of K permutes the cosets of C and the weight 1 vectors of W and thus determines an automorphism of $\mathbb{E}_1$. The set of translations in W/C also determines a group T of automorphisms of $\mathbb{E}_1$. The group G generated by K and T is a semidirect product $T : K$ of type $3^6 : 2 \cdot M_{12}$, necessarily coinciding with the full automorphism group $G := \mathrm{Aut}(\mathbb{E}_1)$ of $\mathbb{E}_1$.

Suppose β is a hyperplane of $\mathrm{PG}(6,3)$ which intersects $\mathrm{PG}(5,3)$ in a 4-dimensional subspace α such that $\alpha \cap \mathcal{K} = \emptyset$. Then $O := \beta \setminus \alpha$ is an ovoid of $\widetilde{\mathbb{E}_1}$. The hyperplane α of $\mathrm{PG}(5,3)$, which is uniquely determined by the ovoid O, will be denoted by Π_O. By De Bruyn [6, Theorem 4.2], every ovoid of $\widetilde{\mathbb{E}_1}$ can be obtained in the above way. There are 12 hyperplanes in $\mathrm{PG}(5,3)$ disjoint from $\mathcal{K}$ and so there are 36 ovoids in total. The relation of being equal or disjoint defines an equivalence relation on the set of ovoids. There are twelve equivalence classes, each containing three ovoids. Two ovoids O_1 and O_2 are equivalent if and only if $\Pi_{O_1} = \Pi_{O_2}$.

By De Bruyn and Vanhove [10, Lemma A.2], every hyperplane of $\mathrm{PG}(5,3)$ intersects $\mathcal{K}$ in either 0, 3 or 6 points. We denote by $\mathcal{K}^*$ the set of hyperplanes of $\mathrm{PG}(5,3)$ disjoint from $\mathcal{K}$. Then $|\mathcal{K}^*| = 12$. The set $\mathcal{K}^*$, regarded as set of points of the dual projective space $\mathrm{PG}^*(5,3)$ of $\mathrm{PG}(5,3)$, is isomorphic to the set $\mathcal{K}$ of points of $\mathrm{PG}(5,3)$. In fact, by [10, Proposition A.3] there exists an orthogonal polarity ζ of $\mathrm{PG}(5,3)$ mapping $\mathcal{K}$ to $\mathcal{K}^*$, points with $\mathcal{K}$-index 2 to

hyperplanes intersecting $\mathcal{K}$ in precisely six points and points with $\mathcal{K}$-index 3 to hyperplanes intersecting $\mathcal{K}$ in precisely three points. This implies the following:

Lemma 1 *No point of $\mathcal{K}$ is contained in an element of $\mathcal{K}^*$, through every point of $\mathrm{PG}(5,3)$ with $\mathcal{K}$-index 2 there are precisely six elements of $\mathcal{K}^*$ and through every point of $\mathrm{PG}(5,3)$ with $\mathcal{K}$-index 3, there are precisely three hyperplanes of $\mathcal{K}^*$. Every $k \in \{1,2,3,4,5\}$ elements of $\mathcal{K}^*$ intersect in a subspace of dimension $5 - k$.*

The subgroup $J \cong M_{12}$ of $PGL(6,3)$ stabilising $\mathcal{K}$ thus coincides with the subgroup of $\mathrm{PG}(6,3)$ stabilising $\mathcal{K}^*$ and acts sharply 5-transitively on $\mathcal{K}^*$. In fact, a Steiner system $S^*(5,6,12) \cong S(5,6,12)$ can be defined on the set $\mathcal{K}^*$ such that every set of six elements of $\mathcal{K}^*$ through a given point with $\mathcal{K}$-index 2 is a block.

3 The ovoid index of a hyperplane

The intention of this section is to show that each hyperplane of $\widetilde{\mathbb{E}_1}$ can be expressed in a unique way as a sum of mutually non-equivalent ovoids.

Lemma 2 *Let $\Pi \in \mathcal{K}^*$ and let O_1, O_2 and O_3 be the three mutually distinct ovoids of $\widetilde{\mathbb{E}_1}$ such that $\Pi = \Pi_{O_1} = \Pi_{O_2} = \Pi_{O_3}$. Then $O_1 + O_2 = O_3$.*

Proof The ovoids O_1 and O_2 are disjoint and O_3 is the complement of $O_1 \cup O_2$.

Lemma 3 *If four mutually distinct hyperplanes α_1, α_2, α_3 and α_4 of $\mathrm{PG}(6,3)$ cover the whole point set of $\mathrm{PG}(6,3)$, then they are the four hyperplanes through a given subspace of co-dimension 2.*

Proof We have $|\alpha_1| = \frac{3^6 - 1}{2}$ and $|\alpha_i \setminus \alpha_1| = 3^5$ for every $i \in \{2,3,4\}$. So, $|\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4| \le \frac{3^6 - 1}{2} + 3 \cdot 3^5 = \frac{3^7 - 1}{2} = |\mathrm{PG}(6,3)|$. So, we must have that $\alpha_3 \cap (\alpha_1 \cup \alpha_2) = (\alpha_3 \cap \alpha_1) \cup (\alpha_3 \cap \alpha_2) = \alpha_3 \cap \alpha_1$, i.e. $\alpha_3 \cap \alpha_1 = \alpha_3 \cap \alpha_2$. A similar argument shows that $\alpha_4 \cap \alpha_1 = \alpha_4 \cap \alpha_2$. It follows that $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ all contain the same subspace $\alpha_1 \cap \alpha_2$ of co-dimension 2.

Lemma 4 *Let $O_1, O_2, \ldots, O_k$ be $k \in \{1, 2, \ldots, 12\}$ ovoids of $\widetilde{\mathbb{E}_1}$ such that $\Pi_{O_1}, \Pi_{O_2}, \ldots, \Pi_{O_k}$ are mutually distinct. Then $O_1 + O_2 + \cdots + O_k$ is distinct from the point set $\widetilde{\mathcal{P}}$ of $\widetilde{\mathbb{E}_1}$.*

Proof Clearly, this is the case if $k = 1$ and $k = 2$ (as $O_1 \ne O_2$). Suppose therefore that $k \ge 3$ and that $O_1 + O_2 + \cdots + O_k = \widetilde{\mathcal{P}}$. Our intention is to derive a contradiction.

By Lemma 1, there exists a point p that is contained in precisely three elements Π_1, Π_2 and Π_3 of $\mathcal{K}^*$. Since $J \cong M_{12}$ acts 3-transitively on $\mathcal{K}^*$, we may without loss of generality suppose that $\Pi_{O1} = \Pi_1$, $\Pi_{O_2} = \Pi_2$ and $\Pi_{O_3} = \Pi_3$. Since Π_1, Π_2 and Π_3 are mutually distinct, they intersect in a

plane (by Lemma 1) and hence $\langle O_1 \rangle \cap \langle O_2 \rangle \cap \langle O_3 \rangle$ is 3-dimensional. Lemma 3 then implies that there is a line $L = \{p, x_1, x_2, x_3\}$ through p not contained in $\mathrm{PG}(5,3) \cup \langle O_1 \rangle \cup \langle O_2 \rangle \cup \langle O_3 \rangle$. For every $i \in \{1, 2, 3\}$, let n_i denote the total number of ovoids of $\{O_1, O_2, \ldots, O_k\}$ containing x_i. Then $n_1 + n_2 + n_3 = k - 3$ since none of the x_i's is contained $O_1 \cup O_2 \cup O_3$ and every O_j with $j \in \{4, 5, \ldots, k\}$ contains precisely one x_i. Since $O_1 + O_2 + \ldots + O_k = \widetilde{\mathcal{P}}$, the number $k - n_i$ of ovoids of $\{O_1, O_2, \ldots, O_k\}$ missing x_i is even for every $i \in \{1, 2, 3\}$. Hence, also $(k - n_1) + (k - n_2) + (k - n_3) = 2k + 3$ would be even, an obvious contradiction.

Lemma 5 *The number of hyperplanes of $\widetilde{\mathbb{E}_1}$ that can be written as a sum of ovoids is equal to $2^{24} - 1$. If H is such a hyperplane, there there exists a unique nonempty set $\{O_1, O_2, \ldots, O_k\}$ of mutually non-equivalent ovoids such that $H = O_1 + O_2 + \cdots + O_k$.*

Proof By Lemma 2, every hyperplane of $\widetilde{\mathbb{E}_1}$ that can be written as the sum of ovoids can be written as $O_1 + O_2 + \cdots + O_k$ where $O_1, O_2, \ldots, O_k$ are mutually non-equivalent ovoids. By Lemmas 2 and 4, the representation of the hyperplane as a sum $O_1 + O_2 + \cdots + O_k$ is unique, up to a permutation of $O_1, O_2, \ldots, O_k$. Now, for fixed $k \in \{1, 2, \ldots, 12\}$, there are $\binom{12}{k}$ possibilities for $\{\Pi_{O_1}, \Pi_{O_2}, \ldots, \Pi_{O_k}\}$ and for fixed $\{\Pi_{O_1}, \Pi_{O_2}, \ldots, \Pi_{O_k}\}$, there are 3^k possibilities for $O_1, O_2, \ldots, O_k$. So, the total number of hyperplanes that can be written as the sum of ovoids is equal to $\sum_{i=1}^{12} \binom{12}{k} \cdot 3^k = (1+3)^{12} - 1 = 2^{24} - 1$. $\square$

Proposition 1 *For every hyperplane H of $\widetilde{\mathbb{E}_1}$ there exists a unique $k \in \{1, 2, \ldots, 12\}$ and a unique set $\{O_1, O_2, \ldots, O_k\}$ of k ovoids such that $H = O_1 + O_2 + \cdots + O_k$ and $\Pi_{O_1}, \Pi_{O_2}, \ldots, \Pi_{O_k}$ are mutually distinct.*

Proof In view of Lemma 5, it suffices to show that $\widetilde{\mathbb{E}_1}$ has precisely $2^{24} - 1$ hyperplanes. As any dense near polygon with three points per line, $\widetilde{\mathbb{E}_1}$ has a full projective embedding ([18, Corollary 2, page 183]). Hence, $\widetilde{\mathbb{E}_1}$ also has a universal full projective embedding, see again [18]. Every hyperplane α of the universal embedding space $\mathrm{PG}(\widetilde{V})$ naturally gives rise to a hyperplane H_α of $\widetilde{\mathbb{E}_1}$. The set H_α consists of those points of $\widetilde{\mathbb{E}_1}$ whose images under the universal embedding are contained in α. By Ronan [18], this natural correspondence defines a bijection between the hyperplanes of $\widetilde{\mathbb{E}_1}$ and those of $\mathrm{PG}(\widetilde{V})$. So, the total number of hyperplanes of $\widetilde{\mathbb{E}_1}$ is equal to $2^{\widetilde{d}} - 1$, where $\widetilde{d}$ is the dimension of the $\mathbb{F}_2$-vector space $\widetilde{V}$. It is known that $\widetilde{d} = 24$, see Brouwer et al. [2, p. 350], De Bruyn [7, Theorem 1.1] or Yoshiara [21, Theorem 1]. $\square$

The number k mentioned in Proposition 1 is called the *ovoid index* of the hyperplane H.

4 Rephrasing of the classification problem

As mentioned above, we will regard the automorphisms of $\widetilde{\mathbb{E}_1}$ as automorphisms of $\mathrm{PG}(6,3)$, and we consider a special point o in $\mathrm{AG}(6,3)$, the origin

of AG$(6,3)$. We put $\mathcal{K}^* = \{\Pi_1, \Pi_2, \ldots, \Pi_{12}\}$. Every automorphism $\theta \in \widetilde{G}$ of $\widetilde{\mathbb{E}_1}$ stabilises $\mathcal{K}^*$ and so there exists a permutation $\pi(\theta)$ of $I := \{1, 2, \ldots, 12\}$ such that $(\Pi_i)^\theta = \Pi_{i^{\pi(\theta)}}$.

For every $i \in I$, we denote by $O_0^{(i)}$, $O_1^{(i)}$ and $O_{-1}^{(i)}$ the three ovoids of $\widetilde{\mathbb{E}_1}$ for which Π_i is the corresponding subspace at infinity, chosen in such a way that $o \in O_0^{(i)}$.

With every vector $\bar{v}$ of AG$(6,3)$, we denote by $\Omega(\bar{v})$ the unique element of $W = \mathbb{F}_3^{12}$ whose ith coordinate $\Omega_i(\bar{v})$ is the element $\epsilon \in \mathbb{F}_3$ for which $\bar{v} + O_0^{(i)} = O_\epsilon^{(i)}$.

Lemma 6 *For every vector $\bar{v}$ of AG$(6,3)$, the vector $\Omega(\bar{v}) \in W$ has weight 0, 6, 9 or 12. The vector $\Omega(\bar{v})$ has weight 0 if and only if $\bar{v} = \bar{o}$.*

Proof If $\bar{v} = \bar{o}$, then $\Omega(\bar{v})$ is the zero vector. We suppose therefore that $\bar{v} \neq \bar{o}$. Let p be the point at infinity of the line $\langle \bar{v} \rangle$ of AG$(6,3)$. Note that if $i \in I$, then $\bar{v} + O_0^{(i)} = O_0^{(i)}$ if and only if $p \in \Pi_i$. So, by Lemma 1, $\Omega(\bar{v})$ has weight $12 - 0 = 12$ if $p \in \mathcal{K}$, weight $12 - 6 = 6$ if p has $\mathcal{K}$-index 2 and weight $12 - 3 = 9$ if p has $\mathcal{K}$-index 3. $\qquad \square$

Lemma 7 *If $\bar{v}_1$ and $\bar{v}_2$ are vectors of AG$(6,3)$, then $\Omega(\bar{v}_1 + \bar{v}_2) = \Omega(\bar{v}_1) + \Omega(\bar{v}_2)$.*

Proof Let $i \in I$. Put $O_0^{(i)} + \bar{v}_1 = O_{\epsilon_1}^{(i)}$ and $O_0^{(i)} + \bar{v}_2 = O_{\epsilon_2}^{(i)}$. A translation by the vector $\bar{v}_2$ either fixes each of $O_0^{(i)}, O_1^{(i)}, O_{-1}^{(i)}$ (if the point p at infinity of $\langle \bar{v}_2 \rangle$ is contained in Π_i) or permutes them according to a cycle of length 3. So, $O_0^{(i)} + \bar{v}_2 = O_{\epsilon_2}^{(i)}$ implies that $O_\lambda^{(i)} + \bar{v}_2 = O_{\lambda + \epsilon_2}^{(i)}$ for every $\lambda \in \mathbb{F}_3$. Hence, $O_0^{(i)} + \bar{v}_1 + \bar{v}_2 = O_{\epsilon_1}^{(i)} + \bar{v}_2 = O_{\epsilon_1 + \epsilon_2}^{(i)}$, implying that $\Omega_i(\bar{v}_1 + \bar{v}_2) = \epsilon_1 + \epsilon_2 = \Omega_i(\bar{v}_1) + \Omega_i(\bar{v}_2)$. Since $i \in I$ was arbitrary, we have $\Omega(\bar{v}_1 + \bar{v}_2) = \Omega(\bar{v}_1) + \Omega(\bar{v}_2)$. $\qquad \square$

The following is an immediate consequence of Lemma 7.

Corollary 1 *Let $\overline{C} \subseteq W$ denote the set of all elements of the form $\Omega(\bar{v})$, where $\bar{v}$ is some vector of AG$(6,3)$. Then $\overline{C}$ is a 6-dimensional subspace of W, every vector of which has weight 0, 6, 9 or 12.*

By Delsarte & Goethals [11] or Pless [16,17], we know that every 6-dimensional subspace of $\mathbb{F}_3^{12}$ with the property that every nonzero vector has weight at least 6 is equivalent with the extended ternary Golay code. So, we have:

Corollary 2 *The subspace $\overline{C}$ is equivalent to the extended ternary Golay code $C \subseteq W$.*

Let $\theta \in \widetilde{K} \cong 2 \cdot M_{12}$. Then $(\Pi_i)^\theta = \Pi_{i^{\pi(\theta)}}$ for every $i \in I$. Since $(O_0^{(i)})^\theta = O_0^{(i^{\pi(\theta)})}$, there exists for every $i \in I$ a $\lambda_\theta(i) \in \mathbb{F}_3^*$ such that $(O_\epsilon^{(i)})^\theta = O_{\epsilon \cdot \lambda_\theta(i)}^{(i^{\pi(\theta)})}$ for every $\epsilon \in \mathbb{F}_3$. Note that if θ is the nontrivial central collineation σ of

PG$(6,3)$ with center o and axis PG$(5,3)$, then $\lambda_\theta(i) = -1$ for every $i \in I$. A straightforward calculation shows that

$$\pi(\theta\theta') = \pi(\theta)\pi(\theta') \text{ and } \lambda_{\theta\theta'}(i) = \lambda_\theta(i) \cdot \lambda_{\theta'}(i^{\pi(\theta)}) \qquad (*)$$

for all $\theta, \theta' \in \widetilde{K}$ and every $i \in I$, hereby following the convention that permutations and automorphisms are composed from left to right.

For each $\theta \in \widetilde{K}$, we associate the element $\bar\theta$ of $GL(W)$ defined by $\bar{e}_i^{\bar\theta} := \lambda_\theta(i) \cdot \bar{e}_{i^{\pi(\theta)}}$, $i \in I$. The condition $(*)$ implies that $\overline{\theta\theta'} = \bar\theta\bar\theta'$ for all $\theta, \theta' \in \widetilde{K}$. If $\bar\theta$ is the trivial element of $GL(W)$ for a certain $\theta \in \widetilde{K}$, then $i^{\pi(\theta)} = i$ and $\lambda_\theta(i) = 1$ for every $i \in I$. The former implies that $\pi(\theta) = 1$ and thus that $\theta \in \widetilde{K} \cap \langle \widetilde{T}, \sigma \rangle = \langle \sigma \rangle$. The latter implies that $\theta \neq \sigma$, so that θ is the trivial automorphism.

We conclude that the map $\widetilde{K} \to GL(W)$ defined by $\theta \mapsto \bar\theta$ is a faithful representation. We denote the image of $\widetilde{K}$ in $GL(W)$ by $\overline{K} \cong 2 \cdot M_{12}$.

Lemma 8 $\overline{K}$ *leaves the subspace* $\overline{C}$ *of* W *invariant.*

Proof Let θ be an arbitrary element of $\widetilde{K}$. It suffices to show that $\overline{C}^{\bar\theta} = \overline{C}$, or equivalently that $\Omega(\bar{v})^{\bar\theta} \in \overline{C}$ for every vector $\bar{v}$ of AG$(6,3)$. The translation by the vector $\bar{v}$ determines an automorphism $t \in \widetilde{T}$. As $\widetilde{T} \trianglelefteq \widetilde{G}$, we have $\widetilde{T}\widetilde{K} = \widetilde{K}\widetilde{T}$ and so there exist (unique) $\theta' \in \widetilde{K}$ and $t' \in \widetilde{T}$ such that $t\theta = \theta't'$. Suppose that t' is induced by the translation by a vector $\bar{v}'$. Then for every $i \in I$ and every $\epsilon \in \mathbb{F}_3$, we have

$$(O_\epsilon^{(i)} + \bar{v})^\theta = (O_\epsilon^{(i)})^{\theta'} + \bar{v}'.$$

The left hand side of this equation is equal to

$$(O_{\epsilon+\Omega_i(\bar{v})}^{(i)})^\theta = O_{(\epsilon+\Omega_i(\bar{v}))\cdot\lambda_\theta(i)}^{(i^{\pi(\theta)})},$$

while the right hand side is equal to

$$O_{\epsilon\cdot\lambda_{\theta'}(i)}^{(i^{\pi(\theta')})} + \bar{v}' = O_{\epsilon\cdot\lambda_{\theta'}(i)+\Omega_{i^{\pi(\theta')}}(\bar{v}')}^{(i^{\pi(\theta')})}.$$

We thus have $\pi(\theta) = \pi(\theta')$, $\lambda_\theta(i) = \lambda_{\theta'}(i)$ and $\Omega_i(\bar{v}) \cdot \lambda_\theta(i) = \Omega_{i^{\pi(\theta')}}(\bar{v}')$ for every $i \in I$. The former two of these equations imply that $\theta = \theta'$ (since $\theta \mapsto \bar\theta$ is faithful), and the third then implies that $\Omega_i(\bar{v}) \cdot \lambda_\theta(i) = \Omega_{i^{\pi(\theta)}}(\bar{v}')$ for every $i \in I$. So, we have that

$$\Omega(\bar{v})^{\bar\theta} = \Big(\sum_{i\in I}\Omega_i(\bar{v})\cdot\bar{e}_i\Big)^{\bar\theta} = \sum_{i\in I}\Omega_i(\bar{v})\cdot\lambda_\theta(i)\cdot\bar{e}_{i^{\pi(\theta)}} = \sum_{i\in I}\Omega_{i^{\pi(\theta)}}(\bar{v}')\cdot\bar{e}_{i^{\pi(\theta)}} = \Omega(\bar{v}') \in \overline{C}.$$

This is precisely what we needed to prove.

The following is an immediate consequence of Lemma 8.

Corollary 3 *The automorphism group of the "extended ternary Golay code"* $\overline{C} \subseteq W$ *is precisely* $\overline{K} \cong 2 \cdot M_{12}$.

If H is a hyperplane of $\widetilde{\mathbb{E}_1}$, then there exists a unique set $\{O_1, O_2, \ldots, O_k\}$ of mutually non-equivalent ovoids such that $H = O_1 + O_2 + \ldots + O_k$. Set $S := \{s_1, s_2, \ldots, s_k\}$, where $\Pi_{O_i} = \Pi_{s_i}$, $i \in \{1, 2, \ldots, k\}$. We call S the *support* of the hyperplane H. For every nonempty $S \subseteq I$, we denote by $\mathcal{H}(S)$ the set of all hyperplanes of $\widetilde{\mathbb{E}_1}$ whose support is S. Then the set $\mathcal{H}$ of all hyperplanes of $\widetilde{\mathbb{E}_1}$ is equal to

$$\mathcal{H} = \bigcup_{\emptyset \neq S \subseteq I} \mathcal{H}(S).$$

A subset $\{s_1, s_2, \ldots, s_6\}$ of size 6 of I is called a *hexad* if $\{\Pi_{s_1}, \Pi_{s_2}, \ldots, \Pi_{s_6}\}$ is a block of the Steiner system $S^*(5, 6, 12)$. By the proof of Lemma 6, we know that the supports of the weight 6 vectors of $\overline{C}$ are precisely the hexads. We call two nonempty subsets S_1 and S_2 of I *equivalent* if either $|S_1| = |S_2| \neq 6$, $|S_1| = |S_2| = 6$ and S_1, S_2 are both hexads, or $|S_1| = |S_2| = 6$ and none of S_1, S_2 are hexads.

As mentioned above, every $\theta \in \widetilde{G}$ determines a permutation $\pi(\theta)$ of I. The group $\widetilde{G}$ thus has an induced action on the set $2^I \setminus \{\emptyset\}$ of all nonempty subsets of I, and two nonempty subsets of I lie in the same orbit if they are equivalent. If $\emptyset \neq S \subseteq I$, then $\widetilde{G}_S$ denotes the subgroup of $\widetilde{G}$ consisting of all $\theta \in \widetilde{G}$ for which $S^{\pi(\theta)} = S$. Observe also that if H is a hyperplane with support S and $\theta \in \widetilde{G}$, then H^θ is a hyperplane with support $S^{\pi(\theta)}$. This allows to conclude the following.

Lemma 9 – *If H_1 and H_2 are two hyperplanes of $\widetilde{\mathbb{E}_1}$ whose supports are nonequivalent, then H_1 and H_2 cannot be isomorphic.*
 – *If S_1 and S_2 are two equivalent nonempty subsets of I, then every hyperplane of $\mathcal{H}(S_1)$ is isomorphic to a hyperplane of $\mathcal{H}(S_2)$.*

In view of Lemma 9, the classification of the isomorphism classes of hyperplanes of $\widetilde{\mathbb{E}_1}$ is equivalent to the following problem.

Let $\emptyset \neq S \subseteq I$. Determine the orbits of the group $\widetilde{G}_S$ on the set $\mathcal{H}(S)$.

Choose $S = \{s_1, s_2, \ldots, s_k\} \subseteq I$, where $k = |S| > 0$. Every hyperplane H of $\mathcal{H}(S)$ is of the form $O_{\epsilon_1}^{(s_1)} + O_{\epsilon_2}^{(s_2)} + \cdots + O_{\epsilon_k}^{(s_k)}$, where $\epsilon_1, \epsilon_2, \ldots, \epsilon_k \in \mathbb{F}_3$. We associated with H the vector $\Delta_H := \epsilon_1 \bar{e}_{s_1} + \epsilon_2 \bar{e}_{s_2} + \cdots + \epsilon_k \bar{e}_{s_k} + W^S$ of the quotient space W/W^S. The map $\phi : H \mapsto \Delta_H$ defines a bijection between $\mathcal{H}(S)$ and W/W^S.

Note that $\widetilde{T} \subseteq \widetilde{G}_S$. If we define $\widetilde{K}_S := \widetilde{G}_S \cap \widetilde{K}$, then $\widetilde{G}_S$ is the semidirect product $\widetilde{T} : \widetilde{K}_S$. The element of $\widetilde{T}$ corresponding to the translation by the vector $\bar{v}$ maps the hyperplane H to the hyperplane H' for which $\Delta_{H'} = \Delta_H + \Omega(\bar{v}) + W^S$. Since the $\Omega(\bar{v})$'s generate $\overline{C}$, we see that the orbits of $\widetilde{T}$ on the set

$\mathcal{H}(S)$ are in one-to-one correspondence with the vectors of the quotient space $W/(W^S + \overline{C})$.

If $\theta \in \widetilde{K}_S$, then $\bar{\theta}$ fixes W^S and $\overline{C}$ and hence also $W^S + \overline{C}$. If $H' = H^\theta$, then by the definition of the map $\bar{\theta} \in GL(W)$, we know that

$$\Delta_H = \epsilon_1 \bar{e}_{s_1} + \epsilon_2 \bar{e}_{s_2} + \cdots + \epsilon_k \bar{e}_{s_k} + W^S,$$

$$\Delta_{H'} = \left(\epsilon_1 \bar{e}_{s_1} + \epsilon_2 \bar{e}_{s_2} + \cdots + \epsilon_k \bar{e}_{s_k} \right)^{\bar{\theta}} + W^S.$$

The set $\{ \bar{\theta} \, | \, \theta \in \widetilde{K}_S \}$ consists of all elements of $\overline{K}$ that leave the subspace $W_S = \langle \bar{e}_i \, | \, i \in S \rangle$ invariant. We denote this set by $\overline{K}_S$. We thus have the following.

Proposition 2 *The orbits of $\widetilde{G}_S$ on the set $\mathcal{H}(S)$ are in one-to-one correspondence with the orbits of $\overline{K}_S$ on $W/(W^S + \overline{C})$.*

Consider now an orbit for the action of $\overline{K}_S$ on $W/(W^S + \overline{C})$. With every vector $\bar{w} + (W^S + \overline{C})$ of this orbit, there corresponds $3^{\dim(W^S + \overline{C}) - \dim(W^S)}$ hyperplanes of $\mathcal{H}(S)$ that lie in the same $\widetilde{G}_S$-orbit, namely the hyperplanes $\phi^{-1}(\bar{w} + \bar{w}' + W^S)$, where $\bar{w}' \in W^S + \overline{C}$. Taking into account Lemma 9, we thus have:

Proposition 3 *Let S be a nonempty subset of I. Then there exists a bijective correspondence between the isomorphism classes of hyperplanes of $\widehat{\mathbb{E}_1}$ whose supports are equivalent with S and the orbits of $\overline{K}_S$ on $W/(W^S + \overline{C})$. Specifically, with each $\overline{K}_S$-orbit of size M, there corresponds an isomorphism class of hyperplanes of size $M \cdot N_S \cdot 3^{\dim(W^S + \overline{C}) - \dim(W^S)}$.*

In Proposition 3, N_S denotes the number of nonempty subsets of I equivalent with S. We have $N_S = \binom{12}{|S|}$ if $|S| \neq 6$, $\frac{1}{6}\binom{12}{5} = 132$ if S is a hexad and $N_S = \binom{12}{6} - 132 = 792$ if $|S| = 6$ and S is not a hexad. In Section 5, we use Proposition 3 to determine the isomorphism classes of hyperplanes of $\widetilde{\mathbb{E}_1}$.

5 Classification of the hyperplanes

In this section, we invoke Proposition 3 to determine all isomorphism classes of hyperplanes of $\widetilde{\mathbb{E}_1}$ and their sizes. We first prove a number of lemmas that provide more information on the subspaces $W^S + \overline{C}$ of W.

Lemma 10 *Let $S \subseteq I$ be of size 5 or 6. Then $\dim(W^S \cap \overline{C}) \in \{0,1\}$ and $\dim(W^S \cap \overline{C}) = 0$ if and only if $|S| = 6$ and S is a not a hexad.*

Proof If $|S| = 5$, then let S' be the unique hexad containing S. If $|S| = 6$, then set $S' := S$. Suppose $\bar{v}$ is a nonzero vector of $W^S \cap \overline{C}$. Since the support $S_{\bar{v}}$ of $\bar{v}$ is contained in $I \setminus S$ and $|I \setminus S| \in \{6, 7\}$, the weight $|S_{\bar{v}}|$ of the vector $\bar{v} \in \overline{C}$

is equal to 6, i.e., $S_{\bar{v}}$ is a hexad. In case $|S| = 6$, this hexad $S_{\bar{v}}$ necessarily coincides with $I \setminus S = I \setminus S'$, implying that both S' and $I \setminus S'$ are hexads. In case $|S| = 5$, the hexads $S_{\bar{v}}$ and $I \setminus S'$ intersect in at least five elements and hence coincide.

In case $|S| = 6$ and S a not a hexad, the above argument implies that $W^S \cap \overline{C} = 0$. Suppose therefore that $|S| = 5$ or that S is a hexad. Then the above discussion implies that $\dim(W^S \cap \overline{C}) = 1$, since there are two vectors of $\overline{C}$ whose supports are equal to $I \setminus S'$, and one of them is the opposite of the other.

Lemma 11 *Suppose S is a nonempty subset of size at most 6 of I such that S is not a hexad. Then $W^S + \overline{C} = W$.*

Proof If $|S| \leq 5$, then let S' be a subset of size 5 of I containing S, and if $|S| = 6$, let $S' := S$. As $W^{S'} \subseteq W^S$, it suffices to show that $W^{S'} + \overline{C} = W$. By Lemma 10, we have $\dim(W^{S'} + \overline{C}) = \dim(W^{S'}) + \dim(\overline{C}) - \dim(W^{S'} \cap \overline{C}) = 6 + \epsilon + 6 - \epsilon = 12$, where $\epsilon = 1$ if $|S'| = 5$ and $\epsilon = 0$ if $|S'| = 6$. We thus have $W^{S'} + \overline{C} = W$ and hence $W^S + \overline{C} = W$.

Lemma 12 *Suppose S is a hexad. Then $\dim(W^S + \overline{C}) = 11$ and $\dim(W/(W^S + \overline{C})) = 1$.*

Proof By Lemma 10, we have $\dim(W^S + \overline{C}) = \dim(W^S) + \dim(\overline{C}) - \dim(W^S \cap \overline{C}) = 6 + 6 - 1 = 11$ and hence $\dim(W/(W^S + \overline{C})) = \dim(W) - \dim(W^S + \overline{C}) = 1$.

Lemma 13 *Let S be a subset of size $k \in \{7, 8, \ldots, 12\}$. Then $\dim(W^S + \overline{C}) = 18 - k$ and $\dim(W/(W^S + \overline{C})) = k - 6$.*

Proof Every vector of W^S has weight at most $12 - k \leq 5$. As every nonzero vector of $\overline{C}$ has weight at least 6, we have $W^S \cap \overline{C} = 0$. Hence, $\dim(W^S + \overline{C}) = \dim(W^S) + \dim(\overline{C}) = 12 - k + 6 = 18 - k$ and $\dim(W/(W^S + \overline{C})) = \dim(W) - \dim(W^S + \overline{C}) = 12 - (18 - k) = k - 6$.

Proposition 3 implies that if S is a nonempty subset of I such that $W = W^S + \overline{C}$, then there exists a unique isomorphism class of hyperplanes of $\widetilde{\mathbb{E}_1}$ whose support is equivalent to S. This isomorphism class contains $N_S \cdot 3^{\dim(W) - \dim(W^S)} = N_S \cdot 3^{|S|}$ hyperplanes. Taking into account Lemma 11, we thus have:

Proposition 4   – *For every $k \in \{1, 2, \ldots, 5\}$, there is a unique isomorphism class of hyperplanes of $\widetilde{\mathbb{E}_1}$ whose supports have size k. This isomorphism class contains $\binom{12}{k} \cdot 3^k$ hyperplanes.*

– *There is a unique isomorphism class of hyperplanes of $\widetilde{\mathbb{E}_1}$ whose supports have size 6 and are not hexads. This isomorphism class contains $792 \cdot 3^6$ hyperplanes.*

One of the orbits of $\overline{K}_S$ on $W/(W^S + \overline{C})$ is the singleton $\{W^S + \overline{C}\}$. We call this the *trivial $\overline{K}_S$-orbit*. Since $-1 \in \overline{K}_S$, we see that the vectors $\bar{v} + (W^S + \overline{C})$ and $-\bar{v} + (W^S + \overline{C})$ belong to the same orbit. Thus, there exists a bijective correspondence between the nontrivial orbits of $\overline{K}_S$ on the vectors of $W/(W^S + \overline{C})$ and the orbits of $\overline{K}_S$ on the 1-spaces of $W/(W^S + \overline{C})$. So, by Proposition 3, we know that if S is a nonempty subset of I such that $W/(W^S + \overline{C})$ has dimension 1, then there are two isomorphism classes of hyperplanes whose supports are equivalent with S. The sizes of these isomorphism classes are respectively equal to $N_S \cdot 3^{(\dim(W)-1)-\dim(W^S)} = N_S \cdot 3^{|S|-1}$ and $2 \cdot N_S \cdot 3^{(\dim(W)-1)-\dim(W^S)} = 2 \cdot N_S \cdot 3^{|S|-1}$. Taking into account Lemmas 12 and 13, we thus have:

Proposition 5   – *There are two isomorphism classes of hyperplanes of $\widetilde{\mathbb{E}_1}$ whose supports are hexads. Their sizes are $132 \cdot 3^5$ and $2 \cdot 132 \cdot 3^5$.*
  – *There are two isomorphism classes of hyperplanes of $\widetilde{\mathbb{E}_1}$ whose supports have size 7. Their sizes are $\binom{12}{7} \cdot 3^6$ and $2 \cdot \binom{12}{7} \cdot 3^6$.*

In the sequel, we will suppose that $k := |S| \in \{8, 9, 10, 11, 12\}$. Then $N_S = \binom{12}{k}$ and $\dim(W^S) = 12 - k$. By Lemma 13, we know that $\dim(W^S + \overline{C}) = 18 - k$. So, by Proposition 3, with each orbit of size M for the action of $\overline{K}_S$ on $W/(W^S + \overline{C})$, there corresponds an isomorphism class of hyperplanes of $\widetilde{\mathbb{E}_1}$ whose size is $\binom{12}{k} \cdot M \cdot 3^6$. The isomorphism class corresponding to the trivial $\overline{K}_S$-orbit has size $\binom{12}{k} \cdot 3^6$ and the remaining isomorphism classes correspond to the orbits of $\overline{K}_S$ on the 1-spaces of $W/(W^S + \overline{C})$.

Proposition 6 *There are two isomorphism classes of hyperplanes of $\widetilde{\mathbb{E}_1}$ whose supports have size 8. Their sizes are $\binom{12}{8}3^6$ and $8\binom{12}{8}3^6$.*

Proof Let S be a given subset of size 8 of I, and set $S' := I \setminus S$. Then $W/(W^S + \overline{C})$ is 2-dimensional by Lemma 13. Clearly, the first isomorphism class comes from the trivial $\overline{K}_S$-orbit on $W/(W^S + \overline{C})$. Hence, we just need to show that all non-zero vectors of $W/(W^S + \overline{C})$ are in the same $\overline{K}_S$-orbit. Equivalently, we can show that $\overline{K}_S$ is transitive on the four 1-spaces from $W/(W^S + \overline{C})$.

    The group $\overline{K}$ acts 5-transitively on $\{W_i \mid i \in I\}$ and so $\overline{K}_S$ transitively permutes the subspaces W_i, $i \in S$. Consider the $\overline{K}_S$-orbit consisting of the images in $W/(W^S + \overline{C})$ of the 1-spaces W_i, $i \in S$. First of all, note that $W_i \le W^S + \overline{C}$ if and only if $(W_i + W^S) \cap \overline{C} \ne 0$. The latter is however impossible, as the elements of $W_i + W^S = W^{S \setminus \{i\}}$ have weight at most 5. Thus, $W_i \nleq W^S + \overline{C}$, and so we indeed have such a $\overline{K}_S$-orbit.

    Next, consider the possibility that the images of W_i and W_j in $W/(W^S + \overline{C})$ coincide, for two distinct $i, j \in S$. This happens exactly when $(W_i + W_j) \cap (W^S + \overline{C}) \ne 0$, or equivalently, $(W_i + W_j + W^S) \cap \overline{C} \ne 0$. Note that $W_i + W_j + W^S = W_{S' \cup \{i,j\}}$. Hence, the intersection is nontrivial if and only if $S' \cup \{i, j\}$ is a hexad. As the 5-set $S' \cup \{i\}$ is contained in a unique hexad, we conclude

that every W_i, $i \in S$, has the same image in $W/(W^S + \overline{C})$ as just one other W_j, $j \in S$. Thus, the images of the eight 1-subspaces W_i, $i \in S$, constitute a $\overline{K}_S$-orbit of size 4, and so indeed all four 1-spaces of $W/(W^S + \overline{C})$ are in the same $\overline{K}_S$-orbit.

The cases $k \geq 9$ again have a common feature. Namely, the images of the 1-spaces W_i, $i \in S$, in $W/(W^S + \overline{C})$ are pairwise distinct and form a $\overline{K}_S$-orbit of size k. The argument is similar to the above. First of all, $\overline{K}_S$ acts transitively on these W_i, since the group $\overline{K}$ acts 5-transitively on $\{W_i \mid i \in I\}$. For the main claim, suppose the images of W_i and W_j coincide for distinct $i, j \in S$. Then $W_i \leq W_j + W^S + \overline{C}$, or equivalently, $0 \neq (W_i + W_j + W^S) \cap \overline{C} = W_{S' \cup \{i,j\}} \cap \overline{C}$, where $S' := I \setminus S$. Since $|S' \cup \{i, j\}| < 6$, we have a contradiction. So indeed, the images of the 1-spaces W_i, $i \in S$, form a $\overline{K}_S$-orbit of size k, and this leads to a hyperplane class of size $2k\binom{12}{k}3^6$. This is in addition to the hyperplane class of size $\binom{12}{k}3^6$ coming from the trivial orbit of $\overline{K}_S$ on $W/(W^S + \overline{C})$.

Proposition 7 *There are three isomorphism classes of hyperplanes of $\widetilde{\mathbb{E}}_1$ whose supports have size 9. Their sizes are $\binom{12}{9}3^6$, $18\binom{12}{9}3^6$ and $8\binom{12}{9}3^6$.*

Proof Let S be a given subset of size 9 of I, and set $S' := I \setminus S$. The first isomorphism class comes from the trivial $\overline{K}_S$-orbit, and the second one comes from the $\overline{K}_S$-orbit consisting of the images of W_i, $i \in S$. Note that $W/(W^S + \overline{C})$ is of dimension 3 (by Lemma 13) and so it has $\frac{3^3 - 1}{2} = 13$ 1-spaces in total. The second orbit accounts for nine of these 1-spaces. So, we just need to show that the remaining four 1-spaces form a single $\overline{K}_S$-orbit.

Take distinct $i, j \in S$, and consider $T = W_{\{i,j\}}$. Clearly, the image of T in $W/(W^S + \overline{C})$ contains the images of W_i and W_j. Does it contain the image of W_s for any third $s \in S$? This happens if and only if $W_s \leq T + W^S + \overline{C}$, or equivalently, $(W_s + T + W^S) \cap \overline{C} \neq 0$. Note that $W_s + T + W^S = W_{\{i,j,s\} \cup S'}$ and the set $\{i, j, s\} \cup S'$ has size 6. Hence the intersection is nontrivial if and only if this set is a hexad. Since the 5-set $\{i, j\} \cup S'$ lies in a unique hexad, such s is unique for given i and j. Hence, out of the four 1-spaces in the image of T, three 1-spaces are the images of W_i, W_j, and W_s, and the fourth one is not of this kind. Let us denote this 1-space by $Y_{i,j}$. Note that $Y_{i,j} = Y_{i,s} = Y_{j,s}$.

We claim that all the 1-spaces $Y_{i,j}$, $i, j \in S$, $i \neq j$, are conjugate under $\overline{K}_S$. Indeed, by 5-transitivity of $\overline{K}$ on $\{W_i \mid i \in I\}$, $\overline{K}_S$ is 2-transitive on the set $\{W_i \mid i \in S\}$. Hence we get here a new orbit, and it suffices now to show that there are at least four different 1-spaces $Y_{i,j}$. For this, fix i and consider possible pairs $\{j, s\}$. Such pairs (where the order of j and s is not important) form a partition of $S \setminus \{i\}$, and so we have exactly $\frac{8}{2} = 4$ of them. Consider two such pairs $\{j, s\}$ and $\{j', s'\}$. If $Y_{i,j} = Y_{i,j'}$, then the images of $W_{i,j}$ and $W_{i,j'}$ in $W/(W^S + \overline{C})$ share the image of W_i and $Y_{i,j} = Y_{i,j'}$, implying that they coincide. This gives a contradiction, as we know by the above that the image of $W_{j'}$ does not lie in the image of $W_{i,j}$. This shows that the four 1-spaces $Y_{i,j}$ with a fixed i are pairwise distinct. Hence the orbit consisting of the 1-spaces

$Y_{i,j}$ has size at least 4, and therefore, its size is exactly 4, as claimed. This $\overline{K}_S$-orbit leads to a hyperplane class of size $8\binom{12}{9}3^6$.

Proposition 8 *There are three isomorphism classes of hyperplanes of $\widetilde{\mathbb{E}_1}$ with supports of size* 10. *Their sizes are* $\binom{12}{10}3^6$, $20\binom{12}{10}3^6$ *and* $60\binom{12}{10}3^6$.

Proof Let S be a given subset of size 10 of I, and set $S' := I \setminus S$. The first isomorphism class comes from the trivial $\overline{K}_S$-orbit and the second isomorphism class comes from the $\overline{K}_S$-orbit of the images of W_i, $i \in S$, in $W/(W^S + \overline{C})$. Note that the total number of 1-spaces in $W/(W^S + \overline{C})$ is $\frac{3^4-1}{2} = 40$. Hence we have thirty 1-spaces unaccounted for. We claim that they form a single $\overline{K}_S$-orbit.

To prove this, let us look again, as in the preceding proposition, at the 1-spaces of W with support of size 2. Let us call them *double 1-spaces*. Recall that each 2-subspace $W_{\{i,j\}}$, $i, j \in S$ with $i \neq j$, contains two such subspaces, in addition to the subspaces W_i and W_j. This gives us the set X of $2\binom{10}{2} = 90$ double 1-spaces with support in S. Since the Schur multiplier of M_{11} is trivial, the stabiliser of any W_i, $i \in I$, in $\overline{K} \cong 2 \cdot M_{12}$ is isomorphic to $M_{11} \times C_2$ and so the subgroup of $\overline{K}_S$ stabilising each W_i with $i \in S'$ is isomorphic to $M_{10} \times C_2$. We will see that M_{10} (regarded as subgroup of $\overline{K}_S$) acts transitively on X. Since M_{10} is 3-transitive on $\{W_i \mid i \in S\}$, it is transitive on the 45 sets $W_{\{i,j\}}$, where i and j are two distinct elements of S, and so the only other possibility is that M_{10} has two orbits of length 45 on X and, for any 2-subset $\{i,j\}$ of S, the two double 1-spaces from $W_{\{i,j\}}$ belong to different orbits.

Let us fix two distinct $i, j \in S$ and let $Y = \langle \bar{e}_i + \bar{e}_j \rangle$ and $Y' = \langle \bar{e}_i - \bar{e}_j \rangle$ be the two elements of X with support $\{i,j\}$. As the images of $W_i = \langle \bar{e}_i \rangle$ and $W_j = \langle \bar{e}_j \rangle$ in $W/(W^S + \overline{C})$ are distinct, the images of W_i, W_j, Y and Y' in $W/(W^S + \overline{C})$ are mutually distinct.

Now, suppose $s \in S \setminus \{i,j\}$. Then the images of W_s, Y and Y' are mutually distinct. For, if this were not the case, then $W_{\{i,j,s\}} \cap (W^S + \overline{C}) \neq 0$, i.e., $W_{S' \cup \{i,j,s\}} \cap \overline{C} \neq 0$, in contradiction with the fact that $S' \cup \{i,j,s\}$ has size 5. A similar argument also shows that if Y'' is a double 1-space with support equal to $\{i,s\}$ or $\{j,s\}$, then the images of Y, Y' and Y'' are mutually distinct.

Let us investigate how many double 1-spaces from X can have their images coinciding with the images of either Y or Y'. Consider $Y'' \in X$ with support $\{s,t\} \neq \{i,j\}$. If the image of Y'' coincides with that of Y or Y', then $\{s,t\} \subset S \setminus \{i,j\}$ by the previous paragraph and $(W_{\{i,j\}} + W_{\{s,t\}}) \cap (W^S + \overline{C}) \neq 0$, i.e., $W^S + W_{\{i,j\}} + W_{\{s,t\}} = W_{S' \cup \{i,j,s,t\}}$ has a nontrivial intersection with $\overline{C}$. It follows that $S' \cup \{i,j,s,t\}$ is a hexad. There are exactly four pairs $\{s,t\}$ satisfying this condition. Note that the images of $W_{\{i,j\}}$ and $W_{\{s,t\}}$ in $W/(W^S + \overline{C})$ cannot coincide and so $W_{\{s,t\}}$ contains at most one double 1-space whose image coincides with that of Y or Y'. Our calculation thus shows that in total there are exactly four double 1-spaces Y'' as above.

If Y and Y' are conjugate under the action of M_{10} then, clearly, they share these Y'' equally, and so the image of Y coincides with the image of exactly

two other double 1-spaces $Y'' \in X$. Hence the image of X in $W/(W^S + \overline{C})$ is an orbit of size $\frac{90}{3} = 30$, which is the claim of the proposition.

Let us now suppose that Y and Y' are not conjugate under the action of M_{10}. Note that the orbit of Y (or Y'), being of length $45 > 30$, cannot map injectively into the set of 1-spaces of $W/(W^S + \overline{C})$. So, it is an m-to-1 mapping, where m divides 45, the length of the orbit. Hence $m \geq 3$, and this means that each of Y and Y' must correspond to at least two Y'' above. Since there is exactly four double 1-spaces Y'' in total, we conclude that Y (respectively, Y') has exactly two Y'' whose image coincides with that of Y (respectively, of Y'). Furthermore, these two Y'' are in the same orbit as Y (respectively, as Y'). This means that the image of the orbit of Y is an orbit of length $\frac{45}{3} = 15$ and the image of the orbit of Y' is a further orbit of size $\frac{45}{3} = 15$. This now is a contradiction as M_{10} has no transitive actions of length 15. Indeed, by the list of maximal subgroups of $M_{10} = A_6.2_3$ given in the ATLAS [4], we know that M_{10} has no subgroups of index 15.

Proposition 9 *There are three isomorphism classes of hyperplanes of $\widetilde{\mathbb{E}_1}$ whose supports have size* 11. *Their sizes are* $\binom{12}{11}3^6$, $22\binom{12}{11}3^6$ *and* $220\binom{12}{11}3^6$.

Proof Let $S \subseteq I$ be of size 11, and set $S' := I \setminus S$. Clearly, the first isomorphism class comes from the trivial $\overline{K}_S$-orbit on $W/(W^S + \overline{C})$, and the second isomorphism class comes from the orbit consisting of the images of the 1-spaces W_i, $i \in S$. We have shown in the preceding proposition that a certain group isomorphic to M_{10} acts transitively on the 90 double 1-spaces in W with support in the suitable 10-element subset of I. It follows that $\overline{K}_S$ is also transitive on the set X of $2\binom{11}{2} = 110$ double 1-spaces with support in S.

We show that two of such 1-spaces, with supports $\{i, j\}$ and $\{s, t\}$, cannot have the same image in $W/(W^S + \overline{C})$. In the case $\{i, j\} = \{s, t\}$, this is (similarly as in Proposition 8) a consequence of the fact that W_i and W_j have distinct images in $W/(W^S + \overline{C})$. In the case where $\{i, j\}$ and $\{s, t\}$ intersect in a singleton, say $\{j\} = \{t\}$, this is a consequence of the fact that $|S' \cup \{i, j, s\}| = 4$, which implies that $W^S + W_{\{i,j,s\}} = W_{S' \cup \{i,j,s\}}$ has trivial intersection with $\overline{C}$. In the case where $\{i, j\}$ and $\{s, t\}$ are disjoint, this is a consequence of the fact that $|S' \cup \{i, j, s, t\}| = 5$, which implies that $W^S + W_{\{i,j\}} + W_{\{s,t\}} = W_{S' \cup \{i,j,s,t\}}$ has trivial intersection with $\overline{C}$. It follows that the set X maps into the set of 1-spaces from $W/(W^S + \overline{C})$ injectively, giving an orbit of length $|X| = 110$.

Since the total number of 1-spaces in $W/(W^S + \overline{C})$ is $\frac{3^5 - 1}{2} = 121$ and the latter is $11 + 110$, the enumeration of $\overline{K}_S$-orbits is complete.

For the last case $k = 12$, we will rely on the known fact that the automorphism group $\overline{K}$ of the "extended ternary Golay code" $\overline{C} \subset W$ has four orbits on $W/\overline{C}$, the trivial orbit $\{\overline{C}\}$, an orbit containing all 24 vectors of the form $\bar{v} + \overline{C}$ where $\bar{v}$ has weight 1, an orbit containing all 264 vectors of the form $\bar{v} + \overline{C}$ where $\bar{v}$ has weight 2 and an orbit of size 440 containing all vectors of the form $\bar{v} + \overline{C}$ where $\bar{v}$ has weight 3.

If $k = |S| = 12$, then $\overline{K}_S = \overline{K}$, $W^S + \overline{C} = \overline{C}$, and so we have:

Proposition 10 *There are four isomorphism classes of hyperplanes of $\widetilde{\mathbb{E}_1}$ whose supports have size 12. Their sizes are 3^6, $24 \cdot 3^6$, $264 \cdot 3^6$ and $440 \cdot 3^6$.*

6 Computer computations and explicit constructions of the hyperplanes

In Section 5, we classified all isomorphism classes of hyperplanes of $\mathbb{E}_1$, and found that there are 25 of them. This classification was achieved without the help of a computer. We have also implemented a computer program to determine explicitly all hyperplanes of $\mathbb{E}_1$. This program confirmed our previous results. We did find the same number of hyperplanes (namely 25) and the same orbit sizes. Once a particular hyperplane is found, we can collect all kinds of combinatorial information about it. In this way, the data in Table 1 (columns 4–8) was obtained.

We started by implementing a computer model of $\mathbb{E}_1$. The automorphism group G of $\mathbb{E}_1$ acts primitively on its point set. The corresponding permutation group on 729 points can easily be retrieved in GAP [13] as the unique primitive permutation group of size $2 \cdot 3^6 \cdot |M_{12}|$ and degree 729. Using this permutation group, the following GAP code builds a computer model for $\mathbb{E}_1$ with point set $\{1, 2, \ldots, 729\}$, line set `lines`, automorphism group `g` and distance function `Distance`. The code below is based on the fact that G acts distance-transitively on the point set, and that $(|\Gamma_0(x)|, |\Gamma_1(x)|, |\Gamma_2(x)|, |\Gamma_3(x)|) = (1, 24, 264, 440)$ for every point x of $\mathbb{E}_1$. Here, $\Gamma_i(x)$ with $i \in \{0, 1, 2, 3\}$ denotes the set of points at distance i from x.

```
v:=729;
size:=Size(MathieuGroup(12))*3^6*2;
g:=AllPrimitiveGroups(DegreeOperation,v,Size,size)[1];
orbs := Orbits(Stabilizer(g,1),[1..v]);
dist0 := Set(Filtered(orbs,x->Size(x)=1)[1]);
dist1 := Set(Filtered(orbs,x->Size(x)=24)[1]);
dist2 := Set(Filtered(orbs,x->Size(x)=264)[1]);
dist3 := Set(Filtered(orbs,x->Size(x)=440)[1]);
perp := Union([1],dist1);
r := RepresentativeAction(g,1,dist1[1]);
line := Intersection(perp,OnSets(perp,r));
lines := Orbit(g,line,OnSets);

Distance:=function(i,j)
 local r,k;
 r:=RepresentativeAction(g,i,1);
 k:=j^r;
 if k in dist0 then return 0; fi;
 if k in dist1 then return 1; fi;
 if k in dist2 then return 2; fi;
 if k in dist3 then return 3; fi;
```

```
end;
```

One type of hyperplane can easily be implemented, namely the singular hyperplanes, as it consists of all points at distance at most 2 from a given point. Once we have implemented the singular hyperplanes, it is easy to implement all remaining hyperplanes. Indeed, the following proposition says that every hyperplane of $\mathbb{E}_1$ can be obtained as sum of singular hyperplanes.

Proposition 11 *For every hyperplane H of $\mathbb{E}_1$, there exist singular hyperplanes $H_1, H_2, \ldots, H_k$ for a certain $k \in \mathbb{N} \setminus \{0\}$ such that $H = H_1 + H_2 + \cdots + H_k$.*

Proof In view of Proposition 1, it suffices to prove this in the case where H is an ovoid. Set $H = \{z_1, z_2, \ldots, z_{243}\}$. If $x \in H$, then the straightforward counting shows that $|\Gamma_0(x) \cap H| = 1$, $|\Gamma_1(x) \cap H| = 0$, $|\Gamma_2(x) \cap H| = 132$ and $|\Gamma_3(x) \cap H| = 110$. If y is a point not belonging to H, then $|\Gamma_0(y) \cap H| = 0$, $|\Gamma_1(y) \cap H| = 12$, $|\Gamma_2(y) \cap H| = 66$ and $|\Gamma_3(y) \cap H| = 165$. For every $z \in \{z_1, z_2, \ldots, z_{243}\}$, we denote by H_z the singular hyperplane with center z. Then $H_{z_1} + H_{z_2} + \cdots + H_{z_{243}}$ contains all points u that lie outside an even number of the singular hyperplanes $H_{z_1}, H_{z_2}, \ldots, H_{z_{243}}$, i.e., all points u for which $|\Gamma_3(u) \cap H|$ is even. It follows that $H_{z_1} + H_{z_2} + \cdots + H_{z_{243}} = H$. $\qquad\square$

The complete GAP code of our computation can be found online [9].

As previously mentioned, some of the hyperplanes of $\mathbb{E}_1$ are special, like the ovoids (which occur as hyperplanes of type 1 in Table 1) and the singular hyperplanes. Based on the discussion given in Sections 4 and 5, we now give explicit constructions for all 25 hyperplanes as sums of ovoids and/or singular hyperplanes. We have also verified these constructions by means of GAP computations.

Let $O_1, O_2, \ldots, O_k$ be a collection of $k \in I$ mutually nonequivalent ovoids of $\widetilde{\mathbb{E}_1}$ such that $O_1, O_2, \ldots, O_{k-\epsilon}$ contain a given point, say, the origin o of $AG(6, 3)$, and $O_{k-\epsilon+1}, O_{k-\epsilon+2}, \ldots, O_k$ do not contain o. Here, $\epsilon = 0$ if $k \in \{1, 2, 3, 4, 5\}$, $\epsilon \in \{0, 1\}$ if $k \in \{6, 7, 8\}$, $\epsilon \in \{0, 1, 2\}$ if $k \in \{9, 10, 11\}$ and $\epsilon \in \{0, 1, 2, 3\}$ if $k = 12$. Set $S = \{s_1, s_2, \ldots, s_k\}$, where $\{\Pi_{s_1}, \Pi_{s_2}, \ldots, \Pi_{s_k}\} = \{\Pi_{O_1}, \Pi_{O_2}, \ldots, \Pi_{O_k}\}$, and $S' := I \setminus S$. We moreover assume that

- $\epsilon = 0$ if $k = |S| = 6$ and S is not a hexad;
- $\{\Pi_{O_1}, \Pi_{O_2}, \ldots, \Pi_{O_6}\}$ is not a block of $S^*(5, 6, 12)$ if $k = 7$ and $\epsilon = 1$;
- if $k = 9$, $\epsilon = 2$ and $O_8 = O_{\lambda_i}^{(i)}$, $O_9 = O_{\lambda_j}^{(j)}$, where i, j are two distinct elements of S and $\lambda_i, \lambda_j \in \mathbb{F}_3 \setminus \{0\}$, then $\langle \lambda_i \bar{e}_i + \lambda_j \bar{e}_j, \bar{e}_s, \bar{e}_k \mid k \in S' \rangle \cap \overline{C} = 0$, where s is the unique element of S such that $\{i, j, s\} \cup S'$ is a hexad.

If either $k \neq 6$ or S is a hexad, then we denote the hyperplane $H := O_1 + O_2 + \cdots + O_k$ also by $H_{k,\epsilon}$. If $k = 6$ and S is not a hexad (and so, $\epsilon = 0$), then we denote H also by $H_{6',0}$. The type of the hyperplane $H_{k,\epsilon}$ is denoted by $T_{k,\epsilon}$ and the type of $H_{6',0}$ by $T_{6',0}$.

From the discussion in Sections 4 and 5, it follows that the hyperplane H is associated with the trivial $\overline{K}_S$-orbit if $\epsilon = 0$, the $\overline{K}_S$-orbit consisting of the images of W_i, $i \in S$ if $\epsilon = 1$ (with i not belonging to the unique hexad containing $I \setminus S$ if $k = 7$), the $\overline{K}_S$-orbit consisting of the images of the double 1-spaces of W_S if $\epsilon = 2$ and $k \neq 9$, the images of certain double 1-spaces of W_S if $\epsilon = 2$ and $k = 9$, and the $\overline{K}$-orbit consisting of all vectors $\bar{v} + \overline{C}$ where $\bar{v} \in W$ has weight 3 if $\epsilon = 3$ (then $k = 12$, $\overline{K}_S = \overline{K}$ and $W^S + \overline{C} = \overline{C}$). So, we have the following values for $T_{k,\epsilon}$ and $T_{6',0}$:

$T_{1,0}$	1	$T_{6,0}$	6	$T_{8,0}$	11	$T_{10,0}$	16	$T_{11,2}$	21
$T_{2,0}$	2	$T_{6,1}$	7	$T_{8,1}$	12	$T_{10,1}$	17	$T_{12,0}$	22
$T_{3,0}$	3	$T_{6',0}$	8	$T_{9,0}$	13	$T_{10,2}$	18	$T_{12,1}$	23
$T_{4,0}$	4	$T_{7,0}$	9	$T_{9,1}$	14	$T_{11,0}$	19	$T_{12,2}$	24
$T_{5,0}$	5	$T_{7,1}$	10	$T_{9,2}$	15	$T_{11,1}$	20	$T_{12,3}$	25

Remarks:

- If we choose $\epsilon = 1$ in the case that $|S| = 6$ and S is not a hexad, then the corresponding hyperplane also has type $T_{6',0}$.
- Suppose $k = 7$ and $\epsilon = 1$. If we had chosen the ovoids $O_1, O_2, \ldots, O_7$ in such a way that $\{\Pi_{O_1}, \Pi_{O_2}, \ldots, \Pi_{O_6}\}$ is a block of $S^*(5, 6, 12)$, then the corresponding hyperplane has type $T_{7,0}$ (instead of $T_{7,1}$).
- Suppose $k = 9$, $\epsilon = 2$ and $O_8 = O_{\lambda_i}^{(i)}$, $O_9 = O_{\lambda_j}^{(j)}$, where i, j are two distinct elements of S and $\lambda_i, \lambda_j \in \mathbb{F}_3 \setminus \{0\}$ such that $\langle \lambda_i \bar{e}_i + \lambda_j \bar{e}_j, \bar{e}_s, \bar{e}_k \,|\, k \in S' \rangle \cap \overline{C} = 0$, where s is the unique element of S such that $\{i, j, s\} \cup S'$ is a hexad. Then $\langle \lambda_i \bar{e}_i - \lambda_j \bar{e}_j, \bar{e}_s, \bar{e}_k \,|\, k \in S' \rangle \cap \overline{C} \neq 0$ and $O_1 + O_2 + \cdots + O_7 + O_{\lambda_i}^{(i)} + O_{-\lambda_j}^{(j)}$ has type $T_{9,1}$ (instead of $T_{9,2}$).

Proposition 12 *Let $O_1, O_2, \ldots, O_{12}$ denote the twelve ovoids of $\widetilde{\mathbb{E}_1}$ through a point x. Then $O_1 + O_2 + \cdots + O_{12}$ is the singular hyperplane with center x. As a consequence, the singular hyperplanes are precisely the hyperplanes isomorphic to $H_{12,0}$ (and hence have type 22).*

Proof Clearly, $x \in O_1 + O_2 + \cdots + O_{12}$.

Suppose $y \in \Gamma_1(x)$. Then $y \notin O_i$ for every $i \in \{1, 2, \ldots, 12\}$ and hence $y \in O_1 + O_2 + \cdots + O_{12}$.

Suppose $y \in \Gamma_2(x)$. Then xy meets $\mathrm{PG}(5,3)$ in a point p with $\mathcal{K}$-index 2. Through p there are precisely six elements of $\mathcal{K}^*$. As y does not belong to an even number of ovoids in the set $\{O_1, O_2, \ldots, O_{12}\}$, we have $y \in O_1 + O_2 + \cdots + O_{12}$.

Suppose $y \in \Gamma_3(x)$. Then xy meets $\mathrm{PG}(5,3)$ in a point p with $\mathcal{K}$-index 3. Through p there are precisely three elements of $\mathcal{K}^*$. As y does not belong to an odd number of the ovoids in the set $\{O_1, O_2, \ldots, O_{12}\}$, we have $y \notin O_1 + O_2 + \cdots + O_{12}$.

So, $O_1 + O_2 + \cdots + O_{12}$ equals the singular hyperplane with center x. $\square$

Proposition 13 *Let x_1 and x_2 be two points of $\widetilde{\mathbb{E}_1}$ and let H_i with $i \in \{1, 2\}$ denote the singular hyperplane with center x_i.*

(1) *If $d(x_1, x_2) = 1$, then $H_1 + H_2$ is the singular hyperplane whose center is the unique point of $x_1 x_2$ distinct from x_1 and x_2.*
(2) *If $d(x_1, x_2) = 2$, then $H_1 + H_2$ is isomorphic to $H_{6,0}$.*
(3) *If $d(x_1, x_2) = 3$, then $H_1 + H_2$ is isomorphic to $H_{9,0}$.*

Proof Let p denote the point of $PG(5, 3)$ contained in the line $x_1 x_2$ and let x_3 denote the fourth point on the line $x_1 x_2$. Set $\mathcal{K}^* = \{\Pi_1, \Pi_2, \ldots, \Pi_{12}\}$. Without loss of generality, we may suppose that there exists a $k \in \{0, 1, 2, \ldots, 12\}$ such that $\Pi_1, \Pi_2, \ldots, \Pi_k$ contain p and $\Pi_{k+1}, \Pi_{k+2}, \ldots, \Pi_{12}$ do not. Then $k = 0$ if $d(x_1, x_2) = i_{\mathcal{K}}(p) = 1$, $k = 6$ if $d(x_1, x_2) = i_{\mathcal{K}}(p) = 2$ and $k = 3$ if $d(x_1, x_2) = i_{\mathcal{K}}(p) = 3$. For every $i \in \{1, 2, \ldots, 12\}$ and every $j \in \{1, 2, 3\}$, let $X_i^{(j)}$ denote the unique ovoid containing x_j for which $\Pi_{X_i^{(j)}} = \Pi_i$. By Lemma 2, $X_i^{(1)} = X_i^{(2)} = X_i^{(3)}$ for every $i \in \{1, 2, \ldots, k\}$ and $X_i^{(1)} + X_i^{(2)} = X_i^{(3)}$ for every $i \in \{k + 1, k + 2, \ldots, 12\}$. Taking into account Proposition 12, this implies that $H_1 + H_2 = (X_1^{(1)} + X_2^{(1)} + \cdots + X_{12}^{(1)}) + (X_1^{(2)} + X_2^{(2)} + \cdots + X_{12}^{(2)}) = (X_{k+1}^{(3)} + X_{k+2}^{(3)} + \cdots + X_{12}^{(3)})$ is isomorphic to $H_{12,0}$ if $d(x_1, x_2) = 1$, isomorphic to $H_{6,0}$ if $d(x_1, x_2) = 2$ and isomorphic to $H_{9,0}$ if $d(x_1, x_2) = 3$. If $d(x_1, x_2) = 1$, then $H_1 + H_2$ must be a singular hyperplane by Proposition 12. Since x_3 is contained in each of the ovoids $X_1^{(3)}, X_2^{(3)}, \ldots, X_{12}^{(3)}$, x_3 is the center of this singular hyperplane.

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