Long properly coloured cycles in edge-coloured graphs

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LONG PROPERLY COLOURED CYCLES IN EDGE-COLOURED GRAPHS.

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Abstract. Let G be an edge-coloured graph. The minimum colour degree $\delta^c(G)$ of G is the largest integer k such that, for every vertex v, there are at least k distinct colours on edges incident to v. We say that G is properly coloured if no two adjacent edges have the same colour. In this paper, we show that, for any $\varepsilon > 0$ and n large, every edge-coloured graph G with $\delta^c(G) \geq (1/2 + \varepsilon)n$ contains a properly coloured cycle of length at least $\min\{n, \lfloor 2\delta^c(G)/3 \rfloor\}$.

1. Introduction

An *edge-coloured* graph is a graph G with an edge-colouring c of G. We say that G is *properly coloured* if no two adjacent edges of G have the same colour. If all edges have the same (or distinct) colour, then G is *monochromatic* (or *rainbow*, respectively).

Finding properly coloured subgraphs in edge-coloured graphs G has a long and rich history. Grossman and Häggkvist [10] are the first to give a sufficient condition on the existence of properly coloured cycles in edge-coloured graphs with two colours. Later on, Yeo [19] extended the result to edge-coloured graphs with any number of colours. A natural question is to ask what guarantees the existence of properly coloured Hamiltonian paths and cycles.

In particular, the case when G is an edge-coloured K_n has been receiving the most attention. Given $k \in \mathbb{N}$, an edge-coloured graph G is *locally k-bounded* if for all vertices $v \in V(G)$, no colour appears more than k times on the edges incident to v for all vertices v. A conjecture of Bollobás and Erdős [4] states that every locally $(\lfloor n/2 \rfloor - 1)$-bounded edge-coloured K_n contains a properly coloured Hamilton cycle. There is a series of partial results toward this conjecture by Bollobás and Erdős [4], Chen and Daykin [6], Shearer [17], and Alon and Gutin [1]. In [15] the author showed that the conjecture of Bollobás and Erdős holds asymptotically, that is, for any $\varepsilon > 0$ and n sufficiently large, every locally $(1/2 - \varepsilon)n$-bounded edge-coloured K_n contains a properly coloured Hamilton cycle. A hypergraph generalisation of finding properly coloured Hamilton cycle in locally k-bounded edge-coloured complete graphs has also been studied by Dudek, Frieze and Ruciński [8] as well as Dudek and Ferrara [7]. Recently, Sudakov and Volec [18] proved that every locally $n/(500r^{3/4})$-bounded edge-coloured K_n contains all properly coloured graphs with at most r paths of length two. This proved a conjecture of Shearer [17] as well as improves results of

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Alon, Jiang, Miller, Pritikin [2] and Böttcher, Kohayakawa and Procacci [5]. For a survey regarding properly coloured subgraphs in edge-coloured graphs, we recommend Chapter 16 of [3]. Also see [11] for a survey regarding monochromatic and rainbow subgraphs in edge-coloured graphs.

Consider an edge-coloured (not necessarily complete) graph G. Given a vertex $v \in V(G)$, the *colour degree* $d^c_G(v)$ is the number of distinct colours of edges incident to v. The *minimum colour degree* $\delta^c(G)$ is the minimum $d^c_G(v)$ over all vertices v in G. Li and Wang [12] showed that every edge-coloured graph G with $\delta^c(G) \geq d$ contains a properly coloured path of length $2d$ or a properly coloured cycle of length at least $2d/3$. In [13], the author improved $2d/3$ to $d+1$, which is best possible. In the same paper, the author conjectured the following.

Conjecture 1.1. *Every edge-coloured connected graph G with $\delta^c(G) \geq d$ contains a properly coloured Hamilton cycle or a properly coloured path of length $\lfloor 3d/2 \rfloor$.*

If this conjecture holds, then the bound is sharp by the following example. Let $d, n \in \mathbb{N}$ with $n \geq 3d/2$. Let $c_1, c_2, \dots, c_d$ be distinct colours. Let X, Y be disjoint sets of vertices such that $X = \{x_1, x_2, \dots, x_d\}$ and $|Y| = n - d$. For each $1 \leq i \leq d$, join x_i to each vertex of Y with colour c_i. For $1 \leq i < j \leq d$, join x_i to x_j with a new distinct colour. Let G be the resulting edge-coloured graph. Note that G has n vertices and $\delta^c(G) = d$. Every properly coloured path in G with both endpoints in Y must contain at least two vertices in X. Thus, every properly coloured path in G is of length at most $|X| + \lfloor |X|/2 \rfloor = \lfloor 3d/2 \rfloor$.

In [14], the author proved that the conjecture holds when $d \geq (2/3+\varepsilon)n$ for $\varepsilon > 0$ and n large, that is, every edge-coloured graph G on n vertices with $\delta^c(G) \geq (2/3+\varepsilon)n$ contains a properly coloured Hamilton cycle.

In this paper, we prove the following results.

Theorem 1.2. *For $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that every edge-coloured graph G on $n \geq n_0$ vertices with $\delta^c(G) \geq (1/2+\varepsilon)n$ contains a properly coloured cycle of length at least* $\min\{\lfloor 3\delta^c(G)/2 \rfloor, n\}$.

Note that Theorem 1.2 implies Conjecture 1.1 when $d \geq (1/2+\varepsilon)n$ and n large. By analysing the proof of Theorem 1.2, one might be able to prove Conjecture 1.1 when $d \geq n/2$. Therefore, it would be interesting to know whether Conjecture 1.1 hold for $d < n/2$.

2. Notation and sketch proof

For a graph G, we denote $V(G)$ and $E(G)$ for the vertex set and edge set of G, respectively. Write $|G|$ for $|V(G)|$. For (edge-coloured) graphs G and H, we write $G - H$ for the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$. For $W \subseteq V(G)$, we write $G \setminus W$ for the subgraph of G induced by the vertex set $V(G) \setminus W$, and write $G \setminus H$ for $G \setminus V(H)$. For disjoint $X, Y \subseteq V(G)$, let $G[X]$ be the (edge-coloured) subgraph induced by X and let $G[X, Y]$ be the induced bipartite subgraph with vertex classes X and Y. For a set of edges E, we write $G \cup E$ for the graph with vertex set $V(G) \cup V(E)$ and edge set $E(G) \cup E$. For a singleton set $\{v\}$, we sometimes write v for short.

For an edge-coloured graph G, let $C(G) := \{c(uv) : uv \in E(G)\}$, that is, the set of colours appeared in G. For a vertex $v \in V(G)$, let $C_G(v) := \{c(uv) : u \in N_G(v)\}$. Thus $d^c_G(v) = |C_G(v)|$. For $V \subseteq V(G)$, define $d^c_G(v, V) := |C_{G[V \cup v]}(v)|$. Let $\mathbf{x} = (x, c_x)$ be a pair with vertex $x \in V(G)$ and colour $c_x \in C_G(x)$. We write $N_G(\mathbf{x})$ be the set of vertices $v \in N_G(x)$ such that $c(xv) \neq c_x$. For distinct $x, y \in V(G)$, we denote by

$\mathrm{dist}_G(x, y)$ the shortest distance between x and y. If x and y are not connected, then we say $\mathrm{dist}_G(x, y) = \infty$. If G is known from the context, then we omit G in the subscript.

For a path $P = x_1x_2 \dots x_k$ from x_1 to x_k and a vertex $y \notin V(P)$, we write Py for the path $x_1x_2 \dots x_ky$. If $P' = y_1 \dots y_\ell$ is a path with $y_1 = x_k$ and $V(P) \cap V(P') = \{x_k\}$, then we write PP' for the concatenated path $x_1x_2 \dots x_ky_2 \dots y_\ell$.

An edge-coloured graph G is *critical*, if for every edge uv, $d^c_G(u) > d^c_{G-uv}(u)$ or $d^c_G(v) > d^c_{G-uv}(v)$. Note that if G is critical, then any monochromatic subgraph H of G is a union of vertex-disjoint stars. Since we are only concerning about properly coloured subgraphs, we may assume further that any two vertex-disjoint monochromatic component in G have distinct colours. Thus, from now on, we assume that every monochromatic subgraph H of any critical edge-coloured graph G is a star.

Let F be a direct graph. For $u, v \in V(F)$, we write uv for the directed edge from u to v. For $Z, W \subseteq V(F)$, denote by $e_F(Z, W)$ the number of directed edges from Z to W in F.

The constants in the hierarchies used to state our results are chosen from right to left. For example, if we claim that a result holds whenever $0 < 1/n \ll a \ll b \ll c \leq 1$ (where n is the order of the graph), then there is a non-decreasing function $f : (0, 1] \to (0, 1]$ such that the result holds for all $0 < a, b, c \leq 1$ and all $n \in \mathbb{N}$ with $b \leq f(c)$, $a \leq f(b)$ and $1/n \leq f(a)$. Hierarchies with more constants are defined in a similar way.

2.1. **Sketch proof of Theorem 1.2.** Here we present an outline of the proof of Theorem 1.2, which naturally splits into three lemmas. First, we consider the case when G is close to the extremal example in Section 3. More precisely, for $\delta, \varepsilon > 0$, we say that an edge-coloured graph G on n vertices is (δ, ε)-*extremal* if there exist disjoint $A, B \subseteq V(G)$ such that

- (A1) $|A| \geq (\delta - \varepsilon)n$ and $|B| \geq (1 - \delta - \varepsilon)n$;
- (A2) for each $a \in A$, there exists a distinct colour c_a such that there are at least $|B| - \varepsilon n$ vertices $b \in B$ such that $c(ab) = c_a$;
- (A3) for each $b \in B$, $d_G(b) \leq (\delta + \varepsilon)n$ and b has at least $|A| - \varepsilon n$ neighbours $a \in A$ such that $c(ab) = c_a$.

Throughout this paper, we will always assume that $\varepsilon \ll \delta$. In this case, we will find a properly coloured cycle (of the desired length) directly (see Section 3).

Lemma 2.1. *Let $0 < 1/n \ll \varepsilon \ll \delta \leq 1$. Let G be a (δ, ε)-extremal critical edge-coloured graph on n vertices with $\delta^c(G) \geq \delta n$. Then G contains a properly coloured cycle of length $\min\{\lfloor 3\delta n/2 \rfloor, n\}$.*

Note that Lemma 2.1 does not require that $\delta \geq 1/2 + \varepsilon$. Thus Lemma 2.1 implies that Conjecture 1.1 holds if G is (δ, ε)-extremal with $1/n \ll \varepsilon \ll \delta \leq 1$.

If G is not close to the extremal, then we proceed using the *absorption technique* introduced by Rödl, Ruciński and Szemerédi [16], which was used to tackle Hamiltonicity problems in hypergraphs. The absorption technique has been adapted for finding properly coloured Hamilton cycles in [14, 15]. First we find a small 'absorbing cycle' C in G using the following lemma, which is proved in Section 4.

Lemma 2.2. *Let $0 < 1/n \ll \gamma \ll \varepsilon < 1/2$. Suppose that G is an edge-coloured graph on n vertices with $\delta^c(G) \geq (1/2 + \varepsilon)n$. Then there exists a properly coloured cycle C of length at most $\varepsilon n/2$ such that for any collection $P_1, \dots, P_k$ of vertex-disjoint properly coloured paths in $G \setminus V(C)$ with $k \leq \gamma n$, there exists a properly coloured cycle with vertex set $V(C) \cup \bigcup_{1 \leq i \leq k} V(P_i)$.*

Remove the vertices of C from G and call the resulting graph G'. Since G is not extremal, neither is G'. (Indeed, if G' is (δ,ε)-extremal with vertex subsets A, B, then G is $(\delta, 2\varepsilon)$-extremal with vertex subsets A, B as $\varepsilon \ll 1$.) We find vertex-disjoint properly coloured paths by the next lemma (which is implied by Lemma 5.1).

Lemma 2.3. *Let $0 < 1/n \ll \beta \ll \varepsilon \ll 1/2 < \delta$. Suppose that G is a critical edge-coloured graph on n vertices with $\delta^c(G) \geq \delta n+1$. If G is not (δ,ε)-extremal, then G contains vertex-disjoint properly coloured paths $P_1, \dots, P_k$ with $k \leq 100\beta^{-1}$ covering* $\min\{(3\delta+\beta)n/2, n\}$ *vertices.*

We now prove Theorem 1.2 using Lemmas 2.1–2.3.

Proof of Theorem 1.2. Without loss of generality, we may assume that G is critical edge-coloured with $\delta^c(G) = \delta n$ and that ε is sufficiently small. Let γ, ε' be such that $1/n \ll \gamma \ll \varepsilon \ll \varepsilon' \ll 1/2 < \delta$.

Apply Lemma 2.2 and obtain a properly coloured cycle C of length at most $\varepsilon n/2$ such that for any collection $P_1, \dots, P_k$ of vertex-disjoint properly coloured paths in $G \setminus V(C)$ with $k \leq \gamma n$, there exists a properly coloured cycle with vertex set $V(C) \cup \bigcup_{1\leq i\leq k} V(P_i)$.

Let $G' := G \setminus C$, $n' := |G'|$ and $\delta' := (\delta n - |C| - 1)/n'$. Note that $\delta^c(G) \geq \delta' n' + 1$ and $1/n' \ll \varepsilon \ll \varepsilon' \ll 1/2 < \delta'$. If G' is not (δ', ε')-extremal, then apply Lemma 2.3 (with $\varepsilon, \varepsilon', \delta', n'$ playing the roles of $\beta, \varepsilon, \delta, n$) and obtain vertex-disjoint properly coloured paths $P_1, \dots, P_k$ such that $k \leq 100\varepsilon^{-1} \leq \gamma n$ and

$$\bigcup_{i\leq k} |V(P_i')| \geq \min\{3(\delta - |C| - 1)n + \varepsilon n')/2, n - |C|\} \geq \min\{3\delta n/2, n\} - |C|$$

as $|C| \leq \varepsilon n/2 \leq \varepsilon n'$. Thus, by the property of C, there exists a properly coloured cycle C' with vertex set $V(C) \cup \bigcup_{i\leq k} V(P_i')$. So $|C'| \geq \min\{3\delta n/2, n\}$ as desired.

On the other hand, if G' is (δ', ε')-extremal, then there exist disjoint $A, B \subseteq V(G') = V(G) \setminus V(C)$ satisfying

(A1) $|A| \geq (\delta' - \varepsilon')n' \geq (\delta - 2\varepsilon')n$ and $|B| \geq (1 - \delta' - \varepsilon')n' \geq (1 - \delta - 2\varepsilon')n$;
(A2) for each $a \in A$, there exists a colour c_a such that there are at least $|B| - \varepsilon' n' \geq |B| - 2\varepsilon' n$ vertices $b \in B$ such that $c(ab) = c_a$;
(A3) for each $b \in B$,
$$d_G(b) \leq d_{G'}(b) + |C| \leq (\delta' + \varepsilon')n' + |C| = \delta n - 1 + \varepsilon' n' < (\delta + 2\varepsilon')n$$
and b has at least $|A| - \varepsilon' n' \geq |A| - 2\varepsilon' n$ neighbours $a \in A$ such that $c(ab) = c_a$.

Therefore G is $(\delta, 2\varepsilon')$-extremal. By Lemma 2.1, G contains a properly coloured cycles of length at least $\min\{\lfloor 3\delta n/2 \rfloor, n\}$. □

3. Extremal case

In this section, we prove Lemma 2.1, that is, Theorem 1.2 when G is critical and (δ,ε)-extremal. We would need the following definition. Let G be an edge-coloured graph on n vertices. Let $A, B \subseteq V(G)$ be disjoint. We say that the ordered pair (A, B) is *ε-extremal* if the following holds:

(E1) for each $a \in A$, there exists a distinct colour c_a;
(E2) for each $a \in A$, there are at least $|B| - \varepsilon n$ vertices $b \in B \cap N(a)$ such that $c(ab) = c_a$, and at least $|A| - \varepsilon n$ vertices $a' \in A \cap N(a)$ such that $c_a \neq c(aa') \neq c_{a'}$;
(E3) for each $b \in B$, there are at least $|A| - \varepsilon n$ vertices $a \in A \cap N(b)$ such that $c(ab) = c_a$.

Next we show that if G is (δ, ε)-extremal, then there exists $4\sqrt{\varepsilon}$-extremal pair in G.

Lemma 3.1. *Let $0 < 1/n \ll \varepsilon \ll 1$ and let $\delta > 4\sqrt{\varepsilon}$. Let G be a critical edge-coloured graph on n vertices with $\delta^c(G) \geq \delta n$. Suppose that G is (δ, ε)-extremal. Then there exist disjoint $A, B \subseteq V(G)$ such that (A, B) is $4\sqrt{\varepsilon}$-extremal, $|A| \geq (\delta - 4\sqrt{\varepsilon})n$, $|B| \geq (1-\delta-\varepsilon)n$ and, for each $b \in B$, $d_G(b) \leq (\delta + \varepsilon)n$.*

Proof. Let $\varepsilon' := 4\sqrt{\varepsilon}$. Since G is (δ, ε)-extremal, there exist disjoint $A^*, B^* \subseteq V(G)$ satisfying (A1)–(A3).

Note that $|V(G) \setminus (A^* \cup B^*)| \leq 2\varepsilon n$. We say that an edge aa' in $G[A^*]$ is *good* if $c_a \neq c(aa') \neq c_{a'}$. We bound the number of good edges from below as follows. Define a directed graph D on A^* such that there is a directed edge from a to a' if and only if $c_a \neq c(aa')$. For each $a \in A^*$, a sends at most $1 + \varepsilon n + |V(G) \setminus (A^* \cup B^*)| \leq 3\varepsilon n + 1$ distinct colours (including the colour c_a) to $V(G) \setminus A^*$ by (A2). So the outdegree of a in D is at least $\delta n - 3\varepsilon n - 1 \geq |A^*| - 5\varepsilon n - 1$. Since the number of good edges equals the number of 2-cycles in D, the number of good edges is at least $(|A^*| - 5\varepsilon n - 1)|A^*| - \binom{|A^*|}{2} = |A^*|(|A^*| - 10\varepsilon n - 1)/2$. Let A' be the set of $a \in A^*$ that is incident with at most $|A^*| - \varepsilon' n$ good edges. Note that $|A'| \leq 3\sqrt{\varepsilon} n$.

Let $A := A^* \setminus A'$. Thus $|A| \geq |A^*| - 3\sqrt{\varepsilon} n \geq (\delta - \varepsilon')n$ by (A1). Moreover, every $a \in A$ is incident with at least $|A| - \varepsilon' n$ good edges in $G[A]$ implying (E2). Set $B := B^*$. So $|B| \geq (1 - \delta - \varepsilon)n$. Also, (A3) implies that (E3) holds and that, for each $b \in B$, $d_G(b) \leq (\delta + \varepsilon)n$. Therefore (A, B) is ε'-extremal. □

In the next two lemma, we find properly coloured cycles spanning $A \cup B$, when (A, B) is ε-extremal.

Lemma 3.2. *Let $\varepsilon < 1/36$. Let G be an edge-coloured graph on $3m$ vertices. Suppose that there is a partition A, B of $V(G)$ such that (A, B) is ε-extremal, $|A| = 2m$ and $|B| = m$. Then G has a properly coloured Hamilton cycle.*

Proof. Partition A into X and Y each of size m. Let H_X be the subgraph of $G[X, B]$ induced by edges of colour in $\{c_a : a \in A\}$. By (E2) and (E3), H_X is a bipartite graph with $\delta(H_X) \geq m - 3\varepsilon m$. Hence by Hall's theorem, there exists a perfect matching M_X in H_X.

Similarly, let H_Y be the subgraph of $G[Y, B]$ induced by edges of colour in $\{c_a : a \in A\}$ and there exists a perfect matching M_Y in H_Y. Note that $M_X \cup M_Y$ is a union of m vertex-disjoint path of length 2 each with midpoint in B. By (E1), $M_X \cup M_Y$ is properly coloured. Let $M_X \cup M_Y = \{x_i b_i y_i : x_i \in X, b_i \in B, y_i \in Y \text{ and } i \leq m\}$.

Now define an oriented graph F on vertex set $Z = \{z_1, \dots, z_m\}$ such that there is a directed edge from z_i to z_j if and only if $y_i x_j$ is an edge (in G) with $c_{y_i} \neq c(y_i x_j) \neq c_{x_j}$. By (E2), each z_i has indegree and outdegree at least $m - 3\varepsilon m \geq m/2$. Therefore F contains a directed Hamilton cycle by a result of Ghouila-Houri [9], $z_1 z_2 \dots z_m z_1$ say. Then $x_1 b_1 y_1 x_2 b_2 y_2 \dots z_m x_1$ is a properly coloured Hamilton cycle in G as desired. □

Lemma 3.3. *Let $\ell \in \mathbb{N}$ and $0 < 1/n \ll \varepsilon \ll \alpha < 1/3$ with $\ell < \alpha n$. Let G be a critical edge-coloured graph on n vertices. Suppose that (A, B) is ε-extremal such that $\alpha n + \ell + 1 \leq |B| \leq |A|/2 + \ell$. Suppose that $\mathcal{P}$ is a union of ℓ vertex-disjoint properly coloured paths such that each path has both of its endpoints in B and $|(A \cup B) \cap V(\mathcal{P})| = 2\ell$. Then G contains a properly coloured cycle with vertex set $V(C) = A \cup B \cup V(\mathcal{P})$.*

Proof. First suppose that $|B| < |A|/2 + \ell$. Let $p := |A| - 2(|B| - \ell - 1)$, so $3 \leq p \leq |A| - 2\alpha n$. By (E2) and a greedy argument, G contains a properly colour path $ba_1 a_2 \dots a_p b'$ such that

$a_1, \dots, a_p \in A$ and $b, b' \in B \setminus V(\mathcal{P})$. We add the path $ba_1a_2 \dots a_pb'$ to $\mathcal{P}$ and call the resulting set $\mathcal{P}'$. Let $A' = A \setminus \{a_1, \dots, a_p\}$, so $|A'| = |A| - p = 2(|B| - \ell - 1)$. Furthermore (A', B) is ε-extremal. Therefore by replacing $A, B, \mathcal{P}$ with $A', B, \mathcal{P}'$, we may assume that without loss of generality that $|A| = 2m$ and $|B| = m + \ell$ for some integer $m \geq \alpha n$ with $\ell \leq m$.

Consider $G[A \cup B] \cup \mathcal{P}$. Suppose that $P_1, \dots, P_\ell$ are the paths of $\mathcal{P}$. We now contract each P_i as follows. Let b_i and b'_i be the end vertices of P_i, so $b_i, b'_i \in B$. Let N_i be the common neighbours $a \in A$ of b_i and b'_i such that $c(ab_i) = c(ab'_i) = c_a \notin C_{P_i}(b_i) \cup C_{P_i}(b'_i)$. Note that $|N_i| \geq |A| - 2\varepsilon n - 2 \geq 2m - 3\varepsilon\alpha^{-1}m \geq 2m - 3\sqrt{\varepsilon}m$ by (E3). We replace each $V(P_i)$ with a new vertex x_i and join x_i to each vertices $a \in N_i$ with colour c_a. Call the resulting graph H. So $A \subseteq H$ and $|H| = 3m$. Note that, for each $i \leq \ell$, $d_H(x_i, A) = |N_i| \geq 2m - 3\sqrt{\varepsilon}m$. Since $V(H) \setminus A = B \setminus V(\mathcal{P}) \cup \{x_1, \dots, x_\ell\}$, it is easy to see that $(A, V(H) \setminus A)$ is $\sqrt{\varepsilon}$-extremal in H. Lemma 3.2 implies that H has a properly coloured Hamiltonian cycle C. By replacing each x_i in C with P_i we obtain a properly coloured cycle in G with vertex set $A \cup B \cup V(\mathcal{P})$ as required. □

By Lemmas 3.1 and 3.3, to prove Lemma 2.1 it suffices to find a union of suitable properly coloured paths. We would need a finer partition $V(G) \setminus (A \cup B)$ into Y and Z as follows. Let Y be the set of $v \in V(G) \setminus (A \cup B)$ such that $d^c_G(v, B) \geq 10\varepsilon n$ or $|\{c(av) : a \in N_G(v) \cap A \text{ and } c(av) \neq c_a\}| \geq 10\varepsilon n$. Let $Z := V(G) \setminus (A \cup B \cup Y)$.

Proposition 3.4. *Let $\varepsilon, \delta > 0$. Let G be a critical edge-coloured graph on n vertices with $\delta^c(G) \geq \delta n$. Suppose that (A, B) is ε-extremal such that $|A| \geq (\delta - \varepsilon)n$ and $|B| \geq (1 - \delta - \varepsilon)n$. Let Y, Z be a partition of $V(G) \setminus (A \cup B)$ as above. For each $v \in Z$, there are at least $|A| - 24\varepsilon n$ vertices $a \in N_G(v) \cap A$ such that $c(av) = c_a$. Moreover, $(A, B \cup Z)$ is 24ε-extremal.*

Proof. Note that $|Y| + |Z| \leq 2\varepsilon n$. Consider any $v \in Z$. Since $d^c_G(v, B) < 10\varepsilon n$, we have

$$d^c_G(v, A) \geq d^c_G(v) - d^c_G(v, B) - |Y| - |Z| \geq (\delta - 12\varepsilon)n \geq |A| - 14\varepsilon n.$$

On the other hand, $|\{c(av) : a \in N_G(v) \cap A \text{ and } c(av) \neq c_a\}| < 10\varepsilon n$. Thus there are at least $|A| - 24\varepsilon n$ vertices $a \in N_G(v) \cap A$ such that $c(av) = c_a$. □

Instead of finding a union of suitable properly coloured paths, the next lemma shows that finding a suitable matching is sufficient.

Lemma 3.5. *Let $0 < 1/n \ll \varepsilon \ll \alpha < 1/3$. Let G be a critical edge-coloured graph on n vertices. Suppose that (A, B) is ε-extremal such that $|A| \geq (2\alpha + 6\varepsilon)n + 2$ and $|B| \geq (\alpha + 4\varepsilon)n + 1$. Let Y be the set of $v \in V(G) \setminus (A \cup B)$ such that $d^c_G(v, B) \geq 10\varepsilon n$ or $|\{c(av) : a \in N_G(v) \cap A \text{ and } c(av) \neq c_a\}| \geq 10\varepsilon n$. Let $Z := V(H) \setminus (A \cup B \cup Y)$. Suppose that M and M' are vertex-disjoint matchings such that*

(i) *there are at most $2\varepsilon n$ edges in $M \cup M'$;*
(ii) *$M \subseteq G \setminus A$;*
(iii) *$M' \subseteq G[A, B \cup Z]$ and for each edges $av \in M'$ with $a \in A$, $c(av) \neq c_a$.*

Then G contains a properly coloured cycle C such that

$$|C| \geq \min\left\{n, \left\lfloor \frac{3|A|}{2} + |M| + \frac{|M'|}{2} + |Y| - \frac{|V(M) \cap Y|}{2} \right\rfloor\right\}.$$

Proof. Note that $(A, B \cup Z)$ is 24ε-extremal by Proposition 3.4. Our aim is to extend $M \cup M'$ into a suitable path system $\mathcal{P}$ (see Claim 3.6 for the precise properties) such that we can apply Lemma 3.3. The key features of $\mathcal{P}$ are that every path is properly coloured with both endpoints in $B \cup Z$ and that $\mathcal{P}$ covers Y. Here, we give a rough outline on how to construct $\mathcal{P}$ from $M \cup M'$ (that is, the proof of Claim 3.6). For simplicity, we assume that $M \subseteq G[B \cup Z]$ (so the edges of M can be already viewed as paths with both endpoints in $B \cup Z$). For each edge $av \in M'$ with $a \in A$, we add the edge ab with $b \in B$ such that $c(ab) = c_a \neq c(av)$. In order to cover Y, consider any $y \in Y$. If $d^c_G(y, B) \geq 10\varepsilon n$, then we extend y to a path byb' with $b, b' \in B$. Otherwise, we have $|\{c(av) : a \in N_G(v) \cap A$ and $c(av) \neq c_a\}| \geq 10\varepsilon n$, so we construct the path $baya'b'$ with $a, a' \in A$ and $b, b' \in B$.

We now give the formal definition of $\mathcal{P}$ in the following claim.

Claim 3.6. *Let $q := |V(M) \cap Y|$. There exists a properly coloured subgraph $\mathcal{P}$ of G such that $M \cup M' \subseteq \mathcal{P}$ and*

- (i′) *$\mathcal{P}$ is a union of ℓ^* vertex-disjoint path such that each path has both endpoints in $B \cup Z$;*
- (ii′) *$\ell^* = |M| + |M'| + |Y| - q \leq 4\varepsilon n$;*
- (iii′) *$\mathcal{P}$ covers Y;*
- (iv′) *$\mathcal{P}$ contains precisely $2\ell^*$ vertices in $B \cup Z$, that is, each vertex in $V(\mathcal{P}) \cap (B \cup Z)$ is an endpoint of some path in $\mathcal{P}$;*
- (v′) *$\mathcal{P}$ contains at most $|M'| + 2|Y| - q$ vertices in A.*

Proof of claim. We construct $\mathcal{P}_0$ as follows. Initially, we set $\mathcal{P}_0 := M \cup M'$. For each edge $av \in M'$ with $a \in A$, we add an edge ab to $\mathcal{P}_0$ such that $b \in B \setminus V(\mathcal{P})$ is distinct and $c(ab) = c_a \neq c(av)$ (which exists by (E2)). Thus $\mathcal{P}_0$ is a union of $|M|+|M'|$ vertex-disjoint paths such that each path has both endpoints in $V(G) \setminus A$,

$$|V(\mathcal{P}_0) \setminus A| = 2|M| + 2|M'|, \quad |V(\mathcal{P}_0) \cap Y| = q \quad \text{and} \quad |V(\mathcal{P}) \cap A| = |M'|.$$

Let $Y := \{y_1, \dots, y_{|Y|}\}$ be such that $V(\mathcal{P}_0) \cap Y = \{y_1, \dots, y_q\}$. Suppose that for some $i \leq |Y|$ we have already constructed $\mathcal{P}_0 \subseteq \dots \subseteq \mathcal{P}_{i-1}$ such that for all $j < i$

- (Q1) $\mathcal{P}_j$ is an union of $|M|+|M'|+\max\{0, j-q\}$ vertex-disjoint properly coloured paths;
- (Q2) $|(B \cup Z) \cap V(\mathcal{P}_j)| = 2|M| + 2|M'| - q + j + \max\{0, j - r\}$ and $|A \cap V(\mathcal{P}_j)| \leq |M'| + j + \max\{0, j - q\}$;
- (Q3) every vertex in $V(\mathcal{P}_j) \cap (B \cup Z)$ is an endpoint of some paths in $\mathcal{P}_j$;
- (Q4) for all $j' \leq j$, $d_{\mathcal{P}_j}(y_{j'}) = 2$ and for all $j' > j$, $d_{\mathcal{P}_j}(y_{j'}) = d_{\mathcal{P}_{j-1}}(y_{j'})$.

We now construct $\mathcal{P}_i$ as follows. By (Q2), $|B \cap V(\mathcal{P}_{i-1})|, |A \cap V(\mathcal{P}_{i-1})| \leq 8\varepsilon n$.

Note that by (Q4)

$$d_{\mathcal{P}_{i-1}}(y_i) = d_{\mathcal{P}_0}(y_i) = d_M(y_i) = \begin{cases} 1 & \text{if } i \leq q \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $i \leq q$. Let c' be the colour of the edge incident with y_i in $\mathcal{P}_{i-1}$. If $d^c_G(y_i, B) \geq 10\varepsilon n$, then there exists an edge by_i such that $b \in B \setminus V(\mathcal{P}_{i-1})$ and $c(by_i) \neq c'$ and set $\mathcal{P}_i := \mathcal{P}_{i-1} \cup by_i$. Thus, we may assume that there exist at least $10\varepsilon n$ vertices $a \in A \cap N_G(y_i)$ such that $c(ay_i) \neq c_a$ and these $c(ay_i)$ are distinct. So there exists a vertex $a \in (A \cap N_G(y_i)) \setminus V(\mathcal{P}_{i-1})$ such that $c_a \neq c(ay_i) \neq c'$. By (E2), there exists a vertex $b \in B \cap N_G(a) \setminus V(\mathcal{P}_{i-1})$ such that $c(ab) = c_a \neq c(ay_i)$. Set $\mathcal{P}_i := \mathcal{P}_{i-1} \cup \{ay_i, ab\}$. A similar argument also holds for the case when $i > q$, where we apply the previous argument twice. Finally, set $\mathcal{P} := \mathcal{P}_{|Y|}$. □

Let $A^* := A \setminus V(\mathcal{P})$. Let B^* be a subset of $B \cup Z$ such that $V(\mathcal{P}) \cap (B \cup Z) \subseteq B^*$ and $|B^*| = \min\{|B| + |Z|, \lfloor |A^*|/2 \rfloor + \ell^*\}$.

Note that $|B| \geq (\alpha + 4\varepsilon n) + 1 \geq \alpha n + \ell^* + 1$, where the last inequality holds by Claim 3.6(ii$'$). Since $|Y| \leq 2\varepsilon n$, together with Claim 3.6(v$'$) and (i), we have

$$|A^*| \geq |A| - (|M'| + 2|Y|) \geq |A| - 6\varepsilon n \geq 2\alpha n + 2.$$

Therefore, we deduce that $|B^*| \geq \alpha n + \ell^* + 1$.

Note that (A^*, B^*) is 24ε-extremal (as $(A, B \cup Z)$ is by Proposition 3.4). By Lemma 3.3, G contains a properly coloured cycle C with vertex set $A^* \cup B^* \cup V(\mathcal{P}) = A \cup B^* \cup Y$ by Claim 3.6(iii$'$). If $|B^*| = |B| + |Z|$, then C is a properly coloured Hamilton cycle of G. If $|B^*| = \lfloor |A^*|/2 \rfloor + \ell^*$, then

$$\begin{aligned} |C| &= |A| + |Y| + |B^*| = |A| + |Y| + \lfloor |A^*|/2 \rfloor + \ell^* \\ &= |A| + |Y| + \lfloor (|A| - |V(\mathcal{P}) \cap A|)/2 \rfloor + \ell^* \\ &\overset{(\text{ii}'),\,(\text{v}')}{\geq} |A| + \left\lfloor \frac{|A| - (|M'| + 2|Y| - q)}{2} \right\rfloor + |M| + |M'| + 2|Y| - q \\ &= \left\lfloor \frac{3|A|}{2} + |M| + \frac{|M'|}{2} + |Y| - \frac{q}{2} \right\rfloor \end{aligned}$$

as required. □

We are ready to prove Lemma 2.1.

Proof of Lemma 2.1. Let $\varepsilon' := 4\sqrt{\varepsilon}$ and without loss of generality (by adjusting ε' slightly), we have $(\delta - \varepsilon')n \in \mathbb{Z}$. Let α such that $\varepsilon \ll \alpha \ll \delta$. Apply Lemma 3.1 and obtain an ε'-extremal pair (A, B) such that $|A| \geq (\delta - \varepsilon')n$,

$$|B| \geq (1 - \delta - \varepsilon')n \geq (\alpha + 8\varepsilon')n + 1.$$

and

$$d_G(b) \leq (\delta + \varepsilon)n \text{ for each } b \in B. \tag{3.1}$$

By removing vertices of A if necessary, we may assume that

$$|A| = (\delta - \varepsilon')n \geq (2\alpha + 12\varepsilon')n + 2. \tag{3.2}$$

Let Y be the set of $v \in V(G) \setminus (A \cup B)$ such that $d^c_G(v, B) \geq 10\varepsilon' n$ or $|\{c(av) : a \in N_G(v) \cap A$ and $c(av) \neq c_a\}| \geq 10\varepsilon' n$. Let $Z := V(G) \setminus (A \cup B \cup Y)$. Let $p := \max\{\varepsilon' n - |Y|, 0\}$, so

$$|Y| \geq \varepsilon' n - p. \tag{3.3}$$

Let $F := G \setminus A$. So $\delta(F) \geq \varepsilon' n$. Let R be the set of vertices $v \in V(F)$ such that $d_F(v) \leq 10\varepsilon' n$ and let $S := V(F) \setminus R$. Note that $|R| \geq (1 - \delta - \varepsilon')n$ as $B \subseteq R$ by (E3) and (3.1). Since $\Delta(F[R]) \leq 10\varepsilon' n$, Vizing's theorem implies that there exists a matching M_R in $F[R]$ such that $|M_R| \geq e(F[R])/(10\varepsilon' n + 1) \geq 8e(F[R])/|R|$. By summing the degrees $d_F(v)$ in $v \in R$, we have

$$\begin{aligned} |R|\varepsilon' n &\leq \sum_{v \in R} d_F(v) = 2e(F[R]) + e(F[R, S]) \leq |R||M_R|/4 + |R||S|, \\ \varepsilon' n &\leq |M_R|/4 + |S|. \end{aligned} \tag{3.4}$$

We now divide the proof into two different cases.

Case 1: $|M_R| + |S| \geq \varepsilon' n + p/2$. We claim that there exists a matching M in $F = G \setminus A$ such that $|M| = \lceil \varepsilon' n + p/2 \rceil$. Indeed, there is nothing to prove if $|M_R| \geq \varepsilon' n + p/2$. If $|M_R| < \varepsilon' n + p/2$, then we can extend M_R into a matching M of size $\lceil \varepsilon' n + p/2 \rceil$ by adding (appropriate) edges incident with S (as $d_F(s) \geq 10\varepsilon' n$ for all $s \in S$ and $p \leq \varepsilon' n$).

Note that $|M| = \lceil \varepsilon' n + p/2 \rceil \leq 2\varepsilon' n$ and

$$\left\lfloor \frac{3|A|}{2} + |M| + |Y| - \frac{|V(M_R) \cap Y|}{2} \right\rfloor \geq \left\lfloor \frac{3|A|}{2} + |M| + \frac{|Y|}{2} \right\rfloor$$
$$\overset{(3.2),(3.3)}{\geq} \left\lfloor \frac{3(\delta - \varepsilon')n}{2} + \varepsilon' n + \frac{p}{2} + \frac{\varepsilon' n - p}{2} \right\rfloor = \lfloor 3\delta n/2 \rfloor.$$

By Lemma 3.5 (with $M, \emptyset, \varepsilon'$ playing the roles of M, M', ε), G contains a properly coloured cycle C such that $|C| \geq \min\{n, \lfloor 3\delta n/2 \rfloor\}$ as desired.

Case 2: $|M_R| + |S| < \varepsilon' n + p/2$. Together with (3.4) we have $|M_R| < 2p/3$ and $p > 0$. Thus $|Y| = \varepsilon' n - p$.

Case 2a: $|S \cap Y| \leq \varepsilon' n - 10p/3$. Note that by (3.3)

$$|Y \setminus (S \cup V(M_R))| \geq |Y| - |S \cap Y| - 2|M_R| \geq \varepsilon' n - p - (\varepsilon' n - 10p/3) - 4p/3 = p.$$

By (3.4), $|M_R| + |S| \geq \varepsilon' n$. We can extend M_R into a matching M in $F = G \setminus A$ such that $|M| = \lceil \varepsilon' n \rceil$ and $|Y \setminus V(M)| \geq p$. Indeed this is possible, by adding appropriate edges between S and $V(F) \setminus Y$ as $d_F(s) \geq 10\varepsilon' n \geq |Y| + 9\varepsilon' n$ for all $s \in S$. Hence

$$\left\lfloor \frac{3|A|}{2} + |M| + |Y| - \frac{|V(M) \cap Y|}{2} \right\rfloor = \left\lfloor \frac{3|A|}{2} + |M| + \frac{|Y| + |Y \setminus V(M_R)|}{2} \right\rfloor$$
$$\overset{(3.2),(3.3)}{\geq} \left\lfloor \frac{3(\delta - \varepsilon')n}{2} + \varepsilon' n + \frac{(\varepsilon' n - p) + p}{2} \right\rfloor = \left\lfloor \frac{3\delta n}{2} \right\rfloor.$$

We are done by Lemma 3.5 (with $M, \emptyset, \varepsilon'$ playing the roles of M, M', ε).

Case 2b: $|S \cap Y| > \varepsilon' n - 10p/3$. Recall that $|M_R| < 2p/3$ and $|M_R| + |S| \leq \varepsilon' n + p/2$. So

$$\begin{aligned} |(S \cup V(M_R)) \cap (B \cup Z)| &= |(S \cup V(M_R)) \setminus Y| \leq |S| + 2|M_R| - |S \cap Y| \\ &\leq \varepsilon' n + p/2 + 2p/3 - (\varepsilon' n - 10p/3) = 9p/2. \end{aligned} \tag{3.5}$$

Let F' be the subgraph $G[A, B \cup Z]$ obtained by removing all edges uv with $c(uv) = c_a$ for some $a \in A$. Note that for each $a \in A$,

$$d_{F'}(a) \geq \delta^c(G) - (1 + |V(G) \setminus (B \cup Z)| - 1) = \delta n - |A| - |Y| = \varepsilon' n - |Y| = p.$$

Hence, $e(F') \geq p|A| \geq p(\delta - \varepsilon')n$ and $\Delta(F') \leq 24\varepsilon' n$ as $(A, B \cup Z)$ is $24\varepsilon'$-extremal by Proposition 3.4. Since $\varepsilon' \ll \delta$, König's theorem implies that there is a matching

$$e(F')/\Delta(F') \geq 11p/2 \overset{(3.5)}{\geq} p + |(S \cup V(M_R)) \cap V(F')|.$$

Thus there is a matching M' in $F' \subseteq G[A, B \cup Z]$ such that $|M'| = p$ and $V(M') \cap (V(M_R) \cup S) = \emptyset$. By adding (appropriate) edges of F incident with S, we can extend M_R into a matching M in $F = G \setminus A$ satisfying $V(M) \cap V(M') = \emptyset$, $|M| = \lceil \varepsilon' n \rceil$. Note that $|M| + |M'| = p + \lceil \varepsilon' n \rceil \leq 2\varepsilon' n + 1$ and

$$\left\lfloor \frac{3|A|}{2} + |M| + \frac{|M'|}{2} + |Y| - \frac{|V(M) \cap Y|}{2} \right\rfloor \geq \left\lfloor \frac{3|A|}{2} + |M| + \frac{|M'|}{2} + \frac{|Y|}{2} \right\rfloor$$
$$\overset{(3.2),(3.3)}{\geq} \left\lfloor \frac{3(\delta - \varepsilon')n}{2} + \varepsilon' n + \frac{p}{2} + \frac{\varepsilon' n - p}{2} \right\rfloor = \left\lfloor \frac{3\delta n}{2} \right\rfloor.$$

Again, we are done by Lemma 3.5 (with $M, M', 2\varepsilon'$ playing the roles of M, M', ε). $\square$

4. Absorbing cycles

In this section, we prove Lemma 2.2. We need the following definitions. Given a vertex x, we say that a path P is an *absorbing path for* x if the following conditions hold:

(i) $P = z_1z_2z_3z_4$ is a properly coloured path of length 3;
(ii) $x \notin V(P)$;
(iii) $z_1z_2xz_3z_4$ is a properly coloured path.

Next we define an absorbing path for two disjoint edges. Given two vertex-disjoint edges x_1x_2, y_1y_2, we say that a path P is an *absorbing path for* $(x_1, x_2; y_1, y_2)$ if the following conditions hold:

(i) $P = z_1z_2z_3z_4$ is a properly coloured path of length 3;
(ii) $V(P) \cap \{x_1, x_2, y_1, y_2\} = \emptyset$;
(iii) both $z_1z_2x_1x_2$ and $y_1y_2z_3z_4$ are properly coloured paths of length 3.

Note that the ordering of $(x_1, x_2; y_1, y_2)$ is important. We would also need the following proposition from [14].

Proposition 4.1. *Let $P' = x_1x_2 \dots x_{\ell-1}x_\ell$ be a properly coloured path with $\ell \geq 4$. Let $P = z_1z_2z_3z_4$ be an absorbing path for $(x_1, x_2; x_{\ell-1}, x_\ell)$ with $V(P) \cap V(P') = \emptyset$. Then $z_1z_2x_1x_2 \dots x_{\ell-1}x_\ell z_3z_4$ is a properly coloured path.*

Given a vertex x, let $\mathcal{L}(x)$ be the set of absorbing paths for x. Similarly, given two vertex-disjoint edges x_1x_2, y_1y_2, let $\mathcal{L}(x_1, x_2; y_1, y_2)$ be the set of absorbing paths for $(x_1, x_2; y_1, y_2)$. The following lemma follows immediately from Lemmas 4.3 and 4.5 of [14].

Lemma 4.2. *Let $0 < 1/n \ll \gamma \ll \varepsilon < 1/2$. Let G be an edge-coloured graph on n vertices with $\delta^c(G) \geq (1/2 + \varepsilon)n$. Then there exists a family $\mathcal{F}$ of vertex-disjoint properly coloured paths each of length 3, which satisfies the following properties:*

$$|\mathcal{F}| \leq \gamma^{1/2}n, \qquad |\mathcal{L}(x) \cap \mathcal{F}| \geq \gamma n, \qquad |\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}| \geq \gamma n$$

for all $x \in V(G)$ and for all distinct vertices $x_1, x_2, y_1, y_2 \in V(G)$ with x_1x_2, $y_1y_2 \in E(G)$.

To prove Lemma 2.2, we aim to join the paths in $\mathcal{F}$ given by Lemma 4.2 into a properly coloured cycle. First, we need the following definition, which are only used in this section.

Let G be an edge-coloured graph on n vertices. Let $x, y \in V(G)$ be distinct and let $\ell \in \mathbb{N}$. Define $\mathcal{P}^G_\ell(x; y)$ to be the set of properly coloured paths P of length ℓ from x to y. Define $\mu^G_\ell(x; y) := |\mathcal{P}^G_\ell(x; y)|/n^{\ell-1}$ and $\mu^G_{\leq\ell}(x; y) := \sum_{\ell' \leq \ell} \mu^G_{\ell'}(x; y)$. For a colour set C_y, let $\mathcal{P}^G_\ell(x; y, C_y)$ be the set of paths $P \in \mathcal{P}_\ell(x; y)$ such that $C_P(y) \in C_y$. Define $\mu^G_\ell(x; y, C_y)$ and $\mu^G_{\leq\ell}(x; y, C_y)$ analogously. For $\ell \in \mathbb{N}$ and $\eta > 0$, we say that y is *$(\leq \ell, \eta)$-reachable from x in G* if $\mu^G_{\leq\ell}(x; y) \geq \eta$. We say that y is *strongly $(\leq \ell, \eta)$-reachable from x in G* if for any colour c_0, y is $(\leq \ell, \eta)$-reachable from x in $G - \{yz \in E(G) : c(yz) = c_0\}$. Equivalently, y is strongly $(\leq \ell, \eta)$-reachable from x in G if $\mu^G_{\leq\ell}(x; y, C(G) \setminus c_0) \geq \eta$ for all colours $c_0 \in C(G)$.

Proposition 4.3. *Let $\ell \in \mathbb{N}$ and let $\eta > 0$. Let G be an edge-coloured graph on n vertices. Let x, y, v be distinct vertices in $V(G)$.*

(i) *If y is strongly $(\leq \ell, \eta)$-reachable from x, then for any colour c_0, we have $\mu^{G \setminus v}_{\leq\ell}(x; y, C(G) \setminus c_0) \geq \eta - \ell^2/n$.*

If y is not strongly $(\leq \ell, \eta)$-reachable from x but is $(\leq \ell, 2\eta)$-reachable, then

(ii) *there exists a unique colour c_y such that $\mu^G_{\leq\ell}(x; y, c_y) \geq \eta$;*

(iii) $\mu^{G\setminus v}_{\leq \ell}(x; y, c_y) \geq \eta - \ell^2/n$.

Proof. For each $\ell' \in \mathbb{N}$, v is in at most $(\ell'-1)n^{\ell'-2}$ paths of length ℓ' from x to y. Hence for all $\ell' \leq \ell$,

$$\begin{aligned}\mu^{G\setminus v}_{\ell'}(x; y, C(G)\setminus c_0) &\geq \mu^{G}_{\ell'}(x; y, C(G)\setminus c_0) - (\ell'-1)/n \\ &\geq \mu^{G}_{\ell'}(x; y, C(G)\setminus c_0) - \ell/n,\end{aligned}$$

so (i) holds. The definitions of $(\leq \ell, 2\eta)$-reachable and strongly $(\leq \ell, \eta)$-reachable implying (ii). The proof of (i) can be adapted to prove (iii). □

Lemma 4.4. *Let $0 < 1/n \ll \varepsilon < 1/2$. Suppose that G is an edge-coloured graph on n vertices with $\delta^c(G) \geq (1/2+\varepsilon)n + 2$. Let $x, y \in V(G)$ be distinct and let c_x, c_y be any two colours. Then there exists a properly coloured path P from x to y of length at most ε^{-2} such that $C_P(x) \neq \{c_x\}$ and $C_P(y) \neq \{c_y\}$.*

Proof. Let $\ell_0 := \lfloor \varepsilon^{-2} \rfloor$ and let η be such that $1/n \ll \eta \ll \varepsilon$. Let G, x, y, c_x, c_y be as defined in the lemma. Remove all edges at x with colour c_x and all edges at y with colour c_y. So $d(x), d(y) \geq (1/2+\varepsilon)n$ and $d^c(v) \geq (1/2+\varepsilon)n$ for all $v \in V(G)\setminus\{x,y\}$. Therefore to prove the lemma, it suffices to show that there exists a properly coloured path from x to y of length at most ℓ_0. Note that for all $v \in V(G)$, all $\ell \leq \ell_0$ and all $P \in P^G_\ell(x; v)$, we may assume that $y \notin V(P)$ or else the lemma holds.

For each $\ell \in \mathbb{N}$, let S_ℓ be the set of vertices $v \in V(G)\setminus x$ that are strongly $(\leq \ell, \eta^\ell)$-reachable from x, and let T_ℓ be the set of vertices $v \in V(G)\setminus(S_\ell \cup x)$ that are $(\leq \ell, 2\eta^\ell)$-reachable from x. Since a $(\leq \ell, 2\eta^\ell)$-reachable vertex from x is also $(\leq \ell+1, 2\eta^{\ell+1})$-reachable from x and a similar statement for strongly reachable, we have

$$S_\ell \subseteq S_{\ell+1} \text{ and } S_\ell \cup T_\ell \subseteq S_{\ell+1}\cup T_{\ell+1} \text{ for all } \ell \in \mathbb{N}. \tag{4.1}$$

Also $S_1 = \emptyset$ and T_1 is the set of vertex $v \in N(x)$, so

$$|T_1| \geq (1/2+\varepsilon)n. \tag{4.2}$$

Suppose that there exists $s \in S_\ell \cap N(y)$. Let $P \in \mathcal{P}^G_\ell(x; s)$ with $c(sy) \notin C_P(s)$ (which exists as s is strongly $(\leq \ell, \eta)$-reachable from x). Note that Py is a properly coloured path from x to y of length at most $\ell+1$. Thus we may assume that $|S_\ell| \leq (1/2-\varepsilon)n$ for all $\ell < \ell_0$. If $2|S_{\ell+1}| + |T_{\ell+1}| \geq 2|S_\ell| + |T_\ell| + \varepsilon^2 n$ for all $1 \leq \ell < \ell_0 - 1$, then together with (4.2) we have $2|S_{\ell_0-1}| + |T_{\ell_0-1}| \geq 3n/2$. Hence $|S_{\ell_0-1}| \geq n/2$, a contradiction. Therefore, we may assume that for some $\ell < \ell_0 - 1$,

$$2|S_{\ell+1}| + |T_{\ell+1}| < 2|S_\ell| + |T_\ell| + \varepsilon^2 n. \tag{4.3}$$

By (4.1), we have

$$|(S_{\ell+1}\cup T_{\ell+1})\setminus(S_\ell \cup T_\ell)| \leq \varepsilon^2 n. \tag{4.4}$$

Let $W := T_\ell \cap T_{\ell+1}$. Recall that $|S_\ell| \leq (1/2-\varepsilon)n$. By (4.1) and (4.2), we have

$$|T_\ell| \geq |S_\ell \cup T_\ell| - |S_\ell| \geq |T_1| - (1/2-\varepsilon)n \geq 2\varepsilon n.$$

Since $T_\ell \setminus W = T_\ell \setminus T_{\ell+1} \subseteq S_{\ell+1}\setminus S_\ell \subseteq (S_{\ell+1}\cup T_{\ell+1})\setminus(S_\ell\cup T_\ell)$ by (4.1), (4.4) implies that

$$|T_\ell \setminus W| \leq \varepsilon^2 n \tag{4.5}$$

and so

$$|W| \geq |T_\ell| - |T_\ell \setminus W| \geq 2\varepsilon n - \varepsilon^2 n \geq \varepsilon n. \tag{4.6}$$

For each $w \in W \subseteq T_\ell$, Proposition 4.3(ii) implies that there exists a unique colour c_w such that $\mu^G_{\le \ell}(x; w, c_w) \ge \eta^\ell$. Define an auxiliary digraph H with on $V(G) \setminus x$ and edge set $E(H) := \{wv : w \in W, v \in N_G(w) \setminus x \text{ and } c(wv) \neq c_w\}$. Note that for each $w \in W$, we have $d^+_H(w) \ge d^c_G(w) - 1 \ge (1+\varepsilon)n/2$ and so

$$e(H) \ge (1+\varepsilon)n|W|/2. \tag{4.7}$$

We now bound $e(H)$ from above (to obtain a contradiction) in the following claim.

Claim 4.5. *Let $e_H(X, Y)$ denote the number of edges from X to Y. Then*

(i) $e_H(W, (S_{\ell+1} \cup T_{\ell+1}) \setminus (S_\ell \cup T_\ell)) < \varepsilon^2 n|W|$;
(ii) $e_H(W, T_\ell \setminus W) < \varepsilon^2 n|W|$;
(iii) $e_H(W, V(G) \setminus (S_{\ell+1} \cup T_{\ell+1} \cup x)) < 4\eta\varepsilon^{-1} n|W|$;
(iv) $e_H(W, S_\ell) < 2\eta n|W|$;
(v) $e_H(W, W) < (1/2 - \varepsilon + 2\eta)n|W|$.

Proof of claim. Note that (i) and (ii) follow from (4.4) and (4.5), respectively. To see (iii), note that if $wv \in E(H)$ with $w \in W$ and $v \in V(G) \setminus x$ and $P \in \mathcal{P}^{G\setminus v}_{\ell'}(x; w, c_w)$, then Pv is a properly coloured path of length $\ell' + 1$ from x to v. By Proposition 4.3(iii), for each $v \in V(G) \setminus x$,

$$\mu^G_{\le \ell+1}(x, v) \ge \frac{1}{n} \sum_{w \in N_H(v)} \mu^{G\setminus x}_{\le \ell}(x; w, c_w) \ge \eta^\ell e_H(W, v)/2n.$$

Therefore, for all $v \in V(G) \setminus (S_{\ell+1} \cup T_{\ell+1} \cup x)$, we have $e_H(W, v) < 4\eta n \le 4\eta\varepsilon^{-1}|W|$, where the last inequality is due to (4.6). Thus (iii) holds.

Consider the edge $ws \in E(H)$ with $w \in W$ and $s \in S_\ell$. If $P \in \mathcal{P}^{G\setminus v}_{\ell'}(x; s, C(G) \setminus c(ws))$, then Pw is a properly coloured path of length $\ell' + 1$ from x to w with $C_P(w) \neq \{c_w\}$. We must have $e_H(w, S_\ell) < 2\eta n$ for all $w \in W$, which in turn implies (iv). Indeed, if $e_H(w, S_\ell) \ge 2\eta n$, then by Proposition 4.3(iii),

$$\begin{aligned}\mu^G_{\le \ell+1}(x; w, C(G) \setminus c_w) &\ge \frac{1}{n} \sum_{s \in N_H(w) \cap S_\ell} \mu^{G\setminus v}_{\le \ell}(x; s, C(G) \setminus c(ws)) \\ &\ge \frac{1}{n} e_H(w, S_\ell)(\eta^\ell - \ell^2/n) \ge \eta^{\ell+1}\end{aligned}$$

and so $w \in S_{\ell+1}$ (as $w \in W \subseteq T_{\ell+1}$ implying that $\mu^G_{\le \ell+1}(x; w, c_w) \ge \eta^{\ell+1}$), a contradiction.

By a similar argument with (T_ℓ playing the role of S_ℓ), we deduce that every $w \in W \subseteq T_{\ell+1}$ has less than $2\eta n$ edges ww' in G such that $w' \in W \subseteq T_\ell$ and $c_w \neq c(ww') \neq c_{w'}$. This means that, in H, each $w \in W$ is contained less than $2\eta n$ 2-cycles. Since each $w \in W$ is incident to at most $(1/2 - \varepsilon)n$ edges of the same colour in G, we have $e_H(W, w) < (1/2 - \varepsilon)n + 2\eta n = (1/2 - \varepsilon + 2\eta)n$ implying (v). □

By Claim 4.5, we deduce that

$$e(H) \le \left(\varepsilon^2 + \varepsilon^2 + 4\varepsilon^{-1}\eta + 2\eta + 1/2 - \varepsilon + 2\eta\right) n|W| < (1+\varepsilon)n|W|/2,$$

contradicting (4.7). This complete the proof of Lemma 4.4. □

We now prove Lemma 2.2.

Proof of Lemma 2.2. Let ε_0 be such that $1/n \ll \varepsilon_0 \ll \varepsilon$. Apply Lemma 4.2 and obtain a family $\mathcal{F}$ of vertex-disjoint properly coloured paths each of length 3 such that for all $x \in V(G)$ and for all distinct vertices $x_1, x_2, y_1, y_2 \in V(G)$ with $x_1x_2,\ y_1y_2 \in E(G)$,

$$|\mathcal{F}| \leq 3\gamma^{1/2}n, \qquad |\mathcal{L}(x) \cap \mathcal{F}| \geq 3\gamma n, \qquad |\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}| \geq 3\gamma n.$$

Let $P_1, \dots, P_{|\mathcal{F}|}$ be paths in $\mathcal{F}$. Let x_i and y_i be endvertices of P_i for all $i \leq |\mathcal{F}|$. Suppose that for $j \leq |\mathcal{F}|$, we have already found $Q_1, \dots, Q_{j-1}$ such that

(a) for all $i < j$, Q_i is a path from y_i to x_{i+1} of length at most ε_0^{-2};
(b) for all $i < j$, $P_iQ_iP_{i+1}$ is a properly coloured path;
(c) $Q_1, \dots, Q_{j-1}, P_{j+1}, \dots, P_{|\mathcal{F}|}$ are disjoint.

We now find Q_j as follows. Let $C_{P_j}(y_j) = \{c_y\}$, let $C_{P_{j+1}}(x_{j+1}) = \{c_x\}$ and let $W := (\bigcup_{i \leq |\mathcal{F}|} V(P_i) \cup \bigcup_{i' < j} V(Q_{i'})) \setminus \{y_j, x_{j+1}\}$, where we take $P_{|\mathcal{F}|+1} = P_1$ and $x_{|\mathcal{F}|+1} = x_1$. Note that $|W| \leq 3\gamma^{1/2}n(4 + \varepsilon_0^{-2}) \leq \varepsilon n/2$. Let $G' = G \setminus W$. So $\delta^c(G') \geq (1/2 + \varepsilon/2)n \geq (1/2 + \varepsilon_0)|G'|$. Apply Lemma 4.4 and obtain a properly coloured path Q_j in G' from y_j to x_{j+1} of length at most ε_0^{-2} such that $C_{Q_j}(y_j) \neq \{c_y\}$ and $C_{Q_j}(x_{j+1}) \neq \{c_x\}$. Thus we have found $Q_1, \dots, Q_{|\mathcal{F}|}$.

Let $C := P_1Q_1P_2 \dots P_{|\mathcal{F}|}Q_{|\mathcal{F}|}$ be a properly coloured cycle in G. Note that $|C| \leq 3\gamma^{1/2}n(4 + \varepsilon_0^{-2}) \leq \varepsilon n/2$. Let $\mathcal{P}$ be any set of k vertex-disjoint properly coloured paths in $G \setminus V(C)$ with $k \leq \gamma n$. Let $\mathcal{P}'$ be the set of properly coloured paths obtained from $\mathcal{P}$ by breaking up every path $P \in \mathcal{P}$ with $|P| \leq 3$ into isolated vertices. Thus $|\mathcal{P}'| \leq 3\gamma n$ and for each $P \in \mathcal{P}'$, $|P| = 1$ or $|P| \geq 4$. For each $P \in \mathcal{P}'$, there exists a distinct $P' \in \mathcal{F}$ such that $P' \in \mathcal{L}(V(P))$ if $|P'| = 1$, and $P' \in \mathcal{L}(u_1, u_2; u_{\ell'}u_{\ell'-1})$ if $P = u_1u_2 \dots u_{\ell'}$. By Proposition 4.1 and the definition of an absorbing path for a vertex, there exists a properly coloured cycle C' with vertex set $V(C) \cup V(\bigcup \mathcal{P})$. □

5. Properly coloured 1-path-cycle

A 1-*path-cycle* is a disjoint union of cycles and at most one path. In this section, we prove the following lemma, which immediately implies Lemma 2.3.

Lemma 5.1. *Let $0 < 1/n \ll \beta \ll \varepsilon \ll 1/2 < \delta$. Suppose that G is a critical edge-coloured graph on n vertices with $\delta^c(G) \geq \delta n + 1$. Then one of the following statements holds*

(i) *G contains a properly coloured 1-path-cycle H such that $|H| \geq \min\{(3\delta + \beta)n/2, n\}$ and every cycle in H has length at least $\beta n/100$;*
(ii) *G is (δ, ε)-extremal.*

To prove Lemma 5.1, we need the following terminology. Let $\mathbf{x} = (x, c_x)$ and $\mathbf{y} = (y, c_y)$ be pairs with vertices $x, y \in V(H)$ and colours c_x, c_y. For $\rho > 0$, we say that H is a 1-*path-cycle with parameters* ρ-$(\mathbf{x}; \mathbf{y})$ if H satisfies the following four properties:

(a) H is a properly coloured 1-path-cycle;
(b) every cycle in H has length at least ρn;
(c) the path component P in H has length at least ρn with endvertices x and y;
(d) $C_H(x) = \{c_x\}$ and $C_H(y) = \{c_y\}$.

Note that c_x and c_y are precisely the colours of the edges in P (and H) incident with x and y, respectively. The order of $\mathbf{x}$ and $\mathbf{y}$ is important. If ρ is known from the context, we simply write $(\mathbf{x}; \mathbf{y})$ instead of ρ-$(\mathbf{x}; \mathbf{y})$.

Orient the cycles of H into directed cycles arbitrarily and orient the path P into a directed path from x to y. For each $v \in V(H) \setminus y$, define $c_+(v)$ to be $c(vv_+)$, where v_+ is the successor of v, and for each $w \in V(H) \setminus x$, define $c_-(w)$ to be $c(ww_-)$, where w_- is the ancestor of w. From now on every 1-path cycle is assumed to be oriented as above. For an oriented cycle C and $u, v \in V(C)$, we write uC^+v for the path $uu_+ \dots v_-v$ in C and uC^-v for the path $uu_- \dots v_+v$ in C.

Lemma 5.2. *Let $\rho > 0$. Let G be an edge-coloured graph on n vertices with $\delta^c(G) \geq \rho n+1$. Suppose that H is a properly coloured 1-path-cycle in G of maximum order such that every cycles in H has length at least ρn, and that $|H| < n$. Then there exists a 1-path-cycle H' with parameters ρ-$(\mathbf{x}; \mathbf{y})$ such that $V(H') = V(H)$.*

Proof. If H contains no path component, then $H + w$ is a properly coloured 1-path-cycle such that every cycle has length at least ρn, where $w \in V(G) \setminus V(H)$. This contradicts the maximality of $|H|$. So we may assume that H contains a path component P.

Suppose that P has length less than ρn. Let x be an endvertex of P. Let $\mathbf{x} = (x, c_x)$ with $C_P(x) = \{c_x\}$ if $|V(P)| \geq 2$, and c_x is an arbitrary colour otherwise. Note that $|N(\mathbf{x})| \geq \delta^c(G) - 1 \geq \rho n \geq |V(P) \setminus x|$. So there exists $w \in N(\mathbf{x}) \setminus V(P)$. If $w \notin V(H)$, then we can extend P thus enlarging H, a contradiction. Hence $w \in V(H) \setminus V(P)$ and let C be the cycle in H containing w. Without loss of generality, we may assume that $c(xw) \neq c_-(w)$. Then $H' = H + xw - ww_-$ is a properly coloured 1-path-cycles on vertex set $V(H)$ such that every cycle in H has length at least ρn and the path component is $P' = w_+C^+wxP$ of length at least $|C| \geq \rho n$. Therefore H' is a 1-path-cycles with parameters $(\mathbf{w}_+; \mathbf{y})$, where $\mathbf{w}_+ = (w_+, c_+(w_+))$ and $\mathbf{y} = (y, c_y)$ such that y is the other endvertex of P' and $C_{P'}(y) = \{c_y\}$. $\square$

In the next proposition, we show how we can change from 1-path-cycle to another one by 'switching edges'.

Proposition 5.3. *Let G be an edge-coloured graph. Let $\rho > 0$. Let H be a 1-path-cycle in G with parameters $(\mathbf{x}; \mathbf{y})$, where $\mathbf{x} = (x, c_x)$ and $\mathbf{y} = (y, c_y)$. Suppose that $w \in V(H) \cup N_G(\mathbf{x})$ such that $\mathrm{dist}_H(w, x), \mathrm{dist}_H(w, y) \geq \rho n + 1$. Then*

(i) *if $c(xw) \neq c_-(w)$, then $H+xw-ww_+$ is a 1-path-cycle with parameters $((w_+, c_+(w_+)); \mathbf{y})$;*
(ii) *if $c(xw) \neq c_+(w)$, then $H+xw-ww_-$ is a 1-path-cycle with parameters $((w_-, c_-(w_-)); \mathbf{y})$.*

A similar statement holds for $w \in V(H) \cup N_G(\mathbf{y})$ with $\mathrm{dist}_H(w, x), \mathrm{dist}_H(w, y) \geq \rho n + 1$.

Proof. Suppose that $c(xw) \neq c_-(w)$. If w is in the path component P of H, then $P + xw - ww_+$ is a properly coloured graph consisting of a cycle $xPwx$ and a path w_+Py (as $c(xw) \neq c_x$). Since $\mathrm{dist}_H(w, x), \mathrm{dist}_H(w, y) \geq \rho n + 1$, both of these components have size at least ρn. Thus $H + xw - ww_+$ is a 1-path-cycle with parameters $((w_+, c_+(w_+)); \mathbf{y})$. If C is the cycle in H containing w, then $P + C + xw - ww_+$ is a properly coloured path w_+C_+wxPy. Hence $H + xw - ww_+$ is a 1-path-cycle with parameters $((w_-, c_-(w_-)); \mathbf{y})$. Therefore (i) holds, and (ii) holds by a similar argument. $\square$

Let H be 1-path-cycle in G with parameters $(\mathbf{x}; \mathbf{y})$ and let H' be an 1-path-cycle with parameters $(\mathbf{z}; \mathbf{y})$ in G obtained from H by switching one edges. Note that we can deduce which edges were involved in the switching by analysing $\mathbf{z}$ as follows. Let $\mathbf{z} = (z, c_z)$ be a pair with vertex $z \in V(H) \setminus \{x, y\}$ and colour $c_z \in C_H(z)$. Define the vertex

$$w_{\mathbf{z}} := \begin{cases} z_- & \text{if } c_z = c_+(z), \\ z_+ & \text{if } c_z = c_-(z). \end{cases}$$

Note that $H' = H + xw_{\mathbf{z}} - w_{\mathbf{z}}z$ by Proposition 5.3.

Let $X_1(H)$ be the set of pairs $\mathbf{z} = (z, c_z)$ with vertex $z \in V(H)$ and colour $c_z \in C_H(z)$ such that

- $H + xw_{\mathbf{z}} - w_{\mathbf{z}}z$ is a 1-path-cycle with parameters $(\mathbf{z}; \mathbf{y})$;
- $\mathrm{dist}_H(w_{\mathbf{z}}, x), \mathrm{dist}_H(w_{\mathbf{z}}, y) \geq 2\rho n$.

Note that $\{(\mathbf{z}; \mathbf{y}) : \mathbf{z} \in X_1(H)\}$ is a subset of possible parameters of the 1-path-cycle that can be obtained from H by switching one edge of H with an edge incident to x. We obtain the following properties of $X_1(H)$.

Proposition 5.4. *Let G be an edge-coloured graph on n vertices and let $\rho > 0$. Suppose that H is a properly coloured 1-path-cycle in G of maximum order, and that H has parameters $\rho - (\mathbf{x}; \mathbf{y})$. Let $z \in N_G(\mathbf{x})$ such that $\mathrm{dist}_H(z, x), \mathrm{dist}_H(z, y) \geq 2\rho n + 1$. Then the following statements hold*

(a) *$N_G(\mathbf{x}) \subseteq V(H)$;*
(b) *if $c(xz) \neq c_-(z)$, then $(z_+, c_+(z_+)) \in X_1(H)$;*
(c) *if $c(xz) \neq c_+(z)$, then $(z_-, c_-(z_-)) \in X_1(H)$;*
(d) *for $\mathbf{z} \in X_1(H)$, $N_G(\mathbf{z}) \subseteq V(H)$.*

Proof. If $z \in N_G(\mathbf{x}) \setminus V(H)$, then $H + xz$ is a 1-path-cycle with parameters $(z, c(xz); \mathbf{y})$ contradicting the maximality of H. Thus (a) holds, and (d) is proved similarly (by considering $H + xw_{\mathbf{z}} - w_{\mathbf{z}}z$ instead of H).

If $c(xz) \neq c_-(z)$, then $H + xz - zz_+$ is a 1-path-cycle with parameters $((z_+, c_+(z_+)); \mathbf{y})$ by Proposition 5.3(i). So $(z_+, c_+(z_+)) \in X_1(H)$ implying (b). A similar argument shows that (c) holds. □

We would also need to consider the set of 1-path-cycles with parameters $(\mathbf{z}; \mathbf{y})$ that can be obtained from H by replacing two edges of H. We now define X_2, which is the analogue of X_1 for replacing two edges of H (with some additional constraints). Let $X_2(H)$ be the set of pairs $\mathbf{z} = (z, c_z)$ with vertex $z \in V(H)$ and colour $c_z \in C_H(z)$ such that there exist at least $10\rho n$ pairs $\mathbf{z}' = (z', c_{z'}) \in X_1(H)$ satisfying

- $\mathrm{dist}_H(z, x), \mathrm{dist}_H(z, y), \mathrm{dist}_H(z', z) \geq 2\rho n$ and
- $H + xw_{\mathbf{z}'} + z'w_{\mathbf{z}} - zw_{\mathbf{z}} - z'w_{\mathbf{z}'}$ is a 1-path-cycle with parameters $(\mathbf{z}; \mathbf{y})$.

In the next lemma, we show that if $|X_1(H) \cup X_2(H)|$ is bounded above, then there exist disjoint $W^*, Z^* \subseteq V(G)$ such that $G[W^* \cup Z^*]$ is extremal with partition W^*, Z^*. The proof relies on analysing the structure of $X_1(H)$, $X_2(H)$ and $N(\mathbf{z})$ for $\mathbf{z} \in X_1(H)$.

Lemma 5.5. *Let $0 < 1/n \ll \rho \leq \alpha/1000 < 1/1000$ and let $1/2 + 3\alpha < \delta \leq 2/3$. Let G be a critical edge-coloured graph on n vertices with $\delta^c(G) \geq \delta n + 1$. Suppose that H is a properly coloured 1-path-cycle in G of maximum order. Suppose that H has parameters $(\mathbf{x}; \mathbf{y})$, that $|X_1(H) \cup X_2(H)| \leq (\delta + \alpha)n$ and that $|H| < n$. Then there exist disjoint $W^*, Z^* \subseteq V(H)$ such that*

(i) *$|W^*| \geq (\delta - 7\sqrt{\alpha})n$ and $|Z^*| \geq (2\delta - 1 - 3\alpha^{1/4})n$;*
(ii) *for each $w \in W^*$, there exists a distinct colour c^*_w such that there are at least $|Z^*| - 3\sqrt{\alpha}n$ vertices $z \in Z^* \cap N_G(w)$ such that $c(zw) = c^*_w$;*
(iii) *for each $z \in Z^*$, $d_G(z) \leq (\delta + 4\alpha^{1/4})n$ and there are at least $(\delta - 6\alpha^{1/4})n$ vertices $w \in W^* \cap N_G(z)$ and $c(zw) = c^*_w$.*

Proof. Write X_1 for $X_1(H)$ and X_2 for $X_2(H)$. Let Z be the set of vertices $z \in V(H)$ such that $\mathrm{dist}_H(z, x), \mathrm{dist}_H(z, y) \geq 2\rho n$ and

($*$) there exists a colour $c_z \in C_H(z)$ such that $\mathbf{z} = (z, c_z) \in X_1$ with $c(zw_{\mathbf{z}}) = c(xw_{\mathbf{z}})$.

Let Z' be the set of vertices $z \in Z$ such that both colours $c_z \in C_H(z)$ satisfy ($*$). Clearly $Z' \subseteq Z$.

We now bound the sizes of Z and Z' from below.

Claim 5.6. $|Z| + |Z'| \geq (\delta - 2\alpha)n \geq n/2$.

Proof of claim. Let

$$N := \{u \in N_G(\mathbf{x})\colon \operatorname{dist}_H(u, x), \operatorname{dist}_H(u, y) > 2\rho n\}, \quad N' := \{u \in N\colon c(xu) \in C_H(u)\}.$$

Thus $|N| \geq \delta^c(G) - 1 - 2 \cdot 2\rho n \geq (\delta - 4\rho)n$ and $N \subseteq V(H)$ by Proposition 5.4(a). By Proposition 5.4(b) and (c),

$$|X_1| \geq |N'| + 2|N \setminus N'| = |N| + |N \setminus N'| \geq (\delta - 4\rho)n + |N \setminus N'|.$$

Since $|X_1 \cup X_2| \leq (\delta + \alpha)n$, we have $|N \setminus N'| \leq (4\rho + \alpha)n$ and so

$$|N'| \geq |N| - |N \setminus N'| \geq (\delta - \alpha - 8\rho)n \geq (\delta - 2\alpha)n.$$

Let X_1' be the subset of X_1 generated by the edges xv with $v \in N'$, that is, $X_1' := \{(x', c_{x'}) \in X_1 : w_{(x', c_{x'})} \in N'\}$. So $|X_1'| \geq (\delta - 2\alpha)n$. Thus if $(z, c_z) \in X_1'$, then $w_{\mathbf{z}} \in N'$ and $c(zw_{\mathbf{z}}) = c(xw_{\mathbf{z}})$. Note that Z contains all vertices $z \in V(H)$ such that $(z, c_z) \in X_1'$ for some colour c_z. Similarly, Z' contains all vertices $z \in V(H)$ such that $(z, c_+(z)), (z, c_-(z)) \in X_1'$. Hence, $|Z| + |Z'| \geq |X_1'| \geq (\delta - 2\alpha)n \geq n/2$ as required. □

Define a directed graph F on $V(H)$ such that there exists a directed edge from z to w if and only if

- $(z, c_z) \in X_1$ and $z \in Z \cap N_H(w)$ and $c(wz) \neq c_z$;
- $\operatorname{dist}_H(w, x), \operatorname{dist}_H(w, y), \operatorname{dist}_H(w, z) \geq 2\rho n$.

We also colour the edges uv (in F) by $c(uv)$. We now establish some properties of F.

Claim 5.7.

(a) $e(F) \geq e_F(Z, V(F)) \geq (\delta - 6\rho)n|Z| + \sum_{z \in Z'}(d_G(z) - \delta n)$.
(b) *If $w \in V(H)$ has $10\rho n$ edges zw in F with $c(zw) \neq c_+(w)$, then $(w_-, c_-(w_-)) \in X_2$.*
(c) *If $w \in V(H)$ has $10\rho n$ edges zw in F with $c(zw) \neq c_-(w)$, then $(w_+, c_+(w_+)) \in X_2$.*

Proof of claim. For $\mathbf{z} \in X_1$, $N_G(\mathbf{z}) \subseteq V(H)$ by Proposition 5.4(d). Hence, for each $z \in Z$, $d^+_F(z) \geq |N_G(\mathbf{z})| - 3 \cdot 2\rho n \geq (\delta - 6\rho)n$. A similar argument implies that, for each $z \in Z'$, $d^+_F(z') \geq d_G(z') - 6\rho n$. Hence (a) holds.

Suppose that zw is an edge in F with $c(zw) \neq c_+(w)$. Thus there is $\mathbf{z} = (z, c_z) \in X_1$ such that $c_z \neq c(zw)$. Note that by the definition of X_1, $H' = H + xw_{\mathbf{z}} - w_{\mathbf{z}}z$ is a 1-path-cycle with parameters $(\mathbf{z}; \mathbf{y})$. Since $\operatorname{dist}_H(w, x), \operatorname{dist}_H(w, y), \operatorname{dist}_H(w, z) \geq 2\rho n$, we have $\operatorname{dist}_{H'}(w, z), \operatorname{dist}_{H'}(w, y) \geq \rho n + 1$. Proposition 5.3(ii) implies that $H' + zw - ww_-$ is a 1-path-cycle with parameters $((w_-, c_-(w_-)); \mathbf{y})$. This implies (b), and (c) is proven similarly. □

Let $W := \{w \in V(F)\colon d^-_F(w) \geq 20\rho n\}$ and $W' := \{w \in V(F)\colon d^-_F(w) \geq (1 - 2\sqrt{\alpha})|Z|\}$. Let W^* be the set of $w \in W'$ such that there exists a colour c^*_w and there are at most $10\rho n$ vertices $z \in N_G(w)$ with $c(zw) \neq c^*_w$.

Claim 5.8. $|W^*| \geq (\delta - 7\sqrt{\alpha})n$, $|W \setminus W^*| \leq 5\sqrt{\alpha}n$ *and*

$$\frac{1}{n}\sum_{z \in Z'}(d_G(z) - \delta n) + |W' \setminus W^*| \leq 4\sqrt{\alpha}n. \tag{5.1}$$

Proof of claim. If $|W \setminus W'| > \sqrt{\alpha}n$, then Claim 5.7(a) implies that

$$\begin{aligned}(\delta - 6\rho)n|Z| &\leq e_F(Z, V(F)) \leq e_F(Z, W) + 20\rho n^2 \leq |Z||W| - 2\sqrt{\alpha}|Z||W \setminus W'| + 20\rho n^2 \\ &\leq |Z||W| - 2\alpha|Z|n + 20\rho n^2 \leq (|W| - 2\alpha n + 80\rho n)|Z|,\end{aligned}$$

where the last inequality holds as $|Z| \geq n/4$ by Claim 5.6. This implies that $|W| > (\delta+\alpha)n$. By Claim 5.7(b) and (c), we have $|X_2| \geq |W|$, a contradiction. Hence,

$$|W \setminus W'| \leq \sqrt{\alpha}n.$$

Thus we have

$$e_F(Z, V(F)) \leq e_F(Z, W) + 20\rho n^2 \leq (|W'| + (\sqrt{\alpha} + 80\rho)n)|Z| \leq (|W'| + 2\sqrt{\alpha}n)|Z|.$$

By Claim 5.7(a), we have

$$\begin{aligned} |W'| &\geq (\delta - 2\sqrt{\alpha} - 6\rho)n + \frac{1}{|Z|}\sum_{z\in Z'}(d_G(z) - \delta n) \\ &\geq (\delta - 3\sqrt{\alpha})n + \frac{1}{n}\sum_{z\in Z'}(d_G(z) - \delta n). \end{aligned} \tag{5.2}$$

Note that if $w \in W' \setminus W^*$, then $(w_-, c_-(w_-)), (w_+, c_+(w_+)) \in X_2$ by Claim 5.7(b) and (c). Thus $|X_2| \geq |W'| + |W' \setminus W^*|$. Since $|X_2| \leq (\delta + \alpha)n$, (5.2) implies that

$$\frac{1}{n}\sum_{z\in Z'}(d_G(z) - \delta n) + |W' \setminus W^*| \leq (\alpha + 3\sqrt{\alpha})n \leq 4\sqrt{\alpha}n,$$

so (5.1) holds. Moreover, $|W' \setminus W^*| \leq 4\sqrt{\alpha}n$, so $|W \setminus W^*| \leq 5\sqrt{\alpha}n$. Together with (5.2), $|W^*| = |W'| - |W' \setminus W^*| \geq (\delta - 7\sqrt{\alpha})n$. □

Recall that for each $w \in W^* \subseteq W'$, $d^-_F(w) \geq (1 - 2\sqrt{\alpha})|Z|$. So for each $w \in W^*$, the number of edges zw of colour c^*_w in G is at least

$$|\{z \in N_G(w)\colon c(zw) = c^*_w\}| \geq (1 - 2\sqrt{\alpha})|Z| - 10\rho n \geq |Z| - 3\sqrt{\alpha}n. \tag{5.3}$$

Since $\delta^c(G) \geq \delta n$, the left hand side of the inequality is bounded above by $(1 - \delta)n$. Thus $|Z| \leq (1 - \delta + 3\sqrt{\alpha})n$ and so Claim 5.6 implies that

$$|Z'| \geq (2\delta - 1 - 4\sqrt{\alpha})n. \tag{5.4}$$

Let Z^* be the set of vertices $z \in Z$ satisfying (iii). We now bound the size of Z^* from below.

Claim 5.9. $|Z^*| \geq (2\delta - 1 - 3\alpha^{1/4})n$.

Proof of claim. Let Z_1 be the set of $z \in Z'$ such that $d_G(z) \geq (\delta + 4\alpha^{1/4})n$. So (5.1) implies that

$$|Z_1| \leq \alpha^{1/4}n.$$

Let Z_2 be the set of $z \in Z$ such that $d_G(z, V(F) \setminus W) \geq 20\sqrt{\rho}n$. Note that

$$|Z_2| \leq e_F(Z, V(F) \setminus W)/20\sqrt{\rho}n \leq \sqrt{\rho}n.$$

Let Z_3 be the set of $z \in Z$ such that there exist at least $4\alpha^{1/4}n$ vertices $w \in W^*$ with $c(zw) \neq c^*_w$. By (5.3), each $w \in W^*$ is incident with at most $3\sqrt{\alpha}n$ edges zw with $z \in Z$ and $c(zw) \neq c^*_w$. Hence

$$|Z_3| \leq 3\sqrt{\alpha}n^2/(4\alpha^{1/4}n) < \alpha^{1/4}n.$$

For each $z \in Z \setminus (Z_2 \cup Z_3)$, the number of edges zw (in both G and F) such that $w \in W^*$ and $c(zw) = c^*_w$ is at least

$$d_G(z, W^*) - 4\alpha^{1/4}n \geq d_G(z) - 20\sqrt{\rho}n - |W \setminus W^*| - 4\alpha^{1/4}n \geq (\delta - 6\alpha^{1/4})n,$$

where the last inequality is due to Claim 5.8. Hence $Z^* \supseteq Z' \setminus (Z_1 \cup Z_2 \cup Z_3)$. Together with (5.4), we have $|Z^*| \geq (2\delta - 1 - 3\alpha^{1/4})n$. □

Note that properties (i) and (ii) holds by Claims 5.8 and 5.9 and (5.3), and (iii) holds by our construction. To complete the proof, it suffices to show that W^* and Z^* are disjoint. For each $w \in W^*$, (ii) and (i) imply that

$$d_G(w) \geq d^c_G(w) - 1 + |Z^*| - 3\sqrt{\alpha}n \geq (3\delta - 1 - 3\alpha^{1/4} - 3\sqrt{\alpha})n > (\delta + 4\alpha^{1/4})n,$$

so $w \notin Z^*$ as required. □

Let G be an edge-coloured graph and let H be 1-path-cycle with parameters $(\mathbf{x};\mathbf{y})$ with path component P. Let H' be the 1-path-cycle with parameters $(\mathbf{y};\mathbf{x})$ obtained from H by reversing the orientations of all edges. Let $Y_1(H) := X_1(H')$ and $Y_2(H) := X_2(H')$. We study the edges between $X_1(H) \cup X_2(H)$ and $Y_1(H) \cup Y_2(H)$ in the following lemma.

Lemma 5.10. *Let G be a critical edge-coloured graph on n vertices and let $\rho > 0$. Suppose that H is a properly coloured 1-path-cycle in G of maximum order. Suppose that H has parameters $(\mathbf{x};\mathbf{y})$ and that $|H| < n$. Then for all $(x', c_{x'}) \in X_1(H) \cup X_2(H)$ and all $(y', c_{y'}) \in Y_1(H) \cup Y_2(H)$ such that $\mathrm{dist}_H(x, y) \geq 2\rho n$, either $xy \notin E(G)$, $c(xy) = c_x$ or $c(xy) = c_y$.*

Proof. Consider any $\mathbf{x}' = (x', c_{x'}) \in X_1(H) \cup X_2(H)$ and any $\mathbf{y}' = (y', c_{y'}) \in Y_1(H) \cup Y_2(H)$ such that $\mathrm{dist}_H(x, y) \geq 2\rho n$. To prove the lemma, it is sufficient to show that there exists a 1-path-cycle H_0 with $V(H_0) = V(H)$ and parameters $(\mathbf{x}';\mathbf{y}')$. To see this suppose that $x'y' \in E(G)$ and $c_{x'} \neq c(xy) \neq c_{y'}$, then $H_0 + x'y'$ is a vertex-disjoint union of cycles each of length at least ρn. For $z \notin V(H)$, $(H_0 + x'y') \cup z$ is a 1-path-cycle contradicting the maximality of $|H|$.

We will only consider the case when $\mathbf{x}' \in X_2(H)$ and $\mathbf{y}' \in Y_2(H)$, since the other cases proved by similar (and simpler) arguments. Choose $\mathbf{z} = (z, c_z) \in X_1(H)$ and $\mathbf{v} = (v, c_v) \in Y_1(H)$ such that

- any pair of $\{x, y, x', y', z, v\}$ are distance at least $\rho n + 10$ apart in H;
- $H' := H + xw_{\mathbf{z}} + zw_{\mathbf{x}'} - zw_{\mathbf{z}} - x'w_{\mathbf{x}'}$ is a 1-path-cycle with parameters $(\mathbf{x}';\mathbf{y})$.
- $H + yw_{\mathbf{v}} + vw_{\mathbf{y}'} - vw_{\mathbf{v}} - y'w_{\mathbf{y}'}$ is a 1-path-cycle with parameters $(\mathbf{x};\mathbf{y}')$.

Note that $\mathbf{z}$ and $\mathbf{v}$ exist since $\mathbf{x}' \in X_2(H)$ and $\mathbf{y}' \in Y_2(H)$. Since $\mathrm{dist}_H(v, x), \mathrm{dist}_H(v, y), \mathrm{dist}_H(v, z) \geq \rho n + 10$, we have $\mathrm{dist}_{H'}(v, x'), \mathrm{dist}_{H'}(v, y) \geq \rho n + 1$. Proposition 5.3 implies that $H'' := H' + yw_{\mathbf{v}} - vw_{\mathbf{v}}$ is a 1-path-cycle with parameters $(\mathbf{x}';\mathbf{v})$. By a similar argument, we deduce that $H'' + vw_{\mathbf{y}'} - y'w_{\mathbf{y}'}$ is a 1-path-cycle with parameters $(\mathbf{x}';\mathbf{y}')$ as required. □

The next lemma plays a key role in the proof of Lemma 5.1.

Lemma 5.11. *Let $\varepsilon, \rho, \alpha$ be such that $1/n \ll \alpha, \varepsilon \ll 1$. Let G be an edge-coloured graph on n vertices with $\delta^c(G) \geq \delta n + 1$. Then one of following statements holds*

(a) *G contains a properly coloured 1-path-cycle such that $|H| \geq \min\{n, (3\delta + \alpha/2)n/2\}$ and every cycle in H has length at least $\alpha n/100$;*
(b) *there exist disjoint $W^*, Z^* \subseteq V(G)$ such that*
  (i) *$|W^*| \geq (\delta - 7\sqrt{\alpha})n$ and $|Z^*| \geq (2\delta - 1 - 3\alpha^{1/4})n$;*
  (ii) *for each $w \in W^*$, there exists a distinct colour c^*_w such that there are at least $|Z^*| - 3\sqrt{\alpha}n$ vertices $z \in Z^*$ such that $c(zw) = c^*_w$;*
  (iii) *for each $z \in Z^*$, $d_G(z) \leq (\delta + 4\alpha^{1/4})n$ and there are at least $(\delta - 6\alpha^{1/4})n$ edges zw such that $w \in W^*$ and $c(zw) = c^*_w$.*

Here we give a brief description of the proof. By Lemma 5.5, we may assume that $|X_1(H) \cup X_2(H)|$ is bounded below (or else (b) holds). Similarly $|Y_1(H) \cup Y_2(H)|$ is also bounded below. Using Lemma 5.10, we then show that $|H| \geq (3\delta + \alpha/2)n/2$ as desired.

Proof of Lemma 5.11. Let $\rho := \alpha/1000$. Let H be a properly coloured 1-path-cycle in G such that every cycle in H has length at least ρn. Suppose that $|H|$ is maximum. We may assume that $|H| < \min\{n, (3\delta + \alpha/2)n/2\}$ or else we are done. By Lemma 5.2, we further assume that H is a 1-path-cycle with parameters ρ-$(\mathbf{x}; \mathbf{y})$.

Let $X := X_1(H) \cup X_2(H)$ and let $Y := Y_1(H) \cup Y_2(H)$. By Lemma 5.5, we may assume that $|X| \geq (\delta + \alpha)n$. Similarly, by reversing all orientation of H and Lemma 5.5, we may also assume that $|Y| \geq (\delta + \alpha)n$. Let S_X be the set of vertices $v \in V(H)$ such that $(v, c_+(v)), (v, c_-(v)) \in X$. Let $R_X := \{(x', c_{x'}) \in X : x' \notin S_X\}$. Note that

$$2|S_X| + |R_X| = |X| \geq (\delta + \alpha)n. \tag{5.5}$$

Consider any $\mathbf{y}' = (y', c_{y'}) \in Y$. Proposition 5.4 and Lemma 5.10 imply that

$$|N_G(\mathbf{y})| \geq \delta n, \qquad N_G(\mathbf{y}) \subseteq V(H), \qquad |N_G(\mathbf{y}) \cap S_X| \leq 4\rho n. \tag{5.6}$$

If $R_X = \emptyset$, then

$$|H| \geq |N_G(\mathbf{y}')| + |S_X| - 4\rho n \geq \delta n + (\delta + \alpha)n/2 - 4\rho n \geq (3\delta + \alpha/2)n/2,$$

a contradiction. Thus $R_X \neq \emptyset$. Similarly, let S_Y be the set of vertices $v \in V(H)$ such that $(v, c_+(v)), (v, c_-(v)) \in Y$ and $R_Y := \{(y', c_{y'}) \in Y : y' \notin S_y\}$.

Define F to be the auxiliary directed bipartite graph on vertex classes R_X and R_Y such that there exists a directed edge from $\mathbf{v} = (v, c_v)$ to $\mathbf{w} = (w, c_w)$ if and only if

- $\mathrm{dist}_G(v, w) \geq 2\rho n$;
- vw is an edge in G with $c(vw) \neq c_v$.

By Lemma 5.10, F is an oriented graph, that is, F has no directed 2-cycle. Consider any $\mathbf{y}' = (y', c_{y'}) \in Y$. We have

$$\begin{aligned} d^+_F(\mathbf{y}') &\geq |N_G(\mathbf{y}') \cap R_X| - 4\rho n \geq |N_G(\mathbf{y}') \cap (R_X \cup S_X)| - 4\rho n - |N_G(\mathbf{y}') \cap S_X| \\ &\overset{(5.6)}{\geq} \delta n + |R_X| + |S_X| - |H| - 8\rho n \overset{(5.5)}{\geq} \frac{(3\delta + \alpha - 16\rho)n + |R_X|}{2} - |H|. \end{aligned}$$

Similarly, for any $\mathbf{x}' \in R_X$, $d^+_F(\mathbf{x}') \geq \frac{(3\delta+\alpha-16\rho)n+|R_Y|}{2} - |H|$. Since F is an oriented graph, we have

$$\begin{aligned} |R_X||R_Y| \geq e(F) &\geq \sum_{\mathbf{x} \in R_X} d^+_F(\mathbf{x}) + \sum_{\mathbf{y} \in R_Y} d^+_F(\mathbf{y}) \\ &\geq |R_X| \left(\frac{(3\delta + \alpha - 16\rho)n + |R_Y|}{2} - |H| \right) + |R_Y| \left(\frac{(3\delta + \alpha - 16\rho)n + |R_X|}{2} - |H| \right), \\ 0 &\geq (|R_X| + |R_Y|)((3\delta + \alpha - 16\rho)n/2 - |H|). \end{aligned}$$

This implies that $|H| \geq (3\delta + \alpha - 16\rho)n/2 \geq (3\delta + \alpha/2)n/2$ as $R_X \cup R_Y \neq \emptyset$, a contradiction. $\square$

When $\delta \geq 2/3$, Lemma 5.11 implies Lemma 5.1. For $1/2 < \delta < 2/3$, we present a rough sketch proof of Lemma 5.1 using Lemma 5.11. Suppose that Lemma 5.1 holds for any δ' with $\delta' > \delta$. Apply Lemma 5.11 and we may assume that Lemma 5.11(b) holds (or else we are done). Thus there exist disjoint $Z^*, W^* \subseteq V(G)$ satisfying Lemma 5.11(b). Let $\delta^* := (\delta - 4\alpha^{1/8})n/|G \setminus Z^*|$. So $\delta^* > \delta$. If $d^c(v, Z^*) \leq 4\alpha^{1/8}n$ for all vertices $v \notin Z^*$, then $\delta^c(G \setminus Z^*) \geq (\delta - 4\alpha^{1/8})n = \delta^*|G \setminus Z^*|$. Since $\delta^* > \delta$, we apply Lemma 5.1 to $G \setminus Z^*$. We have either a large enough properly coloured 1-path-cycle or $G \setminus Z^*$ is $(\delta^*, \varepsilon^*)$-extremal for some small ε^* or both. In the second case, we then show that G is (δ, ε)-extremal. This argument is formalised in the lemma below.

We would need the following notation. For $\phi \geq 0$, let $I_0(\phi) := [2/3 - \phi, 1)$. For $s \in \mathbb{N}$, let $I_s(\phi) := \{p \in [0,1) \setminus \bigcup_{0 \leq i < s} I_i(\phi) : \frac{p-\phi}{3/2-p} \in I_{s-1}(\phi)\}$. Let $s_\phi(\delta)$ be the integer s such that $\delta \in I_s(\phi)$.

Lemma 5.12. *Let $0 < 1/n \ll \alpha_{s_\phi(\delta)} \ll \alpha_{s_\phi(\delta)-1} \ll \cdots \ll \alpha_0 \ll \phi \ll \varepsilon \ll 1/2 \ll \delta \leq \delta^* < 1$. Suppose that $4^{s_\phi(\delta)}\varepsilon \ll \delta - 1/2$, and that G is a critical edge-coloured graph on $n^* \geq 2^{s_\phi(\delta^*)}n$ vertices with $\delta^c(G) \geq \delta^* n^* + 1$. Then one of the following statements holds:*

- (i*) *G contains a properly coloured 1-path-cycle H such that $|H| \geq (3\delta^* + \alpha_{s_\phi(\delta^*)}/2)n^*/2$ and every cycle in H has length at least $\alpha_{s_\phi(\delta^*)}n^*/100$;*
- (ii*) *G is $(\delta^*, 4^{s_\phi(\delta^*)}\varepsilon)$-extremal.*

Proof. Fix δ^* and write s^* and α for $s_\phi(\delta^*)$ and $\alpha_{s_\phi(\delta^*)}$, respectively. Without loss of generality, $\delta^* \leq 2/3$. Suppose that G satisfies the hypothesis. Apply Lemma 5.11 to G with $\rho = \alpha_{s^*}/100$. We may assume that Lemma 5.11(b) holds or else we are done. Thus there exist disjoint $W^*, Z^* \subseteq V(G)$ such that

- (i$'$) $|W^*| \geq (\delta^* - 7\sqrt{\alpha})n^*$ and $|Z^*| \geq (2\delta^* - 1 - 3\alpha^{1/4})n^*$;
- (ii$'$) for each $w \in W^*$, there exists a distinct colour c^*_w such that there are at least $|Z^*| - 3\sqrt{\alpha}n^*$ vertices $z \in Z^* \cap N_G(w)$ such that $c(zw) = c^*_w$;
- (iii$'$) for each $z \in Z^*$, $d_G(z) \leq (\delta^* + 4\alpha^{1/4})n^*$ and there are at least $(\delta^* - 6\alpha^{1/4})n^*$ edges zw such that $w \in W^* \cap N_G(z)$ and $c(zw) = c^*_w$.

First suppose that $s^* = 0$. Since $\delta^* \geq 2/3 - \phi$ and $\alpha, \phi \ll \varepsilon$, (i$'$) implies that

$$|Z^*| \geq (2\delta^* - 1 - 3\alpha^{1/4})n^* = (1 - \delta^* + (3\delta^* - 2) - 3\alpha^{1/4})n^* \geq (1 - \delta^* - \varepsilon)n^*.$$

Thus G is (δ^*, ε)-extremal. So we may assume that $s \geq 1$ and the lemma holds for all $s' < s$.

Let F be the subgraph of G induced by edges zv such that $z \in Z^*$ and either $v \notin W^*$ or $v \in W^*$ with $c(zv) \neq c_v$. Note that by (iii$'$), $e(F) \leq 10\alpha^{1/4}n^*|Z^*|$. Let V_F be the set of vertices v such that $d_F(v) \geq 5\alpha^{1/8}n^*$. So $|V_F| \leq 5\alpha^{1/8}n^*$. For any $w \in W^*$, (i$'$) and (ii$'$) imply that

$$d_G(w) \geq (d^c_G(w) - 1) + |Z^*| - 3\sqrt{\alpha}n^* \geq (3\delta^* - 1 - 4\alpha^{1/4})n^*. \tag{5.7}$$

We split the proof into two cases depending on the value of δ^*.

Case 1: $\delta^* < \frac{3(1-15\alpha^{1/8})}{5(1-10\alpha^{1/8})}$**.** Let Z_1 be a subset of Z^* of size $|Z_1| = (\delta^* - 1/2)n^* - |V_F|$ and let $Z_2 := Z^* \setminus Z_1$. Note that by (i$'$),

$$|Z_2| \geq (\delta^* - 1/2 - 3\alpha^{1/4})n^*. \tag{5.8}$$

Let $G' := G \setminus (Z_1 \cup V_F)$. We claim that

$$\delta^c(G') \geq (\delta^* - 10\alpha^{1/8})n^* + 1 \tag{5.9}$$

If $v \in V \setminus W^*$, then $d^c_G(v, Z_1 \cup V_F) \leq d_G(v, Z^*) + |V_F| \leq d_F(v) + |V_F| \leq 10\alpha^{1/8}n^*$. If $w \in W^*$, then by (ii$'$), $d^c_G(v, Z_1 \cup V_F) \leq d^c_G(w, Z^*) + |V_F| \leq 1 + 3\sqrt{\alpha}n^* + |V_F| \leq 10\alpha^{1/8}n^*$. Hence (5.9) holds.

Let

$$n' := |G'| = (3/2 - \delta^*)n^* \quad and \quad \delta' := \frac{\delta^* - 10\alpha^{1/8}}{3/2 - \delta^*} \geq \frac{\delta^* - \phi}{3/2 - \delta^*}.$$

Note that $s_\phi(\delta') < s^*$, $\alpha n^* \ll \alpha_{s_\phi(\delta')} n'$ and $\delta^c(G') \geq \delta' n' + 1$. Also,

$$\frac{(3\delta' + \alpha_{s_\phi(\delta')}/2)n'}{2} = \frac{3(\delta^* - 10\alpha^{1/8})n^* + \alpha_{s_\phi(\delta')}n'/2}{2} > \frac{3(\delta^* + \alpha/2)n^*}{2}.$$

By our assumption on δ^*, we have $(3\delta' + \alpha'/2)n'/2 < n'$. Clearly, $|G'| \geq n^*/2 \geq 2^{s_\phi(\delta')}n$. Let $\varepsilon' := 4^{s_\phi(\delta')}\varepsilon$. By induction hypothesis, we may assume that G' is (δ', ε')-extremal (or else we are done). Thus there exist disjoint $A', B' \subseteq V(G')$ such that

(A1′) $|A'| \geq (\delta' - \varepsilon')n'$ and $|B'| \geq (1 - \delta' - \varepsilon')n'$;
(A2′) for each $a \in A'$, there exists a distinct colour c'_a such that there are at least $|B'| - \varepsilon' n'$ vertices $b \in B'$ such that $c(ab) = c'_a$;
(A3′) for each $b \in B'$, $d_G(b) \leq (\delta' + \varepsilon')n'$ and b has at least $|A'| - \varepsilon' n'$ neighbours $a \in A'$ such that $c(ab) = c'_a$.

Let $U' := V(G') \setminus (A' \cup B')$, so $|U'| \leq 2\varepsilon' n'$. Recall that $W^* \subseteq V(G')$ and that $\varepsilon', \alpha \ll \delta^* - 1/2$. For any $w \in W^*$,

$$\begin{aligned} d_{G'}(w) &\geq d_G(w) - |Z_1 \cup V_F| \overset{(5.7)}{\geq} (3\delta^* - 1 - 4\alpha^{1/4})n^* - (\delta^* - 1/2)n^* \\ &= (2\delta^* - 1/2 - 4\alpha^{1/4})n^* \geq (\delta^* + \varepsilon')n^* \geq (\delta' + \varepsilon')n'. \end{aligned}$$

Therefore $W^* \cap B' = \emptyset$ by (A3′). Let $A := W^* \cap A'$. So

$$|A| \geq |W^*| - |U'| \overset{(\mathrm{i}')}{\geq} (\delta^* - 7\sqrt{\alpha})n^* - 2\varepsilon' n' \geq (\delta^* - 4^{s^*}\varepsilon)n^* \tag{5.10}$$

and $|A' \setminus A| \leq (\delta' + \varepsilon')n' - |A| \leq 2 \cdot 4^{s^*}\varepsilon n^*$. Since $Z_2 \cap W^* = \emptyset$, we have $Z_2 \cap A' \subseteq A \cap A'$. Hence

$$|Z_2 \cap B'| \geq |Z_2| - |Z_2 \cap A'| - |Z_2 \setminus (A' \cup B')| \geq |Z_2| - |A \cap A'| - |U'| \overset{(5.8)}{>} 3\sqrt{\alpha}n^* + \varepsilon' n'.$$

Consider any $a \in A$. By (ii′) and (A2′), there exists vertex $z \in Z_2 \cap B'$ such that $c^*_a = c(az) = c'_a$. Therefore we have $c^*_a = c'_a$ for all $a \in A$.

Let $B := B' \cup Z_1$. Note that

$$|B| = |V(G) \setminus (A' \cup U' \cup V(F)| \geq n^* - |A'| - |U'| - |V_F| \geq (1 - \delta - 4^{s^*}\varepsilon)n. \tag{5.11}$$

We now claim that G is $(\delta, 4^{s^*}\varepsilon)$-extremal with partition (A, B). Note that (A1) holds by (5.10) and (5.11). Statements (ii′) and (A2′) imply (A2). Similarly, statements (iii′) and (A3′) imply (A3).

Case 2: $\delta^* \geq \frac{3(1-15\alpha^{1/8})}{5(1-10\alpha^{1/8})}$**.** Note that $s^* = 1$. Case 2 is proved via a similar argument used in Case 1, where we let Z_1 be a subset of Z^* of size $|Z_1| = (1 - (3\delta^* + \alpha/2)/2)n^* - |V_F|$. □

We now prove Lemma 5.1 by choosing $\phi, \alpha_0, \alpha_1, \ldots, \alpha_{s_\phi(\delta)}$ appropriately.

Proof of Lemma 5.1. Let $s_0 := s_0(\delta)$ and let $\varepsilon' := 4^{-2s_0}\varepsilon$. Choose $\beta \ll \phi \ll \varepsilon', \delta - 1/2$ such that $s_\phi(\delta) \leq 2s_0$. So $4^{s_\phi(\delta)}\varepsilon' \leq \varepsilon$. Next choose $\beta < \alpha_{s_\phi(\delta)} \ll \alpha_{s_\phi(\delta)-1} \ll \cdots \ll \alpha_0 \ll \phi$. Therefore, Lemma 5.12 with ε' playing the role of ε implies Lemma 5.1. □

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