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On the spherical convexity of quadratic functions

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On the spherical convexity of quadratic functions

O. P. Ferreira · S. Z. Németh

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Abstract In this paper we study the spherical convexity of quadratic functions on spherically convex sets. In particular, conditions characterizing the spherical convexity of quadratic functions on spherical convex sets associated to the positive orthants and Lorentz cone are given.

Keywords Spheric convexity · quadratic functions · positive orthant · Lorentz cone

1 Introduction

In this paper we study the spherical convexity of quadratic functions on spherical convex sets. This problem arises when one tries to make certain fixed point theorems, surjectivity theorems, and existence theorems for complementarity problems and variational inequalities more explicit (see [9–12]). Other results on this subject can also be found in [14]. In particular, some existence theorems could be reduced to optimizing a quadratic function on the intersection of the sphere and a cone. Indeed, consider a closed convex cone $K \subseteq \mathbb{R}^n$ with dual K^*. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping such that $G : \mathbb{R}^n \to \mathbb{R}^n$ defined by $G(x) = \|x\|^2 F(x/\|x\|^2)$ and $G(0) = 0$ is differentiable at 0. Denote by $DG(0)$ the Jacobian matrix of G at 0. By [12, Corollary 8.1] and [22, Theorem18], if $\min_{\|u\|=1, u\in K} \langle DG(0)u, u\rangle > 0$, then the nonlinear complementarity problem defined by $K \ni x \perp F(x) \in K^*$ has a solution. Thus, we need to minimize a quadratic form on the intersection between a cone and the

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sphere. These sets are exactly the spherically convex sets; see [6]. Therefore, this leads to minimizing quadratic functions on spherically convex sets. In fact the optimization problem above reduces to the problem of calculating the scalar derivative, introduced by S. Z. Németh in [18–20], along cones; see [22]. Similar minimizations of quadratic functions on spherically convex sets are needed in the other settings; see [9–11]. Apart from the above, motivation of this study is much wider. For instance, the quadratic constrained optimization problem on the sphere

$$\min\{\langle Qx, x\rangle \ : \ x \in C\}, \qquad C \subseteq \mathbb{S}^n, \tag{1}$$

for a symmetric matrix Q, is a minimal eigenvalue problem, that is, finding the spectral norm of the matrix $-Q$ (see, e.g., [27]). The problem (1) also contains the trust region problem that appears in many nonlinear programming algorithms as a sub-problem, see [3].

It is worth to point out that when a quadratic function is spherically convex (see, for example, [6]), then the spherical local minimum is equal to the global minimum. Furthermore, convex optimization problems posed on the sphere, have a specific underlining algebraic structure that could be exploited to greatly reduce the cost of obtaining the solutions; see [27,28,32,33]. Therefore, it is natural to consider the problem of determining the spherically convex quadratic functions on spherically convex sets. The goal of the paper is to present conditions satisfied by quadratic functions which are spherically convex on spherical convex sets. Besides, we present conditions characterizing the spherical convexity of quadratic functions on spherically convex sets associated to the Lorentz cones and the positive orthant cone.

The remainder of this paper is organized as follows. In Section 2, we recall some notations and basic results used throughout the paper. In Section 3 we present some general properties satisfied by quadratic functions which are spherically convex. In Section 4 we present a condition characterizing the spherical convexity of quadratic functions on the spherical convex set defined by the positive orthant cone. In Section 5 we present a condition characterizing the spherical convexity of quadratic functions on spherical convex sets defined by Lorentz cone. We conclude this paper by making some final remarks in Section 6.

2 Notations and basic results

In this section we present the notations and some auxiliary results used throughout the paper. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the canonical inner product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$. Denote by $\mathbb{R}^n_+$ the nonnegative orthant and by $\mathbb{R}^n_{++}$ the positive orthant. The notation $x \perp y$ means that $\langle x, y \rangle = 0$. Denote by e^i the i-th canonical unit vector in $\mathbb{R}^n$. The *unit sphere* is denoted by

$$\mathbb{S} := \{x \in \mathbb{R}^n \ : \ \|x\| = 1\}.$$

The *dual cone* of a cone $\mathcal{K} \subset \mathbb{R}^n$ is the cone $\mathcal{K}^*{:=}\{x \in \mathbb{R}^n : \langle x, y\rangle \geq 0,\ \forall\, y{\in}\mathcal{K}\}$. Any pointed closed convex cone with nonempty interior will be called *proper cone.* $\mathcal{K}$ is called *subdual* if $\mathcal{K} \subset \mathcal{K}^*$, *superdual* if $\mathcal{K}^* \subset \mathcal{K}$ and *self-dual* if $\mathcal{K}^* = \mathcal{K}$. $\mathcal{K}$ is called *strongly superdual* if $\mathcal{K}^* \subset \operatorname{int}(\mathcal{K})$. The set of all $m \times n$ matrices with real entries is denoted by $\mathbb{R}^{m\times n}$ and $\mathbb{R}^n \equiv \mathbb{R}^{n\times 1}$. In Section 5 we will also use the identification $\mathbb{R}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R}$, which makes the notations much easier. The matrix $\mathrm{I_n}$ denotes the $n \times n$ identity matrix. If $x \in \mathbb{R}^n$ then $\operatorname{diag}(x)$ will denote an $n\times n$ diagonal matrix with (i,i)-th entry equal to x_i, for $i = 1, \ldots, n$. For $a \in \mathbb{R}$ and $B \in \mathbb{R}^{(n-1)\times(n-1)}$ we denote $\operatorname{diag}(a, B) \in \mathbb{R}^{n\times n}$ the matrix defined by

$$\operatorname{diag}(a, B) := \begin{bmatrix} a & 0 \\ 0 & B \end{bmatrix}.$$

Recall that a *Z-matrix* is a matrix with nonpositive off-diagonal elements. Let $\mathcal{K} \subset \mathbb{R}^n$ be a pointed closed convex cone with nonempty interior, the *$\mathcal{K}$-Z-property* of a matrix $A \in \mathbb{R}^{n\times n}$ means that $\langle Ax, y\rangle \leq 0$, for any $(x, y) \in C(\mathcal{K})$, where $C(\mathcal{K}) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \in \mathcal{K},\ y \in \mathcal{K}^*,\ x \perp y\}$. The matrix $A \in \mathbb{R}^{n\times n}$ is said to have the *$\mathcal{K}$-Lyapunov-like property* if A and $-A$ have the $\mathcal{K}$-Z-property, and is said to be *$\mathcal{K}$-copositive* if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{K}$. If $\mathcal{K} = \mathbb{R}^n_+$, then the $\mathcal{K}$-Z-property of a matrix coincides with the matrix being a Z-matrix and the $\mathcal{K}$-Lyapunov-like property with the matrix being diagonal.

The intersection curve of a plane though the origin of $\mathbb{R}^n$ with the sphere $\mathbb{S}$ is called a *geodesic*. A geodesic segment is said to be *minimal* if its arc length is equal to the intrinsic distance between its end points, i.e., if $\ell(\gamma) := \arccos\langle\gamma(a), \gamma(b)\rangle$, where $\gamma : [a, b] \to \mathbb{S}$ is a parametrization of the geodesic segment. *Through the paper we will use the same terminology for a geodesic and its parameterization.* The set $C \subseteq \mathbb{S}$ is said to be *spherically convex* if for any x, $y \in C$ all the minimal geodesic segments joining x to y are contained in C. Let $C \subset \mathbb{S}$ be a spherically convex set and $I \subset \mathbb{R}$ an interval. The following result is proved in [5].

Proposition 1 *Let $K_C := \{tp : p \in C,\ t \in [0, +\infty)\}$ be the cone generated by the set $C \subset \mathbb{S}^n$. The set C is spherically convex if and only if the associated cone K_C is convex and pointed.*

A function $f : C \to \mathbb{R}$ is said to be *spherically convex (respectively, strictly spherically convex)* if for any minimal geodesic segment $\gamma : I \to C$, the composition $f \circ \gamma : I \to \mathbb{R}$ is convex (respectively, strictly convex) in the usual sense. The next result is an immediate consequence of [6, Propositions 8 and 9].

Proposition 2 *Let $\mathcal{K} \subset \mathbb{R}^n$ be a proper cone, $\mathcal{C} = \operatorname{int}(\mathcal{K}) \cap \mathbb{S}$ and $f : \mathcal{C} \to \mathbb{R}$ a differentiable function. Then, the following statements are equivalent:*

(i) *f is spherically convex;*
(ii) $\langle Df(x) - Df(y), x - y\rangle + (\langle x, y\rangle - 1)\,[\langle Df(x), x\rangle + \langle Df(y), y\rangle] \geq 0$, *for all* $x, y \in \mathcal{C}$;
(iii) $\left\langle D^2 f(y)x, x\right\rangle - \langle Df(y), y\rangle \geq 0$, *for all* $y \in \mathcal{C}$, $x \in \mathbb{S}$ *with* $x \perp y$.

It is well known that if $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then Q defines a linear orthogonal mapping, which is an isometry of the sphere. In the following remark we state some important properties of the isometries of the sphere, for that, given $\mathcal{C} \subset \mathbb{S}$ and $Q \in \mathbb{R}^{n\times n}$, we define

$$Q\mathcal{C} := \{Qx \ : \ x \in \mathcal{C}\}.$$

Remark 1 Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix, i.e., $Q^T = Q^{-1}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ be spherically convex sets. Then $\tilde{\mathcal{C}}_2 := Q\mathcal{C}_2$ is a spherically convex set. Hence, if $\tilde{\mathcal{C}}_2 \subset \tilde{\mathcal{C}}_1$ and $f : \tilde{\mathcal{C}}_1 \to \mathbb{R}$ is a spherically convex function, then $h := f \circ Q : \mathcal{C}_2 \to \mathbb{R}$ is also a spherically convex function. In particular, if $\tilde{\mathcal{C}}_2 = \tilde{\mathcal{C}}_1$ then, $f : \tilde{\mathcal{C}}_1 \to \mathbb{R}$ is spherically convex if, only if, $h := f \circ Q : \mathcal{C}_2 \to \mathbb{R}$ is spherically convex.

We will show next a useful property of proper cones which will be used in the Section 5.

Lemma 1 *Let $\mathcal{K} \subset \mathbb{R}^n$ be a proper cone. If $x \in \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ such that $x \perp y$, then $x \notin \operatorname{int}(\mathcal{K}^*) \cup -\operatorname{int}(\mathcal{K}^*)$.*

Proof If $x \in \operatorname{int}(\mathcal{K}^*)$, then $\langle x, y\rangle > 0$ and if $x \in -\operatorname{int}(\mathcal{K}^*)$, then $\langle x, y\rangle < 0$. Hence, $x \in \mathbb{S}$, $y \in \mathcal{K} \cap \mathbb{S}$ and $x \perp y$ imply $x \notin \operatorname{int}(\mathcal{K}^*) \cup -\operatorname{int}(\mathcal{K}^*)$.

Let $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. For a quadratic function $f : \mathcal{C} \to \mathbb{R}$ defined by $f(x) = \langle Ax, x\rangle$, we will simply use the notation f for the function $\tilde{f} : \mathcal{D} \to \mathbb{R}$ defined by $\tilde{f}(x) = \langle Ax, x\rangle$.

3 Quadratic functions on spherical convex sets

In this section we present some general properties satisfied by quadratic functions which are spherically convex.

Proposition 3 *Let $\mathcal{K} \subset \mathbb{R}^n$ be a proper cone, $\mathcal{C} = \operatorname{int}(\mathcal{K}) \cap \mathbb{S}$ and let $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A \in \mathbb{R}^{n\times n}$. Then, the following statements are equivalent:*

(i) The function f is spherically convex;
(ii) $\langle Ax, x\rangle - \langle Ay, y\rangle \geq 0$, for all $x \in \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ with $x \perp y$.

Proof To prove the equivalence of items (i) and (ii), note that $\mathcal{C} = \operatorname{int}(\mathcal{K}) \cap \mathbb{S}$ is an open spherically convex set, $Df(x) = 2Ax$ and $D^2 f(x) = 2A$, for all $x \in \mathcal{C}$. Then, from item (iii) of Proposition 2 we conclude that $\langle Ax, x\rangle \geq \langle Ay, y\rangle$, for all $x \in \mathbb{S}$ and $y \in \mathcal{C}$ with $x \perp y$. Hence, by continuity this inequality extends for all $y \in \mathcal{K} \cap \mathbb{S}$ with $x \perp y$.

Proposition 4 *Let $\mathcal{K} \subset \mathbb{R}^n$ be a proper cone, $\mathcal{C} = \operatorname{int}(\mathcal{K}) \cap \mathbb{S}$ and let $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. The following statements are equivalent:*

(i) The function f is spherically convex;
(ii) $2\langle Ax, y\rangle \leq (\langle Ax, x\rangle + \langle Ay, y\rangle)\langle x, y\rangle$, for all $x, y \in \mathcal{K} \cap \mathbb{S}$.

As a consequence, if $\mathcal{K}$ is superdual and f is spherically convex, then A has the $\mathcal{K}$-Z-property.

Proof First note that, by taking $f(x) = \langle Ax, x\rangle$ the inequality in item (ii) of Proposition 2 becomes $\langle Ax - Ay, x - y\rangle + (\langle x, y\rangle - 1)[\langle Ax, x\rangle + \langle Ay, y\rangle] \geq 0$, for all $x, y \in \mathcal{C}$. Considering that $A = A^T$, some algebraic manipulations show that $2\langle Ax, y\rangle \leq (\langle Ax, x\rangle + \langle Ay, y\rangle)$, for all $x, y \in \mathcal{C}$, and by continuity this inequality extends for all $x, y \in \mathcal{K} \cap \mathbb{S}$. Terefore, the equivalence of items (i) and (ii) follows from item (ii) of Proposition 2. For the second part, let $x \in \mathcal{K} \cap \mathbb{S}$ and $y \in \mathcal{K}^* \cap \mathbb{S} \subset \mathcal{K} \cap \mathbb{S}$ with $x \perp y$. Since f is spherically convex and $x \perp y$, the inequality in item (ii) implies $\langle Ax, y\rangle \leq 0$. Therefore, the result follows from the definition of $\mathcal{K}$-Z-property.

Proposition 5 *Let $\mathcal{K} \subset \mathbb{R}^n$ be a superdual proper cone, $\mathcal{C} = \operatorname{int}(\mathcal{K}) \cap \mathbb{S}$ and $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. If f is spherically convex, then the following statements hold:*

(i) If $x, y \in (\mathcal{K} \cup -\mathcal{K}) \cap \mathbb{S}$ are such that $x \perp y$, then $\langle Ax, x\rangle = \langle Ay, y\rangle$;
(ii) If $x \in \operatorname{int}(\mathcal{K}) \cap \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ are such that $x \perp y$, then $Ax \perp y$;
(iii) If $x \in -\operatorname{int}(\mathcal{K}) \cap \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ are such that $x \perp y$, then $Ax \perp y$.

Proof For proving item (i), we use the equivalence of items (i) and (ii) of Proposition 3 to obtain that $\langle Ax, x\rangle \geq \langle Ay, y\rangle$ and $\langle Ay, y\rangle \geq \langle Ax, x\rangle$, for all $x, y \in (\mathcal{K} \cup -\mathcal{K}) \cap \mathbb{S}$, and the results follows. To prove item (ii), given $x \in \operatorname{int}(\mathcal{K}) \cap \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ such that $x \perp y$, define $u = (1/(m^2+1))(mx - y)$ and $v = (1/(m^2+1))(x + my)$, where m is a positive integer. Since $x \in \operatorname{int}(\mathcal{K}) \cap \mathbb{S}$, if m is large enough, then $(1/m)u \in \mathcal{K}$ and therefore $u \in \mathcal{K}$ too. It is easy to check that $u, v \in \mathcal{K} \cap \mathbb{S}$ such that $u \perp v$. By using item (i) twice, we conclude that $\langle mAx - Ay, mx - y\rangle = \langle Ax + mAy, x + my\rangle$, which after some algebraic transformations, bearing in mind that $A = A^T$, implies $Ax \perp y$. We can prove item (iii) in a similar fashion.

Corollary 1 *Let $\mathcal{K} \subset \mathbb{R}^n$ be a strongly superdual proper cone, $\mathcal{C} = \operatorname{int}(\mathcal{K}) \cap \mathbb{S}$ and let $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. If f is spherically convex, then A is $\mathcal{K}$-Lyapunov-like.*

Proof Let $x \in \mathcal{K} \cap \mathbb{S}$ and $y \in \mathcal{K}^* \cap \mathbb{S} \subset \operatorname{int}(\mathcal{K}) \cap \mathbb{S}$ with $x \perp y$. Then, item (ii) of Proposition 5 implies $Ax \perp y$ and the result follows from the definition of the $\mathcal{K}$-Lyapunov-like property.

Proposition 6 *Let $\mathcal{K} \subset \mathbb{R}^n$ be a superdual proper cone, $\mathcal{C} = \operatorname{int}(\mathcal{K}) \cap \mathbb{S}$ and $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. If A is $\mathcal{K}$-copositive and f is spherically convex, then A is positive semidefinite.*

Proof Since A is $\mathcal{K}$-copositive we have $\langle Ax, x\rangle \geq 0$ for all $x \in (\mathcal{K}^* \cup -\mathcal{K}^*) \cap \mathbb{S} \subset (\mathcal{K} \cup -\mathcal{K}) \cap \mathbb{S}$. Assume that $x \in \mathbb{S} \setminus (\mathcal{K}^* \cup -\mathcal{K}^*)$. We claim that, there exists

$y \in \mathcal{K} \cap \mathbb{S}$ such that $y \perp x$. We proceed to prove the claim. Suppose that there is no such y. Then, we must have that either $\langle u, x\rangle < 0$ for all $u \in \mathcal{K} \setminus \{0\}$, or $\langle u, x\rangle > 0$ for all $u \in \mathcal{K} \setminus \{0\}$. If there exist $u \in \mathcal{K} \setminus \{0\}$ with $\langle u, x\rangle < 0$ and a $v \in \mathcal{K} \setminus \{0\}$ with $\langle v, x\rangle \geq 0$, then $\psi(0) < 0$ and $\psi(1) \geq 0$, where the continuous function $\psi : \mathbb{R} \to \mathbb{R}$ is defined by $\psi(t) = \langle (1-t)u + tv, x\rangle$. Hence, there is an $s \in [0, 1]$ such that $\psi(s) = 0$. By the convexity of $\mathcal{K} \setminus \{0\}$ ($\mathcal{K} \setminus \{0\}$ is spherically convex because $\mathcal{K}$ is pointed), we conclude that $(1-s)u + sv \in \mathcal{K} \setminus \{0\}$. Let $w = (1-s)u + sv$ and $y = w/\|w\|$. Clearly, $y \in \mathcal{K} \cap \mathbb{S}$ and $y \perp x$, which contradicts our assumptions. If $\langle u, x\rangle < 0$ for all $u \in \mathcal{K} \setminus \{0\}$, then $x \in -\mathcal{K}^*$, which is a contradiction. If $\langle u, x\rangle > 0$ for all $u \in \mathcal{K} \setminus \{0\}$, then $x \in \mathcal{K}^*$, which is also a contradiction. Thus, the claim holds. Since f is convex, Proposition 3 implies that $\langle Ax, x\rangle \geq \langle Ay, y\rangle$. Since A is K-copositive, we have $\langle Ay, y\rangle \geq 0$ and hence $\langle Ax, x\rangle \geq 0$. Thus, $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{S}$. In conclusion, A is positive semidefinite.

By using arguments similar to the ones used in the proof of Proposition 6 we can also prove the following result.

Proposition 7 *Let $\mathcal{K} \subset \mathbb{R}^n$ be a subdual proper cone, $\mathcal{C} = \operatorname{int}(\mathcal{K}) \cap \mathbb{S}$ and $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. If A is $\mathcal{K}^*$-copositive and f is spherically convex, then A is positive semidefinite.*

4 Quadratic functions on spherical positive orthant

In this section we present a condition characterizing the spherical convexity of quadratic functions on the spherical convex set associated to the positive orthant cone.

Theorem 1 *Let $\mathcal{C} = \mathbb{S} \cap \mathbb{R}^n_{++}$ and $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. Then, f is spherically convex if and only if there exists $\lambda \in \mathbb{R}$ such that $A = \lambda I_n$. In this case, f is a constant function.*

Proof Assume that there exists $\lambda \in \mathbb{R}$ such that $A = \lambda I_n$. In this case, $f(x) = \lambda$, for all $x \in \mathcal{C}$. Since any constant function is spherically convex this implication is proved. For the converse statement, we suppose that f is spherically convex. From the equivalence of items (i) and (ii) of Proposition 3 we have

$$\langle Ax, x\rangle \geq \langle Ay, y\rangle, \tag{2}$$

for any $y \in \mathbb{R}^n_+$ and any $x \perp y$ with $x, y \in \mathbb{S}$. First take $x = e^i$ and $y = e^j$. Then, (2) implies that $a_{jj} \geq a_{ii}$. Hence, by swapping i and j, we conclude that $a_{ii} = \lambda$ for any i, where $\lambda \in \mathbb{R}$ is a constant. Next take $y = (1/\sqrt{2})(e^i + e^j)$ and $x = (1/\sqrt{2})(e^i - e^j)$. This leads to $a_{ij} \leq 0$, for any i, j. Hence, $A = B + \lambda I_n$, where B is a Z-matrix with zero diagonal. It is easy to see that inequality (2) is equivalent to

$$\langle Bx, x\rangle \geq \langle By, y\rangle, \tag{3}$$

for any $y \in \mathbb{R}^n_+$ and any $x \perp y$ with $x, y \in \mathbb{S}$. Let i, j be arbitrary but different and k different from both i and j. Let $y = e^k$ and $x = (1/\sqrt{2})(e^i + e^j)$. Then, (3) implies that $a_{ij} = b_{ij} \geq 0$. Together with $a_{ij} \leq 0$ this gives $a_{ij} = b_{ij} = 0$. Hence $A = \lambda I_n$ and therefore $f(x) = \lambda$, for any $x \in \mathcal{C}$, and the proof is concluded.

5 Quadratic functions on Lorentz spherical convex sets

In this section we present a condition characterizing the spherical convexity of quadratic functions on spherical convex sets associated to the Lorentz cones. We begin with the following definition: Let $\mathcal{L} \subset \mathbb{R}^n$ be the *Lorentz cone* defined by

$$\mathcal{L} := \left\{ x \in \mathbb{R}^n \ : \ x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2} \right\}. \tag{4}$$

Lemma 2 *Let $\mathcal{L}$ be the Lorentz cone, $x := (x_1, \tilde{x})$ and $y := (y_1, \tilde{y})$ in $\mathbb{S}$. Then the following statements hold:*

(i) $y \in -\mathcal{L} \cup \mathcal{L}$ if and only if $y_1^2 \geq 1/2$. Moreover, $y_1^2 \geq 1/2$ if and only if $\|\tilde{y}\|^2 \leq 1/2$;
(ii) $y \in -\operatorname{int}(\mathcal{L}) \cup \operatorname{int}(\mathcal{L})$ if and only if $y_1^2 > 1/2$. Moreover, $y_1^2 > 1/2$ if and only if $\|\tilde{y}\|^2 < 1/2$;
(iii) $x \notin -\operatorname{int}(\mathcal{L}) \cup \operatorname{int}(\mathcal{L})$ if and only if $x_1^2 \leq 1/2$. Moreover, $x_1^2 \leq 1/2$ if, and only if, $\|\tilde{x}\|^2 \geq 1/2$;
(iv) If $y \in -\mathcal{L} \cup \mathcal{L}$ and $x \perp y$ then $x \notin -\operatorname{int}(\mathcal{L}) \cap \operatorname{int}(\mathcal{L})$. Moreover, $x \notin -\operatorname{int}(\mathcal{L}) \cap \operatorname{int}(\mathcal{L})$ if, and only if $x_1^2 \leq 1/2$. Furthermore, $x_1^2 \leq 1/2$ if and only if $\|\tilde{x}\|^2 \geq 1/2$.

Proof Items (i)-(iii) follow easily from the definitions of $\mathbb{S}$ and $\mathcal{L}$. Item (iv) follows from Lemma 1 and item (iii).

Remark 2 Let $\tilde{Q} \in \mathbb{R}^{(n-1)\times(n-1)}$ be orthogonal. Then, $Q = \operatorname{diag}(1, \tilde{Q})$ is also ortogonal and $Q\mathcal{L} = \mathcal{L}$. Hence, from Remark 1 we conclude that $f : \mathcal{L} \cap \mathbb{S} \to \mathbb{R}$ is spherically convex if, and only if, $g := f \circ Q = \mathcal{L} \cap \mathbb{S} \to \mathbb{R}$ is spherically convex.

Theorem 2 *Let $\mathcal{C} = \operatorname{int}(\mathcal{L}) \cap \mathbb{S}$ and $f : \mathcal{C} \to \mathbb{R}$ be defined by $f(x) = \langle Ax, x\rangle$, where $A = A^T \in \mathbb{R}^{n\times n}$. Then f is spherically convex if and only if there exist $a, \lambda \in \mathbb{R}$ with $\lambda \geq a$ such that $A = \operatorname{diag}(a, \lambda I_{n-1})$.*

Proof Assume that f is spherically convex. Let $x, y \in \mathcal{L} \cap \mathbb{S}$ with $x \perp y$ be defined by

$$x = \frac{1}{\sqrt{2}}e^1 + \frac{1}{\sqrt{2}}e^i, \qquad y = \frac{1}{\sqrt{2}}e^1 - \frac{1}{\sqrt{2}}e^i, \qquad i \in \{2, \ldots, n\}.$$

Hence the item (i) of Proposition 5 implies that $\langle Ax, x\rangle = \langle Ay, y\rangle$. Hence, after computing these inner products, we obtain

$$\frac{1}{2}(a_{11} + a_{1i}) + \frac{1}{2}(a_{i1} + a_{ii}) = \frac{1}{2}(a_{11} - a_{1i}) - \frac{1}{2}(a_{i1} - a_{ii}), \qquad i \in \{2, \ldots, n\}.$$

Since A is a symmetric matrix, the last equality implies that $a_{1i} = 0$, for all $i \in \{2, \dots, n\}$. Thus, by letting $a = a_{11}$, we have $A = \mathrm{diag}(a, \tilde{A})$ with $\tilde{A} \in \mathbb{R}^{(n-1)\times(n-1)}$ a symmetric matrix. Let $\tilde{Q} \in \mathbb{R}^{(n-1)\times(n-1)}$ be an orthogonal matrix such that $\tilde{Q}^T\tilde{A}\tilde{Q} = \Lambda$, where $\Lambda = \mathrm{diag}(\lambda_2, \dots, \lambda_n)$ and λ_i is an eigenvalue of $\tilde{A}$, for all $i \in \{2, \dots, n\}$. Thus, Remark 2 implies that $f : \mathcal{L} \cap \mathbb{S} \to \mathbb{R}$ is spherically convex if, and only if, $g(x) = \langle \mathrm{diag}(a_{11}, \Lambda)x, x\rangle$ is spherically convex. On the other hand, using Proposition 3 we conclude that $g(x) = \langle \mathrm{diag}(a_{11}, \Lambda)x, x\rangle$ is spherically convex if and only if

$$h(x) = \langle [\mathrm{diag}(a_{11}, \Lambda) - a_{11}I_n]x, x\rangle = \langle [\Lambda - a_{11}I_{n-1}]\tilde{x}, \tilde{x}\rangle,$$

where $x := (x_1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, is spherically convex. Since h is spherically convex, from Proposition 3 we have

$$h(x) - h(y) = \langle [\Lambda - a_{11}I_{n-1}]\tilde{x}, \tilde{x}\rangle - \langle [\Lambda - a_{11}I_{n-1}]\tilde{y}, \tilde{y}\rangle \geq 0, \tag{5}$$

for all points $x = (x_1, \tilde{x}) \in \mathbb{S}$, $y = (y_1, \tilde{y}) \in \mathcal{L} \cap \mathbb{S}$ with $x \perp y$. If we assume that $\lambda_2 = \ldots = \lambda_n$, we have $\Lambda = \lambda I_{n-1}$ and then $A = \mathrm{diag}(a, \lambda I_{n-1})$, where $a := a_{11}$ and $\lambda := \lambda_2 = \cdots = \lambda_n$. Thus (5) becomes $[\lambda - a_{11}][\|\tilde{x}\|^2 - \|\tilde{y}\|^2] \geq 0$. Bearing in mind that $\mathcal{L} = \mathcal{L}^*$, Lemma 2 implies $\|\tilde{x}\|^2 - \|\tilde{y}\|^2 \geq 0$, and then we have from the previous two inequalities that $a = a_{11} \leq \lambda$. Therefore, for concluding the proof of this implication it remains to prove that $a_{11} \leq \lambda_2 = \ldots = \lambda_n$. Without loss of generality we can assume that $n \geq 3$. Let $x \in \mathbb{S}$ and $y \in \mathcal{L} \cap \mathbb{S}$ with $x \perp y$ be defined by

$$x = -\left(\frac{1}{\sqrt{2}}\cos\theta\right)e^1 + \left(\frac{1}{2}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)e^i + \left(\frac{1}{2}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)e^j, \tag{6}$$

$$y = \frac{1}{\sqrt{2}}e^1 + \frac{1}{2}e^i + \frac{1}{2}e^j, \tag{7}$$

where $\theta \in (0, \pi)$. From (6) and (7), it is straightforward to check that $x \in \mathbb{S}$, $y \in \mathcal{L} \cap \mathbb{S}$ and $x \perp y$. Hence, (5) becomes

$$\left(\frac{1}{4}\sin^2\theta - \frac{1}{\sqrt{2}}\cos\theta\sin\theta\right)\lambda_i + \left(\frac{1}{4}\sin^2\theta + \frac{1}{\sqrt{2}}\cos\theta\sin\theta\right)\lambda_j \geq 0,$$

or, after dividing by $\sin\theta \neq 0$, that

$$\left(\frac{1}{4}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta\right)\lambda_i + \left(\frac{1}{4}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right)\lambda_j \geq 0.$$

Letting θ goes to 0 in the inequality above, we obtain $\lambda_j \geq \lambda_i$. Hence, by swapping i and j in (6) and (7) we can also prove that $\lambda_i \geq \lambda_j$, and then $\lambda_i = \lambda_j$, for all $i, j \neq 1$. Therefore, $\lambda_2 = \ldots = \lambda_n$ which concludes the implication. Conversely, assume that $A = \mathrm{diag}(a, \lambda I_{n-1})$ and $\lambda \geq a$. Then $f(x) = \langle [\mathrm{diag}(a, \lambda I_{n-1}]x, x\rangle$ and Proposition 3 implies that f is spherically convex if, and only if,

$$h(x) = \langle [\mathrm{diag}(a, \lambda I_{n-1}) - aI_n]x, x\rangle = \langle [\lambda - a]I_{n-1}\tilde{x}, \tilde{x}\rangle,$$

where $x := (x_1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, is spherically convex. Take $x = (x_1, \tilde{x}) \in \mathbb{S}$ and $y = (y_1, \tilde{y}) \in \mathcal{L} \cap \mathbb{S}$ with $x \perp y$. Thus, from Lemma 1 and (4) we have $\|\tilde{x}\|^2 \geq \|\tilde{y}\|^2$. Hence considering that $a \leq \lambda$ we conclude that

$$\langle [\lambda - a] I_{n-1} \tilde{x}, \tilde{x} \rangle - \langle [\lambda - a] I_{n-1} \tilde{y}, \tilde{y} \rangle = [\lambda - a][\|\tilde{x}\|^2 - \|\tilde{y}\|^2] \geq 0.$$

Therefore, Proposition 3 implies that h is spherically convex and then f is also spherically convex.

Remark 3 Assume that f in Theorem 2 is spherically convex in $\mathcal{L} \cap \mathbb{S}$. Hence there exist $a, \lambda \in \mathbb{R}$ with $\lambda \geq a$ such that $A = \operatorname{diag}(a, \lambda I_{n-1})$ and then $f(x) = a x_1^2 + \lambda \|\tilde{x}\|^2 = \lambda - (\lambda - a) x_1^2$, where $x := (x_1, \tilde{x}) \in \mathcal{L} \cap \mathbb{S}$. Hence, it is clear that the minimum of f on $\mathcal{L} \cap \mathbb{S}$ is obtained when x_1 is maximal, that is, when $x_1 = 1$, which happens exactly when $x = e^1$. Similarly, the maximum of f on $\mathcal{L} \cap \mathbb{S}$ is obtained when x_1 is minimal, that is, when $x_1 = 1/\sqrt{2}$ (see item (i) of Lemma 2), which happens exactly when $\|\tilde{x}\| = x_1 = 1/\sqrt{2}$. Hence, $\operatorname{argmin}\{f(x) : x \in \mathcal{L} \cap \mathbb{S}\} = e^1$, $\min\{f(x) : x \in \mathcal{L} \cap \mathbb{S}\} = a$, $\operatorname{argmax}\{f(x) : x \in \mathcal{L} \cap \mathbb{S}\} = \left\{ \frac{1}{\sqrt{2}}(1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : \|\tilde{x}\| = 1 \right\}$ and $\max\{f(x) : x \in \mathcal{L} \cap \mathbb{S}\} = (a + \lambda)/2$.

Remark 4 If $\lambda > a$ then Theorem 2 implies that $f(x) = \langle \operatorname{diag}(a, \lambda I_{n-1}) x, x \rangle$ is spherically convex. However, in this case $\operatorname{diag}(a, \lambda, \ldots, \lambda)$ does not have the $\mathcal{L}$-Lyapunov-like property. Hence, Corollary 1 is not true if we only require that the cone is superdual proper. Indeed, the Lorentz cone $\mathcal{L}$ is self-dual proper, i.e., $\mathcal{L}^* = \mathcal{L}$ and consequently is superdual proper. Moreover, letting $x, y \in \mathcal{L} \cap \mathbb{S}$ with $x \perp y$ be defined by

$$x = \frac{1}{\sqrt{2}} e^1 + \frac{1}{\sqrt{2}} e^i, \qquad y = \frac{1}{\sqrt{2}} e^1 - \frac{1}{\sqrt{2}} e^i, \qquad i \in \{2, \ldots, n\},$$

we have $\langle \operatorname{diag}(a, \lambda I_{n-1}) x, y \rangle = (a - \lambda)/2 < 0$. Therefore, $\operatorname{diag}(a, \lambda I_{n-1})$ does not have the $\mathcal{L}$-Lyapunov-like property, and the strong superduality of the cone is necessary in Corollary 1.

6 Final remarks

This paper is a continuation of [5,6], where we studied some basic intrinsic properties of spherically convex functions on spherically convex sets of the sphere. We expect that the results of this paper can aid in the understanding of the behaviour of spherically convex functions on spherically convex sets of the sphere. In the future we will also study spherically quasiconvex functions [21] (see also [15] for the definition of quasiconvex functions) on spherically convex sets of the sphere.

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