

On the spherical convexity of quadratic function

Ferreira, Orizon; Nemeth, Sandor

DOI:

[10.1007/s10898-018-0710-6](https://doi.org/10.1007/s10898-018-0710-6)

License:

Other (please specify with Rights Statement)

Document Version

Peer reviewed version

Citation for published version (Harvard):

Ferreira, O & Nemeth, S 2019, 'On the spherical convexity of quadratic function', *Journal of Global Optimization*, vol. 73, pp. 537–545. <https://doi.org/10.1007/s10898-018-0710-6>

[Link to publication on Research at Birmingham portal](#)

Publisher Rights Statement:

Checked for eligibility: 28/09/2018

The final authenticated version is available online at: <http://dx.doi.org/10.1007/s1089>

General rights

Unless a licence is specified above, all rights (including copyright and moral rights) in this document are retained by the authors and/or the copyright holders. The express permission of the copyright holder must be obtained for any use of this material other than for purposes permitted by law.

- Users may freely distribute the URL that is used to identify this publication.
- Users may download and/or print one copy of the publication from the University of Birmingham research portal for the purpose of private study or non-commercial research.
- User may use extracts from the document in line with the concept of 'fair dealing' under the Copyright, Designs and Patents Act 1988 (?)
- Users may not further distribute the material nor use it for the purposes of commercial gain.

Where a licence is displayed above, please note the terms and conditions of the licence govern your use of this document.

When citing, please reference the published version.

Take down policy

While the University of Birmingham exercises care and attention in making items available there are rare occasions when an item has been uploaded in error or has been deemed to be commercially or otherwise sensitive.

If you believe that this is the case for this document, please contact UBIRA@lists.bham.ac.uk providing details and we will remove access to the work immediately and investigate.

Journal of Global Optimization

On the spherical convexity of quadratic functions

--Manuscript Draft--

| | | |
|--|---|-----------------------------|
| Manuscript Number: | JOGO-D-18-00061R1 | |
| Full Title: | On the spherical convexity of quadratic functions | |
| Article Type: | Manuscript | |
| Keywords: | Spheric convexity; quadratic functions; positive orthant; Lorentz cone | |
| Corresponding Author: | Orizon Pereira Ferreira, PhD Universidade Federal de Goiás Goiânia, GO BRAZIL | |
| Corresponding Author Secondary Information: | | |
| Corresponding Author's Institution: | Universidade Federal de Goiás | |
| Corresponding Author's Secondary Institution: | | |
| First Author: | Orizon Pereira Ferreira, PhD | |
| First Author Secondary Information: | | |
| Order of Authors: | Orizon Pereira Ferreira, PhD | |
| | Sandor Z Nemeth, PhD | |
| Order of Authors Secondary Information: | | |
| Funding Information: | CNPq (302473/2017-3) | Dr. Orizon Pereira Ferreira |
| | CNPq (408151/2016-1) | Dr. Orizon Pereira Ferreira |
| Abstract: | In this paper we study the spherical convexity of quadratic functions on spherically convex sets. In particular, conditions characterizing the spherical convexity of quadratic functions on spherical convex sets associated to the positive orthants and Lorentz cone are given. | |
| Response to Reviewers: | <p>First, I would like to thank you for taking the time to provide comments to our paper. I truly appreciate your willingness to help in the reviewing process of our paper.</p> <p>I am submitting the revised version of Manuscript JOGO-D-18-00061: "On the spherical convexity of quadratic functions", coauthored by Sándor Z. Neméth, for possible publication in Journal of Global Optimization.</p> <p>we have revised the whole paper. In connection with the changes requested by the Reviewer, I have implemented all of them, exactly as demanded.</p> <p>Once again, thank you for your insightful comments.</p> <p>Yours sincerely,</p> | |

Noname manuscript No.
(will be inserted by the editor)

On the spherical convexity of quadratic functions

O. P. Ferreira · S. Z. Németh

Received: date / Accepted: date

Abstract In this paper we study the spherical convexity of quadratic functions on spherically convex sets. In particular, conditions characterizing the spherical convexity of quadratic functions on spherical convex sets associated to the positive orthants and Lorentz cone are given.

Keywords Spheric convexity · quadratic functions · positive orthant · Lorentz cone

1 Introduction

In this paper we study the spherical convexity of quadratic functions on spherical convex sets. This problem arises when one tries to make certain fixed point theorems, surjectivity theorems, and existence theorems for complementarity problems and variational inequalities more explicit (see [9–12]). Other results on this subject can also be found in [14]. In particular, some existence theorems could be reduced to optimizing a quadratic function on the intersection of the sphere and a cone. Indeed, consider a closed convex cone $K \subseteq \mathbb{R}^n$ with dual K^* . Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping such that $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $G(x) = \|x\|^2 F(x/\|x\|^2)$ and $G(0) = 0$ is differentiable at 0. Denote by $DG(0)$ the Jacobian matrix of G at 0. By [12, Corollary 8.1] and [22, Theorem18], if $\min_{\|u\|=1, u \in K} \langle DG(0)u, u \rangle > 0$, then the nonlinear complementarity problem defined by $K \ni x \perp F(x) \in K^*$ has a solution. Thus, we need to minimize a quadratic form on the intersection between a cone and the

O. P. Ferreira

IME/UFG, Avenida Esperança, s/n, Campus Samambaia, Goiânia, GO, 74690-900, Brazil
E-mail: orizon@ufg.br. This work was supported by CNPq (Grants 302473/2017-3 and 408151/2016-1) and FAPEG.

S. Z. Németh

School of Mathematics, University of Birmingham, Watson Building, Edgbaston, Birmingham B15 2TT, United Kingdom E-mail: s.nemeth@bham.ac.uk.

sphere. These sets are exactly the spherically convex sets; see [6]. Therefore, this leads to minimizing quadratic functions on spherically convex sets. In fact the optimization problem above reduces to the problem of calculating the scalar derivative, introduced by S. Z. Németh in [18–20], along cones; see [22]. Similar minimizations of quadratic functions on spherically convex sets are needed in the other settings; see [9–11]. Apart from the above, motivation of this study is much wider. For instance, the quadratic constrained optimization problem on the sphere

$$\min\{\langle Qx, x \rangle : x \in C\}, \quad C \subseteq \mathbb{S}^n, \quad (1)$$

for a symmetric matrix Q , is a minimal eigenvalue problem, that is, finding the spectral norm of the matrix $-Q$ (see, e.g., [27]). The problem (1) also contains the trust region problem that appears in many nonlinear programming algorithms as a sub-problem, see [3].

It is worth to point out that when a quadratic function is spherically convex (see, for example, [6]), then the spherical local minimum is equal to the global minimum. Furthermore, convex optimization problems posed on the sphere, have a specific underlining algebraic structure that could be exploited to greatly reduce the cost of obtaining the solutions; see [27, 28, 32, 33]. Therefore, it is natural to consider the problem of determining the spherically convex quadratic functions on spherically convex sets. The goal of the paper is to present conditions satisfied by quadratic functions which are spherically convex on spherical convex sets. Besides, we present conditions characterizing the spherical convexity of quadratic functions on spherically convex sets associated to the Lorentz cones and the positive orthant cone.

The remainder of this paper is organized as follows. In Section 2, we recall some notations and basic results used throughout the paper. In Section 3 we present some general properties satisfied by quadratic functions which are spherically convex. In Section 4 we present a condition characterizing the spherical convexity of quadratic functions on the spherical convex set defined by the positive orthant cone. In Section 5 we present a condition characterizing the spherical convexity of quadratic functions on spherical convex sets defined by Lorentz cone. We conclude this paper by making some final remarks in Section 6.

2 Notations and basic results

In this section we present the notations and some auxiliary results used throughout the paper. Let \mathbb{R}^n be the n -dimensional Euclidean space with the canonical inner product $\langle \cdot, \cdot \rangle$, norm $\|\cdot\|$. Denote by \mathbb{R}_+^n the nonnegative orthant and by \mathbb{R}_{++}^n the positive orthant. The notation $x \perp y$ means that $\langle x, y \rangle = 0$. Denote by e^i the i -th canonical unit vector in \mathbb{R}^n . The *unit sphere* is denoted by

$$\mathbb{S} := \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

The *dual cone* of a cone $\mathcal{K} \subset \mathbb{R}^n$ is the cone $\mathcal{K}^* := \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0, \forall y \in \mathcal{K}\}$. Any pointed closed convex cone with nonempty interior will be called *proper cone*. \mathcal{K} is called *subdual* if $\mathcal{K} \subset \mathcal{K}^*$, *superdual* if $\mathcal{K}^* \subset \mathcal{K}$ and *self-dual* if $\mathcal{K}^* = \mathcal{K}$. \mathcal{K} is called *strongly superdual* if $\mathcal{K}^* \subset \text{int}(\mathcal{K})$. The set of all $m \times n$ matrices with real entries is denoted by $\mathbb{R}^{m \times n}$ and $\mathbb{R}^n \equiv \mathbb{R}^{n \times 1}$. In Section 5 we will also use the identification $\mathbb{R}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R}$, which makes the notations much easier. The matrix I_n denotes the $n \times n$ identity matrix. If $x \in \mathbb{R}^n$ then $\text{diag}(x)$ will denote an $n \times n$ diagonal matrix with (i, i) -th entry equal to x_i , for $i = 1, \dots, n$. For $a \in \mathbb{R}$ and $B \in \mathbb{R}^{(n-1) \times (n-1)}$ we denote $\text{diag}(a, B) \in \mathbb{R}^{n \times n}$ the matrix defined by

$$\text{diag}(a, B) := \begin{bmatrix} a & 0 \\ 0 & B \end{bmatrix}.$$

Recall that a *Z-matrix* is a matrix with nonpositive off-diagonal elements. Let $\mathcal{K} \subset \mathbb{R}^n$ be a pointed closed convex cone with nonempty interior, the \mathcal{K} -Z-property of a matrix $A \in \mathbb{R}^{n \times n}$ means that $\langle Ax, y \rangle \leq 0$, for any $(x, y) \in C(\mathcal{K})$, where $C(\mathcal{K}) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \in \mathcal{K}, y \in \mathcal{K}^*, x \perp y\}$. The matrix $A \in \mathbb{R}^{n \times n}$ is said to have the \mathcal{K} -Lyapunov-like property if A and $-A$ have the \mathcal{K} -Z-property, and is said to be \mathcal{K} -copositive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{K}$. If $\mathcal{K} = \mathbb{R}_+^n$, then the \mathcal{K} -Z-property of a matrix coincides with the matrix being a Z-matrix and the \mathcal{K} -Lyapunov-like property with the matrix being diagonal.

The intersection curve of a plane through the origin of \mathbb{R}^n with the sphere \mathbb{S} is called a *geodesic*. A geodesic segment is said to be *minimal* if its arc length is equal to the intrinsic distance between its end points, i.e., if $\ell(\gamma) := \arccos(\langle \gamma(a), \gamma(b) \rangle)$, where $\gamma : [a, b] \rightarrow \mathbb{S}$ is a parametrization of the geodesic segment. *Through the paper we will use the same terminology for a geodesic and its parameterization.* The set $C \subseteq \mathbb{S}$ is said to be *spherically convex* if for any $x, y \in C$ all the minimal geodesic segments joining x to y are contained in C . Let $C \subset \mathbb{S}$ be a spherically convex set and $I \subset \mathbb{R}$ an interval. The following result is proved in [5].

Proposition 1 *Let $K_C := \{tp : p \in C, t \in [0, +\infty)\}$ be the cone generated by the set $C \subset \mathbb{S}^n$. The set C is spherically convex if and only if the associated cone K_C is convex and pointed.*

A function $f : C \rightarrow \mathbb{R}$ is said to be *spherically convex* (respectively, *strictly spherically convex*) if for any minimal geodesic segment $\gamma : I \rightarrow C$, the composition $f \circ \gamma : I \rightarrow \mathbb{R}$ is convex (respectively, strictly convex) in the usual sense. The next result is an immediate consequence of [6, Propositions 8 and 9].

Proposition 2 *Let $\mathcal{K} \subset \mathbb{R}^n$ be a proper cone, $C = \text{int}(\mathcal{K}) \cap \mathbb{S}$ and $f : C \rightarrow \mathbb{R}$ a differentiable function. Then, the following statements are equivalent:*

- (i) *f is spherically convex;*
- (ii) *$\langle Df(x) - Df(y), x - y \rangle + (\langle x, y \rangle - 1) [\langle Df(x), x \rangle + \langle Df(y), y \rangle] \geq 0$, for all $x, y \in C$;*
- (iii) *$\langle D^2 f(y)x, x \rangle - \langle Df(y), y \rangle \geq 0$, for all $y \in C$, $x \in \mathbb{S}$ with $x \perp y$.*

It is well known that if $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then Q defines a linear orthogonal mapping, which is an isometry of the sphere. In the following remark we state some important properties of the isometries of the sphere, for that, given $\mathcal{C} \subset \mathbb{S}$ and $Q \in \mathbb{R}^{n \times n}$, we define

$$Q\mathcal{C} := \{Qx : x \in \mathcal{C}\}.$$

Remark 1 Let $Q \in \mathbb{R}^{n \times n}$ be an orthogonal matrix, i.e., $Q^T = Q^{-1}$, \mathcal{C}_1 and \mathcal{C}_2 be spherically convex sets. Then $\tilde{\mathcal{C}}_2 := Q\mathcal{C}_2$ is a spherically convex set. Hence, if $\tilde{\mathcal{C}}_2 \subset \tilde{\mathcal{C}}_1$ and $f : \tilde{\mathcal{C}}_1 \rightarrow \mathbb{R}$ is a spherically convex function, then $h := f \circ Q : \mathcal{C}_2 \rightarrow \mathbb{R}$ is also a spherically convex function. In particular, if $\tilde{\mathcal{C}}_2 = \tilde{\mathcal{C}}_1$ then, $f : \tilde{\mathcal{C}}_1 \rightarrow \mathbb{R}$ is spherically convex if, only if, $h := f \circ Q : \mathcal{C}_2 \rightarrow \mathbb{R}$ is spherically convex.

We will show next a useful property of proper cones which will be used in the Section 5.

Lemma 1 *Let $\mathcal{K} \subset \mathbb{R}^n$ be a proper cone. If $x \in \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ such that $x \perp y$, then $x \notin \text{int}(\mathcal{K}^*) \cup -\text{int}(\mathcal{K}^*)$.*

Proof If $x \in \text{int}(\mathcal{K}^*)$, then $\langle x, y \rangle > 0$ and if $x \in -\text{int}(\mathcal{K}^*)$, then $\langle x, y \rangle < 0$. Hence, $x \in \mathbb{S}$, $y \in \mathcal{K} \cap \mathbb{S}$ and $x \perp y$ imply $x \notin \text{int}(\mathcal{K}^*) \cup -\text{int}(\mathcal{K}^*)$.

Let $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. For a quadratic function $f : \mathcal{C} \rightarrow \mathbb{R}$ defined by $f(x) = \langle Ax, x \rangle$, we will simply use the notation f for the function $\tilde{f} : \mathcal{D} \rightarrow \mathbb{R}$ defined by $\tilde{f}(x) = \langle Ax, x \rangle$.

3 Quadratic functions on spherical convex sets

In this section we present some general properties satisfied by quadratic functions which are spherically convex.

Proposition 3 *Let $\mathcal{K} \subset \mathbb{R}^n$ be a proper cone, $\mathcal{C} = \text{int}(\mathcal{K}) \cap \mathbb{S}$ and let $f : \mathcal{C} \rightarrow \mathbb{R}$ be defined by $f(x) = \langle Ax, x \rangle$, where $A \in \mathbb{R}^{n \times n}$. Then, the following statements are equivalent:*

- (i) *The function f is spherically convex;*
- (ii) *$\langle Ax, x \rangle - \langle Ay, y \rangle \geq 0$, for all $x \in \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ with $x \perp y$.*

Proof To prove the equivalence of items (i) and (ii), note that $\mathcal{C} = \text{int}(\mathcal{K}) \cap \mathbb{S}$ is an open spherically convex set, $Df(x) = 2Ax$ and $D^2f(x) = 2A$, for all $x \in \mathcal{C}$. Then, from item (iii) of Proposition 2 we conclude that $\langle Ax, x \rangle \geq \langle Ay, y \rangle$, for all $x \in \mathbb{S}$ and $y \in \mathcal{C}$ with $x \perp y$. Hence, by continuity this inequality extends for all $y \in \mathcal{K} \cap \mathbb{S}$ with $x \perp y$.

Proposition 4 *Let $\mathcal{K} \subset \mathbb{R}^n$ be a proper cone, $\mathcal{C} = \text{int}(\mathcal{K}) \cap \mathbb{S}$ and let $f : \mathcal{C} \rightarrow \mathbb{R}$ be defined by $f(x) = \langle Ax, x \rangle$, where $A = A^T \in \mathbb{R}^{n \times n}$. The following statements are equivalent:*

- (i) The function f is spherically convex;
- (ii) $2\langle Ax, y \rangle \leq (\langle Ax, x \rangle + \langle Ay, y \rangle) \langle x, y \rangle$, for all $x, y \in \mathcal{K} \cap \mathbb{S}$.

As a consequence, if \mathcal{K} is superdual and f is spherically convex, then A has the \mathcal{K} -Z-property.

Proof First note that, by taking $f(x) = \langle Ax, x \rangle$ the inequality in item (ii) of Proposition 2 becomes $\langle Ax - Ay, x - y \rangle + (\langle x, y \rangle - 1)[\langle Ax, x \rangle + \langle Ay, y \rangle] \geq 0$, for all $x, y \in \mathcal{C}$. Considering that $A = A^T$, some algebraic manipulations show that $2\langle Ax, y \rangle \leq (\langle Ax, x \rangle + \langle Ay, y \rangle)$, for all $x, y \in \mathcal{C}$, and by continuity this inequality extends for all $x, y \in \mathcal{K} \cap \mathbb{S}$. Therefore, the equivalence of items (i) and (ii) follows from item (ii) of Proposition 2. For the second part, let $x \in \mathcal{K} \cap \mathbb{S}$ and $y \in \mathcal{K}^* \cap \mathbb{S} \subset \mathcal{K} \cap \mathbb{S}$ with $x \perp y$. Since f is spherically convex and $x \perp y$, the inequality in item (ii) implies $\langle Ax, y \rangle \leq 0$. Therefore, the result follows from the definition of \mathcal{K} -Z-property.

Proposition 5 Let $\mathcal{K} \subset \mathbb{R}^n$ be a superdual proper cone, $\mathcal{C} = \text{int}(\mathcal{K}) \cap \mathbb{S}$ and $f : \mathcal{C} \rightarrow \mathbb{R}$ be defined by $f(x) = \langle Ax, x \rangle$, where $A = A^T \in \mathbb{R}^{n \times n}$. If f is spherically convex, then the following statements hold:

- (i) If $x, y \in (\mathcal{K} \cup -\mathcal{K}) \cap \mathbb{S}$ are such that $x \perp y$, then $\langle Ax, x \rangle = \langle Ay, y \rangle$;
- (ii) If $x \in \text{int}(\mathcal{K}) \cap \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ are such that $x \perp y$, then $Ax \perp y$;
- (iii) If $x \in -\text{int}(\mathcal{K}) \cap \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ are such that $x \perp y$, then $Ax \perp y$.

Proof For proving item (i), we use the equivalence of items (i) and (ii) of Proposition 3 to obtain that $\langle Ax, x \rangle \geq \langle Ay, y \rangle$ and $\langle Ay, y \rangle \geq \langle Ax, x \rangle$, for all $x, y \in (\mathcal{K} \cup -\mathcal{K}) \cap \mathbb{S}$, and the results follows. To prove item (ii), given $x \in \text{int}(\mathcal{K}) \cap \mathbb{S}$ and $y \in \mathcal{K} \cap \mathbb{S}$ such that $x \perp y$, define $u = (1/(m^2 + 1))(mx - y)$ and $v = (1/(m^2 + 1))(x + my)$, where m is a positive integer. Since $x \in \text{int}(\mathcal{K}) \cap \mathbb{S}$, if m is large enough, then $(1/m)u \in \mathcal{K}$ and therefore $u \in \mathcal{K}$ too. It is easy to check that $u, v \in \mathcal{K} \cap \mathbb{S}$ such that $u \perp v$. By using item (i) twice, we conclude that $\langle mAx - Ay, mx - y \rangle = \langle Ax + mAy, x + my \rangle$, which after some algebraic transformations, bearing in mind that $A = A^T$, implies $Ax \perp y$. We can prove item (iii) in a similar fashion.

Corollary 1 Let $\mathcal{K} \subset \mathbb{R}^n$ be a strongly superdual proper cone, $\mathcal{C} = \text{int}(\mathcal{K}) \cap \mathbb{S}$ and let $f : \mathcal{C} \rightarrow \mathbb{R}$ be defined by $f(x) = \langle Ax, x \rangle$, where $A = A^T \in \mathbb{R}^{n \times n}$. If f is spherically convex, then A is \mathcal{K} -Lyapunov-like.

Proof Let $x \in \mathcal{K} \cap \mathbb{S}$ and $y \in \mathcal{K}^* \cap \mathbb{S} \subset \text{int}(\mathcal{K}) \cap \mathbb{S}$ with $x \perp y$. Then, item (ii) of Proposition 5 implies $Ax \perp y$ and the result follows from the definition of the \mathcal{K} -Lyapunov-like property.

Proposition 6 Let $\mathcal{K} \subset \mathbb{R}^n$ be a superdual proper cone, $\mathcal{C} = \text{int}(\mathcal{K}) \cap \mathbb{S}$ and $f : \mathcal{C} \rightarrow \mathbb{R}$ be defined by $f(x) = \langle Ax, x \rangle$, where $A = A^T \in \mathbb{R}^{n \times n}$. If A is \mathcal{K} -copositive and f is spherically convex, then A is positive semidefinite.

Proof Since A is \mathcal{K} -copositive we have $\langle Ax, x \rangle \geq 0$ for all $x \in (\mathcal{K}^* \cup -\mathcal{K}^*) \cap \mathbb{S} \subset (\mathcal{K} \cup -\mathcal{K}) \cap \mathbb{S}$. Assume that $x \in \mathbb{S} \setminus (\mathcal{K}^* \cup -\mathcal{K}^*)$. We claim that, there exists

$y \in \mathcal{K} \cap \mathbb{S}$ such that $y \perp x$. We proceed to prove the claim. Suppose that there is no such y . Then, we must have that either $\langle u, x \rangle < 0$ for all $u \in \mathcal{K} \setminus \{0\}$, or $\langle u, x \rangle > 0$ for all $u \in \mathcal{K} \setminus \{0\}$. If there exist $u \in \mathcal{K} \setminus \{0\}$ with $\langle u, x \rangle < 0$ and a $v \in \mathcal{K} \setminus \{0\}$ with $\langle v, x \rangle \geq 0$, then $\psi(0) < 0$ and $\psi(1) \geq 0$, where the continuous function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $\psi(t) = \langle (1-t)u + tv, x \rangle$. Hence, there is an $s \in [0, 1]$ such that $\psi(s) = 0$. By the convexity of $\mathcal{K} \setminus \{0\}$ ($\mathcal{K} \setminus \{0\}$ is spherically convex because \mathcal{K} is pointed), we conclude that $(1-s)u + sv \in \mathcal{K} \setminus \{0\}$. Let $w = (1-s)u + sv$ and $y = w/\|w\|$. Clearly, $y \in \mathcal{K} \cap \mathbb{S}$ and $y \perp x$, which contradicts our assumptions. If $\langle u, x \rangle < 0$ for all $u \in \mathcal{K} \setminus \{0\}$, then $x \in -\mathcal{K}^*$, which is a contradiction. If $\langle u, x \rangle > 0$ for all $u \in \mathcal{K} \setminus \{0\}$, then $x \in \mathcal{K}^*$, which is also a contradiction. Thus, the claim holds. Since f is convex, Proposition 3 implies that $\langle Ax, x \rangle \geq \langle Ay, y \rangle$. Since A is K -copositive, we have $\langle Ay, y \rangle \geq 0$ and hence $\langle Ax, x \rangle \geq 0$. Thus, $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{S}$. In conclusion, A is positive semidefinite.

By using arguments similar to the ones used in the proof of Proposition 6 we can also prove the following result.

Proposition 7 *Let $\mathcal{K} \subset \mathbb{R}^n$ be a subdual proper cone, $\mathcal{C} = \text{int}(\mathcal{K}) \cap \mathbb{S}$ and $f : \mathcal{C} \rightarrow \mathbb{R}$ be defined by $f(x) = \langle Ax, x \rangle$, where $A = A^T \in \mathbb{R}^{n \times n}$. If A is \mathcal{K}^* -copositive and f is spherically convex, then A is positive semidefinite.*

4 Quadratic functions on spherical positive orthant

In this section we present a condition characterizing the spherical convexity of quadratic functions on the spherical convex set associated to the positive orthant cone.

Theorem 1 *Let $\mathcal{C} = \mathbb{S} \cap \mathbb{R}_{++}^n$ and $f : \mathcal{C} \rightarrow \mathbb{R}$ be defined by $f(x) = \langle Ax, x \rangle$, where $A = A^T \in \mathbb{R}^{n \times n}$. Then, f is spherically convex if and only if there exists $\lambda \in \mathbb{R}$ such that $A = \lambda I_n$. In this case, f is a constant function.*

Proof Assume that there exists $\lambda \in \mathbb{R}$ such that $A = \lambda I_n$. In this case, $f(x) = \lambda$, for all $x \in \mathcal{C}$. Since any constant function is spherically convex this implication is proved. For the converse statement, we suppose that f is spherically convex. From the equivalence of items (i) and (ii) of Proposition 3 we have

$$\langle Ax, x \rangle \geq \langle Ay, y \rangle, \quad (2)$$

for any $y \in \mathbb{R}_+^n$ and any $x \perp y$ with $x, y \in \mathbb{S}$. First take $x = e^i$ and $y = e^j$. Then, (2) implies that $a_{jj} \geq a_{ii}$. Hence, by swapping i and j , we conclude that $a_{ii} = \lambda$ for any i , where $\lambda \in \mathbb{R}$ is a constant. Next take $y = (1/\sqrt{2})(e^i + e^j)$ and $x = (1/\sqrt{2})(e^i - e^j)$. This leads to $a_{ij} \leq 0$, for any i, j . Hence, $A = B + \lambda I_n$, where B is a Z -matrix with zero diagonal. It is easy to see that inequality (2) is equivalent to

$$\langle Bx, x \rangle \geq \langle By, y \rangle, \quad (3)$$

for any $y \in \mathbb{R}_+^n$ and any $x \perp y$ with $x, y \in \mathbb{S}$. Let i, j be arbitrary but different and k different from both i and j . Let $y = e^k$ and $x = (1/\sqrt{2})(e^i + e^j)$. Then, (3) implies that $a_{ij} = b_{ij} \geq 0$. Together with $a_{ij} \leq 0$ this gives $a_{ij} = b_{ij} = 0$. Hence $A = \lambda I_n$ and therefore $f(x) = \lambda$, for any $x \in \mathcal{C}$, and the proof is concluded.

5 Quadratic functions on Lorentz spherical convex sets

In this section we present a condition characterizing the spherical convexity of quadratic functions on spherical convex sets associated to the Lorentz cones. We begin with the following definition: Let $\mathcal{L} \subset \mathbb{R}^n$ be the *Lorentz cone* defined by

$$\mathcal{L} := \left\{ x \in \mathbb{R}^n : x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2} \right\}. \quad (4)$$

Lemma 2 *Let \mathcal{L} be the Lorentz cone, $x := (x_1, \tilde{x})$ and $y := (y_1, \tilde{y})$ in \mathbb{S} . Then the following statements hold:*

- (i) $y \in -\mathcal{L} \cup \mathcal{L}$ if and only if $y_1^2 \geq 1/2$. Moreover, $y_1^2 \geq 1/2$ if and only if $\|\tilde{y}\|^2 \leq 1/2$;
- (ii) $y \in -\text{int}(\mathcal{L}) \cup \text{int}(\mathcal{L})$ if and only if $y_1^2 > 1/2$. Moreover, $y_1^2 > 1/2$ if and only if $\|\tilde{y}\|^2 < 1/2$;
- (iii) $x \notin -\text{int}(\mathcal{L}) \cup \text{int}(\mathcal{L})$ if and only if $x_1^2 \leq 1/2$. Moreover, $x_1^2 \leq 1/2$ if, and only if, $\|\tilde{x}\|^2 \geq 1/2$;
- (iv) If $y \in -\mathcal{L} \cup \mathcal{L}$ and $x \perp y$ then $x \notin -\text{int}(\mathcal{L}) \cap \text{int}(\mathcal{L})$. Moreover, $x \notin -\text{int}(\mathcal{L}) \cap \text{int}(\mathcal{L})$ if, and only if $x_1^2 \leq 1/2$. Furthermore, $x_1^2 \leq 1/2$ if and only if $\|\tilde{x}\|^2 \geq 1/2$.

Proof Items (i)-(iii) follow easily from the definitions of \mathbb{S} and \mathcal{L} . Item (iv) follows from Lemma 1 and item (iii).

Remark 2 Let $\tilde{Q} \in \mathbb{R}^{(n-1) \times (n-1)}$ be orthogonal. Then, $Q = \text{diag}(1, \tilde{Q})$ is also orthogonal and $Q\mathcal{L} = \mathcal{L}$. Hence, from Remark 1 we conclude that $f : \mathcal{L} \cap \mathbb{S} \rightarrow \mathbb{R}$ is spherically convex if, and only if, $g := f \circ Q = \mathcal{L} \cap \mathbb{S} \rightarrow \mathbb{R}$ is spherically convex.

Theorem 2 *Let $\mathcal{C} = \text{int}(\mathcal{L}) \cap \mathbb{S}$ and $f : \mathcal{C} \rightarrow \mathbb{R}$ be defined by $f(x) = \langle Ax, x \rangle$, where $A = A^T \in \mathbb{R}^{n \times n}$. Then f is spherically convex if and only if there exist $a, \lambda \in \mathbb{R}$ with $\lambda \geq a$ such that $A = \text{diag}(a, \lambda I_{n-1})$.*

Proof Assume that f is spherically convex. Let $x, y \in \mathcal{L} \cap \mathbb{S}$ with $x \perp y$ be defined by

$$x = \frac{1}{\sqrt{2}}e^1 + \frac{1}{\sqrt{2}}e^i, \quad y = \frac{1}{\sqrt{2}}e^1 - \frac{1}{\sqrt{2}}e^i, \quad i \in \{2, \dots, n\}.$$

Hence the item (i) of Proposition 5 implies that $\langle Ax, x \rangle = \langle Ay, y \rangle$. Hence, after computing these inner products, we obtain

$$\frac{1}{2}(a_{11} + a_{ii}) + \frac{1}{2}(a_{i1} + a_{ii}) = \frac{1}{2}(a_{11} - a_{i1}) - \frac{1}{2}(a_{i1} - a_{ii}), \quad i \in \{2, \dots, n\}.$$

Since A is a symmetric matrix, the last equality implies that $a_{1i} = 0$, for all $i \in \{2, \dots, n\}$. Thus, by letting $a = a_{11}$, we have $A = \text{diag}(a, \tilde{A})$ with $\tilde{A} \in \mathbb{R}^{(n-1) \times (n-1)}$ a symmetric matrix. Let $\tilde{Q} \in \mathbb{R}^{(n-1) \times (n-1)}$ be an orthogonal matrix such that $\tilde{Q}^T \tilde{A} \tilde{Q} = \Lambda$, where $\Lambda = \text{diag}(\lambda_2, \dots, \lambda_n)$ and λ_i is an eigenvalue of \tilde{A} , for all $i \in \{2, \dots, n\}$. Thus, Remark 2 implies that $f : \mathcal{L} \cap \mathbb{S} \rightarrow \mathbb{R}$ is spherically convex if, and only if, $g(x) = \langle \text{diag}(a_{11}, \Lambda)x, x \rangle$ is spherically convex. On the other hand, using Proposition 3 we conclude that $g(x) = \langle \text{diag}(a_{11}, \Lambda)x, x \rangle$ is spherically convex if and only if

$$h(x) = \langle [\text{diag}(a_{11}, \Lambda) - a_{11}I_n]x, x \rangle = \langle [A - a_{11}I_{n-1}]\tilde{x}, \tilde{x} \rangle,$$

where $x := (x_1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, is spherically convex. Since h is spherically convex, from Proposition 3 we have

$$h(x) - h(y) = \langle [A - a_{11}I_{n-1}]\tilde{x}, \tilde{x} \rangle - \langle [A - a_{11}I_{n-1}]\tilde{y}, \tilde{y} \rangle \geq 0, \quad (5)$$

for all points $x = (x_1, \tilde{x}) \in \mathbb{S}$, $y = (y_1, \tilde{y}) \in \mathcal{L} \cap \mathbb{S}$ with $x \perp y$. If we assume that $\lambda_2 = \dots = \lambda_n$, we have $\Lambda = \lambda I_{n-1}$ and then $A = \text{diag}(a, \lambda I_{n-1})$, where $a := a_{11}$ and $\lambda := \lambda_2 = \dots = \lambda_n$. Thus (5) becomes $[\lambda - a_{11}][\|\tilde{x}\|^2 - \|\tilde{y}\|^2] \geq 0$. Bearing in mind that $\mathcal{L} = \mathcal{L}^*$, Lemma 2 implies $\|\tilde{x}\|^2 - \|\tilde{y}\|^2 \geq 0$, and then we have from the previous two inequalities that $a = a_{11} \leq \lambda$. Therefore, for concluding the proof of this implication it remains to prove that $a_{11} \leq \lambda_2 = \dots = \lambda_n$. Without loss of generality we can assume that $n \geq 3$. Let $x \in \mathbb{S}$ and $y \in \mathcal{L} \cap \mathbb{S}$ with $x \perp y$ be defined by

$$x = -\left(\frac{1}{\sqrt{2}} \cos \theta\right) e^1 + \left(\frac{1}{2} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta\right) e^i + \left(\frac{1}{2} \cos \theta + \frac{1}{\sqrt{2}} \sin \theta\right) e^j, \quad (6)$$

$$y = \frac{1}{\sqrt{2}} e^1 + \frac{1}{2} e^i + \frac{1}{2} e^j, \quad (7)$$

where $\theta \in (0, \pi)$. From (6) and (7), it is straightforward to check that $x \in \mathbb{S}$, $y \in \mathcal{L} \cap \mathbb{S}$ and $x \perp y$. Hence, (5) becomes

$$\left(\frac{1}{4} \sin^2 \theta - \frac{1}{\sqrt{2}} \cos \theta \sin \theta\right) \lambda_i + \left(\frac{1}{4} \sin^2 \theta + \frac{1}{\sqrt{2}} \cos \theta \sin \theta\right) \lambda_j \geq 0,$$

or, after dividing by $\sin \theta \neq 0$, that

$$\left(\frac{1}{4} \sin \theta - \frac{1}{\sqrt{2}} \cos \theta\right) \lambda_i + \left(\frac{1}{4} \sin \theta + \frac{1}{\sqrt{2}} \cos \theta\right) \lambda_j \geq 0.$$

Letting θ goes to 0 in the inequality above, we obtain $\lambda_j \geq \lambda_i$. Hence, by swapping i and j in (6) and (7) we can also prove that $\lambda_i \geq \lambda_j$, and then $\lambda_i = \lambda_j$, for all $i, j \neq 1$. Therefore, $\lambda_2 = \dots = \lambda_n$ which concludes the implication. Conversely, assume that $A = \text{diag}(a, \lambda I_{n-1})$ and $\lambda \geq a$. Then $f(x) = \langle [\text{diag}(a, \lambda I_{n-1})]x, x \rangle$ and Proposition 3 implies that f is spherically convex if, and only if,

$$h(x) = \langle [\text{diag}(a, \lambda I_{n-1}) - aI_n]x, x \rangle = \langle [\lambda - a]I_{n-1}\tilde{x}, \tilde{x} \rangle,$$

where $x := (x_1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, is spherically convex. Take $x = (x_1, \tilde{x}) \in \mathbb{S}$ and $y = (y_1, \tilde{y}) \in \mathcal{L} \cap \mathbb{S}$ with $x \perp y$. Thus, from Lemma 1 and (4) we have $\|\tilde{x}\|^2 \geq \|\tilde{y}\|^2$. Hence considering that $a \leq \lambda$ we conclude that

$$\langle [\lambda - a]I_{n-1}\tilde{x}, \tilde{x} \rangle - \langle [\lambda - a]I_{n-1}\tilde{y}, \tilde{y} \rangle = [\lambda - a][\|\tilde{x}\|^2 - \|\tilde{y}\|^2] \geq 0.$$

Therefore, Proposition 3 implies that h is spherically convex and then f is also spherically convex.

Remark 3 Assume that f in Theorem 2 is spherically convex in $\mathcal{L} \cap \mathbb{S}$. Hence there exist $a, \lambda \in \mathbb{R}$ with $\lambda \geq a$ such that $A = \text{diag}(a, \lambda I_{n-1})$ and then $f(x) = ax_1^2 + \lambda\|\tilde{x}\|^2 = \lambda - (\lambda - a)x_1^2$, where $x := (x_1, \tilde{x}) \in \mathcal{L} \cap \mathbb{S}$. Hence, it is clear that the minimum of f on $\mathcal{L} \cap \mathbb{S}$ is obtained when x_1 is maximal, that is, when $x_1 = 1$, which happens exactly when $x = e^1$. Similarly, the maximum of f on $\mathcal{L} \cap \mathbb{S}$ is obtained when x_1 is minimal, that is, when $x_1 = 1/\sqrt{2}$ (see item (i) of Lemma 2), which happens exactly when $\|\tilde{x}\| = x_1 = 1/\sqrt{2}$. Hence, $\text{argmin}\{f(x) : x \in \mathcal{L} \cap \mathbb{S}\} = e^1$, $\min\{f(x) : x \in \mathcal{L} \cap \mathbb{S}\} = a$, $\text{argmax}\{f(x) : x \in \mathcal{L} \cap \mathbb{S}\} = \left\{ \frac{1}{\sqrt{2}}(1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : \|\tilde{x}\| = 1 \right\}$ and $\max\{f(x) : x \in \mathcal{L} \cap \mathbb{S}\} = (a + \lambda)/2$.

Remark 4 If $\lambda > a$ then Theorem 2 implies that $f(x) = \langle \text{diag}(a, \lambda I_{n-1})x, x \rangle$ is spherically convex. However, in this case $\text{diag}(a, \lambda, \dots, \lambda)$ does not have the \mathcal{L} -Lyapunov-like property. Hence, Corollary 1 is not true if we only require that the cone is superdual proper. Indeed, the Lorentz cone \mathcal{L} is self-dual proper, i.e., $\mathcal{L}^* = \mathcal{L}$ and consequently is superdual proper. Moreover, letting $x, y \in \mathcal{L} \cap \mathbb{S}$ with $x \perp y$ be defined by

$$x = \frac{1}{\sqrt{2}}e^1 + \frac{1}{\sqrt{2}}e^i, \quad y = \frac{1}{\sqrt{2}}e^1 - \frac{1}{\sqrt{2}}e^i, \quad i \in \{2, \dots, n\},$$

we have $\langle \text{diag}(a, \lambda I_{n-1})x, y \rangle = (a - \lambda)/2 < 0$. Therefore, $\text{diag}(a, \lambda I_{n-1})$ does not have the \mathcal{L} -Lyapunov-like property, and the strong superduality of the cone is necessary in Corollary 1.

6 Final remarks

This paper is a continuation of [5, 6], where we studied some basic intrinsic properties of spherically convex functions on spherically convex sets of the sphere. We expect that the results of this paper can aid in the understanding of the behaviour of spherically convex functions on spherically convex sets of the sphere. In the future we will also study spherically quasiconvex functions [21] (see also [15] for the definition of quasiconvex functions) on spherically convex sets of the sphere.

Acknowledgments

The authors are grateful to Michal Kočvara and Kay Magaard for many helpful conversations.

References

1. G. DAHL, J. M. LEINAAS, J. MYRHEIM, AND E. OVRUM, *A tensor product matrix approximation problem in quantum physics*, Linear Algebra Appl., 420 (2007), pp. 711–725.
2. P. DAS, N. R. CHAKRABORTI, AND P. K. CHAUDHURI, *Spherical minimax location problem*, Comput. Optim. Appl., 18 (2001), pp. 311–326.
3. J. E. DENNIS, JR. AND R. B. SCHNABEL, *Numerical methods for unconstrained optimization and nonlinear equations*, vol. 16 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.
4. Z. DREZNER AND G. O. WESOŁOWSKY, *Minimax and maximin facility location problems on a sphere*, Naval Res. Logist. Quart., 30 (1983), pp. 305–312.
5. O. P. FERREIRA, A. N. IUSEM, AND S. Z. NÉMETH, *Projections onto convex sets on the sphere*, J. Global Optim., 57 (2013), pp. 663–676.
6. O. P. FERREIRA, A. N. IUSEM, AND S. Z. NÉMETH, *Concepts and techniques of optimization on the sphere*, TOP, 22 (2014), pp. 1148–1170.
7. P. T. FLETCHER, S. VENKATASUBRAMANIAN, AND S. JOSHI, *The geometric median on riemannian manifolds with application to robust atlas estimation*, NeuroImage, 45 (2009), pp. S143–S152.
8. D. HAN, H. H. DAI, AND L. QI, *Conditions for strong ellipticity of anisotropic elastic materials*, J. Elasticity, 97 (2009), pp. 1–13.
9. G. ISAC AND S. Z. NÉMETH, *Scalar derivatives and scalar asymptotic derivatives: properties and some applications*, J. Math. Anal. Appl., 278.
10. G. ISAC AND S. Z. NÉMETH, *Scalar derivatives and scalar asymptotic derivatives. An Altman type fixed point theorem on convex cones and some applications*, J. Math. Anal. Appl., 290 (2004), pp. 452–468.
11. G. ISAC AND S. Z. NÉMETH, *Duality in multivalued complementarity theory by using inversions and scalar derivatives*, J. Global Optim., 33 (2005), pp. 197–213.
12. G. ISAC AND S. Z. NÉMETH, *Duality in nonlinear complementarity theory by using inversions and scalar derivatives*, Math. Inequal. Appl., 9 (2006), pp. 781–795.
13. G. ISAC AND S. Z. NÉMETH, *Duality of implicit complementarity problems by using inversions and scalar derivatives*, J. Optim. Theory Appl., 128 (2006), pp. 621–633.
14. G. ISAC AND S. Z. NÉMETH, *Scalar and asymptotic scalar derivatives*, vol. 13 of Springer Optimization and Its Applications, Springer, New York, 2008. Theory and applications.
15. S. KARAMARDIAN AND S. SCHAIBLE, *Seven kinds of monotone maps*, J. Optim. Theory Appl., 66 (1990), pp. 37–46.
16. M. LAURENT, *Sums of squares, moment matrices and optimization over polynomials*, in Emerging applications of algebraic geometry, vol. 149 of IMA Vol. Math. Appl., Springer, New York, 2009, pp. 157–270.
17. C. LING, J. NIE, L. QI, AND Y. YE, *Biquadratic optimization over unit spheres and semidefinite programming relaxations*, SIAM J. Optim., 20 (2009), pp. 1286–1310.
18. S. Z. NÉMETH, *A scalar derivative for vector functions*, Riv. Mat. Pura Appl., (1992), pp. 7–24.
19. S. Z. NÉMETH, *Scalar derivatives and spectral theory*, Mathematica (Cluj), 35(58) (1993), pp. 49–57.
20. S. Z. NÉMETH, *Scalar derivatives and conformity*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., 40 (1997), pp. 99–105 (1998).
21. S. Z. NÉMETH, *Five kinds of monotone vector fields*, Pure Math. Appl., 9 (1998), pp. 417–428.
22. S. Z. NÉMETH, *Scalar derivatives in Hilbert spaces*, Positivity, 10 (2006), pp. 299–314.
23. L. QI, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Comput., 40 (2005), pp. 1302–1324.
24. L. QI AND K. L. TEO, *Multivariate polynomial minimization and its application in signal processing*, J. Global Optim., 26 (2003), pp. 419–433.
25. L. QI, F. WANG, AND Y. WANG, *Z-eigenvalue methods for a global polynomial optimization problem*, Math. Program., 118 (2009), pp. 301–316.
26. B. REZNICK, *Some concrete aspects of Hilbert’s 17th Problem*, in Real algebraic geometry and ordered structures (Baton Rouge, LA, 1996), vol. 253 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2000, pp. 251–272.

-
27. S. T. SMITH, *Optimization techniques on Riemannian manifolds*, in Hamiltonian and gradient flows, algorithms and control, vol. 3 of Fields Inst. Commun., Amer. Math. Soc., Providence, RI, 1994, pp. 113–136.
 28. A. M.-C. SO, *Deterministic approximation algorithms for sphere constrained homogeneous polynomial optimization problems*, Math. Program., 129 (2011), pp. 357–382.
 29. S. WEILAND AND F. VAN BELZEN, *Singular value decompositions and low rank approximations of tensors*, IEEE Trans. Signal Process., 58 (2010), pp. 1171–1182.
 30. G.-L. XUE, *A globally convergent algorithm for facility location on a sphere*, Comput. Math. Appl., 27 (1994), pp. 37–50.
 31. G. L. XUE, *On an open problem in spherical facility location*, Numer. Algorithms, 9 (1995), pp. 1–12.
 32. L. ZHANG, *On the convergence of a modified algorithm for the spherical facility location problem*, Oper. Res. Lett., 31 (2003), pp. 161–166.
 33. X. ZHANG, C. LING, AND L. QI, *The best rank-1 approximation of a symmetric tensor and related spherical optimization problems*, SIAM J. Matrix Anal. Appl., 33 (2012), pp. 806–821.