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# SHADOWING, ASYMPTOTIC SHADOWING AND S-LIMIT SHADOWING

CHRIS GOOD, PIOTR OPROCHA, AND MATE PULJIZ

ABSTRACT. We study three notions of shadowing property: classical shadowing property, limit (or asymptotic) shadowing, and s-limit shadowing. We show that classical and s-limit shadowing property coincide for tent maps and, more generally, for piecewise linear interval maps with constant slopes, and are further equivalent to the linking property introduced by Chen in [Che91].

We also construct a system which exhibits shadowing but not the limit shadowing property, and we study how shadowing properties transfer to maximal transitive subsystems and inverse limits (sometimes called natural extensions).

Where practicable, we show that our results are best possible by means of examples.

It seems that the first time *shadowing* appeared in the literature was in a paper [Sin72] by Sinai where it is shown that Anosov diffeomorphisms have shadowing and furthermore that any pseudo-orbit has a unique point shadowing it. This type of shadowing lemma subsequently appears in all sorts of hyperbolic systems especially in their relation to Markov partitions and symbolic dynamics. We refer the reader to monographs by Palmer [Pal00] and Pilyugin [Pil99, PS17].

Over time different variations of shadowing appeared in the literature driven by different problems people tried to solve using it. Our interest in shadowing comes from its relation to a type of attractors called  $\omega$ -limit sets. Recall that an  $\omega$ -limit set of a point  $x \in X$  is the set of limit points of its forward orbit. It can be shown that each  $\omega$ -limit set is also *internally chain transitive* (ICT), meaning that under the given dynamics and allowing small perturbations one can form a pseudo-orbit between any two points of that set (precise definitions are given in Section 1).

As the ICT condition is more operational, we are interested in systems for which these two notions coincide. It is easy to see that a form of shadowing condition introduced by Pilyugin et al. [ENP97] called *limit (or asymptotic) shadowing* suffices.

Over the years it was shown that  $\omega_f = ICT(f)$  holds for many other systems. Bowen in [Bow75] proved it for Axiom A diffeomorphisms; and in a series of papers [BDG12, BR15, Bar10, BGO12, BGKR10, BGOR13] Barwell, Davies, Good, Knight, Oprocha, and Raines prove it, amongst others, for shifts of finite type and Julia sets for certain quadratic maps. It soon became apparent that most of these systems satisfy both the classical and the asymptotic notion of shadowing. This led Barwell, Davies, and Good [BDG12, Conjectures 1.2 and 1.3] to conjecture that ~~the~~ classical shadowing alone will imply  $\omega_f = ICT(f)$ . Recently, Meddaugh ~~and Raines in with Raines~~ [MR13] answered this in the affirmative for interval maps ~~resolving thus one~~

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~~of their conjectures.~~<sup>1</sup> and with Good [GM18] has shown that, for general compact metric  $X$ ,  $\omega_f = ICT(f)$  if and only if  $f$  satisfies Pilyugin's notion of *orbital limit shadowing* [Pil07]. In Section 3 we disprove the other conjecture by constructing a system with shadowing but for which  $\omega_f \subsetneq ICT(f)$  and thus without limit shadowing (Theorem 3.3). This is the content of Theorem 3.3.

**Theorem 3.3.** *There exists a dynamical system  $(X, f)$  on a compact metric space which exhibits shadowing (SP) but for which  $\omega_f \neq ICT(f)$ .*

Interestingly, Meddaugh and Raines's result did not answer whether for interval maps classical shadowing implies limit shadowing. Trying to resolve this question forms the second line of inquiry in this paper. Sadly, we are not yet able to provide a definite answer for all interval maps but a great deal can be said for systems given by piecewise linear maps with constant slopes. Indeed, Chen in [Che91] showed that the *linking condition* is strong enough to completely resolve whether such a system has shadowing.

Using the same condition we were able to show that for these systems shadowing and linking are equivalent to a condition tightly related to limit shadowing called *s-limit shadowing*. This notion stronger than either classical or limit shadowing was introduced in [Sak03] where Sakai extended the definition of limit shadowing to account for the fact that many systems have limit shadowing but not shadowing [KO11, Pil99]. Below, we quote the two main results we obtain.

**Theorem 2.2.** *Let  $f: I \rightarrow I$  be a continuous piecewise linear map with a constant slope  $s > 1$ . Then the following are equivalent:*

- (1)  $f$  has  $s$ -limit shadowing,
- (2)  $f$  has shadowing,
- (3)  $f$  has the linking property.

*If furthermore the map is transitive, all of the above are additionally equivalent to:*

- (4)  $f$  has limit shadowing.

**Theorem 2.7.** *Let  $f_s: [0, 1] \rightarrow [0, 1]$  be a tent map with  $s \in (\sqrt{2}, 2]$  and denote its core by  $C = [f_s^2(1/2), f_s(1/2)]$ . The following conditions are equivalent:*

- (1)  $f_s$  has  $s$ -limit shadowing property
- (2)  $f_s$  has shadowing property
- (3)  $f_s$  has limit shadowing property
- (4)  $f_s|_C$  has  $s$ -limit shadowing property
- (5)  $f_s|_C$  has shadowing property
- (6)  $f_s|_C$  has limit shadowing property

The rest of the article is organised as follows. In Section 1 we introduce all the basic notions we use. In Section 2 we prove our main results, Theorems 2.2 and 2.7. Section 3 clarifies the distinction between three notions of shadowing: limit,  $s$ -limit, and classical shadowing. In there we also construct a system which exhibits shadowing but not the limit shadowing property (Theorem 3.3). In Section 4 we include few results providing sufficient conditions for shadowing on maximal limit

<sup>1</sup>It is worth noting here that neither classical nor limit shadowing implies the other — more on this in Section 3.

sets of interval maps. And finally Section 5 contains a result on shadowing of the shift map in inverse limit spaces.

## 1. PRELIMINARIES

An interval map  $f: [a, b] \rightarrow [a, b]$  is *piecewise monotone*, if there are  $p_0 = a < p_1 < \dots < p_k < p_{k+1} = b$  such that  $f|_{[p_i, p_{i+1}]}$  is monotone for  $i = 0, \dots, k$  and  $f$  is *piecewise linear* if each  $f|_{[p_i, p_{i+1}]}(x) = a_i + b_i x$  for some  $a_i, b_i \in \mathbb{R}$ . If further each  $a_i, b_i$  is in absolute value equal to some  $s \geq 0$  we say that  $f$  is *piecewise linear with constant slope  $s$* .

A point  $x \in [a, b]$  is a *critical point* of  $f$  if  $x = a$ ,  $x = b$ , or  $f$  is not differentiable at  $x$ , or  $f'(x) = 0$ . The set of critical *point-points* of  $f$  is denoted by  $\text{Crit}(f)$ .

For every  $s \in (1, 2]$  define a tent map  $f_s: [0, 1] \rightarrow [0, 1]$  by:

$$f_s(x) = \begin{cases} sx & \text{if } x \in [0, 1/2], \\ s(1-x) & \text{if } x \in [1/2, 1]. \end{cases}$$

**Definition 1.1** (Pseudo-orbit). A sequence  $\langle x_0, x_1, x_2, \dots \rangle$  is said to be a  $\delta$ -*pseudo-orbit* for some  $\delta > 0$  provided that  $d(f(x_i), x_{i+1}) < \delta$  for each  $i \in \mathbb{N}_0$ . A *finite  $\delta$ -pseudo-orbit of length  $l \geq 1$*  is a finite sequence  $\langle x_0, x_1, x_2, \dots, x_l \rangle$  satisfying  $d(f(x_i), x_{i+1}) < \delta$  for  $0 \leq i < l$ . We also say that it is a  $\delta$ -*pseudo-orbit between  $x_0$  and  $x_l$* .

We say that the sequence  $\langle x_0, x_1, x_2, \dots \rangle$  is an *asymptotic pseudo-orbit* provided that  $\lim_{i \rightarrow \infty} d(f(x_i), x_{i+1}) = 0$ .

**Definition 1.2.** A point  $z \in X$  is said to  $\epsilon$ -*shadow* a sequence  $\langle x_0, x_1, x_2, \dots \rangle$  for some  $\epsilon > 0$  if  $d(x_i, f^i(z)) < \epsilon$  for each  $i \in \mathbb{N}_0$ . It is *asymptotically shadowing* the sequence if  $\lim_{i \rightarrow \infty} d(x_i, f^i(z)) = 0$ .

**Definition 1.3** (Shadowing). A dynamical system  $f: X \rightarrow X$  is said to have *shadowing* provided that for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that every  $\delta$ -pseudo-orbit is  $\epsilon$ -shadowed by some point in  $X$ .

If a sequence  $\langle x_n \rangle_{n \in \mathbb{N}}$  is a  $\delta$ -pseudo-orbit and an asymptotic pseudo-orbit then we simply say that it is an *asymptotic  $\delta$ -pseudo-orbit*. Similarly, if a point  $z$  is  $\epsilon$ -tracing and asymptotically tracing pseudo-orbit  $\langle x_n \rangle_{n \in \mathbb{N}}$ , then we say that  $z$  is *asymptotically  $\epsilon$ -tracing  $\langle x_n \rangle_{n \in \mathbb{N}}$* .

**Definition 1.4** (s-limit shadowing). Let  $f: X \rightarrow X$  be a continuous map on a compact metric space  $X$ . We say that  $f$  has *s-limit shadowing* if and only if for every  $\epsilon > 0$  there is  $\delta > 0$  such that the following two conditions hold:

- (1) for every  $\delta$ -pseudo-orbit  $\langle x_n \rangle_{n \in \mathbb{N}} \subset X$  of  $f$ , there is  $y \in X$  such that  $y$   $\epsilon$ -shadows  $\langle x_n \rangle_{n \in \mathbb{N}}$ , and
- (2) for every asymptotic  $\delta$ -pseudo-orbit  $\langle z_n \rangle_{n \in \mathbb{N}} \subset X$  of  $f$ , there is  $y \in X$  such that  $y$  asymptotically  $\epsilon$ -shadows  $\langle z_n \rangle_{n \in \mathbb{N}}$ .

**Remark 1.5.** Note that s-limit shadowing implies both classical and limit shadowing.

**Definition 1.6.** Let  $f: X \rightarrow X$  be continuous and let  $\epsilon > 0$ . A point  $x \in X$  is  $\epsilon$ -*linked to a point  $y \in X$  by  $f$*  if there exists an integer  $m \geq 1$  and a point  $z$  such that  $f^m(z) = y$  and  $d(f^j(x), f^j(z)) \leq \epsilon$  for  $j = 0, \dots, m$ .

We say  $x \in X$  is linked to  $y \in X$  by  $f$  if  $x$  is  $\varepsilon$ -linked to  $y$  by  $f$  for every  $\varepsilon > 0$ . A set  $A \subset X$  is linked by  $f$  ~~is-if~~ every  $x \in A$  is linked to some  $y \in A$  by  $f$ .

**Definition 1.7.** Let  $f: [0, 1] \rightarrow [0, 1]$  be a continuous piecewise monotone map and let  $\text{Crit}(f)$  be a finite set of critical points of  $f$ . We say  $f$  has the *linking property* if  $\text{Crit}(f)$  is linked by  $f$ .

The following theorem is the main result in [Che91].

**Theorem 1.8** (Chen [Che91]). *Suppose  $f: [0, 1] \rightarrow [0, 1]$  is a map that is conjugate to a continuous piecewise linear map with a constant slope<sup>2</sup>  $s > 1$ . Then  $f$  has the shadowing property if and only if it has the linking property.*

In the proof of Proposition 15 in [Che91], Chen shows the following implication.

**Lemma 1.9** (Chen [Che91]). *Let  $f: [0, 1] \rightarrow [0, 1]$  be a map with constant slope  $s > 1$ . If  $f$  has linking property then there is  $\hat{\varepsilon} > 0$  such that for every  $\varepsilon \in (0, \hat{\varepsilon})$  there is  $N = N(\varepsilon) > 0$  such that for every  $x \in X$  there is an integer  $n = n(x, \varepsilon) < N$  such that*

$$B(f^n(x), s\varepsilon) \subset f^{n+1}(B(x, \varepsilon) \cap f^{-1}(B_{n-1}(f(x), s^2\varepsilon)))$$

where  $B_k(x, \varepsilon) = \{y \in [0, 1] : |f^i(x) - f^i(y)| < \varepsilon \text{ for } i = 0, \dots, k\}$ .

Finally, we define two important notions whose relation with shadowing will soon become apparent.

**Definition 1.10** (Internally Chain Transitive sets). An  $f$ -invariant and closed set  $A \subseteq X$  is said to be *internally chain transitive* if given any  $\delta > 0$  there exists a  $\delta$ -pseudo-orbit between any two points of  $A$  that is completely contained inside  $A$ . By  $ICT(f)$  we denote the minimal set containing all ICT subsets of  $X$  as its elements. This is a subset of the hyperspace of all closed non-empty subsets of  $X$  which we denote by  $2^X$ .

**Definition 1.11** ( $\omega$ -limit sets). The  $\omega$ -limit set of a point  $x \in X$  is the set of limit points of its orbit:

$$\omega_f(x) = \bigcap_{i=1}^{\infty} \overline{\bigcup_{j=i}^{\infty} \{f^j(x)\}}.$$

It can be shown that each  $\omega$ -limit set is a non-empty, closed, and  $f$ -invariant subset of  $X$ , see e.g. [BC92, Chapter IV]. In particular, this means that all of them also belong to  $2^X$ . By

$$\omega_f = \{\omega_f(x) \mid x \in X\} \subseteq 2^X,$$

we denote the set of all  $\omega$ -limit sets in the system.

It is known (see e.g. [BGOR13]) that any  $\omega$ -limit set is also internally chain transitive. We thus have the following inclusion of sets in the hyperspace  $2^X$ :

$$\omega_f \subseteq ICT(f).$$

For some systems this is a strict inclusion and it is not hard to find such examples. It is much more interesting to try to characterise systems in which  $\omega_f$  and  $ICT(f)$  coincide. This would be useful as it is easier to check if a given set is ICT than if it is an  $\omega$ -limit set.

<sup>2</sup>Note that the slope is actually  $\pm s$  but throughout the text for definiteness we always take  $s$  to be the positive value.

## 2. SHADOWING CONDITION

**Lemma 2.1.** *Let  $f: X \rightarrow X$  be a continuous map on the compact metric space  $X$ . Assume that there exist constants  $\lambda \geq 1$  and  $\hat{\varepsilon} > 0$  such that for every  $\varepsilon \in (0, \hat{\varepsilon})$ , there exist a positive integer  $N = N(\varepsilon)$  and  $\eta = \eta(\varepsilon) > 0$  such that for each  $x \in X$ , there exist a positive integer  $n = n(x, \varepsilon) \leq N$  satisfying*

$$(2.1) \quad f(B(f^n(x), \varepsilon + \eta)) \subset \{f^{n+1}(y) : d(x, y) \leq \varepsilon, \\ d(f^i(x), f^i(y)) \leq \lambda\varepsilon, 1 \leq i \leq n\}.$$

Then  $f$  has  $s$ -limit shadowing property.

*Proof.* By [CKY88, Lemma 2.4] (see also [CKY88, Lemma 2.3]) assumptions are sufficient for shadowing, therefore it is enough to prove that for every sufficiently small  $\varepsilon > 0$  (we may take  $\varepsilon < \hat{\varepsilon}$ ) there is  $\delta > 0$  such that every asymptotic  $\delta$ -pseudo-orbit is asymptotically  $\varepsilon$ -traced.

Fix any  $\varepsilon > 0$ , put  $\varepsilon_0 = \varepsilon/3\lambda$ , let  $\eta_0 = \min \langle \eta(\varepsilon_0), \varepsilon_0 \rangle$  and  $\delta$  be such that  $d(f^k(x_0), x_k) < \eta_0$  for any  $\delta$ -pseudo-orbit  $\langle x_i \rangle_{i=0}^k$ , where  $k = 1, 2, \dots, N(\varepsilon_0)$ . We will show that the above  $\delta$  is as desired.

Fix any asymptotic  $\delta$ -pseudo-orbit  $\langle x_i \rangle_{i=0}^\infty$ . Denote  $\delta_0 = \delta$  and for every  $m > 0$  we define  $\varepsilon_m = \varepsilon_0/2^m$ , let  $\eta_m = \min \langle \eta(\varepsilon_m), \varepsilon_m \rangle$  and let  $0 < \delta_m < \varepsilon_m$  be such that  $d(f^k(x_0), x_k) < \eta_m$  for any  $\delta_m$ -pseudo-orbit  $\langle x_i \rangle_{i=0}^k$ , where  $k = 1, 2, \dots, N(\varepsilon_m)$ .

For any  $x \in X$  denote

$$A(x, n, \gamma) = \{y : d(x, y) \leq \gamma, d(f^i(x), f^i(y)) \leq \lambda\gamma, 1 \leq i \leq n\}.$$

We define integers  $m_k, n_k, j_k$ , and sets  $W_k$  for all  $k \geq 0$  as follows.

$$m_0 = 0, \quad j_0 = 0, \quad n_0 = n(x_0, \varepsilon_{j_0}), \quad W_0 = A(x_{m_0}, n_0, \varepsilon_0).$$

Next, for  $k \geq 1$  we put

$$m_k = m_{k-1} + n_{k-1} = n_0 + \dots + n_{k-1}.$$

By the definition of  $j_{k-1}$  the sequence  $\langle x_i \rangle_{i=m_k}^\infty$  is an asymptotic  $\delta_{j_{k-1}}$ -pseudo-orbit. If it is also  $\delta_{j_{k-1}+1}$ -pseudo-orbit then we put  $j_k = j_{k-1} + 1$  and put  $j_k = j_{k-1}$  otherwise. Finally, let

$$n_k = n(x_{m_k}, \varepsilon_{j_k}), \\ W_k = W_{k-1} \cap f^{-m_k-1}(f(A(x_{m_k}, n_k, \varepsilon_{j_k}))).$$

First, we claim that for every  $k \geq 0$  we have

$$(2.2) \quad f^{m_k+1}(W_k) \subset f(A(x_{m_k}, n_k, \varepsilon_{j_k})).$$

We will prove the claim by induction on  $k$ . For  $k = 0$  the claim holds just by the definition, since

$$f^{m_0+1}(W_0) = f(W_0) = f(A(x_{m_0}, n_0, \varepsilon_0))$$

Next, fix any  $s \geq 0$  and suppose that the claim holds for all  $0 \leq k \leq s$ . Since, by definition we have

$$f^{m_{s+1}+1}(W_{s+1}) = f^{m_{s+1}+1}(W_s \cap f^{-m_{s+1}-1}(f(A(x_{m_{s+1}}, n_{s+1}, \varepsilon_{j_{s+1}})))) \\ = f^{m_{s+1}+1}(W_s) \cap f(A(x_{m_{s+1}}, n_{s+1}, \varepsilon_{j_{s+1}}))$$

it remains to prove that

$$f(A(x_{m_{s+1}}, n_{s+1}, \varepsilon_{j_{s+1}})) \subset f^{m_{s+1}+1}(W_s).$$

Observe that by the choice of  $j_s$  we have that  $\langle x_i \rangle_{i=m_s}^\infty$  is a  $\delta_{j_s}$ -pseudo-orbit and hence

$$d(f^j(x_{m_s}), x_{m_s+j}) < \eta_{j_s} \quad \text{for every } 0 \leq j \leq N(\varepsilon_{j_s}).$$

In particular  $d(f^{n_s}(x_{m_s}), x_{m_s+n_s} = x_{m_{s+1}}) < \eta_{j_s}$  and thus

$$B(x_{m_{s+1}}, \varepsilon_{j_s}) \subset B(f^{n_s}(x_{m_s}), \varepsilon_{j_s} + \eta_{j_s}).$$

By the definition of  $n_s$  we also have that

$$\begin{aligned} f(B(f^{n_s}(x_{m_s}), \varepsilon_{j_s} + \eta_{j_s})) &\subset \{f^{n_s+1}(y) : d(x_{m_s}, y) \leq \varepsilon_{j_s}, \\ &\quad d(f^i(x_{m_s}), f^i(y)) \leq \lambda \varepsilon_{j_s}, 1 \leq i \leq n_s\} \\ &= f^{n_s+1}(A(x_{m_s}, n_s, \varepsilon_{j_s})). \end{aligned}$$

Combining the two above observations with the assumptions of induction we obtain that

$$\begin{aligned} f(A(x_{m_{s+1}}, n_{s+1}, \varepsilon_{j_{s+1}})) &\subset f(B(x_{m_{s+1}}, \varepsilon_{j_s})) \subset f(B(f^{n_s}(x_{m_s}), \varepsilon_{j_s} + \eta_{j_s})) \\ &\subset f^{n_s+1}(A(x_{m_s}, n_s, \varepsilon_{j_s})) = f^{n_s}(f^{m_s+1}(W_s)) \\ &= f^{m_{s+1}+1}(W_s). \end{aligned}$$

This completes the induction, so the claim is proved.

Note that since every set  $A(x, n, \gamma)$  is closed, we have a nested sequence of closed non-empty sets  $W_0 \supset W_1 \supset \dots$  and therefore there is at least one point  $z \in \bigcap_{k=0}^\infty W_k$ . We claim that for  $k = 0$  and  $0 \leq i \leq m_1$  or for  $k > 0$  and each  $m_k < i \leq m_{k+1}$  we have

$$d(f^i(z), x_i) \leq 2\lambda \varepsilon_{j_k}.$$

Again we prove it by induction on  $k$ . First, let  $k = 0$  and fix any  $0 \leq i \leq n_0 = n(x_0, \varepsilon_0) < N(\varepsilon_0)$ . Since  $z \in W_0 = A(x_0, n_0, \varepsilon_0)$  we have that  $d(f^i(z), f^i(x_0)) \leq \lambda \varepsilon_0$  and additionally, by the definition of  $\delta_{j_0} = \delta_0$  we have that  $d(f^i(x_0), x_i) < \varepsilon_0$ . Hence  $d(f^i(z), x_i) \leq (\lambda + 1)\varepsilon_0 \leq 2\lambda \varepsilon_0$ . The first step of induction is complete.

Now fix any  $k > 0$  and any  $m_k < i \leq m_{k+1}$ . Denote  $t = i - m_k - 1$  and observe that  $0 \leq t < n_k = n(x_{m_k}, \varepsilon_{j_k})$ . Observe that by (2.2)

$$f^i(z) = f^t(f^{m_k+1}(z)) \in f^t(f^{m_k+1}(W_k)) \subset f^{t+1}(A(x_{m_k}, n_k, \varepsilon_{j_k}))$$

and so  $d(f^i(z), f^{t+1}(x_{m_k})) < \lambda \varepsilon_{j_k}$ . Additionally, by the choice of  $j_k$  and  $\delta_{j_k}$  we have that  $d(f^{t+1}(x_{m_k}), x_{m_k+t+1}) < \varepsilon_{j_k}$ . Combining these two inequalities we obtain that

$$d(f^i(z), x_i) \leq \lambda \varepsilon_{j_k} + \varepsilon_{j_k} \leq 2\lambda \varepsilon_{j_k}.$$

The claim is proved.

Observe that since  $\langle x_i \rangle_{i=0}^\infty$  is an asymptotic pseudo-orbit the sequence  $\langle j_k \rangle_{k=0}^\infty$  is unbounded (i.e.  $\lim_{k \rightarrow \infty} \varepsilon_{j_k} = 0$ ) and hence  $z$  asymptotically traces  $\langle x_i \rangle_{i=0}^\infty$ . Additionally,

$$d(f^i(z), x_i) \leq 2\lambda \varepsilon_{j_k} \leq 2\lambda \varepsilon_0 < \varepsilon$$

and so, in fact,  $z$  is asymptotically  $\varepsilon$ -tracing the sequence  $\langle x_i \rangle_{i=0}^\infty$  which ends the proof.  $\square$

The equivalence of shadowing and linking for piecewise linear maps with a constant slope was demonstrated by Chen in [Che91]. In fact, in the proof of [Che91, Proposition 15] it is shown that for these maps the linking property implies a stronger condition (2.1) with  $\lambda = s^2$  and  $\eta = (s - 1)\varepsilon$  where  $s$  is the slope of the



map. This observation together with Lemma 2.1 immediately gives the following theorem.

**Theorem 2.2.** *Let  $f: I \rightarrow I$  be a continuous piecewise linear map with a constant slope  $s > 1$ . Then the following are equivalent:*

- (1)  *$f$  has  $s$ -limit shadowing,*
- (2)  *$f$  has shadowing,*
- (3)  *$f$  has the linking property.*

*If furthermore the map is transitive, all of the above are additionally equivalent to:*

- (4)  *$f$  has limit shadowing.*

**Remark 2.3.** It is worth noting that this is the best one could hope for as a simple example shows that it is possible to construct a (non-transitive) constant slopes piecewise linear map with limit shadowing but not the other properties. Indeed, take a tent map  $f = f_s$  for which the critical point  $c = 1/2$  forms a three cycle  $c \mapsto f(c) \mapsto f^2(c) \mapsto c$ . This corresponds to the slope being equal to the golden ratio  $s \approx 1.6180$  and the map is shown on Figure 1. Both the map  $f$  over  $[0, 1]$  and the restriction  $f|_{[f^2(c), f(c)]}$  have the linking property and thus all the shadowing properties we are consider in this article.

Now take a point  $d = 1/(s + s^2) \in (0, f^2(c))$  which is a preimage of the interior fixed point under two iterates. The map  $f|_{[d, f(c)]}$  no longer has linking as the prefixed point  $d$  is clearly not  $\varepsilon$ -linked to any of the critical points  $\{d, 1/2, f(c)\}$  for sufficiently small  $\varepsilon$ . This implies that this restriction does not have the shadowing property. We claim, however, that it has limit shadowing.

To this end, take any asymptotic pseudo-orbit  $\langle x_i \rangle_{i=0}^\infty$  in  $[d, f(c)]$ . Substitute any  $x_i \in [d, f^2(c))$  in this sequence with  $f^2(c)$  and denote the new sequence of points in  $[f^2(c), f(c)]$  by  $\langle y_i \rangle_{i=0}^\infty$ . It is easy to check that this is still an asymptotic pseudo-orbit for the map  $f|_{[f^2(c), f(c)]}$  which, we established, has  $s$ -limit shadowing. If we now take a point  $z \in [f^2(c), f(c)]$  which asymptotically shadows the pseudo-orbit  $\langle y_i \rangle_{i=0}^\infty$ , then it is easy to show that it must also asymptotically shadow the original pseudo-orbit  $\langle x_i \rangle_{i=0}^\infty$ .

To conclude,  $f|_{[d, f(c)]} \sim f|_{[d, f(c)]}$  is indeed a piecewise linear map with constant slopes which has limit shadowing but not the linking property and consequentially it does not have the shadowing property.  $\square$

We now proceed to prove Theorem 2.2.

*Proof of Theorem 2.2.* Equivalence of (2) and (3) is provided by Theorem 1.8. If  $f$  has linking property then assumptions of Lemma 2.1 are satisfied, which shows that (1) is a consequence of (3). Remaining implication (1)  $\implies$  (2) is trivial.

In the transitive case the implication (4)  $\implies$  (2) follows from a result by Kulczycki, Kwietniak, and Oprocha [KKO14, Theorem 7.3] whereas the converse (1)  $\implies$  (4) is immediate by definition.  $\square$

In [Par66] Parry showed that any transitive continuous piecewise monotone map of a compact interval is conjugate to a piecewise linear map with a constant slope of the same entropy. By a result of Blokh [Blo82] (for a proof see [BC87]) the entropy  $h$  of a transitive interval map is strictly positive (actually  $h \geq \log \sqrt{2}$ ) and thus this slope is  $e^h > 1$ . As each of the four properties in the theorem above is preserved under conjugations, this argumentation allows us to substitute the constant slopes assumption in the previous theorem with transitivity of the map.



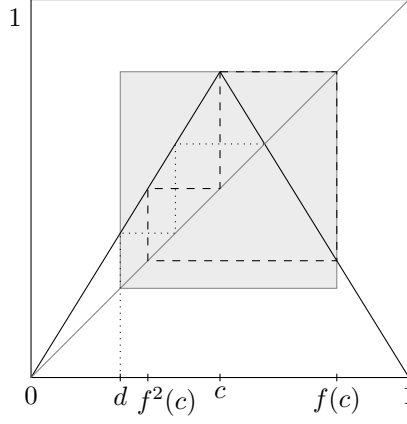


FIGURE 1. A map with limit shadowing but without the classical shadowing property

**Corollary 2.4.** *For every transitive continuous piecewise monotone map of a compact interval, all four properties:  $s$ -limit shadowing, limit shadowing, classical shadowing, and the linking property are equivalent.*

It is natural to ask at this point if the characterisation in Theorem 2.2 extends to maps with countably many monotone pieces. It turns out that the answer is no as the following example shows.

**Example 2.5.** We construct a piecewise linear map with a constant slope  $s > 1$  and countably many pieces of monotonicity which has the linking property but does not have shadowing.

The key steps of the construction are represented in Figure 2. We first construct a nucleus of the map depicted in Figure 2(a) where each critical point is mapped onto another critical point in the rescaled version of the map. To be precise, the critical point denoted by  $C_1$  in Figure 2(a) is mapped to  $C_2^+$  — the rescaled (by a factor  $\mu < 1$ ) copy of the critical point  $C_2$  which is in turn mapped to the original point  $C_2$ . The slope is everywhere the same and is denoted by  $s > 3$ . If we denote by  $a$  the length of the first (and third) piece of monotonicity of the nucleus one easily deduces that the parameters  $a, s, \mu$  have to satisfy the following set of equations:

$$\begin{aligned} sa &= 1 + \mu(1 - a) \\ s\mu a &= \mu + a \\ 4sa &= s + 1. \end{aligned}$$

This system has a solution and approximate numerical values of the parameters are:  $a \approx 0.301696$ ,  $\mu \approx 0.657298$ ,  $s \approx 4.83598$ .

Once the nucleus is constructed, one takes two  $\mu$ -scaled copies of it and glues them to either ends of the nucleus. Then another two  $\mu^2$ -scaled copies are added, and so on ad infinitum. This process gives us the map depicted on Figure 2(b) which we denote by  $f: [0, 1] \rightarrow [0, 1]$  after it has been rescaled to the unit interval.

Note that each critical point of  $f$  (except 0 and 1 which are fixed) is pre-periodic and is eventually mapped onto the cycle  $C_1 \mapsto C_2^+ \mapsto C_2 \mapsto C_1^- \mapsto C_1$  on the

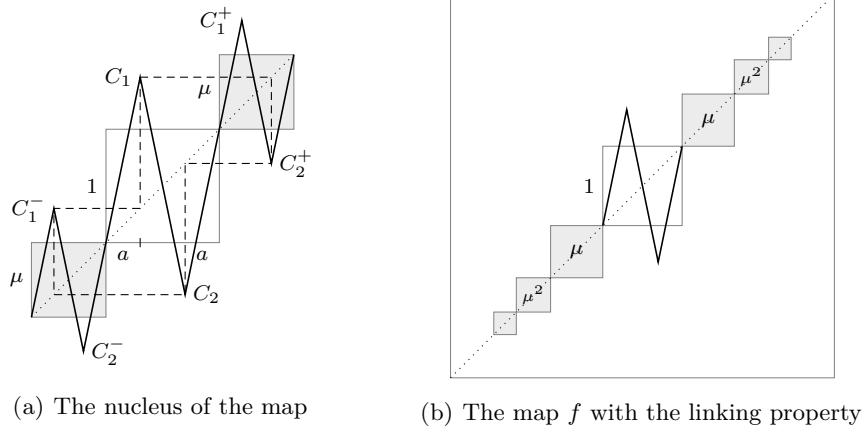


FIGURE 2. Construction of the map which has linking but not shadowing

nucleus. Thus  $f$  has linking. It is not hard to see that  $f$  is also topologically mixing. Namely its slope exceeds 2 and so every open interval  $U \subset [0, 1]$  must eventually cover two critical points (i.e.  $f^n(U)$  contains two consecutive critical points), which in turn implies that for every  $\varepsilon > 0$  there is  $m > 0$  such that  $[\varepsilon, 1 - \varepsilon] \subset f^m(U)$ . This proves topological mixing. Unfortunately, it is not completely clear if  $f$  has shadowing, and to ensure this we make a simple modification.

Let  $g: [-1, 1] \rightarrow [-1, 1]$  be a map given by:

$$g(x) = \begin{cases} -f(x) & \text{if } x \geq 0, \\ f(-x) & \text{if } x < 0. \end{cases}$$

It is clear that  $g$  also possesses the linking property and constant slopes but it cannot have shadowing. The reason is that the map  $g$  is transitive but not mixing as  $g^2$  has invariant intervals  $[-1, 0]$  and  $[0, 1]$ , thus no point can have a dense orbit in  $[-1, 1]$  under  $g^2$ . On the other hand,  $g^2$  is chain transitive on  $[-1, 1]$  as for any  $\delta$ -pseudo-orbit jumping across on the other side of the origin poses no problem. If  $g$  had shadowing then so would  $g^2$  and this would contradict a result from [MR13, Corollary 6] which says that for any interval map with shadowing any ICT set is also an  $\omega$ -limit set.  $\triangleleft$

**Example 2.6.** By [KO10] the standard tent map  $f_2$  can be perturbed to a map  $f: [0, 1] \rightarrow [0, 1]$  such that:

- (1)  $f$  has shadowing property and is topologically mixing,
- (2) inverse limit of  $[0, 1]$  with  $f$  as a unique bonding map is the pseudo-arc, and as a consequence  $f$  has infinite topological entropy (see [Mou12]).

By the above  $f$  is not conjugated to a piecewise-linear map with constant slope, since all these maps have finite entropy.

The answer to the question whether  $f$  has s-limit shadowing is unknown to the authors.  $\triangleleft$

**Theorem 2.7.** *Let  $f_s: [0, 1] \rightarrow [0, 1]$  be a tent map with  $s \in (\sqrt{2}, 2]$  and denote its core by  $C = [f_s^2(1), f_s(1)]$ . The following conditions are equivalent:*

- (1)  $f_s$  has  $s$ -limit shadowing property
- (2)  $f_s$  has shadowing property
- (3)  $f_s$  has limit shadowing property
- (4)  $f_s|_C$  has  $s$ -limit shadowing property
- (5)  $f_s|_C$  has shadowing property
- (6)  $f_s|_C$  has limit shadowing property

*Proof.* The equivalence (1)  $\iff$  (2) is provided by Theorem 2.2. Similarly, since  $f_s|_C$  is an unimodal map with a constant slope, we obtain the equivalence (4)  $\iff$  (5). In fact by Theorem 2.2 we know that for maps  $f_s$  and  $f_s|_C$  shadowing is equivalent to linking.

Let  $c = 1/2$  be the unique critical point of  $f_s$  in  $(0, 1)$ . Note that  $\text{Crit}(f_s) = \{0, c, 1\}$  and  $\text{Crit}(f_s|_C) = \{f_s^2(c), c, f_s(c)\}$ . As  $f_s(1) = 0$  and  $f_s(0) = 0$ , both 0 and 1 are linked to 0 by  $f_s$ . Similarly, both  $c$  and  $f_s(c)$  are linked to  $f_s^2(c)$  by  $f_s|_C$ . Thus for  $s < 2$  checking for the linking property in  $f_s$  (resp.  $f_s|_C$ ) boils down to checking whether  $c$  is linked to itself (resp. whether  $f_s^2(c)$  is linked to itself).

But if  $c$  is linked to itself then clearly  $f_s^2(c)$  is also linked to  $c$  and thus also to  $f_s^2(c)$ . For the converse note that if  $s < 2$  the only pre-image of  $f_s^2(c)$  under  $f_s|_C$  is  $f_s(c)$  and in turn the only pre-image of  $f_s(c)$  is  $c$ . It is now straightforward to check that if  $f_s^2(c)$  is linked to itself then  $c$  must also be linked to itself. Thus we have showed (1)  $\iff$  (2)  $\iff$  (4)  $\iff$  (5) (note that for  $s = 2$  this is trivially satisfied).

If  $f$  has  $s$ -limit shadowing then by definition it has limit shadowing, therefore the implications (1)  $\implies$  (3) and (4)  $\implies$  (6) are trivially satisfied.

It is proved in [KKO14] that if a map is transitive and has limit shadowing then it also has shadowing. But it is well known (e.g. see [BB04, Remark 3.4.17]) that each  $f_s|_C$  is transitive, hence the implication (6)  $\implies$  (5) is also valid.

To close the circle of implications it now only remains to show (3)  $\implies$  (6). To this end we fix any asymptotic pseudo-orbit  $\langle x_i \rangle_{i=0}^\infty \subset C$  and let  $z \in [0, 1]$  be a point which asymptotically traces it under action of  $f_s$ . Observe that the set  $\Lambda = \omega(z, f_s)$  is  $f_s$ -invariant and thus  $\Lambda \subset C$ . Note that for  $\sqrt{2} < s < 2$  (again for  $s = 2$  the implication (3)  $\implies$  (6) trivially holds) the  $f_s$ -invariant set  $\Lambda$  cannot be contained in  $\{f_s^2(c), f_s(c)\}$  but must intersect the interior of  $C$ . Hence there exist an integer  $n \geq 0$  such that  $f^n(z) \in C$  and so  $f^{n+j}(z) \in C$  for every  $j \geq 0$ . If we fix any point  $y \in C$  such that  $f^n(y) = f^n(z)$  then clearly  $y$  is asymptotically tracing  $\langle x_i \rangle_{i=0}^\infty$ . This completes the proof.  $\square$

### 3. HOW DO LIMIT SHADOWING, S-LIMIT SHADOWING, AND CLASSICAL SHADOWING RELATE TO EACH OTHER

In this section we describe the distinction between three notions of shadowing: limit ( $LmSP$ ),  $s$ -limit ( $s-LmSP$ ), and classical shadowing property ( $SP$ ).

Pilyugin in [Pil99, Theorem 3.1.3] showed that for circle homeomorphisms the property  $SP$  implies  $LmSP$ . In fact, he gave efficient characterisations of both  $SP$  and  $LmSP$  for orientation preserving circle homeomorphisms which fix a nowhere dense set containing at least two points (see [Pil99, Theorems 3.1.1 and 3.1.2]).

Pilyugin's results roughly state that such a homeomorphism with a hyperbolic (either repelling or attracting) fixed point has  $LmSP$ , and it further has  $SP$  if the repelling and attracting fixed points alternate. We abstain from stating the full characterisation precisely and instead refer the interested reader to the book [Pil99].

Using this, it is not hard to construct maps that land in areas denoted by (a), (b), and (c) in the Venn diagram in Figure 3. The corresponding graphs are depicted in Figure 4. (Note that the circle is represented as the interval  $[-1, 1]$  with endpoints identified.)

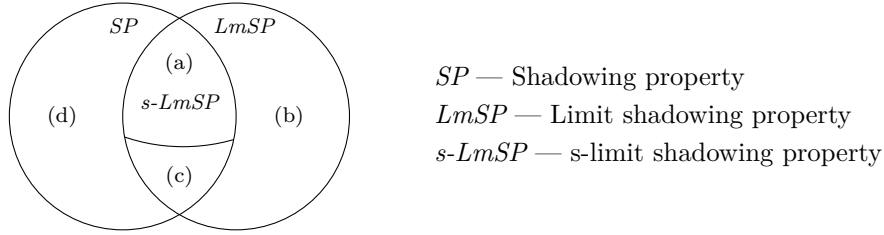


FIGURE 3. Shadowing properties

The map in Figure 4(a) has exactly one repelling (at  $\pm 1$ ) and one attracting fixed point (at 0). They are alternating and thus this circle homeomorphism has  $SP$  and  $LmSP$ . It trivially has  $s-LmSP$  as any point other than  $\pm 1$  is attracted to 0, and therefore for  $0 < \varepsilon < 1/2$  any orbit that  $\varepsilon$ -shadows an asymptotic  $\delta$ -pseudo-orbit (where  $\delta = \delta(\varepsilon)$  is implied by  $SP$ ) is in fact asymptotically shadowing it.

The map in Figure 4(b) has a repelling fixed point at  $\pm 1$  and so it has  $LmSP$ . But it does not have  $SP$  because the fixed point at 0 is non-hyperbolic.

The last graph in Figure 4(c) is an extension to the interval  $[-1, 1]$  of the map given by Barwell, Good, and Oprocha in [BGO12, Example 3.5]. Our map is given by the following equation:

$$\Phi(x) = \begin{cases} x + \frac{1}{2\pi\sqrt{2}}x \sin(2\pi \ln |x|) & \text{if } x \in [-1, 0), \\ x^3 & \text{if } x \in [0, 1]. \end{cases}$$

One readily checks that this is indeed a strictly increasing map on  $[-1, 1]$ . It has a sequence of fixed points converging to the fixed point at 0 and each of them is hyperbolic (by convention 0 is also hyperbolic as the limit of hyperbolic fixed points). The repelling and attracting fixed points alternate, thus this homeomorphism (call it  $\phi$ ) has both  $SP$  and  $LmSP$ . It, however, does not have  $s-LmSP$  and the argument is essentially the one in [BGO12]. We present it below for completeness.

For  $\varepsilon = 1/2$  and any small  $\delta > 0$  one can find a fixed point  $x_\delta \in (0, \delta)$  and an integer  $N \in \mathbb{N}$  large enough such that

$$\langle 1/2, \phi(1/2), \dots, \phi^N(1/2), 0, x_\delta, x_\delta, x_\delta, \dots \rangle$$

is an asymptotic  $\delta$ -pseudo-orbit. Any point that could potentially  $\varepsilon$ -shadow this pseudo-orbit would have to be in  $(0, 1)$ . The orbit of such a point would eventually converge to 0 and would not asymptotically shadow the pseudo-orbit above.

We have already said that Pilyugin ruled out the possibility of a circle homeomorphism in the region marked by (d) on Figure 3. If one drops the bijectivity

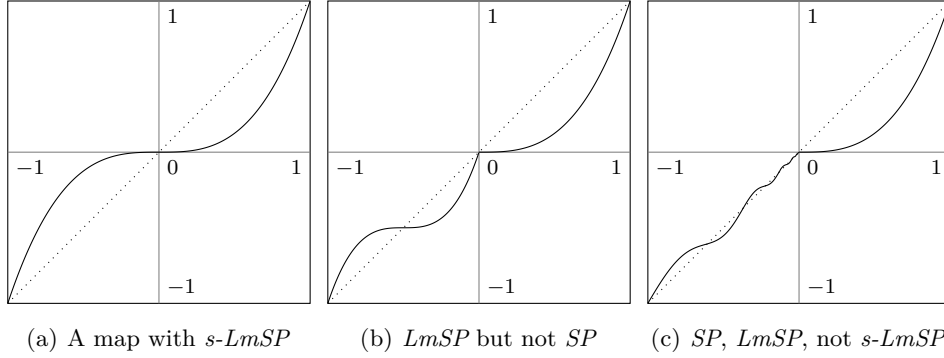


FIGURE 4. Circle homeomorphisms with various shadowing properties

requirement and asks only for continuity, the answer seems to be unknown. Until recently it was unknown whether such a system exists on any compact metric space.

To see why this might be interesting, recall that any  $\omega$ -limit set is also an ICT set. As any ICT set can be traced in the limit by an asymptotic pseudo-orbit, for systems with  $LmSP$ , ICT sets and  $\omega$ -limit sets coincide. If  $SP$  necessarily implied  $LmSP$  then this would give a positive resolution to a problem posed by Barwell, Davies, and Good [BDG12, Conjectures 1.2 and 1.3]. They asked if for a tent map (or more generally any dynamical system on a compact metric space) with  $SP$ ,  $ICT(f) = \omega_f$ .

The answer to [BDG12, Conjecture 1.2] came from Meddaugh and Raines in [MR13]. They showed that the set  $ICT(f)$  must be closed in  $2^X$ , the hyperspace of closed non-empty subsets of  $X$  furnished with the Hausdorff distance. Assuming  $SP$  they further show that the closure of  $\omega_f$  in  $2^X$  is equal to  $ICT(f)$ :

$$\overline{\omega_f} = ICT(f).$$

When combined with a result by Blokh, Bruckner, Humke, and Smítal [BBHS96] that the set  $\omega_f$  is already closed for any continuous map  $f: [0, 1] \rightarrow [0, 1]$ , this yields the positive answer for all interval maps.

Their resolution did not, however, resolve whether  $SP$  implies  $LmSP$  even for interval maps. Indeed, we believe that this question is still open.

**Question 3.1.** Is it true that for interval maps  $SP$  implies  $LmSP$ ?

Surprisingly, the answer to [BDG12, Conjecture 1.3] is negative as we shall now show. We shall construct a system (Theorem 3.3) on the Cantor set with  $SP$  but for which  $ICT(f) \neq \omega_f$ . This in particular implies that this system does not have  $LmSP$  and thus lies in the region (d). Independently, Gareth Davies found a similar example which remains unpublished.

**3.1. A system in  $SP \cap (LmSP)^c$ .** Let  $\mathcal{A}$  be an alphabet, i.e. a finite discrete set of symbols. Recall that a *word* over  $\mathcal{A}$  is a finite sequence of elements in  $\mathcal{A}$ . If one can find a finite collection of words  $W$  such that a shift space  $X$  is precisely the set of sequences in which none of the words from  $W$  appears, then that shift space is said to be of *finite type*. Walters in [Wal78] showed that shifts of finite type are precisely those shift spaces with the shadowing property.

**Proposition 3.2** (Walters [Wal78]). *A shift space over a finite alphabet is of finite type if and only if it has the shadowing property.*

We can now proceed with the construction. We fix the alphabet  $\mathcal{A} = \{0, 1\}$  and for each  $k \in \mathbb{N}_0$  we set

$$X_k = \{\xi \in \Sigma_2 \mid \text{any two 1s in } \xi \text{ are separated with at least } (k+1) \text{ 0s}\},$$

and

$$X_\infty = \{\xi \in \Sigma_2 \mid \xi \text{ has at most one symbol } 1\}.$$

Note that each  $X_k$  is in fact a shift space of finite type where the set of forbidden words is exactly

$$\{\underbrace{10 \dots 01}_{l \text{ zeros}} \mid 0 \leq l \leq k\} = \{11, 101, 1001, \dots, \underbrace{10 \dots 01}_{k \text{ zeros}}\}.$$

We also set  $N = \{1/2^k \mid k \in \mathbb{N} \cup \{0, \infty\}\}$  where  $1/2^\infty = 0$  by the convention. The topology on  $N$  is taken to be inherited from the real line, and the observant reader might realise that  $N$  and  $X_\infty$  are in fact homeomorphic. The space  $N \times \Sigma_2$  is a compact metric space equipped with the max-distance:

$$d((a_1, \xi_1), (a_2, \xi_2)) = \max\{|a_1 - a_2|, d_{\Sigma_2}(\xi_1, \xi_2)\},$$

where  $d_{\Sigma_2}$  is the standard metric on the full shift  $\Sigma_2$ .

On  $N \times \Sigma_2$  we define a continuous map  $f$  as the product of the identity on  $N$  and the shift map  $\sigma$  on  $\Sigma_2$ :

$$f(a, \xi) = (a, \sigma(\xi)).$$

This is easily seen continuous. Finally, we take

$$X = \{(a, \xi) \in N \times \Sigma_2 \mid a = \frac{1}{2^k} \text{ and } \xi \in X_k, \text{ for some } k \in \mathbb{N} \cup \{0, \infty\}\},$$

or equivalently

$$X = \{0\} \times X_\infty \cup \bigcup_{k=0}^{\infty} \{1/2^k\} \times X_k.$$

This is clearly an  $f$ -invariant subset of  $N \times \Sigma_2$ . We below show that  $X$  is also closed and that the map  $f$  restricted to  $X$  provides the counter-example we have been looking for.

**Theorem 3.3.**  *$(X, f)$  is a dynamical system on a compact metric space which exhibits shadowing (SP) but for which  $\omega_f \neq ICT(f)$ .*

Let us briefly describe the idea behind the construction. The map  $f$  on each space  $\{1/2^k\} \times X_k$  is conjugated to a shift of finite type and hence, by Proposition 3.2,  $f$  has shadowing on those subspaces. The space  $\{0\} \times X_\infty$  on the other hand is not of finite type and does not have shadowing. In the construction we exploit the fact that the sequence of spaces  $\langle \{1/2^k\} \times X_k \rangle_{k \in \mathbb{N}_0}$  converge to  $\{0\} \times X_\infty$  in the hyperspace  $2^X$  as  $k \rightarrow \infty$ . This allows us to shadow pseudo-orbits in the subspace  $\{0\} \times X_\infty$  using real orbits in the space  $\{1/2^k\} \times X_k$  for  $k$  large enough. In this way we succeeded (Lemma 3.5) to impose shadowing on  $f$  in the whole space by having shadowing on a family of proper subspaces approximating  $X$  in the limit.

It remains to be seen that for this system  $\omega_f \neq ICT(f)$ . The counter-example is the set  $\{0\} \times X_\infty$  which is not an  $\omega$ -limit set of any of the points in  $X$  (Theorem 3.3). Yet, it is the limit of the sequence of subspaces  $\{1/2^k\} \times X_k$  as  $k \rightarrow \infty$ , each of which is an  $\omega$ -limit set of a point in  $X$ . This, when combined with the result of Meddaugh and Raines, implies that  $\{0\} \times X_\infty$  is an ICT set but not an  $\omega$ -limit set. We shall now proceed with proving these claims.

**Lemma 3.4.**  *$X$  is a closed and hence a compact subset of  $N \times \Sigma_2$ .*

*Proof.* Let  $(a, \xi)$  be a point in  $N \times \Sigma_2 \setminus X$ . If  $a = 1/2^k > 0$ , this means that  $\xi \in \Sigma_2 \setminus X_k$ . Since  $X_k$  is closed in  $\Sigma_2$  there is an open set  $V$  around  $\xi$  that does not intersect  $X_k$ . Since  $U = \{a\}$  is an open (and closed) set in  $N$ ,  $U \times V$  is an open neighbourhood containing  $(a, \xi)$  that does not intersect  $X$ .

If  $a = 0$  and  $\xi \notin X_\infty$  then there exists a  $k \in \mathbb{N}_0$  so that word  $\underbrace{10 \dots 01}_{k \text{ zeros}}$  occurs somewhere in  $\xi$ . Take  $V$  to be the set of all 0-1 sequences in  $\Sigma_2$  which have this word at the same position as  $\xi$  does. This set is easily seen to be clopen. It is indeed what is called a cylinder set in  $\Sigma_2$  (see e.g. [LM95]). Setting  $U = [0, 1/2^k] \cap N$  one readily checks that  $U \times V$  is an open neighbourhood containing  $(a, \xi)$  but not intersecting  $X$ .  $\square$

**Lemma 3.5.**  *$(X, f)$  has shadowing.*

*Proof.* Let  $\varepsilon > 0$  and additionally assume  $\varepsilon < 1$ . Choose a  $k \in \mathbb{N}$  so that  $\varepsilon/2 \leq 1/2^k < \varepsilon$ . Set  $\delta = \min\{\varepsilon/4, \delta_1(\varepsilon), \dots, \delta_k(\varepsilon)\} > 0$ , where each  $\delta_j(\varepsilon)$  for  $1 \leq j \leq k$  is a positive number chosen so that every  $\delta_j(\varepsilon)$ -pseudo-orbit in  $X_j$  is  $\varepsilon$ -shadowed. This can be done by Proposition 3.2. We claim that for this  $\delta$ , every  $\delta$ -pseudo-orbit in  $X$  is  $\varepsilon$ -shadowed by a real orbit.

Let  $\langle (a_n, \xi_n) \rangle_{n \in \mathbb{N}_0}$  be a  $\delta$ -pseudo-orbit in  $X$ . We shall distinguish two cases.

Case 1. We first suppose that  $a_0 > \varepsilon/2$ . If  $a_1 > a_0$  then  $a_1 \geq 2a_0$  and hence  $|a_1 - a_0| > \varepsilon/2 > \delta$ . On the other hand, if  $a_1 < a_0$  then  $2a_1 \leq a_0$  and hence  $|a_0 - a_1| \geq a_0/2 > \varepsilon/4 \geq \delta$ . Therefore, it must be that  $a_1 = a_0$  and inductively  $a_n = a_0$  for all  $n \in \mathbb{N}$ . Which means that in this case the whole pseudo orbit is actually contained in the same subspace  $\{1/2^m\} \times X_m$  where  $a_0 = 1/2^m$ .

Clearly  $m \leq k$ . Since  $\delta \leq \delta_m(\varepsilon)$ , we have that  $\langle \xi_n \rangle_{n \in \mathbb{N}_0}$  is a  $\delta_m(\varepsilon)$ -pseudo-orbit in  $X_m$ , hence we can choose a point  $\xi^*$  that  $\varepsilon$ -shadows it. But then the point  $(a_0, \xi^*)$  clearly  $\varepsilon$ -shadows the initial pseudo-orbit.

Case 2. We now suppose  $a_0 \leq \varepsilon/2$ . A similar argument to the one above shows that  $a_n \leq \varepsilon/2$ , and hence  $a_n \leq 1/2^k$  for all  $n \in \mathbb{N}$ . Since  $(X_n)_{n \in \mathbb{N}_0}$  form a decreasing sequence of sets, each  $\xi_n$  is contained in the space  $X_k$ . The sequence  $\langle \xi_n \rangle_{n \in \mathbb{N}_0}$  is a  $\delta_k(\varepsilon)$ -pseudo-orbit in  $X_k$ , hence there exists a point  $\xi^*$  that  $\varepsilon$ -shadows it. Again, it is readily checked that  $(1/2^k, \xi^*)$   $\varepsilon$ -shadows the initial pseudo-orbit.  $\square$

~~$(X, f)$  is a dynamical system on a compact metric space which exhibits shadowing (SP) but for which  $\omega_f \neq ICT(f)$ .~~

*Proof of Theorem 3.3.* It suffices to note that  $\{0\} \times X_\infty$  is an ICT set that is not an  $\omega$ -limit set of any of the points in  $X$ . If it were an  $\omega$ -limit set of some point  $(a, \xi) \in X$ , it would have to be that  $a = 0$ . But it is not hard to see that the  $\omega$ -limit set of any point in  $\{0\} \times X_\infty$  is the singleton  $\{(0, 0^\infty)\}$  as they are all pre-fixed points. Here by  $0^\infty$  we denote the sequence in  $\Sigma_2$  consisting only of zeros. Therefore the set  $\{0\} \times X_\infty$  is not in  $\omega_f$ .

It remains to be shown that  $\{0\} \times X_\infty$  is in  $ICT(f)$ . To simplify notation we shall instead show that the set  $X_\infty$  is ICT under the shift map  $\sigma$ . This is clearly an equivalent statement. Let  $\delta > 0$  and let  $\xi$  and  $\eta$  be any two points in  $X_\infty$ . We can



always choose  $k \in \mathbb{N}$  such that  $\sigma^k(\xi) = 0^\infty$ . If  $\eta = 0^\infty$  we are done as

$$\langle \xi, \sigma(\xi), \dots, \sigma^k(\xi) = \eta \rangle$$

is a  $\delta$ -pseudo-orbit between  $\xi$  and  $\eta$ .

If otherwise  $\eta = 0^m 10^\infty$  for some  $m \in \mathbb{N}_0$ , choose  $n > m$  large enough so that the point  $\zeta = 0^n 10^\infty$  is  $\delta$ -close to  $0^\infty$ . Then one can check that

$$\langle \xi, \sigma(\xi), \dots, \sigma^k(\xi) = 0^\infty, \zeta, \sigma(\zeta), \dots, \sigma^{n-m}(\zeta) = \eta \rangle$$

is a  $\delta$ -pseudo-orbit between  $\xi$  and  $\eta$ .  $\square$

Dynamical system  $(X, f)$  in Theorem 3.3 is not transitive but it is possible to modify it and produce a transitive system on the Cantor set with  $SP$  but not  $LmSP$ .

**Theorem 3.6.** *There exists a topologically transitive dynamical system on a compact metric space which exhibits shadowing (SP) but for which  $\omega_f \neq ICT(f)$  and which, in particular, does not exhibit limit shadowing (LmSP).*

*Below we shall give a sketch of the proof but the details are left to the reader. We shall need the following lemma.*

**Lemma 3.7.** *Let  $(X, f)$  be a dynamical system and let  $F_1 \subset F_2 \subset \dots \subset X$  be an increasing sequence of closed  $f$ -invariant subsets converging to  $X$ ,  $\bigcup_{i=1}^\infty F_i = X$ . Further assume that for each  $n \in \mathbb{N}$  there exists a continuous map  $\pi_n: X \rightarrow F_n$  which is:*

- (1) *non-expanding, i.e.  $d(\pi_n(x), \pi_n(y)) \leq d(x, y)$ ,*
- (2) *commuting with  $f$ , i.e.  $\pi_n \circ f = f \circ \pi_n$ ,*
- (3) *and is a nearest point projection, i.e.  $d(x, \pi_n(x)) = \min\{d(x, y) : y \in F_n\}$*   
*(note that such  $\pi_n$  is a retraction of  $X$  onto  $F_n$ ).*

*If  $(F_n, f|_{F_n})$  has shadowing for each  $n \in \mathbb{N}$  then so does  $(X, f)$ .*

*Proof.* Let  $\varepsilon > 0$ . Choose  $n \in \mathbb{N}$  large enough so that  $F_n$  and  $X$  are  $\varepsilon/2$ -close when measured in Hausdorff distance on  $2^X$ . As  $\pi_n$  is a nearest point projection, this implies that

$$(3.1) \quad d(x, \pi_n(x)) \leq \varepsilon/2, \text{ for all } x \in X.$$

Now let  $\delta = \delta_n(\varepsilon/2) > 0$  be provided by the shadowing property in  $(F_n, f|_{F_n})$  associated to  $\varepsilon/2$ . We claim that this  $\delta$  suffice.

To this end, let  $\langle x_0, x_1, \dots \rangle$  be a  $\delta$ -pseudo-orbit in  $X$ . The properties of  $\pi_n$  ensure that  $\langle \pi_n(x_0), \pi_n(x_1), \dots \rangle$  is still a  $\delta$ -pseudo-orbit in  $F_n$ . This pseudo-orbit can be  $\varepsilon/2$ -shadowed by a point  $y \in F_n$ . But now using (3.1) one easily checks that  $y$  is  $\varepsilon$ -shadowing the original pseudo-orbit  $\langle x_0, x_1, \dots \rangle$ . This completes the proof.  $\square$

This lemma can be seen as a generalisation of Lemma 3.5. Each of the sets  $F_n = \bigcup_{k=0}^{n-1} \{1/2^k\} \times X_k$  is a disjoint union of  $n$  shifts of finite type and therefore has shadowing. It is not hard to check that the projections  $\pi_n: (a, \xi) \mapsto (\max\{a, 1/2^{n-1}\}, \xi)$  satisfy all the required properties.

Let us now look at another way to represent the system from Theorem 3.3. Let  $a_k = 1/3^{k+1}$  and  $b_k = 1 - 1/3^{k+1}$  (any two other sequences converging to 0 and 1 respectively would work equally well). For each  $k \in \mathbb{N}_0$  the system  $(\{1/2^k\} \times X_k, f|_{\{1/2^k\} \times X_k})$  is naturally isomorphic to a shift of finite type  $X_k$  which in turn is isomorphic to a shift of finite type where all the occurrences of symbol 0

are replaced by symbol  $a_k$  and where all the occurrences of symbol 1 are replaced by  $b_k$ . Now the disjoint union  $F_n = \bigcup_{k=0}^{n-1} \{1/2^k\} \times X_k$  can be simply represented as a shift of finite type over symbols  $\{a_0, \dots, a_{n-1}; b_0, \dots, b_{n-1}\}$  where any word containing  $a_k/b_k$  and  $a_l/b_l$  with  $k \neq l$  is forbidden and, furthermore, any  $b_k$  must be preceded (if possible) and followed by at least  $(k+1)$   $a_k$ s. The set  $X$  is then the closure of the union  $\bigcup_{n=1}^{\infty} F_n$  inside the countable product  $\{a_0, \dots; 0; b_0, \dots; 1\}^{\mathbb{N}}$  where the base set is inheriting the topology from the interval  $[0, 1]$ . The map providing dynamics is the shift map  $\sigma$ .

So far we only gave a different description of the example from Theorem 3.3. The advantage being that we disposed of the first coordinate and are now able to represent our system as a shift space albeit over a countable alphabet. The next step is to make the system transitive. We introduce an extra symbol 2 and for each  $n \in \mathbb{N}$  we let  $F_n$  be a shift of finite type over  $\{2; a_0, \dots, a_{n-1}; b_0, \dots, b_{n-1}\}$  where as before any  $b_k$  must be preceded (if possible) and followed by at least  $(k+1)$   $a_k$ s, and any two symbols  $a_k$  and  $a_l$  with  $k \neq l$  need to be separated by at least  $(k+l)$  2s. Now taking the closure of  $\bigcup_{n=1}^{\infty} F_n$  inside  $\{2; a_0, \dots; 0; b_0, \dots; 1\}^{\mathbb{N}}$  gives a transitive example with shadowing where the ICT set coinciding with  $X_{\infty}$  is not  $\omega$ -limit set of the shift map. We leave the details to the reader.

Related to these results, Good and Meddaugh in [GM18] very recently obtained a characterisation of systems in which  $\omega_f = ICT(f)$ . They show that this is equivalent to another technical property named *orbital limit shadowing*, a definition of which can be found in their paper. Our example thus also shows that a system can exhibit shadowing without having orbital limit shadowing.

#### 4. SHADOWING ON MAXIMAL $\omega$ -LIMIT SETS

In Theorem 2.7 we saw that for tent maps shadowing implies shadowing on the core. ~~Recall that~~ For  $s \in (\sqrt{2}, 2]$  the core of a tent map  $f_s$  is its maximal  $\omega$ -limit set. It is an  $\omega$ -limit set as it is clearly an invariant set and the map  $f_s$  is transitive on its core (e.g. see [BB04, Remark 3.4.17]), hence there exists a point whose orbit is dense in the core. The core is a maximal  $\omega$ -limit set as the trajectory of any point whose  $\omega$ -limit set contains the core would eventually have to enter the interior of the core, and therefore its  $\omega$ -limit set would after all have to be contained in the core.

The results we prove below can thus be seen as ~~generalisations of that result~~ generalising the implication (2)  $\implies$  (5) of Theorem 2.7.

**Theorem 4.1.** *Let  $f: [0, 1] \rightarrow [0, 1]$  be an interval map with shadowing property and further assume that there are only finitely many maximal  $\omega$ -limit sets for  $f$ . Then the restriction of  $f$  to any of these maximal  $\omega$ -limit sets also has shadowing.*

*Proof.* Let  $A_1, \dots, A_k$  be the collection of all maximal  $\omega$ -limit sets for  $f$ . By a result of Meddaugh and Raines [MR13, Corollary 6] we know that for a map with shadowing  $\omega$ -limit sets and internally chain transitive (ICT) sets coincide. It is clear that any two maximal ICT sets must be disjoint, as otherwise their union would again be an ICT set contradicting their maximality. One can also see this directly as follows.

Assume that  $A_i \cap A_j \neq \emptyset$  for some  $i \neq j$ . Then there exists an asymptotic pseudo-orbit  $\langle z_i \rangle_{i=1}^{\infty}$  such that set of its accumulation points contains  $A_i \cup A_j$ . But then, constructing appropriate periodic pseudo-orbits it is not hard to see that for

every  $n$  there is a point  $y_n$  such that  $B(\omega(y_n, f), 1/n) \supset A_i \cup A_j$ . Since the space of all  $\omega$ -limit sets of an interval map is closed due to a result by Blokh, Bruckner, Humke, and Smítal [BBHS96] and since the hyperspace  $2^X$  is compact, this would imply that there exists an  $\omega$ -limit set  $A \supset A_i \cup A_j$  which is a contradiction. Indeed  $A_i \cap A_j = \emptyset$  for  $i \neq j$ .

Denote  $\eta = \min_{i \neq j} \text{dist}(A_i, A_j)/2 > 0$ , fix any  $\varepsilon > 0$  and let  $\delta = \delta(\hat{\varepsilon})$  be provided by the shadowing property of  $f$  to  $\hat{\varepsilon} = \min\{\varepsilon/2, \eta\}$ . Fix any  $A_i$  and any finite  $\delta$ -pseudo-orbit  $\langle x_0, \dots, x_n \rangle \subset A_i$ . Since  $A_i$  is an  $\omega$ -limit set, it is internally chain transitive and therefore there are  $x_{n+1}, \dots, x_{m-1} \in A_i$  such that the sequence  $\langle x_0, \dots, x_{m-1}, x_0 \rangle$  is a (periodic)  $\delta$ -pseudo-orbit. Set  $y_i = x_{(i \bmod m)}$  for all  $i \geq 0$  and let  $z$  be a point that is  $\varepsilon/2$ -tracing  $\delta$ -pseudo-orbit  $\langle y_i \rangle_{i=0}^\infty$ . The set  $\omega(z, f)$  must be contained in some maximal  $\omega$ -limit set, and from our choice of  $\eta$  it is not hard to see that this must be  $A_i$ .

Now let  $q \in \omega(z, f) \subset A_i$  be any limit point of the sequence  $\langle f^{nm}(z) \rangle_{n=0}^\infty$ . It is readily checked that  $q$  is  $\varepsilon$ -tracing the pseudo-orbit  $\langle y_i \rangle_{i=0}^\infty$  and in particular the first portion of it  $\langle x_0, \dots, x_n \rangle$ . This shows that any finite  $\delta$ -pseudo-orbit consisting of points in  $A_i$  can be  $\varepsilon$ -traced by a point from  $A_i$ . This shows that  $f|_{A_i}$  has the shadowing property which completes the proof.  $\square$

In [Blo95, Theorem 5.4] Blokh proved that for interval maps the set  $\omega(f) = \bigcup_{x \in X} \omega(x, f)$  has a spectral decomposition into a family of maximal  $\omega$ -limit sets:

$$\omega(f) = X_f \cup \left( \bigcup_{\alpha} S_{\omega}^{(\alpha)} \right) \cup \left( \bigcup_i B_i \right)$$

where  $X_f$  is a collection of periodic orbits that are also maximal  $\omega$ -limit set (sets of genus 0 in Blokh's notation); each of  $S_{\omega}^{(\alpha)}$ s is a solenoidal limit set and none of them contains a periodic orbit (genus 1); finally each  $B_i$  is a so-called basic set, an infinite maximal limit set containing a periodic orbit (genus 2). It turns out [Blo95, Theorem PM6] (also [Blo86]) that for piecewise linear maps with a constant slope  $s > 1$  there are no sets of genus 1 in this decomposition and that the number of maximal  $\omega$ -limit sets of genus 0 and 2 is finite. Combining this with the previous theorem we obtain the following result.

**Corollary 4.2.** *Let  $f: [0, 1] \rightarrow [0, 1]$  be a piecewise linear map with a constant slope  $s > 1$  and the shadowing property. Then the restriction of  $f$  to any of its maximal  $\omega$ -limit sets has shadowing.*

**Remark 4.3.** The assumption of maximality here is crucial as the ICT set  $W = \bigcup_{i=0}^\infty \{\frac{1}{2^i}\} \cup \{0\}$  is an  $\omega$ -limit set of the full tent map  $f_2$ . Note that  $f_2$  has shadowing but its restriction  $f_2|_W$  does not.

**Theorem 4.4.** *Let  $f: [0, 1] \rightarrow [0, 1]$  be a piecewise monotone map with shadowing property. Then the restriction of  $f$  to any of its  $\omega$ -limit sets with non-empty interior has  $s$ -limit shadowing property.*

*Proof.* We first observe that any such limit set  $\omega(x)$  must in fact be a cycle of disjoint closed intervals (see e.g. [BC92, IV. Lemma 5]). Namely  $\omega(x) = \bigcup_{i=0}^{p-1} J_i$  where  $p \in \mathbb{N}$ , sets  $J_i$  are  $p$  pairwise disjoint segments contained in  $[0, 1]$ , and  $f(J_i) = J_{(i+1 \bmod p)}$ . Furthermore, the map  $f^p|_{J_0}: J_0 \rightarrow J_0$  is transitive (as it has a dense orbit) and it is also piecewise monotone. The result will now follow from Corollary 2.4 if we could only show that  $f^p|_{J_0}$  has shadowing. We remind the reader

that both properties: shadowing and s-limit shadowing hold for a power  $f^p$  of the map  $f$  if and only if they hold for the map itself.

Let  $\varepsilon > 0$  and without loss of generality we assume that  $\varepsilon < \text{diam}(J_0)/2$ . Let  $\delta = \delta(\varepsilon) > 0$  be provided by the shadowing property of  $f^p$  over the whole interval  $[0, 1]$ . We claim that this  $\delta$  will suffice. To this end, take a finite  $\delta$ -pseudo-orbit  $\langle x_0, \dots, x_n \rangle$  contained in  $J_0$ . As  $f^p$  is transitive on  $J_0$  we can extend this pseudo-orbit to a longer  $\delta$ -pseudo-orbit  $\langle x_{-m}, x_{-m+1}, \dots, x_0, \dots, x_n \rangle$  where  $x_{-m} \in J_0$  denotes the midpoint of the segment  $J_0$ . Let  $z \in I$  be a point that is  $\varepsilon$ -tracing this pseudo-orbit. But since  $d(z, x_{-m}) < \varepsilon$  it must be that  $z \in J_0$ . Therefore the  $f^p$ -orbit of  $z$  is actually contained in  $J_0$  and the point  $f^{mp}(z)$  is  $\varepsilon$ -tracing  $\langle x_0, \dots, x_n \rangle$  under the action of  $f^p$ . This completes the proof.  $\square$

**Remark 4.5.** The converse to Theorem 4.4 does not hold as the double tent map shows. The double tent map  $f_{2,2}: [-1, 1] \rightarrow [-1, 1]$  is defined as a symmetric extension of the standard tent map  $f_2$ :

$$f_{2,2}(x) = \begin{cases} f_2(x) & \text{if } x \in [0, 1], \\ -f_2(-x) & \text{if } x \in [-1, 0]. \end{cases}$$

This map is piecewise monotone with a constant slope  $s = 2$  and does not have shadowing as the set of critical points  $\{-1, -1/2, 1/2, 1\}$  is not linked (1 is mapped to a fixed point 0). The restrictions  $f_{2,2}|_{[0,1]}$  and  $f_{2,2}|_{[-1,0]}$  to its two maximal  $\omega$ -limit sets both have linking (as 0 is now also a critical point) and thus by Theorem 2.2 s-limit shadowing.

## 5. INVERSE LIMIT SPACE

**Theorem 5.1.** *Let  $X$  be a compact metric space and  $f: X \rightarrow X$  a continuous onto map. Let  $\sigma$  be the shift map on the inverse limit space  $\varprojlim (X, f)$ . Then:*

- (1)  $\sigma$  has shadowing if and only if  $f$  has shadowing,
- (2)  $\sigma$  has limit shadowing if and only if  $f$  has limit shadowing,
- (3)  $\sigma$  has s-limit shadowing if and only if  $f$  has s-limit shadowing.

*Proof.* The first equivalence was proven by Chen and Li [CL92, Theorem 1.3], the second by Gu and Sheng [GS06, Theorem 3.2], and we shall now see that the third one also holds. While the idea is clear, one needs to be careful with indexing everything properly.

First we will show that if  $\sigma$  has s-limit shadowing then  $f$  has s-limit shadowing. Without loss of generality we may assume that  $\text{diam}(X) = 1$ . Fix any  $\varepsilon > 0$  and let  $\delta > 0$  be provided to  $\varepsilon/2$  by s-limit shadowing of  $\sigma$ . Let  $N$  be such that  $\sum_{i=N}^{\infty} 2^{-i} < \delta/2$  and let  $\gamma \in (0, \frac{\varepsilon}{4N})$  be such that if  $x, y \in X$  satisfy  $d(x, y) < \gamma$  then  $d(f^i(x), f^i(y)) < \min\{\frac{\delta}{4}, \frac{\varepsilon}{4N}\}$  for  $i = 0, \dots, N$ . We claim that this  $\gamma$  suffices.

Fix any asymptotic  $\gamma$ -pseudo orbit  $\langle x_n \rangle_{n=0}^{\infty}$  for  $f$ . There exists a strictly increasing sequence of positive integers  $\langle u_k \rangle_{k=1}^{\infty}$  such that for all  $n \geq u_k - N$  we have  $d(f^{i+1}(x_n), f^i(x_{n+1})) < \gamma/(k+1)^2$  for  $i = 0, \dots, k + N - 1$ . For technical reasons we put  $u_0 = 0$ . Define a sequence  $\langle z_n \rangle_{n=0}^{\infty} \subset X$  by taking any  $y \in f^{-N}(x_0)$ , putting  $x_{-i} = f^{N-i}(y)$  for  $0 \leq i \leq N$  and next:

$$z_n = \begin{cases} x_{n-N} & \text{if } n \leq u_1, \\ f^k(x_{n-k-N}) & \text{if } u_k + k \leq n \leq u_{k+1} + k. \end{cases}$$

We define a sequence of points  $\langle y^{(i)} \rangle_{i=0}^\infty \subset \varprojlim (X, f)$  by the following rule. For  $n \leq N$  and  $i \geq 0$  we put  $y_n^{(i)} = f^{N-n}(z_i)$ . For  $n > N$  we have two cases when defining  $y^{(i)}$ . If  $i \in [u_k + k, u_{k+1} + k]$  for some  $k \geq 0$  and if  $n \leq N + k$  then we put  $y_n^{(i)} = f^{k-n+N}(x_{i-k-N})$  and in the other case we take any  $y_n^{(i)} \in f^{-1}(y_{n-1}^{(i)})$  which is possible, since  $f$  is onto. Note that if  $n \leq N + k$  then we have already defined  $y_N^{(i)} = z_i = f^k(x_{i-k-N})$  and we also put  $y_{N+1}^{(i)} = f^{k-1}(x_{i-k-N})$  and therefore  $f(y_{n+1}^{(i)}) = y_n^{(i)}$  for every  $n \geq 0$ , so with this definition indeed  $y^{(i)} \in \varprojlim (X, f)$ . We claim that the sequence  $y^{(i)}$  constructed by the above procedure is an asymptotic  $\delta$ -pseudo orbit for  $\sigma$ .

Firstly, we claim that  $\langle z_n \rangle_{n=0}^\infty \subset X$  is an asymptotic  $\gamma$ -pseudo orbit. Fix any  $n \in \mathbb{N}$  and assume that both  $n$  and  $n+1$  belong to  $[u_k + k, u_{k+1} + k]$ . Then  $d(f(z_n), z_{n+1}) = d(f^{k+1}(x_{n-k-N}), f^k(x_{n-k-N+1})) < \gamma/(k+1)^2$ . In the second case  $n = u_{k+1} + k$  we have  $d(f(z_n), z_{n+1}) = d(f^{k+1}(x_{n-k-N}), f^{k+1}(x_{n-k-N})) = 0$ . Indeed, the claim holds.

By the choice of  $\gamma$  for  $n \leq N$  we have

$$d(f(y_n^{(i)}), y_n^{(i+1)}) = d(f^{N-n}(f(z_i)), f^{N-n}(z_{i+1})) < \delta/4$$

and so

$$\begin{aligned} d(\sigma(y^{(i)}), y^{(i+1)}) &= \sum_{n=0}^{\infty} 2^{-n} d(f(y_n^{(i)}), y_n^{(i+1)}) \leq \sum_{n=0}^N 2^{-n-2} \delta + \sum_{n=N+1}^{\infty} 2^{-n} \\ &< \delta/2 + \delta/2 \end{aligned}$$

showing that  $y^{(i)}$  is indeed a  $\delta$ -pseudo orbit for  $\sigma$ . To show that it is an asymptotic pseudo-orbit it suffices to show that  $\langle y_n^{(i)} \rangle_{i=0}^\infty$  is an asymptotic pseudo-orbit for any sufficiently large  $n$ . To this end, fix any  $n > N$ . Then if  $k$  is such that  $k \geq n - N$ , then for all  $i \in [u_k + k, u_{k+1} + k]$  we have  $i - k - N \geq u_k - N$  and so:

$$d(f(y_n^{(i)}), y_n^{(i+1)}) = d(f^{k-n+N}(f(x_{i-k-N})), f^{k-n+N}(x_{i-k-N+1})) < \frac{\gamma}{(k+1)^2},$$

while if  $i = u_{k+1} + k$  then:

$$d(f(y_n^{(i)}), y_n^{(i+1)}) = d(f^{k-n+N}(f(x_{i-k-N})), f^{k-n+N+1}(x_{i-k-N})) = 0.$$

This immediately implies that  $\langle y^{(i)} \rangle_{i=0}^\infty$  is an asymptotic pseudo orbit for  $\sigma$ .

Now, let  $p \in \varprojlim (X, f)$  be a point asymptotically  $\varepsilon/2$ -tracing pseudo-orbit  $\langle y^{(i)} \rangle_{i=0}^\infty \subset \varprojlim (X, f)$ . Then by the definition of metric in  $\varprojlim (X, f)$  the coordinate  $p_0 \in X$  of  $p$  is asymptotically  $\varepsilon/2$ -tracing the sequence  $\langle y_0^{(i)} \rangle_{i=0}^\infty \subset X$ . We then have by the definition of  $\gamma$  that:

$$\begin{aligned} d(f^i(p_0), z_{i+N}) &\leq d(f^i(p_0), f^N(z_i)) + \sum_{j=0}^{N-1} d(f^{N-j}(z_{i+j}), f^{N-j-1}(z_{i+j+1})) \\ &\leq d(f^i(p_0), y_0^{(i)}) + N \frac{\varepsilon}{4N} < 3\varepsilon/4. \end{aligned}$$

The same calculation yields also that  $\lim_{i \rightarrow \infty} d(f^i(p_0), z_{i+N}) = 0$ . Furthermore, if  $i \in [u_k + k - N, u_{k+1} + k - N]$  then  $z_{i+N} = f^k(x_{i-k})$  and so

$$\begin{aligned} d(x_i, z_{i+N}) &= d(x_i, f^N(z_i)) = d(x_i, f^k(x_{i-k})) \\ &\leq \sum_{j=0}^{k-1} d(f^j(x_{i-j}), f^{j+1}(x_{i-j-1})) \\ &\leq k \frac{\gamma}{(k+1)^2} \leq \frac{\gamma}{k+1} < \varepsilon/4. \end{aligned}$$

The same calculation, in particular the bound  $\frac{\gamma}{k+1}$ , yields  $\lim_{i \rightarrow \infty} d(x_i, z_{i+N}) = 0$ . Putting together these two calculations shows that  $p_0$  is asymptotically  $\varepsilon$ -tracing the pseudo-orbit  $\langle x_n \rangle_{n=0}^\infty \subset X$  and so the proof of the first implication is completed.

Next we prove that if  $f$  has s-limit shadowing then  $\sigma$  has s-limit shadowing. Fix any  $\varepsilon > 0$  and let  $N$  be such that  $\sum_{i=N}^\infty 2^{-i} < \varepsilon/2$ . There is  $\delta > 0$  such that if  $d(x, y) < \delta$  then  $d(f^i(x), f^i(y)) < \frac{\varepsilon}{2N}$  for  $i = 0, \dots, N$ . There is  $\gamma > 0$  such that if  $\langle x_n \rangle_{n=0}^\infty$  is an asymptotic  $\gamma$ -pseudo-orbit then it is  $\delta$ -traced. Let  $\langle y^{(i)} \rangle_{i=0}^\infty \subset \varprojlim (X, f)$  be an asymptotic  $2^{-N}\gamma$ -pseudo orbit. Then the sequence  $\langle y_N^{(i)} \rangle_{i=0}^\infty$  is an asymptotic  $\gamma$ -pseudo orbit, so let  $z \in X$  be a point which asymptotically  $\delta$ -traces it. Define  $q \in \varprojlim (X, f)$  by putting  $q_i = f^{N-i}(z)$  for  $0 \leq i \leq N$ , and for  $i > N$  let  $q_i$  be any point in  $f^{-1}(q_{i-1})$ . We claim that  $q$  is asymptotically  $\varepsilon$ -tracing  $\langle y^{(i)} \rangle_{i=0}^\infty$ . Note that  $\varepsilon$ -tracing is almost obvious, because for  $n \leq N$  we have

$$d(y_n^{(i)}, f^i(q_n)) = d(f^{N-n}(y_N^{(i)}), f^{N-n}(f^i(q_N))) < \frac{\varepsilon}{2N}$$

and therefore  $d(y^{(i)}, \sigma^i(q)) \leq \frac{\varepsilon}{2} + \sum_{n=0}^N d(y_n^{(i)}, f^i(q_n)) < \varepsilon$ . By the same argument  $\lim_{i \rightarrow \infty} d(y_n^{(i)}, f^i(q_n)) = 0$  for every  $n \leq N$ . But if we fix any  $n > N$  then for  $i \geq n - N$  we have

$$\begin{aligned} d(y_n^{(i)}, f^i(q_n)) &= d(y_n^{(i)}, f^{i-(n-N)}(q_N)) \\ &\leq \sum_{j=1}^{n-N} d(y_{n-j}^{(i-(j-1))}, y_{n-j}^{(i-j)}) + d(y_N^{(i-(n-N))}, f^{i-(n-N)}(q_N)) \\ &= \sum_{j=1}^{n-N} d(y_{n-j+1}^{(i-j+1)}, f(y_{n-j+1}^{(i-j)})) + d(y_N^{(i-(n-N))}, f^{i-(n-N)}(q_N)) \end{aligned}$$

and so also in this case  $\lim_{i \rightarrow \infty} d(y_n^{(i)}, f^i(q_n)) = 0$ , where we use the fact that  $z = q_N$  is asymptotically tracing  $\langle y_N^{(i)} \rangle_{i=0}^\infty$  and that  $\langle y_k^{(i)} \rangle_{i=0}^\infty$  is an asymptotic pseudo-orbit for each  $k \geq 0$ . Indeed,  $q$  is asymptotically tracing  $\langle y^{(i)} \rangle_{i=0}^\infty$  which completes the proof.  $\square$

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## REFERENCES

- [Bar10] Andrew D. Barwell, *A characterization of  $\omega$ -limit sets of piecewise monotone maps of the interval*, Fund. Math. **207** (2010), no. 2, 161–174. MR2586009 (2011k:37059)
- [BB04] Karen M. Brucks and Henk Bruin, *Topics from one-dimensional dynamics*, London Mathematical Society Student Texts, vol. 62, Cambridge University Press, Cambridge, 2004. MR2080037
- [BBHS96] Alexander Blokh, A. M. Bruckner, P. D. Humke, and J. Smítal, *The space of  $\omega$ -limit sets of a continuous map of the interval*, Trans. Amer. Math. Soc. **348** (1996), no. 4, 1357–1372. MR1348857 (96j:58089)
- [BC87] Louis Block and Ethan M. Coven, *Topological conjugacy and transitivity for a class of piecewise monotone maps of the interval*, Trans. Amer. Math. Soc. **300** (1987), no. 1, 297–306. MR871677
- [BC92] L. S. Block and W. A. Coppel, *Dynamics in one dimension*, Lecture Notes in Mathematics, vol. 1513, Springer-Verlag, Berlin, 1992. MR1176513 (93g:58091)
- [BDG12] Andrew D. Barwell, Gareth Davies, and Chris Good, *On the  $\omega$ -limit sets of tent maps*, Fund. Math. **217** (2012), no. 1, 35–54. MR2914921
- [BGKR10] Andrew Barwell, Chris Good, Robin Knight, and Brian E. Raines, *A characterization of  $\omega$ -limit sets in shift spaces*, Ergodic Theory Dynam. Systems **30** (2010), no. 1, 21–31. MR2586343 (2011e:37026)
- [BGO12] Andrew D. Barwell, Chris Good, and Piotr Oprocha, *Shadowing and expansivity in subspaces*, Fund. Math. **219** (2012), no. 3, 223–243. MR3001240
- [BGOR13] Andrew D. Barwell, Chris Good, Piotr Oprocha, and Brian E. Raines, *Characterizations of  $\omega$ -limit sets in topologically hyperbolic systems*, Discrete Contin. Dyn. Syst. **33** (2013), no. 5, 1819–1833. MR3002729
- [Blo82] A. M. Blokh, *Sensitive mappings of an interval*, Uspekhi Mat. Nauk **37** (1982), no. 2(224), 189–190. MR650765
- [Blo86] ———, *Some properties of mappings of constant slope of the interval*, Mathematical physics, functional analysis (Russian), 1986, pp. 127–137, 147. MR906090
- [Blo95] Alexander M. Blokh, *The “spectral” decomposition for one-dimensional maps*, Dynamics reported, 1995, pp. 1–59. MR1346496
- [Bow75] Rufus Bowen,  *$\omega$ -limit sets for axiom A diffeomorphisms*, J. Differential Equations **18** (1975), no. 2, 333–339. MR0413181
- [BR15] Andrew D. Barwell and Brian E. Raines, *The  $\omega$ -limit sets of quadratic Julia sets*, Ergodic Theory Dynam. Systems **35** (2015), no. 2, 337–358. MR3316915
- [Che91] Liang Chen, *Linking and the shadowing property for piecewise monotone maps*, Proc. Amer. Math. Soc. **113** (1991), no. 1, 251–263. MR1079695
- [CKY88] Ethan M. Coven, Ittai Kan, and James A. Yorke, *Pseudo-orbit shadowing in the family of tent maps*, Trans. Amer. Math. Soc. **308** (1988), no. 1, 227–241. MR946440
- [CL92] Liang Chen and Shi Hai Li, *Shadowing property for inverse limit spaces*, Proc. Amer. Math. Soc. **115** (1992), no. 2, 573–580. MR1097338
- [ENP97] Timo Eirola, Olavi Nevanlinna, and Sergei Yu. Pilyugin, *Limit shadowing property*, Numer. Funct. Anal. Optim. **18** (1997), no. 1-2, 75–92. MR1442020
- [GM18] Chris Good and Jonathan Meddaugh, *Orbital shadowing, internal chain transitivity and  $\omega$ -limit sets*, Ergodic Theory Dynam. Systems **38** (2018), no. 1, 143–154. MR3742540
- [GS06] Rongbao Gu and Yeqing Sheng, *APOTP for the inverse limit spaces*, Appl. Math. J. Chinese Univ. Ser. B **21** (2006), no. 4, 473–478. MR2270985
- [KKO14] Marcin Kulczycki, Dominik Kwietniak, and Piotr Oprocha, *On almost specification and average shadowing properties*, Fund. Math. **224** (2014), no. 3, 241–278. MR3194417
- [KO10] Piotr Kościelniak and Piotr Oprocha, *Shadowing, entropy and a homeomorphism of the pseudoarc*, Proc. Amer. Math. Soc. **138** (2010), no. 3, 1047–1057. MR2566570
- [KO11] Marcin Kulczycki and Piotr Oprocha, *Properties of dynamical systems with the asymptotic average shadowing property*, Fund. Math. **212** (2011), no. 1, 35–52. MR2771587
- [LM95] Douglas Lind and Brian Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995. MR1369092



- [Mou12] Christopher Mournon, *Entropy of shift maps of the pseudo-arc*, Topology Appl. **159** (2012), no. 1, 34–39. MR2852946
- [MR13] Jonathan Meddaugh and Brian E. Raines, *Shadowing and internal chain transitivity*, Fund. Math. **222** (2013), no. 3, 279–287. MR3104074
- [Pal00] Ken Palmer, *Shadowing in dynamical systems*, Mathematics and its Applications, vol. 501, Kluwer Academic Publishers, Dordrecht, 2000. Theory and applications. MR1885537
- [Par66] William Parry, *Symbolic dynamics and transformations of the unit interval*, Trans. Amer. Math. Soc. **122** (1966), 368–378. MR0197683
- [Pil07] Sergei Yu. Pilyugin, *Sets of dynamical systems with various limit shadowing properties*, J. Dynam. Differential Equations **19** (2007), no. 3, 747–775. MR2350246
- [Pil99] ———, *Shadowing in dynamical systems*, Lecture Notes in Mathematics, vol. 1706, Springer-Verlag, Berlin, 1999. MR1727170
- [PS17] Sergei Yu. Pilyugin and Kazuhiro Sakai, *Shadowing and hyperbolicity*, Lecture Notes in Mathematics, vol. 2193, Springer, Cham, 2017. MR3618888
- [Sak03] Kazuhiro Sakai, *Various shadowing properties for positively expansive maps*, Topology Appl. **131** (2003), no. 1, 15–31. MR1982811
- [Sin72] Ja. G. Sinaĭ, *Gibbs measures in ergodic theory*, Uspehi Mat. Nauk **27** (1972), no. 4(166), 21–64. MR0399421
- [Wal78] Peter Walters, *On the pseudo-orbit tracing property and its relationship to stability*, The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977), 1978, pp. 231–244. MR518563 (80d:58055)

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