Finite orbits of the pure braid group on the monodromy of the 2-variable Garnier system

Calligaris, Pierpaolo; Mazzocco, Marta

DOI:
10.1093/integr/xyy005

Journal of Integrable Systems (2018) **3**, 1–35
doi: 10.1093/integr/xyy005

Finite orbits of the pure braid group on the monodromy of the 2-variable Garnier system

P. CALLIGARIS
Department of Mathematical Sciences, Loughborough University, Leicestershire LE11 3TU, UK

AND

M. MAZZOCCO[†]
School of Mathematics, The University of Birmingham, Edgbaston, Birmingham B15 2TT, UK
[†]Corresponding author. Email: m.mazzocco@bham.ac.uk

Communicated by: Nalini Joshi

[Received on 26 January 2018; editorial decision on 28 April 2018; accepted on 23 May 2018]

In this article, we realize the $SL_2(\mathbb{C})$ character variety of he Riemann sphere Σ_5 with five boundary components as a 5-parameter family of affine varieties of dimension 4. We show that the action of the mapping class group corresponds to certain action of the braid group on this family of affine varieties and classify exceptional finite orbits. This action represents the nonlinear monodromy of the 2 variable Garnier system and finite orbits correspond to its algebraic solutions.

Keywords: Braid group; Garnier system; Character variety.

1. Introduction

The Garnier system $\mathcal{G}_2$ is the isomonodromy deformation of the following two-dimensional Fuchsian system:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi = \left(\sum_{k=1}^{4} \frac{\mathcal{A}_k}{\lambda - a_k}\right)\Phi, \qquad \lambda \in \mathbb{C}, \tag{1}$$

$a_1, \dots, a_4$, being pairwise distinct complex numbers. The residue matrices $\mathcal{A}_j$ satisfy the following conditions:

$$\text{eigen}\left(\mathcal{A}_j\right) = \pm\frac{\theta_j}{2} \quad \text{and} \quad -\sum_{k=1}^{n+2} \mathcal{A}_k = \mathcal{A}_\infty,$$

where $\theta_j \in \mathbb{C}, j = 1, \dots, 4$ and we assume

$$\mathcal{A}_\infty := \frac{1}{2}\begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix},$$

with $\theta_\infty \in \mathbb{C} \setminus \{0\}$.

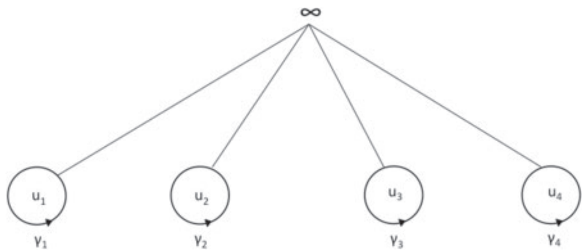

FIG. 1. The basis of loops for $\pi_1(\Sigma_5)$.

The Riemann–Hilbert correspondence associates to each Fuchsian system (1) its monodromy representation class, or in other words, a point in the moduli space of rank two linear monodromy representations over the two-dimensional sphere Σ_5 with five boundary components:

$$\mathcal{M}_{\mathcal{G}_2} := \mathrm{Hom}(\pi_1(\Sigma_5), \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C}),$$

also called $SL_2(\mathbb{C})$ *character variety of* Σ_5.

After fixing a basis of oriented loops $\gamma_1, \dots, \gamma_4, \gamma_\infty$ for $\pi_1(\Sigma_5)$ such that $\gamma_\infty^{-1} = \gamma_1 \cdots \gamma_4$, as in Fig. 1, an equivalence class of an homomorphism in the character variety $\mathcal{M}_{\mathcal{G}_2}$ is determined by the five matrices $M_1, \dots, M_4, M_\infty \in \mathrm{SL}_2(\mathbb{C})$, that are images of $\gamma_1, \dots, \gamma_4, \gamma_\infty$. These matrices must satisfy the relation:

$$M_\infty M_4 M_3 M_2 M_1 = \mathbb{I}. \tag{2}$$

In this article, we assume that M_∞ is diagonalizable:

$$\mathrm{eigen}(M_\infty) = e^{\pm \pi i \theta_\infty}.$$

As a consequence the character variety $\mathcal{M}_{\mathcal{G}_2}$ is identified with the quotient space $\hat{\mathcal{M}}_{\mathcal{G}_2}$, defined as:

$$\hat{\mathcal{M}}_{\mathcal{G}_2} := \left\{(M_1, \dots, M_4) \in \mathrm{SL}_2(\mathbb{C}) | \mathrm{eigen}(M_4 M_3 M_2 M_1) = e^{\pm \pi i \theta_\infty}\right\} / \sim, \tag{3}$$

where $\sim$ is equivalence up to simultaneous conjugation of $M_1, \dots, M_4$ by a matrix in $\mathrm{SL}_2(\mathbb{C})$.

As the pole positions $a_1, \dots, a_4$ in (1) vary in the configuration space of 4 points, the monodromy matrices $M_1, \dots, M_4$ of (1) remain constant if and only if (see [1]) the residue matrices $\mathcal{A}_1, \dots, \mathcal{A}_4$ are solutions of the Schlesinger equations [2] which in the 2×2 case reduce to the Garnier system $\mathcal{G}_2$ [3, 4]. The structure of analytic continuation of the solutions of the Garnier system is described by a certain action of the pure braid group P_4 [5] (see also [6]) that can be deduced from the following action of the braid group B_4:

$$B_4 \times \hat{\mathcal{M}}_{\mathcal{G}_2} \longmapsto \hat{\mathcal{M}}_{\mathcal{G}_2}, \tag{4}$$

defined in terms of the following generators:

$$\begin{aligned} \sigma_1 : (M_1, M_2, M_3, M_4) &\mapsto (M_2, M_2 M_1 M_2^{-1}, M_3, M_4), \\ \sigma_2 : (M_1, M_2, M_3, M_4) &\mapsto (M_1, M_3, M_3 M_2 M_3^{-1}, M_4), \\ \sigma_3 : (M_1, M_2, M_3, M_4) &\mapsto (M_1, M_2, M_4, M_4 M_3 M_4^{-1}), \end{aligned} \tag{5}$$

so that M_∞ is preserved.

Our aim in this article is to classify the finite orbits of this action. In our classification, we exclude the case when the monodromy group $\langle M_1,\dots,M_4\rangle$ is reducible because in this case the Garnier system for which algebraic solutions are classified in [7] (indeed in this case the Garnier system can be solved in terms of Lauricella hypergeometric functions [8]), and the case in which one of the monodromy matrices is a root of the identity because in this case the Garnier system reduces to the sixth Painlevé equation [8] for which all algebraic solutions are classified in [9]. Therefore, we restrict to the following open set:

$$\begin{aligned}\mathcal{U}=\Big\{(M_1,\dots,M_4)\in\hat{\mathcal{M}}_{\mathcal{G}_2}\,|\,\langle M_1,\dots,M_4\rangle \text{ irreducible},\\ M_i\neq\pm\mathbb{I},\forall i=1,\dots,4,\infty\Big\}/\sim,\end{aligned}\tag{6}$$

To explain our classification result, we firstly identify the open set $\mathcal{U}$ with an affine variety (see Section 2):

LEMMA 1.1 Let the functions p_i, p_{ij}, p_{ijk} be defined as:

$$\begin{array}{llll} p_i &= \operatorname{Tr} M_i, & i=1,\dots,4, & \\ p_{ij} &= \operatorname{Tr} M_iM_j, & i,j=1,\dots,4, & i>j,\\ p_{ijk} &= \operatorname{Tr} M_iM_jM_k, & i,j,k=1,\dots,4, & i>j>k,\\ p_\infty &= \operatorname{Tr} M_4M_3M_2M_1, & & \end{array}\tag{7}$$

then, for every choice of $p_1,\dots,p_4,p_\infty$, the open set of monodromy matrices $\mathcal{U}$ is isomorphic to a four dimensional affine variety:

$$\mathcal{A}:=\mathbb{C}[p_{21},p_{31},p_{32},p_{41},p_{42},p_{43},p_{321},p_{432},p_{431},p_{421}]/I,\tag{8}$$

where I is the ideal generated by the algebraically dependent polynomials $f_1,\dots,f_{15}$ defined in (47)–(61).

Therefore, we think of $p_1,\dots,p_4,p_\infty$ as a set of parameters and of p_{ij},p_{ijk} as an over-determined system of coordinates on $\mathcal{U}$, and we express the action (4) in terms of p_i, p_{ij}, p_{ijk} as follows (see Section 3):

LEMMA 1.2 The following maps $\sigma_i:\mathcal{A}\longrightarrow\mathcal{A}$, $i=1,2,3$, acting on the coordinates

$$p:=(p_1,p_2,p_3,p_4,p_\infty,p_{21},p_{31},p_{32},p_{41},p_{42},p_{43},p_{321},p_{432},p_{431},p_{421})\in\mathbb{C}^{15},\tag{9}$$

as follows:

$$\begin{array}{ll}\sigma_1: & p\mapsto(p_2,p_1,p_3,p_4,p_\infty,p_{21},p_{32},p_1p_3-p_{31}-p_{21}p_{32}+p_2p_{321},p_{42},\\ & p_1p_4-p_{41}-p_{21}p_{42}+p_2p_{421},p_{43},p_{321},p_1p_{43}-p_{431}-p_{21}p_{432}+p_2p_\infty,\\ & p_{432},p_{421}),\\ \sigma_2: & p\mapsto(p_1,p_3,p_2,p_4,p_\infty,p_{31},p_1p_2-p_{21}-p_{31}p_{32}+p_3p_{321},p_{32},p_{41},p_{43},\\ & p_2p_4-p_{42}-p_{32}p_{43}+p_3p_{432},p_{321},p_{432},p_2p_{41}-p_{421}-p_{32}p_{431}+p_3p_\infty,\\ & p_{431}),\\ \sigma_3: & p\mapsto(p_1,p_2,p_4,p_3,p_\infty,p_{21},p_{41},p_{42},p_1p_3-p_{31}-p_{41}p_{43}+p_4p_{431},\\ & p_2p_3-p_{32}-p_{42}p_{43}+p_4p_{432},p_{43},p_{421},p_{432},p_{431},\\ & p_{21}p_3-p_{321}-p_{421}p_{43}+p_4p_\infty),\end{array}\tag{10}$$

define an action of the braid group B_4 on $\mathcal{A}$.

Therefore, our problem is to find all points $p \in \mathcal{A}$ such that their orbit under the action of the pure braid group P_4 induced by the action (10) of the braid group B_4 is finite. Incidentally, the action (10) can also be interpreted as the Mapping Class Group action on the character variety $\mathcal{M}_{\mathcal{G}_2}$.

Our approach is based on the simple observation that given $p \in \mathcal{A}$ such that it generates a finite orbit under the action of the pure braid group P_4, then for any subgroup $H \subset P_4$ the action of H over $p \in \mathcal{A}$ must also produce a finite orbit (this is a well-known fact, see for example [7]). We select four subgroups $H_1, H_2, H_3, H_4 \subset P_4$ such that the restricted action is isomorphic to the action of the pure braid group P_3 on the $SL_2(\mathbb{C})$ character variety of the Riemann sphere with four boundary components $\mathcal{M}_{PVI}$ that can be identified with:

$$\hat{\mathcal{M}}_{PVI} := \{(N_1, N_2, N_3) \in \mathrm{SL}_2(\mathbb{C}) | N_\infty N_3 N_2 N_1 = \mathbb{I}, \\ N_\infty = \exp(i\pi\theta_\infty\sigma_3),\ \theta_\infty \in \mathbb{C}\ \} / \sim . \tag{11}$$

In other words, we show that in order for a point $p \in \mathcal{A}$ to belong to a finite orbit of the pure braid group P_4, it must have four projections on points $q = (q_1, q_2, q_3, q_\infty, q_{21}, q_{31}, q_{32})$ that have a finite orbit under the pure braid group P_3.

We then invert this way of thinking: since all finite orbits of the pure braid group P_3 on $q = (q_1, q_2, q_3, q_\infty, q_{21}, q_{31}, q_{32})$ have been classified in Lisovyy and Tykhyy's work [9], we start from their list and reconstruct *candidate points* $p \in \mathcal{A}$ that satisfy the necessary conditions to belong to a finite orbit. We then classify all candidate points that indeed produce finite orbits. In order to avoid redundant solutions to this classification problem, we introduce the symmetry group G of the affine variety (8) and factorize our classification modulo the action of G. The action of the symmetry group G on $\mathcal{A}$ is calculated in the Appendix using known results about Bäcklund transformations of Schlesinger equations [10].

In order to produce our candidate points we use the classification result in [9] that shows that there are four types of finite orbits of the braid group B_3:

(1) Fixed points corresponding to Okamoto's Riccati solutions [11].

(2) *Dubrovin–Kitaev orbits,* corresponding to algebraic solutions of type II, III and IV in [9].

(3) *Picard orbits,* corresponding to algebraic solutions obtained in terms of the Weierstrass elliptic function (see [12, 13]).

(4) 45 exceptional finite orbits [9].

REMARK 1.1 Orbits of type II, III and IV in [9] where first obtained by Dubrovin in [14]. Later Kitaev showed that these solutions satisfy parametric families of the sixth Painlevé equations and re-obtained them by the pull-back of the hypergeometric equation (see [15, 16]).

To keep down the number of pages and of technical lemmata, we restrict our classification to *exceptional orbits,* namely orbits for which the corresponding monodromy group is not reducible, none of the monodromy matrices is a multiple of the identity and at most one projection giving either a Dubrovin–Kitaev or a Picard orbit is allowed. Therefore, our classification does not include the solutions found by Tsuda [17] by calculating fixed points of bi-rational canonical transformations, nor the ones found by Diarra in [18] using the method of pull-back introduced in [15, 16], nor the ones found in [7] as they correspond to reducible monodromy groups, nor the families of algebraic solutions obtained by Girand in [19] by restricting a logarithmic flat connection defined on the complement of a quintic curve on $\mathbb{P}^2$

on generic lines of the projective plane—indeed these algebraic solutions have at least two projections giving Dubrovin–Kitaev orbits.

Our final classification result consists in a list of 54 exceptional finite orbits of the action (10) obtained up to the action of the group of symmetries G (see Table 2). One orbit (element 25 in Table 2) corresponds to an infinite monodromy group despite the fact that all of its projections to points corresponding to PVI generate finite monodromy groups. The other 53 of these orbits correspond to finite monodromy groups.[1] We believe that these 53 orbits are also interesting because even if it is obvious that for finite monodromy groups the braid group orbits must be finite, the problem of classifying the representations of the $SL_2(\mathbb{C})$ character variety of the Riemann sphere with five boundary components on finite groups is not trivial.

From the monodromy data $M_1, \ldots, M_4$, it is in principle possible to recover the explicit formulation of the associated solution of $\mathcal{G}_2$ using the method developed by Lisovyy and Gavrylenko in [20] of Fredholm determinant representation for isomonodromic tau functions of Fuchsian systems of the form (1). However, the shortest finite orbit classified in our paper has length 36, for this reason the associated algebraic solution of $\mathcal{G}_2$ has eventually 36 branches, and we doubt that the expression of this solution can have a nice and compact form.

All the algorithms necessary to produce this classification can be found in [21].

2. Co-adjoint coordinates on $\mathcal{M}_{\mathcal{G}_2}$

As explained in the Section 1, we identify the character variety $\mathcal{M}_{\mathcal{G}_2}$ with the quotient space $\hat{\mathcal{M}}_{\mathcal{G}_2}$ defined in (3). Following [22, 23], the first step to endow $\hat{\mathcal{M}}_{\mathcal{G}_2}$ with a system of co-adjoint coordinates is to introduce a parameterization of the monodromy matrices in terms of their traces and traces of their products. The following result is a generalization of a result proved by Iwasaki for the case of the sixth Painlevé equation [24]:[2]

THEOREM 2.1 Let $(M_1, \ldots, M_4) \in \mathcal{U}$, $p \in \mathcal{A}$ as in Lemma 1.1 and $g(x, y, z) := x^2 + y^2 + z^2 - xyz - 4$, then in the open set:

$$\mathcal{U}_{jk}^{(0)} := \hat{\mathcal{M}}_{\mathcal{G}_2} \cap \{(p_{jk}^2 - 4) g(p_{jk}, p_l, p_{jkl}) \neq 0\}, \tag{12}$$

there exists a global conjugation $P \in SL_2(\mathbb{C})$ such that the matrices $M_1, \ldots, M_4$ can be parametrized as follows (up to conjugation by P):

$$M_l = \begin{pmatrix} \frac{p_{jkl} - p_l \lambda_{jk}^{-}}{r_{jk}} & -\frac{g(p_{jk}, p_l, p_{jkl})}{r_{jk}^2} \\ 1 & -\frac{p_{jkl} - p_l \lambda_{jk}^{+}}{r_{jk}} \end{pmatrix}, \quad M_k = \begin{pmatrix} -\frac{p_j - p_k \lambda_{jk}^{+}}{r_{jk}} & -\frac{y_{kl} - y_{jl} \lambda_{jk}^{-}}{r_{jk}^2} \\ \frac{y_{kl} - y_{jl} \lambda_{jk}^{+}}{g(p_{jk}, p_l, p_{jkl})} & \frac{p_j - p_k \lambda_{jk}^{-}}{r_{jk}} \end{pmatrix},$$

[1] We are grateful to Gael Cousin for asking us this question.

[2] We thank the referee for pointing out that some of these results should be known to experts in geometric invariant theory, indeed the recent paper [25] provides an algorithm, implemented in Mathematica, SageMath and in Python, that takes a finite presentation for a finitely presentable discrete group F and produces a finite presentation of the coordinate ring of the G-character variety of F where G is a rank 1 complex affine algebraic group. We did not try to use this algorithm as we had already obtained our coordinates by hands.

$$M_j = \begin{pmatrix} -\frac{p_k - p_j\lambda^+_{jk}}{r_{jk}} & -\frac{y_{jl} - y_{kl}\lambda^+_{jk}}{r^2_{jk}} \\ \frac{y_{jl} - y_{kl}\lambda^-_{jk}}{g(p_{jk}, p_l, p_{jkl})} & \frac{p_k - p_j\lambda^-_{jk}}{r_{jk}} \end{pmatrix}, \quad M_i = \begin{pmatrix} \frac{p_{ijk} - p_i\lambda^-_{jk}}{r_{jk}} & -\frac{y_{il} + y_{ijkl}\lambda^+_{jk}}{r^2_{jk}} \\ \frac{y_{il} + y_{ijkl}\lambda^-_{jk}}{g(p_{jk}, p_l, p_{jkl})} & -\frac{p_{ijk} - p_i\lambda^+_{jk}}{r_{jk}} \end{pmatrix}. \tag{13}$$

Alternatively on the open set:

$$\mathcal{U}^{(1)}_{jk} := \hat{\mathcal{M}}_{\mathcal{G}_2} \cap \{(p^2_{jk} - 4)g(p_j, p_k, p_{jk}) \neq 0\}, \tag{14}$$

the matrices $M_1, \ldots, M_4$ can be parametrized as follows (up to conjugation by P):

$$M_l = \begin{pmatrix} \frac{p_{jkl} - p_l\lambda^-_{jk}}{r_{jk}} & -\frac{y_{kl} - y_{jl}\lambda^+_{jk}}{r^2_{jk}} \\ \frac{y_{kl} - y_{jl}\lambda^-_{jk}}{g(p_{jk}, p_j, p_k)} & -\frac{p_{jkl} - p_l\lambda^+_{jk}}{r_{jk}} \end{pmatrix}, \quad M_k = \begin{pmatrix} -\frac{p_j - p_k\lambda^+_{jk}}{r_{jk}} & -\frac{g(p_{jk}, p_j, p_k)}{r^2_{jk}} \\ 1 & \frac{p_j - p_k\lambda^-_{jk}}{r_{jk}} \end{pmatrix},$$

$$M_j = \begin{pmatrix} -\frac{p_k - p_j\lambda^+_{jk}}{r_{jk}} & \frac{g(p_{jk}, p_j, p_k)\lambda^+_{jk}}{r^2_{jk}} \\ -\lambda^-_{jk} & \frac{p_k - p_j\lambda^-_{jk}}{r_{jk}} \end{pmatrix}, \quad M_i = \begin{pmatrix} \frac{p_{ijk} - p_i\lambda^-_{jk}}{r_{jk}} & -\frac{y_{ik} - y_{ij}\lambda^+_{jk}}{r^2_{jk}} \\ \frac{y_{ik} - y_{ij}\lambda^-_{jk}}{g(p_{jk}, p_j, p_k)} & -\frac{p_{ijk} - p_i\lambda^+_{jk}}{r_{jk}} \end{pmatrix}. \tag{15}$$

Finally, on the open set:

$$\mathcal{U}^{(2)}_{jk} := \hat{\mathcal{M}}_{\mathcal{G}_2} \cap \{(p^2_{jk} - 4)g(p_{jk}, p_i, p_{ijk}) \neq 0\}, \tag{16}$$

the matrices $M_1, \ldots, M_4$ can be parametrized as follows (up to conjugation by P):

$$M_l = \begin{pmatrix} \frac{p_{jkl} - p_l\lambda^-_{jk}}{r_{jk}} & -\frac{y_{il} + y_{ijkl}\lambda^-_{jk}}{r^2_{jk}} \\ \frac{y_{il} + y_{ijkl}\lambda^+_{jk}}{g(p_{jk}, p_i, p_{ijk})} & -\frac{p_{jkl} - p_l\lambda^+_{jk}}{r_{jk}} \end{pmatrix}, \quad M_k = \begin{pmatrix} -\frac{p_j - p_k\lambda^+_{jk}}{r_{jk}} & -\frac{y_{ik} - y_{ij}\lambda^-_{jk}}{r^2_{jk}} \\ \frac{y_{ik} - y_{ij}\lambda^+_{jk}}{g(p_{jk}, p_i, p_{ijk})} & \frac{p_j - p_k\lambda^-_{jk}}{r_{jk}} \end{pmatrix},$$

$$M_j = \begin{pmatrix} -\frac{p_k - p_j\lambda^+_{jk}}{r_{jk}} & -\frac{y_{ij} - y_{ik}\lambda^+_{jk}}{r^2_{jk}} \\ \frac{y_{ij} - y_{ik}\lambda^-_{jk}}{g(p_{jk}, p_i, p_{ijk})} & \frac{p_k - p_j\lambda^-_{jk}}{r_{jk}} \end{pmatrix}, \quad M_i = \begin{pmatrix} \frac{p_{ijk} - p_i\lambda^-_{jk}}{r_{jk}} & -\frac{g(p_{jk}, p_i, p_{ijk})}{r^2_{jk}} \\ 1 & -\frac{p_{ijk} - p_i\lambda^+_{jk}}{r_{jk}} \end{pmatrix}, \tag{17}$$

where:

$$r_{jk} := \sqrt{p^2_{jk} - 4}, \qquad \lambda^{\pm}_{jk} = \frac{p_{jk} \pm r_{jk}}{2}, \tag{18}$$

$$y_{kl} := 2p_{kl} + p_{jk}p_{jl} - p_j p_{jkl} - p_k p_l,$$

$$y_{jl} := 2p_{jl} + p_{jk}p_{kl} - p_k p_{jkl} - p_j p_l, \tag{19}$$

$$y_{ik} := 2p_{ik} + p_{ij}p_{jk} - p_j p_{ijk} - p_i p_k, \tag{20}$$

$$y_{ij} := 2p_{ij} + p_{ik}p_{jk} - p_k p_{ijk} - p_i p_j, \tag{21}$$

$$y_{il} := 2p_{il} + p_{ijk}p_{jkl} - p_{jk}p_{ijkl} - p_i p_l,$$

$$y_{ijkl} := 2p_{ijkl} - p_{il}p_{jk} - p_i p_{jkl} - p_{ijk}p_l + p_i p_{jk} p_l. \tag{22}$$

Proof. Consider $(M_1,\dots,M_4) \in \mathcal{U}$. We only prove the statement for the open subset $\mathcal{U}_{jk}^{(0)}$. For the parametrizations on the open subsets $\mathcal{U}_{jk}^{(1)}$ and $\mathcal{U}_{jk}^{(2)}$ a similar proof applies. Under the hypothesis that $p_{jk} \neq \pm 2$, there exists a matrix $P \in \mathrm{SL}_2(\mathbb{C})$ such that the product matrix M_jM_k can be brought into diagonal form:

$$\Lambda_{jk} := P(M_jM_k)P^{-1} = \mathrm{diag}\{\lambda_{jk}^+, \lambda_{jk}^-\}, \tag{23}$$

where the eigenvalues $\lambda_{jk}^{\pm}$ are given in (18), where the positive branch of the square root is chosen. Consequently, we conjugate by P the matrices M_l, M_k, M_j, M_i as follows:

$$P(M_l, M_k, M_j, M_i)P^{-1} = (U, V, W, T). \tag{24}$$

Since, $W = \Lambda_{jk}V^{-1}$, we only need to produce the parametrization of the matrices U, V, T. Solving the equations $\mathrm{Tr}\,U = p_l$, $\mathrm{Tr}\,\Lambda_{jk}U = p_{jkl}$ and $\mathrm{Tr}\,V = p_k$, $\mathrm{Tr}\,\Lambda_{jk}V^{-1} = p_j$ and $\mathrm{Tr}\,T = p_i$ and $\mathrm{Tr}\,TWV = \mathrm{Tr}\,T\Lambda_{jk} = p_{ijk}$ we obtain the diagonal elements of U, V and T, respectively:

$$u_{11} = \frac{p_{jkl} - p_l\lambda_{jk}^-}{r_{jk}}, \qquad u_{22} = -\frac{p_{jkl} - p_l\lambda_{jk}^+}{r_{jk}}, \tag{25}$$

$$v_{11} = -\frac{p_j - p_k\lambda_{jk}^+}{r_{jk}}, \qquad v_{22} = \frac{p_j - p_k\lambda_{jk}^-}{r_{jk}}, \tag{26}$$

$$t_{11} = \frac{p_{ijk} - p_i\lambda_{jk}^-}{r_{jk}}, \qquad t_{22} = -\frac{p_{ijk} - p_i\lambda_{jk}^+}{r_{jk}}. \tag{27}$$

We now calculate the off-diagonal elements. Since $\det U = 1$, then the following identity holds:

$$u_{12}u_{21} = -\frac{g(p_{jk}, p_l, p_{jkl})}{r_{jk}^2}, \tag{28}$$

and in $\mathcal{U}_{jk}^{(0)}$ $g(p_{jk}, p_l, p_{jkl}) \neq 0$. Since P is unique up to left multiplication by a diagonal matrix $D \in \mathrm{SL}_2(\mathbb{C})$, we are allowed to fix $u_{21} = 1$. Then equation (28) gives us the element u_{12}.

The system of equations $\mathrm{Tr}\,VU = p_{kl}$ and $\mathrm{Tr}\,\Lambda_{jk}V^{-1}U = p_{jl}$ gives us a parametrization for the off-diagonal elements of V:

$$v_{12} = -\frac{y_{ik} - y_{ij}\lambda_{jk}^-}{r_{jk}^2}, \qquad v_{21} = \frac{y_{ik} - y_{ij}\lambda_{jk}^+}{g(p_{jk}, p_i, p_{ijk})}, \tag{29}$$

where y_{ik} and y_{ij} are defined in (20) and (21), respectively. Finally, consider the system of equations $\mathrm{Tr}\,TU = p_{il}$ and $\mathrm{Tr}\,TWVU = \mathrm{Tr}\,T\Lambda_{jk}U = p_{ijkl}$, then we have the following parametrization for t_{12} and t_{21}:

$$t_{12} = -\frac{y_{il} + y_{ijkl}\lambda_{jk}^+}{r_{jk}^2}, \qquad t_{21} = \frac{y_{il} + y_{ijkl}\lambda_{jk}^-}{g(p_{jk}, p_l, p_{jkl})}, \tag{30}$$

where y_{il} and y_{ijkl} are defined in (19) and (22), respectively. This concludes the proof. $\square$

Theorem 2.1 shows that $(p_1,\ldots,p_4,p_{21},\ldots,p_{43},p_{321},\ldots,p_{421})$ parameterize the following open subset of $\mathcal{U}$:

$$\bigcup_{j>k}\mathcal{U}_{jk}^{(0)}\cup\mathcal{U}_{jk}^{(1)}\cup\mathcal{U}_{jk}^{(2)}. \tag{31}$$

We now show that it is possible to parameterize the monodromy matrices in terms of $p\in\mathcal{A}$ also outside of this open subset.

LEMMA 2.2 Let $(M_1,\ldots,M_4)\in\mathcal{U}$ and $p\in\mathcal{A}$. Assume that $p_{jk}\neq\pm 2$ for at least one choice of $j\neq k$, $j,k=1,\ldots,4$ and

$$g(p_{jk},p_l,p_{jkl})=g(p_j,p_k,p_{jk})=g(p_{jk},p_i,p_{ijk})=0, \tag{32}$$

where $g(x,y,z):=x^2+y^2+z^2-xyz-4$, then there exists at least an index l for which $p_{lk}\neq\lambda_l\lambda_k+\frac{1}{\lambda_l\lambda_k}$ and a global conjugation $P\in SL_2(\mathbb{C})$ such that:

$$PM_kP^{-1}=\begin{pmatrix}\lambda_k & 1\\ 0 & \frac{1}{\lambda_k}\end{pmatrix},\qquad PM_jP^{-1}=\begin{pmatrix}\lambda_j & -\lambda_j\lambda_k\\ 0 & \frac{1}{\lambda_j}\end{pmatrix}, \tag{33}$$

$$PM_lP^{-1}=\begin{pmatrix}\lambda_l & 0\\ p_{lk}-\lambda_l\lambda_k-\frac{1}{\lambda_l\lambda_k} & \frac{1}{\lambda_l}\end{pmatrix}, \tag{34}$$

$$PM_iP^{-1}=\begin{cases}\begin{pmatrix}\lambda_i & 0\\ p_{ik}-\lambda_i\lambda_k-\frac{1}{\lambda_i\lambda_k} & \frac{1}{\lambda_i}\end{pmatrix}, & \text{for } p_{il}=\lambda_i\lambda_l+\frac{1}{\lambda_i\lambda_l},\\ \begin{pmatrix}\lambda_i & \frac{p_{il}-\lambda_i\lambda_l-\frac{1}{\lambda_i\lambda_l}}{p_{lk}-\lambda_l\lambda_k-\frac{1}{\lambda_l\lambda_k}}\\ 0 & \frac{1}{\lambda_i}\end{pmatrix}, & \text{for } p_{il}\neq\lambda_i\lambda_l+\frac{1}{\lambda_i\lambda_l},\end{cases} \tag{35}$$

where $\lambda_s+\frac{1}{\lambda_s}=p_s$, $\forall s=1,\ldots,4$.

Proof. Proceeding as in the proof of Theorem 2.1, we bring the product matrix M_jM_k into the diagonal form. Condition (32) implies that the following equations must be satisfied (we have absorbed the global conjugation P in the matrices $M_1,\ldots,M_4$, here):

$$(M_1)_{12}(M_1)_{21}=(M_2)_{12}(M_2)_{21}=(M_3)_{12}(M_3)_{21}=(M_4)_{12}(M_4)_{21}=0.$$

By global conjugation by a permutation matrix, we can assume that $(M_k)_{12}\neq 0$ and then by global diagonal conjugation we can put M_k in Jordan normal form. Then, since $M_j=\Lambda_{jk}M_k^{-1}$ we immediately obtain (33). Since the monodromy group must be irreducible, one of the two remaining matrices, call it M_l, must have non-zero 21 entry. Then since $\mathrm{Tr}(M_lM_k)=p_{lk}$, we obtain $(M_l)_{21}=p_{lk}-\lambda_l\lambda_k-\frac{1}{\lambda_l\lambda_k}\neq 0$, and therefore (34).

Now if the last matrix is also lower triangular, by imposing $\mathrm{Tr}\,M_iM_k=p_{ik}$, we obtain the first formula in (35), and it is immediate to check that then $p_{il}=\lambda_i\lambda_l+\frac{1}{\lambda_i\lambda_l}$. Otherwise, if M_i is upper triangular, by imposing $\mathrm{Tr}\,M_iM_l=p_{il}$, we obtain the second formula (35), and it is immediate to check that then $p_{il}\neq\lambda_i\lambda_l+\frac{1}{\lambda_i\lambda_l}$. □

PROPOSITION 2.3 Let $(M_1,\dots,M_4)\in\mathcal{U}$ and $p\in\mathcal{A}$. Assume that $p_{jk}=2\epsilon_{jk}$ for all $j,k=1,\dots,4$, where $\epsilon_{jk}=\pm1$. Then, if that at least one matrix M_i is diagonalizable there exists a choice of the ordering of the indices $i,j,k,l\in\{1,2,3,4\}$ such that the following parameterization holds true:

$$PM_iP^{-1}=\begin{pmatrix}\lambda_i & 0\\ 0 & \frac{1}{\lambda_i}\end{pmatrix},\qquad \lambda_i\neq\pm1,\quad \lambda_i+\frac{1}{\lambda_i}=p_i, \tag{36}$$

$$PM_kP^{-1}=\begin{pmatrix}-\frac{p_k-2\epsilon_{ki}\lambda_i}{\lambda_i^2-1} & -\frac{(p_k\lambda_i-\epsilon_{ki}(\lambda_i^2+1))^2}{(\lambda_i^2-1)^2}\\ 1 & \frac{\lambda_i(p_k\lambda_i-2\epsilon_{ki})}{\lambda_i^2-1}\end{pmatrix}, \tag{37}$$

and for $p_k\neq\epsilon_{ki}p_i$:

$$PM_jP^{-1}=\begin{pmatrix}-\frac{p_j-2\epsilon_{ji}\lambda_i}{\lambda_i^2-1} & -\frac{\lambda_i^2(p_k\lambda_i-2\epsilon_{ki})(p_j\lambda_i-2\epsilon_{ji})+\lambda_i(\lambda_i^2-1)(p_{ikj}-2\epsilon_{kj}\lambda_i)}{(\lambda_i^2-1)^2}\\ \frac{(\lambda_i^2-1)(2\epsilon_{kj}-p_{ikj}\lambda_i)+(p_k\lambda_i-2\epsilon_{ki})(p_j\lambda_i-2\epsilon_{ji})}{(p_k\lambda_i-\epsilon_{ki}(\lambda_i^2+1))^2} & \frac{\lambda_i(p_j\lambda_i-2\epsilon_{ji})}{\lambda_i^2-1}\end{pmatrix} \tag{38}$$

$$PM_lP^{-1}=\begin{pmatrix}-\frac{p_l-2\epsilon_{li}\lambda_i}{\lambda_i^2-1} & -\frac{\lambda_i^2(p_k\lambda_i-2\epsilon_{ki})(p_l\lambda_i-2\epsilon_{li})+\lambda_i(\lambda_i^2-1)(p_{ikl}-2\epsilon_{kl}\lambda_i)}{(\lambda_i^2-1)^2}\\ \frac{(\lambda_i^2-1)(2\epsilon_{kl}-p_{ikl}\lambda_i)+(p_k\lambda_i-2\epsilon_{ki})(p_l\lambda_i-2\epsilon_{li})}{(p_k\lambda_i-\epsilon_{ki}(\lambda_i^2+1))^2} & \frac{\lambda_i(p_l\lambda_i-2\epsilon_{li})}{\lambda_i^2-1}\end{pmatrix} \tag{39}$$

and if $p_k=\epsilon_{ki}p_i$, then $p_{ikj}(\lambda_i^2+1)\neq2\lambda_i(\epsilon_{ki}\epsilon_{ji}+\epsilon_{kj})$ and $p_{ikl}(\lambda_i^2+1)\neq2\lambda_i(\epsilon_{ki}\epsilon_{li}+\epsilon_{kl})$ and

$$PM_jP^{-1}=\begin{pmatrix}\frac{\lambda_i(p_{ikj}\lambda_i-2\epsilon_{kj})}{\epsilon_{ki}(\lambda_i^2-1)} & \lambda_i(p_{ikj}(\lambda_i^2+1)-2\lambda_i(\epsilon_{ki}\epsilon_{ji}+\epsilon_{kj}))\\ \frac{\lambda_i^4(p_{ikj}\lambda_i-2\epsilon_{kj})^2-2\epsilon_{kj}\epsilon_{ji}\lambda_i^2(p_{ikj}\lambda_i-2\epsilon_{kj})(\lambda_i^2-1)+(\lambda_i^2-1)^2}{(\lambda_i^2-1)^2\lambda_i(2\lambda_i(\epsilon_{ki}\epsilon_{ji}+\epsilon_{kj})-p_{ikj}(\lambda_i^2+1)^2)} & \frac{2\epsilon_{ki}\epsilon_{ji}\lambda_i(\lambda_i^2-1)-\lambda_i^3(p_{ikj}\lambda_i-2\epsilon_{kj})}{\epsilon_{ki}(\lambda_i^2-1)}\end{pmatrix} \tag{40}$$

$$PM_lP^{-1}=\begin{pmatrix}\frac{\lambda_i(p_{ikl}\lambda_i-2\epsilon_{kl})}{\epsilon_{ki}(\lambda_i^2-1)} & \lambda_i(p_{ikl}(\lambda_i^2+1)-2\lambda_i(\epsilon_{ki}\epsilon_{li}+\epsilon_{kl}))\\ \frac{\lambda_i^4(p_{ikl}\lambda_i-2\epsilon_{kl})^2-2\epsilon_{kl}\epsilon_{li}\lambda_i^2(p_{ikl}\lambda_i-2\epsilon_{kl})(\lambda_i^2-1)+(\lambda_i^2-1)^2}{(\lambda_i^2-1)^2\lambda_i(2\lambda_i(\epsilon_{ki}\epsilon_{li}+\epsilon_{kl})-p_{ikl}(\lambda_i^2+1)^2)} & \frac{2\epsilon_{ki}\epsilon_{li}\lambda_i(\lambda_i^2-1)-\lambda_i^3(p_{ikl}\lambda_i-2\epsilon_{kl})}{\epsilon_{ki}(\lambda_i^2-1)}\end{pmatrix} \tag{41}$$

If none of the monodromy matrices is diagonalizable, then there exists a choice of the ordering of the indices $i,j,k,l\in\{1,2,3,4\}$ such that the following parameterization holds true:

$$PM_iP^{-1}=\begin{pmatrix}\epsilon_i & 1\\ 0 & \epsilon_i\end{pmatrix},\qquad PM_jP^{-1}=\begin{pmatrix}\epsilon_j & 0\\ 4\epsilon_{ij} & \epsilon_j\end{pmatrix}, \tag{42}$$

$$PM_kP^{-1}=\begin{pmatrix}\frac{p_{ijk}-2\epsilon_{ik}\epsilon_j-2\epsilon_{jk}\epsilon_i+2\epsilon_i\epsilon_j\epsilon_k}{4\epsilon_{ij}} & \frac{\epsilon_{jk}-\epsilon_j\epsilon_k}{2\epsilon_{ij}}\\ 2(\epsilon_{ik}-\epsilon_i\epsilon_k) & \frac{2\epsilon_{ik}\epsilon_j+2\epsilon_{jk}\epsilon_i+8\epsilon_{ij}\epsilon_k-2\epsilon_i\epsilon_j\epsilon_k-p_{ijk}}{4\epsilon_{ij}}\end{pmatrix} \tag{43}$$

$$PM_lP^{-1}=\begin{pmatrix}\frac{p_{ijl}-2\epsilon_{il}\epsilon_j-2\epsilon_{jl}\epsilon_i+2\epsilon_i\epsilon_j\epsilon_l}{4\epsilon_{ij}} & \frac{\epsilon_{jl}-\epsilon_j\epsilon_l}{2\epsilon_{ij}}\\ 2(\epsilon_{il}-\epsilon_i\epsilon_l) & \frac{2\epsilon_{il}\epsilon_j+2\epsilon_{jl}\epsilon_i+8\epsilon_{ij}\epsilon_l-2\epsilon_i\epsilon_j\epsilon_l-p_{ijl}}{4\epsilon_{ij}}\end{pmatrix} \tag{44}$$

Proof. First let us assume that at least one matrix M_i is diagonal and work in the basis in which M_i assumes the form (36) with $\lambda_i\neq\pm1$.

Let $j \neq i$, then we have a set of linear equations in the diagonal elements of M_j:

$$\mathrm{Tr}\left(M_i M_j\right) = 2\epsilon_{ji}, \qquad \mathrm{Tr}\, M_j = p_j, \quad \epsilon_{ji} = \pm 1,$$

that it is solved by:

$$(M_j)_{11} = -\frac{p_j - 2\epsilon_{ji}\lambda_i}{\lambda_i^2 - 1}, \qquad (M_j)_{22} = \frac{\lambda_i(p_j\lambda_i - 2\epsilon_{ji})}{\lambda_i^2 - 1}, \tag{45}$$

for $j = 1, \dots, 4, j \neq i$.

Since the monodromy group is not reducible, there is at least one matrix M_k, $k \neq i$ such that in the chosen basis, $(M_k)_{21} \neq 0$, then we use the freedom of global diagonal conjugation to set $(M_k)_{21} = 1$. Since $\det(M_k) = 1$, we obtain the formula (37).

We now deal with the other two matrices in the case $p_k \neq \epsilon_{ki} p_i$—we only need to find the off-diagonal elements of these matrices. To this aim we use the following equations for $s = j, l$:

$$\mathrm{Tr}(M_s M_k) = 2\epsilon_{sk}, \qquad \mathrm{Tr}(M_i M_k M_s) = p_{iks},$$

which, combined with (45) lead to (38) and (37). One can treat the case $p_k = \epsilon_{ki} p_i$ similarly, we omit the proof for brevity. This concludes the proof of the first case.

To prove the second case, assume none of the matrices $M_1, \dots, M_4$ are diagonalizable, then $\mathrm{eigen}(M_i) = \{\epsilon_i, \epsilon_i\}$, $\forall i = 1, \dots, 4$, where $\epsilon_i = \pm 1$. Let us choose a global conjugation such that one of the matrices M_i is in upper triangular form as in (42).

Now, since the monodromy group is not reducible, there exists at least one j such that $(M_j)_{21} \neq 0$. From $\mathrm{Tr}\, M_i M_j = 2\epsilon_{ij}$, we have $2\epsilon_i\epsilon_j + (M_j)_{21} = 2\epsilon_{ij}$, so that $(M_j)_{21} \neq 0$ implies $\epsilon_i\epsilon_j = -\epsilon_{ij}$. We perform a conjugation by a unipotent upper triangular matrix to impose $(M_j)_{12} = 0$, so we obtain the second equation in (42).

For all other matrices, we use $\mathrm{Tr}\, M_i M_s = 2\epsilon_{is}$ and $\mathrm{Tr}\, M_j M_s = 2\epsilon_{js}$, $s = k, l$ to find:

$$(M_s)_{21} = 2(\epsilon_{is} - \epsilon_i\epsilon_s), \qquad (M_s)_{12} = \frac{\epsilon_{js} - \epsilon_j\epsilon_s}{2\epsilon_{ij}},$$

From $\mathrm{Tr}\, M_s = 2\epsilon_s$ and $\mathrm{Tr}(M_i M_j M_s) = p_{ijs}$, we find the final formula (43) for $s = k$ and (44) for $s = l$, respectively. □

In the following Theorem characterizes the space of parameters as an affine variety in the polynomial ring

$$\mathbb{C}[p_1, p_2, p_3, p_4, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}]. \tag{46}$$

THEOREM 2.4 Consider $m := (M_1, \dots, M_4) \in \mathcal{U}$.
(i) The co-adjoint coordinates of m defined in (9) and (7) belong to the zero locus of the following 15 polynomials in the ring (46):

$$\begin{aligned}
f_1(p) := {} & p_{32}p_{31}p_{21} + p_{32}^2 + p_{31}^2 + p_{21}^2 \\
& - (p_1p_{321} + p_2p_3)p_{32} - (p_2p_{321} + p_1p_3)p_{31} \\
& - (p_3p_{321} + p_1p_2)p_{12} + p_3^2 + p_2^2 + p_1^2 + p_{321}^2 + p_3p_2p_1p_{321} - 4,
\end{aligned} \tag{47}$$

$$\begin{aligned}
f_2(p) := {} & p_{42}p_{41}p_{21} + p_{42}^2 + p_{41}^2 + p_{21}^2 \\
& - (p_1p_{421} + p_2p_4)p_{42} - (p_2p_{421} + p_1p_4)p_{41} \\
& - (p_4p_{421} + p_1p_2)p_{12} + p_4^2 + p_2^2 + p_1^2 + p_{421}^2 + p_4p_2p_1p_{421} - 4,
\end{aligned} \tag{48}$$

$$\begin{aligned}
f_3(p) := {} & p_{43}p_{41}p_{31} + p_{43}^2 + p_{41}^2 + p_{31}^2 \\
& - (p_1p_{431} + p_3p_4)p_{43} - (p_3p_{431} + p_1p_4)p_{41} \\
& - (p_4p_{431} + p_1p_3)p_{13} + p_4^2 + p_3^2 + p_1^2 + p_{431}^2 + p_4p_3p_1p_{431} - 4,
\end{aligned} \tag{49}$$

$$\begin{aligned}
f_4(p) := {} & p_{43}p_{42}p_{32} + p_{43}^2 + p_{42}^2 + p_{32}^2 \\
& - (p_2p_{432} + p_3p_4)p_{43} - (p_3p_{432} + p_2p_4)p_{42} \\
& - (p_4p_{432} + p_2p_3)p_{23} + p_4^2 + p_3^2 + p_2^2 + p_{432}^2 + p_4p_3p_2p_{432} - 4,
\end{aligned} \tag{50}$$

$$\begin{aligned}
f_5(p) := {} & -2p_\infty + p_1p_2p_3p_4 + p_1p_{432} + p_2p_{431} + p_3p_{421} + p_{321}p_4 \\
& + p_{21}p_{43} + p_{32}p_{41} - p_1p_2p_{43} - p_1p_4p_{32} - p_2p_3p_{41} - p_3p_4p_{21} \\
& - p_{42}p_{31},
\end{aligned} \tag{51}$$

$$\begin{aligned}
f_6(p) := {} & p_2p_3p_4 - p_{32}p_4 - p_{21}p_3p_{41} + p_{321}p_{41} - p_3p_{42} + p_1p_3p_{421} \\
& - p_{31}p_{421} - p_2p_{43} + p_{21}p_{431} + 2p_{432} - p_1p_\infty,
\end{aligned} \tag{52}$$

$$\begin{aligned}
f_7(p) := {} & -p_1p_4 + 2p_{41} + p_{21}p_{42} - p_2p_{421} + p_{31}p_{43} + p_{21}p_{32}p_{43} \\
& - p_2p_{321}p_{43} - p_3p_{431} - p_{21}p_3p_{432} + p_{321}p_{432} + p_2p_3p_\infty \\
& - p_{32}p_\infty,
\end{aligned} \tag{53}$$

$$\begin{aligned}
f_8(p) := {} & -p_1p_2p_3 + p_{21}p_3 + p_2p_{31} + p_1p_{32} - 2p_{321} + p_2p_{41}p_{43} \\
& - p_{421}p_{43} - p_2p_4p_{431} + p_{42}p_{431} - p_{41}p_{432} + p_4p_\infty,
\end{aligned} \tag{54}$$

$$\begin{aligned}
f_9(p) := {} & -p_1p_2 + 2p_{21} + p_{31}p_{32} - p_3p_{321} + p_{41}p_{42} - p_4p_{421} \\
& + p_{32}p_{41}p_{43} - p_{32}p_4p_{431} - p_3p_{41}p_{432} + p_{431}p_{432} + p_3p_4p_\infty \\
& - p_{43}p_\infty,
\end{aligned} \tag{55}$$

$$\begin{aligned}
f_{10}(p) := {} & -p_1p_2p_4 + p_{21}p_4 + p_2p_{41} + p_1p_{42} - 2p_{421} + p_1p_{32}p_{43} \\
& - p_{321}p_{43} - p_{32}p_{431} - p_1p_3p_{432} + p_{31}p_{432} + p_3p_\infty,
\end{aligned} \tag{56}$$

$$\begin{aligned}
f_{11}(p) := {} & p_1p_3p_4 - p_{31}p_4 - p_{21}p_{32}p_4 + p_2p_{321}p_4 - p_3p_{41} - p_{321}p_{42} \\
& + p_{32}p_{421} - p_1p_{43} + 2p_{431} + p_{21}p_{432} - p_2p_\infty,
\end{aligned} \tag{57}$$

$$f_{12}(p) := -p_2p_4 + p_{21}p_{41} + 2p_{42} - p_1p_{421} + p_{32}p_{43} - p_{321}p_{431} \\ - p_3p_{432} + p_{31}p_\infty, \tag{58}$$

$$f_{13}(p) := p_1p_3 - 2p_{31} - p_{21}p_{32} + p_2p_{321} - p_{41}p_{43} + p_4p_{431} \\ + p_{421}p_{432} - p_{42}p_\infty, \tag{59}$$

$$f_{14}(p) := p_2p_3 - p_{21}p_{31} - 2p_{32} + p_1p_{321} - p_{21}p_{41}p_{43} - p_{42}p_{43} \\ p_1p_{421}p_{43} + p_{21}p_4p_{431} - p_{421}p_{431} + p_4p_{432} - p_1p_4p_\infty \\ + p_{41}p_\infty, \tag{60}$$

$$f_{15}(p) := -p_3p_4 + p_{31}p_{41} + p_{21}p_{32}p_{41} - p_2p_{321}p_{41} + p_{32}p_{42} \\ - p_1p_{32}p_{421} + p_{321}p_{421} + 2p_{43} - p_1p_{431} - p_2p_{432} + p_1p_2p_\infty \\ - p_{21}p_\infty. \tag{61}$$

(ii) For every given generic $p_1, \dots, p_4, p_\infty$, the affine variety $\mathcal{A}$ defined in (8) with $I = \langle f_1, \dots, f_{15}\rangle$, is four-dimensional.

Proof. To prove the relations (47), …, (61), we use iterations of the *skein relation*:

$$\operatorname{Tr} AB + \operatorname{Tr} A^{-1}B = \operatorname{Tr} A \operatorname{Tr} B, \ \forall A, B \in \mathrm{SL}_2(\mathbb{C}), \tag{62}$$

together with (2).

To prove statement (ii), we used Macaulay2 [26], in order to compute the dimension of the affine variety defined in (8). The result is that (8) has dimension four. □

COROLLARY 2.5 The quantities $(p_{21}, \dots, p_{43}, p_{321}, \dots, p_{421})$ give a set of over-determined coordinates on the open subset $\mathcal{U} \subset \hat{\mathcal{M}}_{\mathcal{G}_2}$ defined in (6).

Proof. Thanks to Theorem 2.1, Lemma 2.2 and Proposition 2.3 the quantities p_i, p_{ij}, p_{ijk} parameterize the monodromy matrices up to global conjugation. Thanks to Theorem 2.4 for every fixed choice of $p_1, p_2, p_3, p_4, p_\infty$ only four among the quantities p_{ij}, p_{ijk} for $i, j, k = 1, \dots, 4$ are independent. This concludes the proof. □

3. Braid group action on $\mathcal{M}_{\mathcal{G}_2}$

We start this section by proving Lemma 1.2.

Proof. First we prove that that action (10) is well defined, or in other words that the $I = \langle \mathcal{F}\rangle = \{f_1, \dots, f_{15}\}$ is invariant under the action (10). To this aim, we need to show that for each generator σ_i, $i = 1, 2, 3,$

$\sigma_i(I) = I$. We carry out the computation for σ_1 only, the other computations are similar.

$$
\begin{gathered}
f_1(\sigma_1(p)) = f_1(p), \qquad f_2(\sigma_1(p)) = f_2(p), \qquad f_3(\sigma_1(p)) = f_4(p), \\
f_4(\sigma_1(p)) = f_3(p) + (p_{21}p_{42} - p_2p_{421})f_6(p) + (p_{21}p_{431} - p_2p_\infty)f_{11}(p) \\
+(p_2p_{321} - p_{21}p_{32})f_{13}(p), \\
f_5(\sigma_1(p)) = f_1(p) - p_2f_7(p), \qquad f_6(\sigma_1(p)) = f_8(p) - p_{21}f_2(p), \\
f_7(\sigma_1(p)) = f_3(p) + p_2f_9(p), \qquad f_9(\sigma_1(p)) = f_5(p) - p_2f_2(p), \\
f_8(\sigma_1(p)) = f_4(p) - p_{42}f_2(p) - p_{432}f_7(p) + p_{32}f_9(p), \\
f_{10}(\sigma_1(p)) = -f_7(p), \qquad f_{11}(\sigma_1(p)) = f_6(p) - p_{21}f_7(p), \\
f_{12}(\sigma_1(p)) = -f_2(p), \qquad f_{14}(\sigma_1(p)) = -f_9(p), \\
f_{13}(\sigma_1(p)) = -p_{21}f_9(p) + f_{10}(p), \qquad f_{15}(\sigma_1(p)) = f_{11}(p) + p_1f_7(p).
\end{gathered}
$$

Similar formulae can be proved for all other generators of the braid group. This shows that the action (10) is well defined on $\mathcal{A}$.

In order to prove that σ_i for $i = 1, 2, 3$, defined in (10), is indeed an action of the braid group B_4, we recall that the braid group B_n in Artin's presentation is given by:

$$
\begin{aligned}
B_n = \big\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ 1 \le i \le n-2, \\
\sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| > 1 \big\rangle,
\end{aligned} \tag{63}
$$

so, we need to prove that the following relations are satisfied:

$$
\sigma_1\sigma_3(m) = \sigma_3\sigma_1(m), \quad \sigma_1\sigma_2\sigma_1(m) = \sigma_2\sigma_1\sigma_2(m), \quad \sigma_2\sigma_3\sigma_2(m) = \sigma_3\sigma_2\sigma_3(m). \tag{64}
$$

The first relation is straightforward, while the last two follow from the fact that polynomials (57)–(60) are zero for every $p \in \mathcal{A}$. □

This lemma allows us to reformulate our classification problem as follows:
classify all finite orbits:

$$
\mathrm{O}_{P_4}(p) = \{\beta(p) | \beta \in P_4\},
$$

where p is the following 15-tuple of complex quantities:

$$
p = (p_1, p_2, p_3, p_4, p_\infty, p_{21}, p_{31}, p_{32}, p_{41}, p_{42}, p_{43}, p_{321}, p_{432}, p_{431}, p_{421}) \in \mathbb{C}^{15},
$$

defined in (7), and P_4 is the pure braid group $P_4 = \langle \beta_{21}, \beta_{31}, \beta_{32}, \beta_{41}, \beta_{42}, \beta_{43} \rangle$ where:

$$
\begin{gathered}
\beta_{21} = \sigma_1^2, \qquad \beta_{31} = \sigma_2^{-1}\sigma_1^2\sigma_2, \qquad \beta_{32} = \sigma_2^2, \\
\beta_{41} = \sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2\sigma_3, \qquad \beta_{42} = \sigma_3^{-1}\sigma_2^2\sigma_3, \qquad \beta_{43} = \sigma_3^2,
\end{gathered} \tag{65}
$$

and the generators satisfy the following relations:

$$\beta_{rs}\beta_{ij}\beta_{rs}^{-1} = \begin{cases} \beta_{ij}, & \text{if } j < s < r < i, \\ & \text{or } s < r < j < i, \\ \beta_{rj}^{-1}\beta_{ij}\beta_{rj}, & s < j = r < i, \\ \beta_{rj}^{-1}\beta_{sj}^{-1}\beta_{ij}\beta_{sj}\beta_{rj}, & j = s < r < i, \\ \beta_{rj}^{-1}\beta_{sj}^{-1}\beta_{rj}\beta_{sj}\beta_{ij}\beta_{sj}^{-1}\beta_{rj}^{-1}\beta_{sj}\beta_{rj}, & s < j < r < i. \end{cases} \tag{66}$$

4. Restrictions

In this section, we select subgroups $H \subset P_4$ such that the restricted action is isomorphic to the action of the pure braid group P_3 on the quotient space (11).

THEOREM 4.1 The following four subgroups $H_i \subset P_4$ with $i = 1, \ldots, 4$:

$$H_1 := \langle \beta_{32}, \beta_{43}, \beta_{42} \rangle, \qquad H_2 = \langle \beta_{43}, \beta_{31}, \beta_{41} \rangle,$$
$$H_3 = \langle \beta_{21}, \beta_{42}, \beta_{41} \rangle, \qquad H_4 = \langle \beta_{21}, \beta_{32}, \beta_{31} \rangle,$$

where the generators β_{jk}, $1 \leq k < j \leq 4$, are defined in (65), are isomorphic to the pure braid group P_3. Moreover given any ordered 4-tuple of matrices $(M_1, M_2, M_3, M_4) \in \mathcal{U}$, each H_i, for $i = 1, \ldots, 4$, acts as pure braid group P_3 on a certain triple of matrices $(N_1, N_2, N_3) \in \hat{\mathcal{M}}_{PVI}$ given by:

$$H_1 : \hat{N}_1 = M_2, \ \hat{N}_2 = M_3, \ \hat{N}_3 = M_4, \ \hat{N}_\infty = (M_4 M_3 M_2)^{-1}, \tag{67}$$

$$H_2 : \bar{N}_1 = M_1, \ \bar{N}_2 = M_3, \ \bar{N}_3 = M_4, \ \bar{N}_\infty = (M_4 M_3 M_1)^{-1}, \tag{68}$$

$$H_3 : \check{N}_1 = M_1, \ \check{N}_2 = M_2, \ \check{N}_3 = M_4, \ \check{N}_\infty = (M_4 M_2 M_1)^{-1}, \tag{69}$$

$$H_4 : \tilde{N}_1 = M_1, \ \tilde{N}_2 = M_2, \ \tilde{N}_3 = M_3, \ \tilde{N}_\infty = (M_3 M_2 M_1)^{-1}. \tag{70}$$

Proof. To prove that each H_i, $i = 1, \ldots, 4$ is isomorphic to P_3 we need to prove that its generators satisfy the relations (66), for $n = 3$. This can be checked by direct computations.

We now prove the second statement explicitly for the subgroup H_1, for the other subgroups a similar proof applies. First of all, thanks to (2) we have immediately:

$$\hat{N}_\infty \hat{N}_3 \hat{N}_2 \hat{N}_1 = \mathbb{I},$$

so that $\hat{n} = (\hat{N}_1, \hat{N}_2, \hat{N}_3) \in \hat{\mathcal{M}}_{PVI}$. To show that the subgroup H_1 acts as pure braid group P_3 on $\hat{\mathcal{M}}_{PVI}$, we use the fact that the generators of H_1 are defined in terms of generators σ_2 and σ_3 of the full braid group B_4, so it is enough to prove that σ_2 and σ_3 act as generators of the braid group B_3 on $\hat{\mathcal{M}}_{PVI}$. Consider (67), then the following relations hold:

$$\sigma_2(m) = (M_1, M_3, M_3 M_2 M_3^{-1}, M_4) = (\hat{N}_2, \hat{N}_2 \hat{N}_1 \hat{N}_2^{-1}, \hat{N}_3) = \sigma_1^{(PVI)}(\hat{n}),$$
$$\sigma_3(m) = (M_1, M_2, M_4, M_4 M_3 M_4^{-1}) = (\hat{N}_1, \hat{N}_3, \hat{N}_3 \hat{N}_2 \hat{N}_3^{-1}) = \sigma_2^{(PVI)}(\hat{n}). \tag{71}$$

This concludes the proof. □

TABLE 1. *Matching using traces: elements on the same columns must be equal*

	p_1	p_2	p_3	p_4	p_∞	p_{21}	p_{31}	p_{32}	p_{41}	p_{42}	p_{43}	p_{321}	p_{432}	p_{431}	p_{421}
H_1		$\hat{q}_1$	$\hat{q}_2$	$\hat{q}_3$				$\hat{q}_{21}$		$\hat{q}_{31}$	$\hat{q}_{32}$		$\hat{q}_\infty$		
H_2	$\bar{q}_1$		$\bar{q}_2$	$\bar{q}_3$			$\bar{q}_{21}$		$\bar{q}_{31}$		$\bar{q}_{32}$			$\bar{q}_\infty$	
H_3	$\check{q}_1$	$\check{q}_2$		$\check{q}_3$		$\check{q}_{21}$			$\check{q}_{31}$	$\check{q}_{32}$					$\check{q}_\infty$
H_4	$\tilde{q}_1$	$\tilde{q}_2$	$\tilde{q}_3$			$\tilde{q}_{21}$	$\tilde{q}_{31}$	$\tilde{q}_{32}$				$\tilde{q}_\infty$			

We now consider the action of the subgroups H_i for $i = 1, \ldots, 4$ in terms of co-adjoint coordinates on $\hat{\mathcal{M}}_{PVI}$:

$$\begin{aligned} \hat{q} &:= (\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_\infty, \hat{q}_{21}, \hat{q}_{31}, \hat{q}_{32}), \quad \bar{q} := (\bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{q}_\infty, \bar{q}_{21}, \bar{q}_{31}, \bar{q}_{32}), \\ \check{q} &:= (\check{q}_1, \check{q}_2, \check{q}_3, \check{q}_\infty, \check{q}_{21}, \check{q}_{31}, \check{q}_{32}), \quad \tilde{q} := (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_\infty, \tilde{q}_{21}, \tilde{q}_{31}, \tilde{q}_{32}), \end{aligned} \tag{72}$$

where $\hat{q}_i = \operatorname{Tr} \hat{N}_i$ for $i = 1, 2, 3, \infty$ and $\hat{q}_{jk} = \operatorname{Tr} \hat{N}_j \hat{N}_k$ for $j > k,\ j, k = 1, 2, 3$ and similar formulae for $\bar{q}$, $\check{q}$ and $\tilde{q}$. Then identifications (67)–(70) imply the identities summarized in Table 1, where p_i,p_{ij},p_{ijk} are defined in (7) and elements in the same column are identical.

We define the following four projections:

$$\tilde{\pi}, \hat{\pi}, \check{\pi}, \bar{\pi} : \mathcal{A} \mapsto \hat{\mathcal{M}}_{PVI}, \tag{73}$$

as follows

$$\begin{aligned} \tilde{\pi}(p) &:= (p_1, p_2, p_3, p_{321}, p_{21}, p_{31}, p_{32}) = \tilde{q}, \\ \hat{\pi}(p) &:= (p_2, p_3, p_4, p_{432}, p_{32}, p_{42}, p_{43}) = \hat{q}, \\ \check{\pi}(p) &:= (p_1, p_2, p_4, p_{421}, p_{21}, p_{41}, p_{42}) = \check{q}, \\ \bar{\pi}(p) &:= (p_1, p_3, p_4, p_{431}, p_{31}, p_{41}, p_{43}) = \bar{q}. \end{aligned} \tag{74}$$

Viceversa, given four 7-ples $\tilde{q}$,$\hat{q}$,$\check{q}$,$\bar{q}$, such that they satisfy the equalities in the columns of Table 1, we can lift them to a point $p \in \mathcal{A}$, in which the value of p_∞ can be recovered using relation (56). We call this *matching procedure.*

5. Input set

The classification result by Lisovyy and Tykhyy produced a list of all finite orbits under the action of the braid group B_3 modulo the action of the group F_4 of Okamoto transformations acting on $\mathcal{M}_{PVI}$. However, points q that are equivalent modulo the action of the group F_4 of Okamoto transformations, and of the pure braid group P_3, don't necessarily produce candidate points p that are equivalent modulo the action of the symmetry group G of the Garnier system $\mathcal{G}_2$ nor by the action of the pure braid group P_4. Therefore, we need to expand the list of input points q by considering all images under F_4 and P_3. In this section, we define an expansion algorithm that applies the action of F_4 and P_3 to the 45 exceptional orbits of [9]. Thanks to the fact that the action of F_4 over these 45 finite orbits is finite, the result is a finite set that we call E_{45}. This set does not include points that correspond to solutions of Okamoto type nor solutions corresponding to Picard or Dubrovin–Kitaev orbits—we will deal with these points in Section 6.

5.1 *The classification result by Lisovyy and Tykhyy*

In order to expand Lisovyy and Tykhyy list of 45 finite orbits (see Table 5 in [9]) it is best to introduce the following quantities:

$$\begin{aligned}&\omega_1 := q_1 q_\infty + q_3 q_2, \qquad \omega_2 := q_2 q_\infty + q_3 q_1, \qquad \omega_3 := q_3 q_\infty + q_2 q_1,\\ &\omega_4 := q_3^2 + q_2^2 + q_1^2 + q_\infty^2 + q_3 q_2 q_1 q_\infty.\end{aligned} \tag{75}$$

The group F_4 of Okamoto transformations of the sixth Painlevé equation acts as $K_4 \rtimes S_3$ on $(\omega_1, \dots, \omega_4)$ [9]. Extending this action to the q_{ij}s, namely acting on $(\omega_1, \dots, \omega_4, q_{21}, q_{31}, q_{32})$ it is straightforward to prove the following:

Proposition 5.1 The group F_4 of the Okamoto transformations of the sixth Painlevé equation is generated by the following transformations that act on $(\omega_1, \dots, \omega_4, q_{21}, q_{31}, q_{32})$ as follows:

$$\begin{aligned}
&s_i(q_{21}, q_{31}, q_{32}, \omega_1, \omega_2, \omega_3, \omega_4) = (q_{21}, q_{31}, q_{32}, \omega_1, \omega_2, \omega_3, \omega_4),\ i = 1, 2, 3, \infty, \delta,\\
&r_1(q_{21}, q_{31}, q_{32}, \omega_1, \omega_2, \omega_3, \omega_4) = (-q_{21}, -q_{31}, q_{32}, \omega_1, -\omega_2, -\omega_3, \omega_4),\\
&r_2(q_{21}, q_{31}, q_{32}, \omega_1, \omega_2, \omega_3, \omega_4) = (-q_{21}, q_{31}, -q_{32}, -\omega_1, \omega_2, -\omega_3, \omega_4),\\
&r_3(q_{21}, q_{31}, q_{32}, \omega_1, \omega_2, \omega_3, \omega_4) = (q_{21}, -q_{31}, -q_{32}, -\omega_1, -\omega_2, \omega_3, \omega_4),\\
&P_{13}(q_{21}, q_{31}, q_{32}, \omega_1, \omega_2, \omega_3, \omega_4) = (q_{32}, \omega_2 - q_{31} - q_{21}q_{32}, q_{21}, \omega_3, \omega_2, \omega_1, \omega_4),\\
&P_{23}(q_{21}, q_{31}, q_{32}, \omega_1, \omega_2, \omega_3, \omega_4) = (\omega_2 - q_{31} - q_{21}q_{32}, q_{21}, q_{32}, \omega_1, \omega_3, \omega_2, \omega_4).
\end{aligned}$$

Proof. The proof of this is a consequence of the results of [10, 27]. □

In particular we observe that P_{13} and P_{23} are elements of the braid group B_3—since we act only on points that have finite orbits under the action of the braid group, the action of the whole group F_4 produces a finite set of values. All these values will be in the form $(\omega_1, \dots, \omega_4, q_{21}, q_{31}, q_{32})$; in order to extract q_1, q_2, q_3 and q_∞ we use the fact that we can consider the relations (75) as a system of equations in q_1, q_2, q_3 and q_∞ and that each q_i has the form:

$$q_i = 2\cos \pi\theta_i, \quad i = 1, 2, 3, \infty.$$

One particular solution of equations (75) is listed in [9] in terms of $\theta_1, \theta_2, \theta_3, \theta_\infty$ for each point in the Table 5 in [9]. We can then compute all other solutions q_1, q_2, q_3 and q_∞ by using the following:

Lemma 5.2 Suppose $\omega_1, \omega_2, \omega_3, \omega_4$ are given and consider system (75) in the variables q_1, q_2, q_3, q_∞, then this system admits at most 24 solutions. Any two such solutions are related by the following elements of F_4:

$$\begin{aligned}
&id, \quad \alpha, \quad \beta, \quad \gamma, \quad \alpha\cdot\beta, \quad \alpha\cdot\gamma, \quad \beta\cdot\gamma,\\
&\alpha\cdot\beta\cdot\gamma \quad s_\delta, \quad \alpha\cdot s_\delta, \quad \beta\cdot s_\delta, \quad \gamma\cdot s_\delta, \quad \alpha\cdot\beta\cdot s_\delta,\\
&\alpha\cdot\gamma\cdot s_\delta, \quad \beta\cdot\gamma\cdot s_\delta, \quad \alpha\cdot\beta\cdot\gamma\cdot s_\delta, \quad s_\delta s_1, \quad \alpha\cdot s_\delta\cdot s_1,\\
&\beta\cdot s_\delta\cdot s_1, \quad \gamma\cdot s_\delta\cdot s_1 \quad \alpha\cdot\beta\cdot s_\delta\cdot s_1, \quad \alpha\cdot\gamma\cdot s_\delta\cdot s_1,\\
&\beta\cdot\gamma\cdot s_\delta\cdot s_1, \quad \alpha\cdot\beta\cdot\gamma\cdot s_\delta\cdot s_1.
\end{aligned}$$

where α, β, γ, s_δ, s_1 act as follows on the parameters θ_i:

$$\alpha(\theta_1,\theta_2,\theta_3,\theta_\infty) = (1+\theta_1, 1+\theta_2, 1+\theta_3, 1+\theta_\infty)$$

$$\beta(\theta_1,\theta_2,\theta_3,\theta_\infty) = (\theta_2,\theta_1,\theta_\infty-2,\theta_3), \qquad \gamma(\theta_1,\theta_2,\theta_3,\theta_\infty) = (\theta_3,\theta_\infty-2,\theta_1,\theta_2)$$

$$s_\delta(\theta_1,\theta_2,\theta_3,\theta_\infty) = (\theta_1-\delta,\theta_2-\delta,\theta_3-\delta,\theta_\infty-\delta), \quad \delta = \frac{\theta_1+\theta_2+\theta_3+\theta_\infty}{2},$$

$$s_1(\theta_1,\theta_2,\theta_3,\theta_\infty) = (-\theta_1,\theta_2,\theta_3,\theta_\infty).$$

Proof. It is an immediate consequence of Proposition 10 in [9]. □

This lemma allows us to calculate all the solutions of the system (75) in terms of the given $\omega_1, \omega_2, \omega_3, \omega_4$ starting from only one solution q_1, q_2, q_3 and q_∞. We are therefore able to set up our expansion algorithm:

Algorithm 1 For every line of Table 5 in [9], take the values $(\omega_1, \dots, \omega_4,\ q_{21}, q_{31}, q_{32})$ and the corresponding $(q_1, q_2, q_3, q_\infty)$ given in [9].

(1) Apply to $(\omega_1, \dots, \omega_4, q_{21}, q_{31}, q_{32})$ all 48 transformations of the group $K_4 \rtimes S_3$. For each new set of values $(\omega_1', \dots, \omega_4', q_{21}', q_{31}', q_{32}')$ obtained in this way, compute the corresponding $(q_1', \dots, q_\infty')$ as the result of the same transformation on $(q_1, q_2, q_3, q_\infty)$.

(2) For every element $(\omega_1', \dots, \omega_4', q_{21}', q_{31}', q_{32}')$ obtained in step 1, generate their orbit under the action of the braid group B_3. For each new set of values $(\omega_1'', \dots, \omega_4'', q_{21}'', q_{31}'', q_{32}'')$ obtained in this way, compute the corresponding $(q_1'', \dots, q_\infty'')$ as the result of the same braid on $(q_1', q_2', q_3', q_\infty')$.

(3) For every element $(\omega_1'', \dots, \omega_4'', q_{21}'', q_{31}'', q_{32}'')$ and $(q_1'', \dots, q_\infty'')$ obtained in step 2, find all other solutions $(q_1''', q_2''', q_3''', q_\infty''')$ of the system (75) for $(\omega_1'', \dots, \omega_4'')$ by applying the transformations in Lemma 5.2 to $(q_1'', \dots, q_\infty'')$.

(4) Merge $(q_1''', q_2''', q_3''', q_\infty'')$ and $(q_{21}'', q_{31}'', q_{32}'')$ into:

$$q''' = (q_1''', q_2''', q_3''', q_\infty''', q_{21}'', q_{31}'', q_{32}'').$$

(5) Generate the P_3-orbit of q''' and save the result in the set E_{45}

Once this algorithm ends, the set E_{45} will contain only a finite number of orbits. This set contains 86,768 points.

6. Matching procedure

In this section, we propose a procedure to construct all *candidate* points $p \in \mathcal{A}$:

Definition 6.1 A point p such that its four projections $\hat{q}, \check{q}, \bar{q}, \tilde{q}$, defined in (74), generate finite orbits under the action of P_3 and such that at most one projections is a Picard or Dubrovin–Kitaev orbit, is said to be a *candidate* point.

Note that, to generate a *candidate* point p, it is not necessary to know all four projections $\hat{q}, \check{q}, \bar{q}, \tilde{q}$. Indeed, looking at Table 1, we see that if we give three projections, then only the value of p_∞ and one value p_{ijk} will be undetermined, but we can calculate these values from (51) and by choosing appropriately one of the four relations $f_1, \ldots, f_4$, defined in (47)–(50), respectively. So, in order to obtain the set $\mathcal{C}$ of all *candidate* points, we can set up three matching procedures, each of them based on the knowledge of only three projections. We denote by $\tilde{C}$, $\hat{C}$, $\check{C}$ and $\bar{C}$ the sets obtained by matching three projections and missing $\tilde{q}$, $\hat{q}$, $\check{q}$ or $\tilde{q}$, respectively.

In order to construct the set $\mathcal{C}$, the union of all the above four sets $\tilde{\mathcal{C}}, \hat{\mathcal{C}}, \check{\mathcal{C}}, \bar{\mathcal{C}}$ must be taken:

$$\mathcal{C} = \tilde{\mathcal{C}} \cup \hat{\mathcal{C}} \cup \check{\mathcal{C}} \cup \bar{\mathcal{C}}. \tag{76}$$

As we are going to show in the next Lemma, it is enough to know only one of the sets $\tilde{\mathcal{C}}, \hat{\mathcal{C}}, \check{\mathcal{C}}, \bar{\mathcal{C}}$ to generate the whole set $\mathcal{C}$:

LEMMA 6.1 Consider $m \in \mathcal{U}$ and the permutation $\pi_{(1234)}$ that acts on the co-adjoint coordinates of m as follows:

$$\pi_{(1234)}(p) = (p_4, p_1, p_2, p_3, p_\infty, p_{41}, p_{42}, p_{21}, p_{43}, p_{31}, p_{32}, p_{421}, p_{321}, p_{432}, p_{431}),$$

then:

$$\pi_{(1234)}(\tilde{\mathcal{C}}) = \check{\mathcal{C}}, \quad \pi_{(1234)}(\check{\mathcal{C}}) = \bar{\mathcal{C}}, \quad \pi_{(1234)}(\bar{\mathcal{C}}) = \hat{\mathcal{C}}, \quad \pi_{(1234)}(\hat{\mathcal{C}}) = \tilde{\mathcal{C}}. \tag{77}$$

Proof. We only prove the first of (77), the other relations can be proved in a similar way. Thanks to Theorem 2.1, a point $p \in \tilde{\mathcal{C}}$ parameterizes a quadruple m of monodromy matrices $m := (M_1, M_2, M_3, M_4)$ up to global diagonal conjugation. Analogously, the three projections $\hat{q}, \check{q}, \bar{q} \in \hat{\mathcal{M}}_{PVI}$ parameterize three triples of monodromy matrices $\hat{n}, \check{n}, \bar{n} \in \hat{\mathcal{M}}_{PVI}$, such that, up to global diagonal conjugation:

$$\begin{aligned}
&\hat{N}_1 = M_2,\ \hat{N}_2 = M_3,\ \hat{N}_3 = M_4,\ \hat{N}_\infty = (M_4M_3M_2)^{-1},\\
&\bar{N}_1 = M_1,\ \bar{N}_2 = M_3,\ \bar{N}_3 = M_4,\ \bar{N}_\infty = (M_4M_3M_1)^{-1},\\
&\check{N}_1 = M_1,\ \check{N}_2 = M_2,\ \check{N}_3 = M_4,\ \check{N}_\infty = (M_4M_2M_1)^{-1}.
\end{aligned}$$

Now take the point $p' = \pi_{(1234)}(p)$, this parameterizes the triple $m' = \pi_{(1234)}(m)$ up to global diagonal conjugation. Consider now the three projections $\hat{q}', \tilde{q}', \bar{q}' \in \hat{\mathcal{M}}_{PVI}$ of p'. They parameterize three triples of monodromy matrices $\hat{n}', \tilde{n}', \bar{n}' \in \hat{\mathcal{M}}_{PVI}$, such that, up to global diagonal conjugation:

$$\begin{aligned}
&\hat{N}'_1 = M'_2 = M_1,\ \hat{N}'_2 = M'_3 = M_2,\ \hat{N}'_3 = M'_4 = M_3,\\
&\qquad \hat{N}'_\infty = (M'_4M'_3M'_2)^{-1} = (M_3M_2M_1)^{-1},\\
&\bar{N}'_1 = M'_1 = M_4,\ \bar{N}'_2 = M'_3 = M_2,\ \bar{N}'_3 = M'_4 = M_3,\\
&\qquad \bar{N}'_\infty = (M'_4M'_3M'_1)^{-1} = (M_3M_2M_4)^{-1},\\
&\tilde{N}'_1 = M'_1 = M_4,\ \tilde{N}'_2 = M'_2 = M_1,\ \tilde{N}'_3 = M'_3 = M_2,\\
&\qquad \tilde{N}'_\infty = (M'_3M'_2M'_1)^{-1} = (M_2M_1M_4)^{-1}.
\end{aligned}$$

These relations show that

$$\hat{n}' = \tilde{n}, \quad \bar{n}' = \pi_{(123)}\hat{n}, \quad \tilde{n}' = \pi_{(123)}\check{n},$$

where

$$\pi_{(123)}(q) = (q_3, q_1, q_2, q_\infty, q_{32}, q_{21}, q_{31}),$$

Now since $\tilde{n}, \pi_{(123)}\hat{n}, \pi_{(123)}\check{n} \in \hat{\mathcal{M}}_{PVI}$, this shows that $p' \in \check{\mathcal{C}}$. Viceversa, we can prove in a similar way that given $p' \in \check{\mathcal{C}}$, then $p = \pi^{-1}_{(1234)} p' \in \tilde{\mathcal{C}}$. This concludes the proof. □

We are now ready to describe how to implement the matching algorithmically.

6.1 *Matching with the PVI 45 exceptional algebraic solutions*

In this section, we give an algorithm that produces the *finite* set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}}$ of all *candidate* points p such that three over four projections $\hat{q}, \check{q}, \bar{q}, \tilde{q}$, defined in (74), are in the set E_{45}.

Algorithm 2

(1) Consider $(\hat{q}, \check{q}, \bar{q}) \in \mathrm{E}_{45} \times \mathrm{E}_{45} \times \mathrm{E}_{45}$.

(2) Check if $\hat{q}, \check{q}, \bar{q}$ satisfy relations given by the columns of Table 1, then go to the next step, otherwise go to Step 1.

(3) Determine the two roots $p^{(i)}_{321}$, for $i = 1, 2$, using equation (47).

For each $i = 1, 2$:

(4) Calculate the values of $p^{(i)}_\infty$ using equation (51).

(5) Use Table 1 to determine all the other components of $p^{(i)}$.

(6) If $p^{(i)}$ satisfies equations (52)–(61) then go to the next Step, otherwise go to Step 1.

(7) Save $p^{(i)}$ in the set $\tilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}}$, eliminate $(\hat{q}, \check{q}, \bar{q})$ from $\mathrm{E}_{45} \times \mathrm{E}_{45} \times \mathrm{E}_{45}$ and go to Step 1.

Since E_{45} is a finite set, this algorithm terminates and produces a finite set $\tilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}}$. Finally the big set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}}$ can be generated by Lemma 6.1 as follows:

$$\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}} = \tilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}} \bigcup_{i=1}^{3} \pi^{i}_{(1234)}(\tilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}}).$$

The Algorithm 2 together with the action of the permutations producing the set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{E}_{45}}$ can be found [21]. This set contains all *candidate* points $p \in \mathcal{A}$ such that three projections (74) are in the set E_{45} and consists of 3,355,200 points.

6.2 *Matching with Okamoto's Riccati solutions*

We call *Okamoto-type solutions* the algebraic solutions of the PVI equation belonging to Okamoto's Riccati solutions. The set O of all finite orbits corresponding to Okamoto-type solutions is an infinite set, therefore to construct candidate points with projections in this set is not a straightforward adaptation of Algorithm 2.

Definition 6.2 A point p is called *not relevant* if the associated monodromy group is reducible or there exists an index $i = 1, \ldots, 4, \infty$ such that $M_i = \pm\mathbb{I}$. A point p is called *relevant* otherwise.

In this subsection, we are going to prove a few lemmata that show that in order for p to be relevant, the number of projections corresponding to solutions of Okamoto type is limited. We will then characterize these projections and formulate algorithms that exploit these characterizations to classify candidate points with projections of Okamoto type.

Proposition 6.2 If a point $p \in \mathcal{A}$ is such that any three of its four projections $\hat{q}, \check{q}, \bar{q}, \tilde{q}$ are in the set O of all finite orbits corresponding to algebraic solutions of Okamoto type then the point p is *not relevant*.

Consequently, all points p satisfying hypotheses of Proposition 6.2 will be irrelevant to our classification (and then excluded from it).

Before proving this result, we will need the following two definitions:

Definition 6.3 The set $\mathrm{O_{ID}}$ is the set of all the $q \in \mathrm{O}$ such that the associated triple of monodromy matrices $n \in \hat{\mathcal{M}}_{PVI}$ admits one matrix equals to $\pm\mathbb{I}$.

Definition 6.4 The set $\mathrm{O_{RED}}$ is the set of all the $q \in \mathrm{O}$ such that if we consider the associated triple of monodromy matrices $n \in \hat{\mathcal{M}}_{PVI}$ then the monodromy group $\langle N_1, N_2, N_3 \rangle$ is *reducible*.

Proof of Proposition 6.2: In order to prove the statement, we distinguish three cases:
(i) Assume p has three projections in $\mathrm{O_{ID}}$. It is enough to consider $m \in \hat{\mathcal{M}}_{\mathcal{G}_2}$ and the following three projections:

$$\tilde{n} = (M_1, M_2, M_3), \qquad \hat{n} = (M_2, M_3, M_4), \qquad \check{n} = (M_1, M_2, M_4), \tag{78}$$

because all other cases differ from this case only by a permutation of the matrices M_i, see Lemma 6.1. If any of $M_i = \pm\mathbb{I}$, then we conclude. If not, we are left with the following case:

$$\tilde{N}_\infty = M_3 M_2 M_1 = \tilde{\epsilon}\mathbb{I}, \quad \hat{N}_\infty = M_4 M_3 M_2 = \hat{\epsilon}\mathbb{I}, \quad \check{N}_\infty = M_4 M_2 M_1 = \check{\epsilon}\mathbb{I},$$

where $\tilde{\epsilon}, \hat{\epsilon}, \check{\epsilon} = \pm 1$. Combining these relations we obtain:

$$M_4 = \tilde{\epsilon}\hat{\epsilon} M_1, \quad M_3 = \tilde{\epsilon}\check{\epsilon} M_4, \text{ and therefore } M_3 = \hat{\epsilon}\check{\epsilon} M_1, \quad M_2 = \tilde{\epsilon}\hat{\epsilon}\check{\epsilon} M_1^{-2},$$

so that finally $m = (M_1, \tilde{\epsilon}\hat{\epsilon}\check{\epsilon} M_1^{-2}, \hat{\epsilon}\check{\epsilon} M_1, \tilde{\epsilon}\hat{\epsilon} M_1)$ which is reducible. Therefore p is not relevant.
(ii) Suppose p is such that three projections over four are in the set $\mathrm{O_{RED}}$. Again it is enough to consider the three projections (78). Since the three monodromy groups defined by the triples $\tilde{n}, \hat{n}, \check{n}$ are reducible, these triples have each a common eigenvector, let us denote them $\tilde{v}$, $\hat{v}$ and $\check{v}$, respectively. Now the matrix M_2 that appears in all the three projections, has three eigenvectors $\tilde{v}, \hat{v}$ and $\check{v}$, which implies that one of the following identities must hold: $\tilde{v} = \hat{v}$ or $\tilde{v} = \check{v}$ or $\hat{v} = \check{v}$. Therefore the monodromy group is reducible and the point p is not relevant.
(iii) When there are three projections in O, not all of the same type, we apply Lemma 6.3. This concludes the proof.

LEMMA 6.3 If a point $p \in \mathcal{A}$ is such that one of its four projections $\hat{q}, \check{q}, \bar{q}, \tilde{q}$ is in the set O_{ID} and another one projection is in the set $\mathrm{O}_{\mathrm{RED}}$, then such point p is *not relevant*.

Proof. Consider $m \in \hat{\mathcal{M}}_{\mathcal{G}_2}$ and the following two distinct *generic* projections:

$$(M_i, M_j, M_k) \in \mathrm{O}_{\mathrm{ID}}, \; i > j > k, \; i,j,k = 1, \dots, 4, \tag{79}$$

$$(M_{i'}, M_{j'}, M_{k'}) \in \mathrm{O}_{\mathrm{RED}}, \; i' > j' > k', \; i',j',k' = 1, \dots, 4. \tag{80}$$

If either M_i, M_j, M_k is equal to $\pm\mathbb{I}$, then we conclude. Otherwise suppose:

$$M_i M_j M_k = \pm\mathbb{I}. \tag{81}$$

Moreover, suppose the monodromy group associated to the triple $(M_{i'}, M_{j'}, M_{k'})$ is reducible, then the matrices $M_{i'}, M_{j'}, M_{k'}$ have a common eigenvector v. In (79) and in (80), at least two indices i, j, k that are equal to two indices i', j', k', without loss of generality, suppose $i \neq i'$, $j = j'$ and $k = k'$, then equation (81) implies $M_i = \pm(M_{j'} M_{k'})^{-1}$, which shows that v is also an eigenvector for M_j and therefore the monodromy group $\langle M_i, M_{i'}, M_j, M_k \rangle$ is reducible as we wanted to prove. □

LEMMA 6.4 Let p be a relevant point such that one of its projections q is in the set O_{ID}, then q satisfies:

$$q_{21} = \pm q_3, \quad q_{31} = \pm q_2, \quad q_{32} = \pm q_1, \quad q_\infty = \pm 2. \tag{82}$$

Proof. Consider the triple of matrices $n = (N_1, N_2, N_3)$ determined by $q \in \mathrm{O}_{\mathrm{ID}}$. If any of the N_i is equal to $\pm\mathbb{I}$, by the matching procedure, we end up with a point p that is not relevant, therefore, we avoid this case. Otherwise, assume $N_\infty = N_3 N_2 N_1 = \pm\mathbb{I}$, then:

$$N_1 = \pm(N_3 N_2)^{-1}, \; N_2 = \pm(N_1 N_3)^{-1}, \; N_3 = \pm(N_2 N_1)^{-1}. \tag{83}$$

By taking the traces we obtain (82). This concludes the proof. □

LEMMA 6.5 Let q be the co-adjoint coordinates on $\hat{\mathcal{M}}_{PVI}$. If q is in the set $\mathrm{O}_{\mathrm{RED}}$, then q satisfies:

$$\begin{cases} q_{ij} = \frac{1}{2}(q_i q_j - \epsilon_i \epsilon_j s_i s_j), \; i > j, \; i,j = 1,2,3, \\ q_\infty = \frac{1}{4}(q_1 q_2 q_3 - \epsilon_1 \epsilon_2 s_1 s_2 q_3 - \epsilon_1 \epsilon_3 s_1 s_3 q_2 - \epsilon_2 \epsilon_3 s_2 s_3 q_1) \end{cases} \tag{84}$$

where $s_k = \sqrt{4 - q_k^2}$ for some choice of the signs $\epsilon_k = \pm 1$ for $k = 1, 2, 3$.

Proof. Consider the triple of matrices $n = (N_1, N_2, N_3)$ determined by $q \in \mathrm{O}_{\mathrm{RED}}$, they define a reducible monodromy group. Therefore, we can choose a basis in which they are all upper triangular. Then their diagonal elements are given by their eigenvalues $\mathrm{eigen}v(N_i) = \exp(\epsilon_l \pi \theta_i)$, where $\epsilon_l = \pm 1$, so that:

$$\mathrm{Tr}(N_i) = 2\cos \pi\theta_i, \; i = 1, 2, 3, \infty, \tag{85}$$

$$\mathrm{Tr}(N_i N_j) = 2\cos(\pi(\epsilon_i \theta_i + \epsilon_j \theta_j)), \; i,j = 1, 2, 3, \; i > j, \tag{86}$$

$$\mathrm{Tr}(N_3 N_2 N_1) = 2\cos(\pi(\epsilon_1 \theta_1 + \epsilon_2 \theta_2 + \epsilon_3 \theta_3)). \tag{87}$$

Applying trigonometric identities we obtain relations (84). This concludes the proof. □

An obvious consequence of this result is:

LEMMA 6.6 Suppose $p \in \mathcal{A}$ is a relevant point such that any two of its four projections $\hat{q}, \check{q}, \bar{q}, \tilde{q}$, defined in (74), are in the set $\mathrm{O_{RED}}$. Denote by q one of the remaining projections, then there exists a couple of indices (i,j),(i',j') with one index in (i,j) equal to one index in (i',j') such that:

$$\begin{cases} q_{ij}^2 + q_i^2 + q_j^2 - q_{ij}q_iq_j - 4 = 0, & i > j,\ i,j = 1,2,3, \\ q_{i'j'}^2 + q_{i'}^2 + q_{j'}^2 - q_{i'j'}q_{i'}q_{j'} - 4 = 0, & i' > j',\ i',j' = 1,2,3. \end{cases} \tag{88}$$

Lemmata 6.4, 6.5 and 6.6 lead to the development of additional matching algorithms in order to complete our classification for the cases when these points are included. Thanks to Lemma 6.3, in order to complete our classification of candidate points, we need to construct only the following four sets: $\mathcal{C}_{\mathrm{E_{45}\times O_{ID}\times O_{ID}}}$, the set of all candidate points with at least two projections in $\mathrm{O_{ID}}$ and one in $\mathrm{E_{45}}$, the set $\mathcal{C}_{\mathrm{E_{45}\times O_{RED}\times O_{RED}}}$, the set of all candidate points with at least two projections in $\mathrm{O_{RED}}$ and one in $\mathrm{E_{45}}$, $\mathcal{C}_{\mathrm{E_{45}\times E_{45}\times O_{ID}}}$, the set of all candidate points with at least two projections in $\mathrm{E_{45}}$ and one in $\mathrm{O_{ID}}$, and $\mathcal{C}_{\mathrm{E_{45}\times E_{45}\times O_{RED}}}$, the set of all candidate points with at least two projections in $\mathrm{E_{45}}$ and one in $\mathrm{O_{RED}}$. The set $\mathcal{C}_{\mathrm{E_{45}\times O_{RED}\times O_{RED}}}$ turns out to be empty.

To construct the set $\mathcal{C}_{\mathrm{E_{45}\times O_{ID}\times O_{ID}}}$, we proceed as follows: firstly we construct the set $\tilde{\mathcal{C}}_{\mathrm{E_{45}\times O_{ID}\times O_{ID}}}$, where one over the three projections $\hat{q}, \check{q}, \bar{q}$ is in the set $\mathrm{E_{45}}$ and two of the remaining projections are in the set $\mathrm{O_{ID}}$, then, applying Lemma 6.1 we generate the whole set $\mathcal{C}_{\mathrm{E_{45}\times O_{ID}\times O_{ID}}}$.

The set $\tilde{\mathcal{C}}_{\mathrm{E_{45}\times O_{ID}\times O_{ID}}}$ is the union of the following three sets of *candidate* points p:

(A2.1) $\bar{\tilde{\mathcal{C}}}_{\mathrm{E_{45}\times O_{ID}\times O_{ID}}}$: *candidate* points p with $\hat{q}, \check{q} \in \mathrm{O_{ID}}$, $\bar{q} \in \mathrm{E_{45}}$.

(A2.2) $\check{\tilde{\mathcal{C}}}_{\mathrm{E_{45}\times O_{ID}\times O_{ID}}}$: *candidate* points p with $\hat{q}, \bar{q} \in \mathrm{O_{ID}}$, $\check{q} \in \mathrm{E_{45}}$.

(A2.3) $\hat{\tilde{\mathcal{C}}}_{\mathrm{E_{45}\times O_{ID}\times O_{ID}}}$: *candidate* points p with $\bar{q}, \check{q} \in \mathrm{O_{ID}}$, $\hat{q} \in \mathrm{E_{45}}$.

Here, we state only the algorithm that generates the subset (A2.1), the other algorithms for the subsets (A2.2) and (A2.3) can be derived in a similar way. The algorithm is based on the following result, which is an obvious consequence of Lemma 6.4:

LEMMA 6.7 If a point $p \in \mathcal{A}$, is such that $\hat{q}, \check{q} \in \mathrm{O_{ID}}$, then $\bar{q}$ must satisfy:

$$\bar{q}_2 = \hat{\epsilon}\check{\epsilon}\bar{q}_1,\ \bar{q}_{32} = \hat{\epsilon}\check{\epsilon}\bar{q}_{31}, \tag{89}$$

and p is such that:

$$\begin{gathered} p_1 = \bar{q}_1,\ p_2 = \check{\epsilon}\bar{q}_{31},\ p_3 = \hat{\epsilon}\check{\epsilon}\bar{q}_1,\ p_4 = \bar{q}_3,\ p_{21} = \check{\epsilon}\bar{q}_3,\ p_{31} = \bar{q}_{21},\ p_{32} = \hat{\epsilon}\bar{q}_3, \\ p_{41} = \bar{q}_{31},\ p_{42} = \check{\epsilon}\bar{q}_1,\ p_{43} = \hat{\epsilon}\check{\epsilon}\bar{q}_{31},\ p_{432} = \hat{\epsilon}2,\ p_{431} = \bar{q}_\infty,\ p_{421} = \check{\epsilon}2. \end{gathered} \tag{90}$$

Algorithm 3

(1) Take $\bar{q} \in \mathrm{E_{45}}$.

(2) Check if $\bar{q}$ satisfies:

$$\bar{q}_2 = \hat{\epsilon}\check{\epsilon}\bar{q}_1, \quad \text{and} \quad \bar{q}_{32} = \hat{\epsilon}\check{\epsilon}\bar{q}_{31},$$

then go to the next Step, otherwise go to Step 1.

(3) Determine the components of p involved in identities (90).

(4) Determine the values $p_{321}^{(i)}$, for $i = 1, 2$, using equation (47).

For each $i = 1, 2$:

(5) Calculate the values of $p_\infty^{(i)}$ using equation (51).

(6) Use identities given by the columns of Table 1 in order to determine the other components of $p^{(i)}$.

(7) If $p^{(i)}$ satisfies equations (52)–(61) then go to the next Step, otherwise Step 1.

(8) Save $p^{(i)}$ in the set $\bar{\tilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}}$, and go to Step 1.

When Algorithm 3 and the algorithms for subsets (A2.2) and (A2.3) end, the following set is obtained:

$$\tilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}} = \bar{\tilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}} \cup \check{\tilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}} \cup \hat{\tilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}},$$

then, by Lemma 6.1, we generate the set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}}$ as:

$$\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}} = \tilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}} \bigcup_{i=1}^{3} \pi_{(1234)}^{i}(\tilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{ID}}\times\mathrm{O}_{\mathrm{ID}}}), \tag{91}$$

where permutation $\pi_{(1234)}$ is defined in Lemma 6.1. This set contains 6,385 points and Algorithm 3 can be found in [21].

We proceed in a similar way to construct the set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$ of all *candidate* points $p \in \hat{\mathcal{M}}_{\mathcal{G}_2}$ such that one over the four projections $\hat{q}, \check{q}, \bar{q}, \tilde{q}$ is in the set $\mathrm{O}_{\mathrm{RED}}$ and two of the remaining projections are in the set E_{45}. We give here only the algorithm such that $\hat{q}, \check{q} \in \mathrm{E}_{45}$, $\bar{q} \in \mathrm{O}_{\mathrm{RED}}$—all other cases can be derived in similar way.

Algorithm 4

(1) Consider $\hat{q}, \check{q} \in \mathrm{E}_{45} \times \mathrm{E}_{45}$.

(2) Check if $\hat{q}, \check{q}$ satisfy relations given by the columns of the first and third rows of Table 1 then go to the next step, otherwise go to Step 1.

(3) Calculate p_{31} and p_{431} using Table 1 and conditions (84).

(4) Determine the values $p_{321}^{(i)}$, for $i = 1, 2$, using equation (47).

For each $i = 1, 2$:

(5) Calculate the values of $p_\infty^{(i)}$ using equation (51).

(6) Use identities given by the columns of Table 1 in order to determine the other components of $p^{(i)}$.

(7) If $p^{(i)}$ satisfies equations (52)–(61) then go to the next step, otherwise Step 1.

(8) Save $p^{(i)}$ in the set $\bar{\tilde{\mathcal{C}}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$, and go to Step 1.

When Algorithm 4 and the analogous algorithms for $\bar{q}, \check{q} \in \mathrm{E}_{45}$, $\hat{q} \in \mathrm{O}_{\mathrm{RED}}$ and for $\bar{q}, \hat{q} \in \mathrm{E}_{45}$, $\check{q} \in \mathrm{O}_{\mathrm{RED}}$, respectively end, the set $\tilde{\mathcal{C}}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$ is obtained. Then, as before, the set $\mathcal{C}_{\mathrm{E}_{45}\times\mathrm{E}_{45}\times\mathrm{O}_{\mathrm{RED}}}$ is given by:

$$\mathcal{C}_{E_{45}\times E_{45}\times O_{RED}} = \tilde{\mathcal{C}}_{E_{45}\times E_{45}\times O_{RED}} \bigcup_{i=1}^{3} \pi^{i}_{(1234)}(\tilde{\mathcal{C}}_{E_{45}\times E_{45}\times O_{RED}}).$$

This set contains 342,368 points and Algorithm 4 can be found in [21].

We now produce the algorithm that generates the set $\mathcal{C}_{E_{45}\times E_{45}\times O_{ID}}$ of all *candidate* points $p \in \mathcal{A}$ such that one projection is in the set O_{ID} and two of the remaining three projections are in the set E_{45}. We give here only the algorithm such that $\hat{q}, \check{q} \in E_{45}$, $\bar{q} \in O_{ID}$ - all other cases can be derived in similar way. This is a simple adaptation of Algorithm 4 in which we substitute Steps (2) and (8):

Algorithm 5

(1), (2), (4), (5), (6), (7) see Algorithm 4.

(3) Calculate p_{31} and p_{431} using Table 1 and conditions (82).

(8) Save $p^{(i)}$ in the set $\tilde{\tilde{\mathcal{C}}}_{E_{45}\times E_{45}\times O_{ID}}$, and go to Step 1.

When Algorithm 5 and the analogous algorithms for $\bar{q}, \check{q} \in E_{45}$, $\hat{q} \in O_{ID}$ and for $\bar{q}, \hat{q} \in E_{45}$, $\check{q} \in O_{ID}$, respectively end, we obtain $\tilde{\mathcal{C}}_{E_{45}\times E_{45}\times O_{ID}}$, then as before:

$$\mathcal{C}_{E_{45}\times E_{45}\times O_{ID}} = \tilde{\mathcal{C}}_{E_{45}\times E_{45}\times O_{ID}} \bigcup_{i=1}^{3} \pi^{i}_{(1234)}(\tilde{\mathcal{C}}_{E_{45}\times E_{45}\times O_{ID}}).$$

This set contains 245,760 points, and Algorithm 5 can be found in [21].

Finally the set of all candidate points is:

$$\mathcal{C} = \mathcal{C}_{E_{45}\times E_{45}\times E_{45}} \cup \mathcal{C}_{E_{45}\times O_{ID}\times O_{ID}} \cup \mathcal{C}_{E_{45}\times E_{45}\times O_{RED}} \cup \mathcal{C}_{E_{45}\times E_{45}\times O_{ID}}. \tag{92}$$

This is a finite set consisting of 3,461,273 points (duplicated points are erased). We re-define this set by throwing away all points that produce $M_\infty = \pm\mathbb{I}$, so that the resulting set $\mathcal{C}$ has 3,287,140 elements.

7. Extracting finite orbits

Now, we need to determine which points in $\mathcal{C}$ lead to a finite orbit of the P_4-action. The following result is fundamental to achieve this:

LEMMA 7.1 Let $p \in \mathcal{C}$ a *candidate* point, then its orbit is finite if and only if $\beta(p) \in \mathcal{C}$ for every braid $\beta \in P_4$.

Proof. Suppose $\beta(p) \in \mathcal{C}$ for every $\beta \in P_4$, then the orbit is finite since $\mathcal{C}$ is finite too. Vice versa, suppose p has a finite P_4-orbit, then for every β, $\beta(p)$ must have a finite orbit. Hence, $\beta(p)$ must be an element of $\mathcal{C}$. □

Therefore, to select the finite orbits is equivalent to find the subset $\mathcal{C}_0 \subset \mathcal{C}$ such that:

$$\mathcal{C}_0 = \{p \in \mathcal{C} \mid \beta(p) \in \mathcal{C},\ \beta \in P_4\}. \tag{93}$$

To construct the set $\mathcal{C}_0$, we use the following:

Algorithm 6

(1) Consider $p \in \mathcal{C}$.

(2) Apply to it all the generators (65) of P_4.

(3) If there exists an $i = 1, \dots, 6$ such that $p^{(i)} \notin \mathcal{C}$ then delete p from the set $\mathcal{C}$ and go to Step 1, otherwise save p in $\mathcal{C}_0$ and go to Step 1.

This algorithm is designed in such a way that points already considered are not considered again, or in other words, we order points in $\mathcal{C}$ and proceed in order. This algorithm ends when in the set $\mathcal{C}$ there are no more elements to delete. The final set $\mathcal{C}_0$ contains 1,270,050 points and Algorithm 6 can be found in [21].

Note that $\mathcal{C}_0$ contains only elements that generate finite orbits under the P_4-action. In fact, assume by contradiction that $p \in \mathcal{C}_0$ has an infinite orbit. Then there exists a braid β such that $\beta(p) \notin \mathcal{C}$. Now every braid $\beta \in P_4$ can be thought as an ordered combination of generators β_{ij}:

$$\beta = \underbrace{\beta_{i'j'} \dots \beta_{ij}}_{n}, \tag{94}$$

where n indicates the length of the word. Let us introduce the following notation:

$$p^{(0)} = p, \quad p^{(1)} = \beta_{ij}(p^{(0)}), \dots, p^{(n)} = \beta(p) = \beta_{i'j'}(p^{(n-1)}) = \underbrace{\beta_{i'j'} \dots \beta_{ij}}_{n}(p^{(0)}). \tag{95}$$

Since we supposed $p^{(n)} \notin \mathcal{C}$, Algorithm 6 deletes $p^{(n-1)}$ from the set $\mathcal{C}$. In the next iteration it deletes $p^{(n-2)}$ and so on, till when $p^{(0)} = p$ is deleted from $\mathcal{C}$, and therefore p is not in $\mathcal{C}_0$, contradicting our hypothesis.

8. Extracting non-equivalent orbits

In this section, we quotient the set C_0 of all points p giving rise to a finite orbit with respect to the action of the pure braid group, so that we select only one representative point for every finite orbit, and by the action of the symmetry group G of $\hat{\mathcal{M}}_{\mathcal{G}_2}$ described in the next theorem proved in the Appendix.

THEOREM 8.1 The group

$$G := \langle P_{13}, P_{23}, P_{34}, P_{1\infty}, \mathrm{sign}_1, \dots, \mathrm{sign}_4, \pi_{(12)(34)}, \pi_{(1234)} \rangle \tag{96}$$

where

$$P_{13}(p) = \sigma_2\sigma_1^{-1}\sigma_2^{-1}(p), \tag{97}$$

$$P_{23}(p) = \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^{-1}(p), \tag{98}$$

$$\begin{aligned}
P_{34}(p) &= \sigma_3\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_2\sigma_1^{-1}(p), && (99)\\
P_{1\infty}(p) &= (-p_\infty, p_2, p_3, p_4, -p_1, p_2p_\infty - p_{432}p_{21} + p_{43}p_1 - p_{431},\\
&\quad p_3p_\infty - p_{43}p_{321} + p_4p_{21} - p_{421}, p_{32}, p_{321}, p_{42}, p_{43},\\
&\quad p_{32}p_\infty - p_{432}p_{321} + p_4p_1 - p_{41}, p_{432}, p_{21},\\
&\quad p_2p_{321} - p_{32}p_{21} + p_3p_1 - p_{31}), && (100)\\
\mathrm{sign}_1(\mathrm{p}) &= (-p_1, p_2, p_3, p_4, -p_\infty, -p_{21}, -p_{31}, p_{32}, -p_{41}, p_{42}, p_{43}, -p_{321}, p_{432},\\
&\qquad - p_{431}, -p_{421}), && (101)\\
\mathrm{sign}_2(\mathrm{p}) &= (p_1, -p_2, p_3, p_4, -p_\infty, -p_{21}, p_{31}, -p_{32}, p_{41}, -p_{42}, p_{43}, -p_{321}, -p_{432},\\
&\qquad p_{431}, -p_{421}), && (102)\\
\mathrm{sign}_3(\mathrm{p}) &= (p_1, p_2, -p_3, p_4, -p_\infty, p_{21}, -p_{31}, -p_{32}, p_{41}, p_{42}, -p_{43}, -p_{321}, -p_{432},\\
&\qquad - p_{431}, p_{421}), && (103)\\
\mathrm{sign}_4(\mathrm{p}) &= (p_1, p_2, p_3, -p_4, -p_\infty, p_{21}, p_{31}, p_{32}, -p_{41}, -p_{42}, -p_{43}, p_{321}, -p_{432}, -\\
&\qquad - p_{431}, p_{421}). && (104)\\
\pi_{(12)(34)}(p) &= (p_2, p_1, p_4, p_3, p_\infty, p_{21}, p_{42}, p_{41}, p_{32}, p_{31}, p_{43}, p_{421}, p_{431}, p_{432}, p_{321}), && (105)\\
\pi_{(1234)}(p) &= (p_4, p_1, p_2, p_3, p_\infty, p_{41}, p_{42}, p_{21}, p_{43}, p_{31}, p_{32}, p_{421}, p_{321}, p_{432}, p_{431}). && (106)
\end{aligned}$$

is a group of symmetries for $\hat{\mathcal{M}}_{\mathcal{G}_2}$.

8.1 *Points belonging to the same orbit*

In this subsection, we explain how to take the following quotient:

$$\mathcal{C}_1 := \mathcal{C}_0/P_4.$$

Algorithm 7 For every $p \in \mathcal{C}_0$:

(1) Calculate $\mathrm{O}_{P_4}(p)$.
(2) Save $p \in \mathcal{C}_1$ and delete $\mathrm{O}_{P_4}(p)$ from $\mathcal{C}_0$.

Since the set $\mathcal{C}_0$ is finite, the algorithm ends. This algorithm produces the set $\mathcal{C}_1$, that contains 17,946 finite orbits of the P_4-action.

8.2 *Quotient under the symmetry group G*

Our aim is to quotient $\mathcal{C}_1$ by the action of the symmetry group G, see Appendix. Note that G is an infinite group; however, it acts as a finite group on $(p_1, p_2, p_3, p_4, p_\infty)$ and preserves the length of a P_4-orbit. Thanks to this fact we are able to set up a finite factorization algorithm.

We proceed as follows: we factorize by the action of the finite subgroup:

$$\langle \mathrm{sign}_1, \ldots, \mathrm{sign}_4, \pi_{(12)(34)}, \pi_{(1234)} \rangle \subset \mathrm{G}, \tag{107}$$

to obtain the set $\mathcal{C}_2'$.

Algorithm 8

(1) Consider $p \in \mathcal{C}_1$.
(2) Remove from $\mathcal{C}_1$ the set $\mathrm{O}_{P_4}(p)$ and save p in the set $\mathcal{C}_2'$.
(3) Apply to p all transformations in $\langle \mathrm{sign}_1, \ldots, \mathrm{sign}_4 \rangle$ and save the result in the set A_0.
For every $p' \in A_0$:
(4) Apply to p' all transformations in $\langle \pi_{(12)(34)}, \pi_{(1234)} \rangle$ and save the result in the set A_1.
For every $p'' \in A_1$:
(5) If p'' is in $\mathcal{C}_1$, then $\mathrm{O}_{P_4}(p)$ and $\mathrm{O}_{P_4}(p'')$ are equivalent. Remove $\mathrm{O}_{P_4}(p'')$ from $\mathcal{C}_1$. If p'' is not in $\mathcal{C}_1$, apply again the current Step to the next p'' in A_1.
(6) If all possible choices of p'' in A_1 are exhausted go to Step 1.

This algorithm ends when all choices of points p in the finite set $\mathcal{C}_1$ are exhausted. The set $\mathcal{C}_2'$, created in this way, contains 122 points, therefore this factorization reduces dramatically the number of orbits to be processed from 17,946 to 122.

Next, we subdivide the set $\mathcal{C}_2'$ into subsets that contain orbits of the same length and have the same $(p_1, p_2, p_3, p_4, p_\infty)$ modulo change of signs or permutations. This is useful because, since the action of G preserves the length of an orbit and that the $(p_1, p_2, p_3, p_4, p_\infty)$ remain invariant up to permutations and sign flips under the G action, only points within the same subset can be related by a transformation in G.

Algorithm 9

(1) Consider $p \in \mathcal{C}_2'$, with $|\mathrm{O}_{P_4}(p)| = N$, $N \in \mathbb{N}$.
(2) Save p in a set A_N.
(3) Remove p from $\mathcal{C}_2'$.
For every $p' \in \mathcal{C}_2'$:
(4) If p' is such that:

- $|\mathcal{O}_{P_4}(p')| = N$.
- $(p_1, p_2, p_3, p_4, p_\infty)$ and $(p_1', p_2', p_3', p_4', p_\infty')$ differ by change of signs or permutations.

Save p' in A_N and remove p' from $\mathcal{C}_2'$, otherwise apply again this Step to another $p' \in \mathcal{C}_2'$.

Since the set $\mathcal{C}_2'$ is finite, this algorithm ends when there are no more elements in $\mathcal{C}_2'$. This algorithm generates a finite list of 54 subsets A_N, where N is such that for every $p \in A_N$ we have $|\mathcal{O}_{P_4}(p)| = N$ and $(p_1, p_2, p_3, p_4, p_\infty)$ and $(p_1', p_2', p_3', p_4', p_\infty')$ differ by change of signs or permutations.

Then, within each subset A_N, for all the elements in the subset, we apply a transformation in G in such a way that every element p in the subset will have the same ordered $(p_1, p_2, p_3, p_4, p_\infty)$ and check if there is a P_4 transformation linking the points in the same A_N. This is done in the following:

Algorithm 10 For every subset A_N:

(1) Choose a point $p \in A_N$ and save it in the set $\mathcal{C}_2$.
(2) Remove p from A_N.
(3) Act with G on each element in the set A_N, producing a new set A'_N in such a way that every element p' in A'_N will have:

$$(p'_1, p'_2, p'_3, p'_4, p'_\infty) = (p_1, p_2, p_3, p_4, p_\infty).$$

For every $p' \in A'_N$:

(4) Generate the orbit of p' under the action of $\langle P_{13}, P_{23}, P_{34}\rangle$; if p is in this orbit, then $\mathrm{O}_{P_4}(p)$ and $\mathrm{O}_{P_4}(p')$ are equivalent, otherwise save p' in $\mathcal{C}_2$ and apply again this Step to another $p' \in A'_N$.
(5) When all choices of $p' \in A'_N$ are exhausted, go to Step 1.

Since the number of subsets A_N is 54, and each subset has a finite number of elements, this algorithm ends when there are no more subsets A_N to process. It turns out that for each set A_N there is only one class of equivalence under the action of the group G. This completes our classification of all finite orbits. We summarize the content of the set $\mathcal{C}_2$, in Table 2.

8.3 *Finite monodromy groups*

Here we show that solution 25 in Table 2 corresponds to an infinite monodromy group and there is no symmetry mapping it to an orbit with finite monodromy group. We also calculate the order of the monodromy groups generated by all other orbits. The results about monodromy group orders are resumed in Table 3.

To prove these statements, we calculate the monodromy matrices by using the parameterization formulae in Section 2 with the corresponding values of p_i, $i = 1, \dots, 4, \infty$ and p_{ij}, $i, j = 1, \dots, 4$ from Table 2. Since none of our groups are cyclic and they are subgroups of $SL_2(\mathbb{C})$, by Klein classification result only binary polyhedral and binary dihedral group are allowed. The order of the binary polyhedral groups is bounded by 120, therefore we wrote a C^+ program that generates group elements up to 121 distinct elements. In this way we characterize the orders of the monodromy groups associated to all orbits except the 25th one. Since for orbit 25 all generating matrices $M_1, \dots, M_4$ are not diagonalizable and therefore not idempotent, the group is automatically infinite. This property is clearly preserved by the action of the symmetry group G defined in Theorem A.5. For completeness we list here the monodromy matrices associated to the 25th orbit (in the basis of M_3M_2 diagonal):

$$M_1 = \begin{pmatrix} 1 - i\sqrt{\frac{2}{5+\sqrt{5}}} & \frac{1}{10}\left(5 - \sqrt{5}\right) \\ 1 & 1 + i\sqrt{\frac{2}{5+\sqrt{5}}} \end{pmatrix}$$

TABLE 2.

#	sz.	p_1	p_2	p_3	p_4	p_∞	p_{21}	p_{31}	p_{32}	p_{41}	p_{42}	p_{43}
1	36	1	0	$\sqrt{2}$	0	0	-1	0	$-\sqrt{2}$	0	$\sqrt{2}$	1
2	36	1	0	1	0	0	1	0	1	1	0	1
3	40	-1	1	$\sqrt{2}$	1	$-\sqrt{2}$	-1	$-\sqrt{2}$	0	1	1	$\sqrt{2}$
4	40	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0	1	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0
5	40	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	1	1	-1
6	45	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	-1	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
7	45	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	1	2	1
8	48	$\sqrt{2}$	0	0	0	$\sqrt{2}$	$\sqrt{2}$	-1	$\sqrt{2}$	0	0	1
9	72	0	0	-1	0	0	$\sqrt{2}$	$-\sqrt{2}$	1	-1	0	0
10	72	$-\sqrt{2}$	0	0	-1	$-\sqrt{2}$	0	-1	-1	$\sqrt{2}$	$-\sqrt{2}$	0
11	81	$\frac{-1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	1	-1	$\frac{1-\sqrt{5}}{2}$	-1	0
12	81	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	-1	$-\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	-1	-1	1
13	96	$\sqrt{2}$	0	0	0	0	1	$-\sqrt{2}$	$\sqrt{2}$	1	-2	$\sqrt{2}$
14	96	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
15	96	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	2	1	-1
16	96	0	0	1	0	-1	2	0	0	$-\sqrt{2}$	$\sqrt{2}$	-1
17	105	$-\frac{1+\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	-1	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	-1
18	105	1	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	-1	0	0
19	108	$\frac{1+\sqrt{5}}{2}$	1	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	-2	0	2
20	108	$\frac{-1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	1	$\frac{1-\sqrt{5}}{2}$	-1	$\frac{1-\sqrt{5}}{2}$	-2	$\frac{-1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0
21	120	1	0	-1	0	-1	0	-1	$\sqrt{2}$	$-\sqrt{2}$	-1	0
22	144	$\frac{-1+\sqrt{5}}{2}$	1	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1	$\frac{-1+\sqrt{5}}{2}$	-1	$\frac{1-\sqrt{5}}{2}$	-1	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$
23	144	-1	$-\frac{1+\sqrt{5}}{2}$	-1	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	1	1	$\frac{-1+\sqrt{5}}{2}$	1
24	144	0	1	0	0	$\sqrt{2}$	0	2	0	1	$-\sqrt{2}$	-1
25	192	2	2	-2	-2	-2	$\frac{1-\sqrt{5}}{2}$	-1	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	1
26	192	0	0	0	0	0	$-\sqrt{2}$	-2	$-\sqrt{2}$	-1	$-\sqrt{2}$	-1
27	200	0	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	1	1	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
28	200	$\frac{1+\sqrt{5}}{2}$	0	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0	$\frac{1+\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$

29	205	-1	1	1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0	-1	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0
30	216	-1	0	0	0	0	0	$\sqrt{2}$	1	$-\sqrt{2}$	0	1
31	220	-1	1	$\frac{-1+\sqrt{5}}{2}$	-1	$\frac{-1+\sqrt{5}}{2}$	-1	$-\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0	$-\frac{1+\sqrt{5}}{2}$
32	220	$-\frac{1+\sqrt{5}}{2}$	-1	-1	$-\frac{1+\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
33	240	1	-1	$\frac{1-\sqrt{5}}{2}$	0	$\frac{-1+\sqrt{5}}{2}$	0	$-\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	1	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
34	240	$\frac{1-\sqrt{5}}{2}$	0	$\frac{1-\sqrt{5}}{2}$	0	$\frac{1-\sqrt{5}}{2}$	0	1	0	$-\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0
35	240	1	-1	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0	-1	$\frac{1-\sqrt{5}}{2}$	0	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$
36	240	0	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	0	$-\frac{1+\sqrt{5}}{2}$	1	0	-1	$\frac{1+\sqrt{5}}{2}$	-1	0
37	300	$\frac{1+\sqrt{5}}{2}$	1	1	1	1	1	0	1	1	$\frac{1+\sqrt{5}}{2}$	1
38	300	1	$\frac{-1+\sqrt{5}}{2}$	1	1	-1	-1	$\frac{1-\sqrt{5}}{2}$	-1	0	0	$\frac{1-\sqrt{5}}{2}$
39	360	0	$\frac{-1+\sqrt{5}}{2}$	0	-1	$\frac{-1+\sqrt{5}}{2}$	-1	-1	-1	1	1	0
40	360	$\frac{1-\sqrt{5}}{2}$	0	0	$-\frac{1+\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	2	1	0	0
41	360	1	0	$-\frac{1+\sqrt{5}}{2}$	0	$-\frac{1+\sqrt{5}}{2}$	-1	0	0	0	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
42	480	1	-1	1	1	-1	-1	0	0	$\frac{1-\sqrt{5}}{2}$	-1	1
43	480	0	0	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	0	1	0	1	-1
44	480	0	0	$\frac{1+\sqrt{5}}{2}$	0	$\frac{1+\sqrt{5}}{2}$	0	1	0	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
45	580	$\frac{1-\sqrt{5}}{2}$	0	0	0	$\frac{1+\sqrt{5}}{2}$	0	$\frac{1+\sqrt{5}}{2}$	-1	0	-2	-1
46	600	0	-1	0	$\frac{1-\sqrt{5}}{2}$	-1	0	$\frac{1-\sqrt{5}}{2}$	1	$-\frac{1+\sqrt{5}}{2}$	0	-1
47	600	$-\frac{1+\sqrt{5}}{2}$	1	0	0	1	-1	$\frac{-1+\sqrt{5}}{2}$	-1	$\frac{-1+\sqrt{5}}{2}$	-1	-2
48	900	0	0	0	-1	$\frac{-1+\sqrt{5}}{2}$	0	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	1
49	900	0	0	0	-1	$-\frac{1+\sqrt{5}}{2}$	0	1	$\frac{-1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
50	1200	0	0	$\frac{1-\sqrt{5}}{2}$	0	0	$\frac{-1+\sqrt{5}}{2}$	1	1	-1	-1	1
51	1200	0	$\frac{1+\sqrt{5}}{2}$	0	0	0	$\frac{1-\sqrt{5}}{2}$	1	$\frac{-1+\sqrt{5}}{2}$	1	0	1
52	2160	1	0	0	0	-1	0	0	2	-1	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
53	2160	0	0	0	-1	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	-2	0	1	1
54	3072	0	0	0	0	0	$-\frac{1+\sqrt{5}}{2}$	0	-1	$-\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	1

TABLE 3.

Orbits	Group order
1,3, 8, 9, 10, 13, 16, 21, 24, 26, 30	24
2	12
25	Infinite
All others	60

$$M_2 = \begin{pmatrix} 1 - \frac{i\left(3+\sqrt{5}\right)}{\sqrt{2\left(5+\sqrt{5}\right)}} & \frac{2+2\sqrt{5}-i\sqrt{2\left(5+\sqrt{5}\right)}+i\sqrt{10\left(5+\sqrt{5}\right)}}{4\left(\sqrt{5}-5\right)} \\ \frac{i\left(4i\left(2+\sqrt{5}\right)+\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{10\left(5+\sqrt{5}\right)}\right)}{8} & 1 + \frac{i\left(3+\sqrt{5}\right)}{\sqrt{2\left(5+\sqrt{5}\right)}} \end{pmatrix}$$

$$M_3 = \begin{pmatrix} \frac{i\left(3+\sqrt{5}+i\sqrt{2\left(5+\sqrt{5}\right)}\right)}{\sqrt{2\left(5+\sqrt{5}\right)}} & \frac{-1+\sqrt{5}-i\sqrt{2\left(5+\sqrt{5}\right)}}{2\left(\sqrt{5}-5\right)} \\ -\frac{i\left(-2i\left(1+\sqrt{5}\right)+3\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{10\left(5+\sqrt{5}\right)}\right)}{8} & -1 - \frac{i\left(3+\sqrt{5}\right)}{\sqrt{2\left(5+\sqrt{5}\right)}} \end{pmatrix}$$

$$M_4 = \begin{pmatrix} -1 - \frac{i\left(\sqrt{5}-3\right)}{\sqrt{2\left(5+\sqrt{5}\right)}} & \frac{-1+\sqrt{5}+2i\sqrt{2\left(5+\sqrt{5}\right)}-i\sqrt{10\left(5+\sqrt{5}\right)}}{2\left(5+\sqrt{5}\right)} \\ \frac{\left(-1+\sqrt{5}-2i\sqrt{2\left(5+\sqrt{5}\right)}+i\sqrt{10\left(5+\sqrt{5}\right)}\right)}{4} & \frac{i\left(-3+\sqrt{5}+i\sqrt{2\left(5+\sqrt{5}\right)}\right)}{\sqrt{2\left(5+\sqrt{5}\right)}} \end{pmatrix}$$

9. Outlook

From the parameterization results of Section 2, it is clear that we could reconstruct all monodromy matrices (up to global conjugation) by matching only two points, and therefore completely reconstruct the candidate point in that way. This means that we could in fact classify all finite orbits up to two projections to Picard or Dubrovin–Kitaev orbits. This computation is theoretically possible but extremely technical and would require covering so many sub-cases that we felt it is best to postpone it to further publications.

Another direction of research is to classify all finite orbits of the pure braid group P_n on the moduli space of $SL_2(\mathbb{C})$ monodromy representations over the $n+1$-punctured Riemann sphere for $n > 4$, or in other words all algebraic solutions of the Garnier system $\mathcal{G}_{n-2}$. We expect the matching procedure to work in this case too: now there will be $\begin{pmatrix} n \\ 3 \end{pmatrix}$ restrictions to PVI, so many more necessary conditions to be satisfied in order to produce a candidate point. We have seen that for $n = 4$, we start from an extended list of 86,768 to produce only 54 orbits. As n increases, the starting list is the same, but the number of necessary conditions increases—therefore we expect that there will be less and less exceptional orbits as n increases.

Acknowledgments

The authors are grateful to P. A. Clarkson, G. Cousin, D. Guzzetti, O. Lisovyy, A. Nakamura and V. Rubtsov for helpful conversations.

Funding

EPSRC DTA allocation to the Mathematical Sciences Department at Loughborough University to P.C.

A. The symmetry group G of $\hat{\mathcal{M}}_{\mathcal{G}_2}$

The general theory of the bi-rational transformations of the Garnier systems was developed in [28], where Kimura proved that the symmetric group S_5 acts as a group of bi-rational transformations on the Garnier system (see also [17, 29, 30]). These bi-rational transformations map algebraic solutions to algebraic solutions with the same number of branches. This means that the corresponding action on the co-adjoint coordinates maps finite orbits to finite orbits with the same number of points. To compute this action, we use the following result proved in [10]:

LEMMA A.1 The symmetric group S_5 giving rise to Kimura's bi-rational transformations of the Garnier system acts on $\mathcal{M}_{\mathcal{G}_2}$ as the group $\langle P_{13}, P_{23}, P_{34}, P_{1\infty}\rangle$ where the transformations P_{13},P_{23},P_{34} act on the monodromy matrices as follows:

$$\begin{aligned}
P_{13} &: (M_1, M_2, M_3, M_4) \mapsto (M_1^{-1}M_2^{-1}M_3M_2M_1, M_2, M_2M_1M_2^{-1}, M_4),\\
P_{23} &: (M_1, M_2, M_3, M_4) \mapsto ((M_2^{-1}M_3M_2M_1)^{-1}M_1M_2^{-1}M_3M_2M_1,\\
&\qquad (M_2M_1)^{-1}M_3M_2M_1, M_2, M_4),\\
P_{34} &: (M_1, M_2, M_3, M_4) \mapsto (M_\infty M_3M_2M_1(M_\infty M_3M_2)^{-1}, M_2,\\
&\qquad (M_3M_2M_1M_2^{-1})^{-1}M_4(M_3M_2M_1M_2^{-1}), M_3),
\end{aligned} \tag{A.1}$$

while transformation $P_{1\infty}$ acts on the monodromy matrices as:

$$\begin{aligned}
P_{1\infty} &: (M_1, M_2, M_3, M_4) \mapsto (-C_1M_\infty C_1^{-1}, C_1^{-1}M_2C_1, C_1^{-1}M_3C_1,\\
&\qquad C_1^{-1}M_4C_1),
\end{aligned} \tag{A.2}$$

where C_1 is the diagonalizing matrix of M_1.

COROLLARY A.2 The group $\langle P_{13}, P_{23}, P_{34}, P_{1\infty}\rangle$ acts on the co-adjoint coordinates as in (97)–(100).

Proof. This is a straightforward computation relying on the definition of the co-adjoint coordinates and the skein relation. □

We wish to extend the class of transformations satisfying this property by adding to $\langle P_{13}, P_{23}, P_{34}, P_{1\infty}\rangle$ the following set of transformations that also map finite orbits to finite orbits with the same number of points (see Theorem A.5):

(i) Sign flips, or transformations that change signs to matrices M_i for $i = 1, \ldots, 4$, corresponding to the so-called Schlesinger transformations introduced by Jimbo–Miwa in [31]:

$$\begin{aligned}
\mathrm{sign}_{(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)} &: (M_1, M_2, M_3, M_4, M_\infty) \mapsto (\epsilon_1M_1, \epsilon_2M_2, \epsilon_3M_3, \epsilon_4M_4,\\
&\qquad \epsilon_1\epsilon_2\epsilon_3\epsilon_4(M_4M_3M_2M_1)^{-1}),
\end{aligned} \tag{A.3}$$

where $\epsilon_i = \pm 1$ for $i = 1, \ldots, 4$.

(ii) Permutations of the matrices M_i for $i = 1, \dots, 4$ generated by:

$$\pi_{(12)(34)} : (M_1, M_2, M_3, M_4, M_\infty) \mapsto (M_2^{-1}, M_1^{-1}, M_4^{-1}, M_3^{-1}, M_2M_1M_4M_3), \tag{A.4}$$

$$\pi_{(1234)} : (M_1, M_2, M_3, M_4, M_\infty) \mapsto (M_4, M_1, M_2, M_3, (M_3M_2M_1M_4)^{-1}). \tag{A.5}$$

The following two results give the action of the sign flips and permutations on the co-adjoint coordinates and can be proved by straightforward computations:

PROPOSITION A.3 The sign flips are invertible maps generated by the four basic elements:

$$\begin{aligned} \mathrm{sign}_1 &:= \mathrm{sign}_{(-1,1,1,1)}, \quad \mathrm{sign}_2 := \mathrm{sign}_{(1,-1,1,1)}, \\ \mathrm{sign}_3 &:= \mathrm{sign}_{(1,1,-1,1)} \quad \mathrm{sign}_4 := \mathrm{sign}_{(1,1,1,-1)} \end{aligned} \tag{A.6}$$

that act as follow on the co-adjoint coordinates (9) as in (101)–(104).

PROPOSITION A.4 The generators $\pi_{(12)(34)}$ and $\pi_{(1234)}$ act on the co-adjoint coordinates (9) as in (105) and (106).

Finally, we characterize the group G of symmetries of $\hat{\mathcal{M}}_{\mathcal{G}_2}$:

DEFINITION A.1 A *symmetry* for $\hat{\mathcal{M}}_{\mathcal{G}_2}$ is an invertible map $\Phi : \hat{\mathcal{M}}_{\mathcal{G}_2} \mapsto \hat{\mathcal{M}}_{\mathcal{G}_2}$ such that given an element $p \in \hat{\mathcal{M}}_{\mathcal{G}_2}$ and its orbit $\mathcal{O}(p)$, the following is true:

$$|\mathcal{O}(\Phi(p))| = |\mathcal{O}(p)|. \tag{A.7}$$

THEOREM A.5 The group

$$G := \langle P_{13}, P_{23}, P_{34}, P_{1\infty}, \mathrm{sign}_1, \dots, \mathrm{sign}_4, \pi_{(12)(34)}\pi_{(1234)} \rangle \tag{A.8}$$

is a group of symmetries for $\hat{\mathcal{M}}_{\mathcal{G}_2}$.

Proof. The statement is true for the subgroup $\langle P_{13}, P_{23}, P_{34}, P_{1\infty} \rangle$ by construction. We now prove that each generator Φ in $\langle \mathrm{sign}_1, \dots, \mathrm{sign}_4, \pi_{(12)(34)}\pi_{(1234)} \rangle$ satisfies (A.7). It is straightforward to prove the following relations:

$$\begin{aligned} &\sigma_1 \mathrm{sign}_2 = \mathrm{sign}_1 \sigma_1, \quad \sigma_1 \mathrm{sign}_3 = \mathrm{sign}_3 \sigma_1, \quad \sigma_1 \mathrm{sign}_4 = \mathrm{sign}_4 \sigma_1, \\ &\sigma_2 \mathrm{sign}_1 = \mathrm{sign}_1 \sigma_2, \quad \sigma_2 \mathrm{sign}_2 = \mathrm{sign}_3 \sigma_2, \quad \sigma_2 \mathrm{sign}_3 = \mathrm{sign}_2 \sigma_2, \\ &\sigma_2 \mathrm{sign}_4 = \mathrm{sign}_4 \sigma_2, \quad \sigma_3 \mathrm{sign}_1 = \mathrm{sign}_1 \sigma_3, \quad \sigma_3 \mathrm{sign}_2 = \mathrm{sign}_2 \sigma_3, \\ &\sigma_3 \mathrm{sign}_3 = \mathrm{sign}_4 \sigma_3, \quad \sigma_3 \mathrm{sign}_4 = \mathrm{sign}_3 \sigma_3. \end{aligned}$$

so that all sign flips are indeed symmetries. Regarding the permutations, it is straightforward to prove the following relations:

$$\begin{aligned}
&\sigma_2\pi_{(1234)} = \pi_{(1234)}\sigma_1, \quad \sigma_1\pi_{(12)(34)} = \pi_{(12)(34)}\sigma_1^{-1},\\
&\sigma_2\pi_{(12)(34)} = \pi_{(12)(34)}(1234)^3\sigma_2\sigma_3, \quad \sigma_3\pi_{(12)(34)} = \pi_{(12)(34)}\sigma_3^{-1},\\
&\sigma_1\pi_{(1234)} = \pi_{(1234)}\pi_{(1234)}\sigma_2^{-1}\sigma_1^{-1}, \quad \sigma_2\pi_{(1234)} = \pi_{(1234)}\sigma_1,\\
&\sigma_3\pi_{(1234)} = \pi_{(1234)}\sigma_2.
\end{aligned}$$

This conclude the proof. □

REFERENCES

1. BOLIBRUCH, A. (1997) On isomonodromic deformations of Fuchsian systems. *J. Dynam. Control Syst.,* **3**, 589–604.
2. SCHLESINGER, L. (1912) Uber eine Klasse von Differentsial System Beliebliger Ordnung mit Festen Kritischer Punkten. *J. fur Math.,* **141**, 96–145.
3. GARNIER, R. (1912) Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes. *Ann. Sci. École Norm.. Sup.,* **29**, 1–126.
4. GARNIER, R. (1926) Solution du probleme de Riemann pour les systemes différentielles linéaires du second ordre. *Ann. Sci. École Norm.. Sup.,* **43**, 239–352.
5. DUBROVIN, B. & MAZZOCCO, M. (2000) Monodromy of certain Painlevé VI transcendents and reflection groups. *Invent. Math.,* **141**, 55–147.
6. COUSIN, G. (2017) Algebraic isomonodromic deformations of logarithmic connections on the Riemann sphere and finite braid group orbits on character varieties. *Math. Ann.,* **367**, 965–1005.
7. COUSIN, G. & MOUSSARD, D. (2016) Finite braid group orbits in Aff(C)-character varieties of the punctured sphere. *ArXiv1604.04234*.
8. MAZZOCCO, M. (2001) The geometry of the classical solutions of the Garnier systems. *Int. Math. Res. Notices,* **12**, 613–646.
9. LISOVYY, O. & TYKHYY, Y. (2014) Algebraic solutions of the sixth Painlevé equation. *J. Geom. Phys.,* **85**, 124–163.
10. DUBROVIN, B. & MAZZOCCO, M. (2007) Canonical structure and symmetries of the Schlesinger equations. *Commun. Math. Phys.,* **271**, 289–373.
11. OKAMOTO, K. (1987) Studies on the Painlevé equations I, sixth Painlevé equation. *Ann. Mat. Pura Appl.,* **146**, 337–381.
12. MAZZOCCO, M. (2001) Picard and Chazy solutions to the Painlevé VI equation. *Math. Ann.,* **321**, 157–195.
13. PICARD, E. (1889) Mémoire sur la théorie des fonctions algébriques de deux variables. *J. Math. Pures Appl.,* **5**, 135–320.
14. DUBROVIN, B. (1996) Geometry of 2D topological field theories. *Integrable systems and quantum groups (Montecatini Terme, 1993)* (M. Francaviglia & S. Greco eds). Lecture Notes in Mathematics, vol. 1620, Fond. CIME/CIME Found. Subseries. Berlin: Springer, pp. 120–348.
15. ANDREEV, F. & KITAEV, A. (2002) Transformations $RS_4^2(3)$ of the ranks ≤ 4 and algebraic solutions of the sixth Painlevé equation. *Commun. Math. Phys.,* **228**, 151–176.
16. DORAN, C. (2001) Algebraic and geometric isomonodromic deformations. *J. Diff. Geom.,* **59**, 33–85.
17. TSUDA, T. (2006) Toda equation and special polynomials associated with the Garnier system. *Adv. Math.,* **206**, 657–683.

18. Diarra, K. (2013) Construction et classification de certaines solutions algébriques des systèmes de Garnier. *Bull. Braz. Math. Soc. (N.S.),* **44**, 129–154.
19. Girand, A. (2016) A new two-parameter family of isomonodromic deformations over the five punctured sphere. *Bull. Soc. Math. France,* **144**, 339–368.
20. Gavrylenko, P. & Lisovyy, O. (2016) Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions. *ArXiv 1608.00958*.
21. Calligaris, P. & Mazzocco, M. Algorithms for finite orbits of the pure braid on the monodromy of the 2-variable Garnier system. Available at https://doi.org/10.17028/rd.lboro.4924181.
22. Procesi, C. (1976) The invariant theory of $n \times n$ matrices. *Adv. Math.,* **19**, 306–381.
23. Procesi, C. (2007) *Lie Groups, An Approach through Invariants and Representations, Universitext*. New York: Springer.
24. Iwasaki, K. (2003) An Area-Preserving Action of the Modular Group on Cubic Surfaces and the Painlevé VI Equation. *Commun. Math. Phys.,* **242**, 185–219.
25. Ashley, C., Burelle, J. P. & Lawton, S. (2017) Rank 1 character varieties of finitely presented groups. *arXiv1703.08241*.
26. Grayson, D. & Stillman, M. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.
27. Inaba, M., Iwasaki, K. & Saito, M. (2004) Bäcklund transformations of the sixth Painlevé equation in terms of Riemann–Hilbert correspondence. *Int. Math. Res. Not.,* **2004**, 1–30.
28. Kimura, H. (1990) Symmetries of the Garnier system and of the associated polynomial Hamiltonian system. *Proc. Jpn. Acad. Ser. A Math. Sci.,* **66**, 176–178.
29. Iwasaki, K., Kimura, H., Shimomura, S. & Yoshida, M. (1991) From Gauss to Painlevé. *Aspects of Mathematics, E16.* Braunschweig: Friedr. Vieweg & Sohn, pp. xii–347.
30. Mazzocco, M. (2004) Irregular isomonodromic deformations for Garnier systems and Okamoto's canonical transformations. *J. Lond. Math. Soc. (2),* **70**, 405–419.
31. Jimbo, M. & Miwa, T. (1981) Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II. *Phys. D,* **2**, 306–352.