Unavoidable trees in tournaments

Mycroft, Richard; Naia Dos Santos, Tássio

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Richard Mycroft[*] and Tássio Naia[†]

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Abstract

An oriented tree T on n vertices is unavoidable if every tournament on n vertices contains a copy of T. In this paper we give a sufficient condition for T to be unavoidable, and use this to prove that almost all labelled oriented trees are unavoidable, verifying a conjecture of Bender and Wormald. We additionally prove that every tournament on $n + \mathrm{o}(n)$ vertices contains a copy of every oriented tree T on n vertices with polylogarithmic maximum degree, improving a result of Kühn, Mycroft and Osthus.

1 Introduction

An oriented graph H on n vertices is *unavoidable* if every tournament on n vertices contains a copy of H; otherwise, we say that H is *avoidable*. In particular, if H contains a directed cycle then H must be avoidable, since a transitive tournament contains no directed cycles and hence no copy of H. It is therefore natural to ask which oriented trees are unavoidable. A classical result of Rédei [18] states that every directed path is unavoidable. More recently, Thomason showed that all orientations of sufficiently long cycles are unavoidable except for those which yield directed cycles [22]. In particular this implies that all orientations of sufficiently long paths are unavoidable. Havet and Thomassé [7] then gave a complete answer for paths: with three exceptions, every orientation of a path is unavoidable (the exceptions are antidirected paths of length 3, 5 and 7, which are not contained in the directed cycle of length 3, the regular 5-vertex tournament and the Paley tournament on 7 vertices respectively). Significant attention has also been focused on the unavoidability of claws (a claw is an oriented graph formed by identifying the initial vertices of a collection of vertex-disjoint directed paths). Indeed Saks and Sós [20] conjectured that every claw on n vertices with maximum degree at most $n/2$ is unavoidable. Lu [12, 13] gave a counterexample to this conjecture, but in the other direction showed that every claw with maximum degree at most $3n/8$ is unavoidable. Lu, Wang and Wong [14] then extended these results by showing that every claw with maximum degree at most $19n/50$ is unavoidable, but that there exist claws with maximum degree approaching $11n/23$ which are avoidable. Finding the supremum of all $c > 0$ for which every claw with maximum degree at most cn is unavoidable remains an open problem.

Some oriented trees are far from being unavoidable. For example, the outdirected star S on n vertices (whose edges are oriented from the central vertex to each of the $n - 1$ leaves) is not contained in a regular tournament on $2n - 3$ vertices, since each vertex of the latter has only $n - 2$ outneighbours. That is, there exist tournaments with almost twice as many vertices as S which do not contain a copy of S. On the other hand, Bender and Wormald [1] proved that almost all oriented trees are 'almost unavoidable', in the sense that they are contained in almost all tournaments on the same number of vertices.
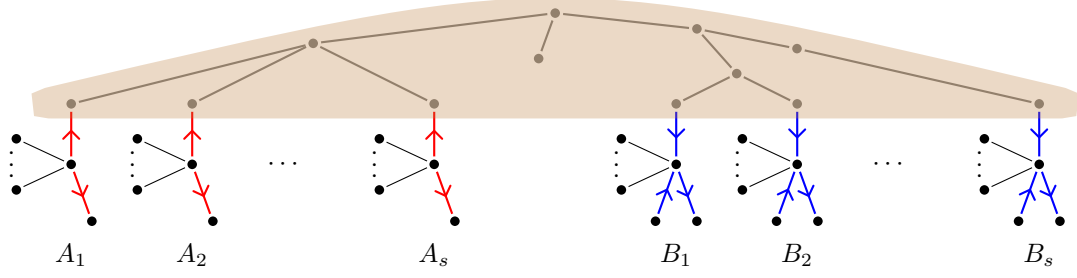
Figure 1: An α-nice tree T has $s = \lceil \alpha n \rceil$ pendant stars $A_1, \ldots A_s$ which contain an out-leaf of T such that the edge between $T - A_i$ and A_i is directed away from A_i, and also s pendant stars $B_1, \ldots, B_s$ which contain both an in-leaf of T and an out-leaf of T such that the edge between $T - B_i$ and B_i is directed towards B_i. In this illustration we only indicate the orientations of edges specified by this definition. The shaded area is the subtree $T - \bigcup_{i \in [s]} (V(A_i) \cup V(B_i))$.

Theorem 1.1 (Bender and Wormald, [1, Theorem 4.4]). *Let $\mathcal{T}_n$ denote the set of all labelled oriented trees on n vertices. Then there is a subset $\mathcal{T}'_n \subseteq \mathcal{T}_n$ of size $(1 - \mathrm{o}(1))|\mathcal{T}_n|$ such that a uniformly-random labelled tournament on n vertices asymptotically almost surely contains every tree in $\mathcal{T}'_n$.*

In particular, it follows that if T is chosen uniformly at random from the set of all labelled oriented trees on n vertices, and G is chosen uniformly at random from the set of all labelled tournaments on n vertices, then asymptotically almost surely G contains a copy of T. In the same paper Bender and Wormald conjectured that this holds for every tournament G, or, in other words, that almost all labelled oriented trees are unavoidable. The main result of this paper is to prove this conjecture.

Theorem 1.2. *Let T be chosen uniformly at random from the set of all labelled oriented trees on n vertices. Then asymptotically almost surely T is unavoidable.*

The following definitions are crucial for the proof of Theorem 1.2. We say that a subtree T' of a tree T is *pendant* if $T - T'$ is connected. Next, we define 'nice' oriented trees, whose properties are useful for embedding in tournaments, as follows (see Figure 1).

Definition 1.3. For $\alpha > 0$ we say that an oriented tree T on n vertices is *α-nice* if, writing $s := \lceil \alpha n \rceil$, T contains $2s$ vertex-disjoint pendant oriented stars $A_1, \ldots, A_s$ and $B_1, \ldots, B_s$ such that for each $i \in [k]$

   (i) A_i is a subtree of T which contains an out-leaf of T and the edge between A_i and $T - A_i$ is oriented away from A_i, and
   (ii) B_i is a subtree of T which contains both an in-leaf of T and an out-leaf of T and the edge between B_i and $T - B_i$ is oriented towards B_i.

Most of the work involved in proving Theorem 1.2 is in the proof of the following theorem, which states that large nice oriented trees with polylogarithmic maximum degree are unavoidable.

Theorem 1.4. *For every $\alpha, C > 0$ there exists n_0 such that if T is an oriented tree on $n \geq n_0$ vertices such that*
   (i) $\Delta(T) \leq (\log n)^C$ *and*
   (ii) T *is α-nice,*
then T is unavoidable.

Almost all labelled trees satisfy condition (i) of Theorem 1.4, as proved by Moon.

Theorem 1.5 ([16, Corollaries 1 and 2])**.** *For every $\varepsilon > 0$, if T is chosen uniformly at random from the set of all labelled trees on n vertices, then asymptotically almost surely*

$$(1 - \varepsilon)\frac{\log n}{\log \log n} \leq \Delta(T) \leq (1 + \varepsilon)\frac{\log n}{\log \log n}.$$

Since a uniformly-random orientation of a uniformly-random labelled tree yields a uniformly-random labelled oriented tree, Theorem 1.5 remains valid if we replace 'labelled tree' by 'labelled oriented tree'. We prove that almost all labelled oriented trees satisfy condition (ii) of Theorem 1.4.

Theorem 1.6. *Let T be chosen uniformly at random from the set of all labelled oriented trees on n vertices. Then asymptotically almost surely T is $\frac{1}{250}$-nice.*

Combining Theorems 1.4, 1.5 and 1.6 (with $C = \varepsilon = 1$ and $\alpha = \frac{1}{250}$) immediately proves Theorem 1.2.

Another natural question is to find, for a given oriented tree T, the smallest integer $g(T)$ such that every tournament on $g(T)$ vertices contains a copy of T. In particular, T is unavoidable if and only if $g(T) = |T|$. Sumner conjectured that for every oriented tree T on n vertices we have $g(T) \leq 2n - 2$, and the example of an outdirected star described above demonstrates that this bound would be best possible. Kühn, Mycroft and Osthus [10, 11] used a randomised embedding algorithm to prove that Sumner's conjecture holds for every sufficiently large n; previous upper bounds on $g(T)$ had been established by Chung [3], Wormald [23], Häggkvist and Thomason [4], Havet [5], Havet and Thomassé [6] and El Sahili [19]. In particular, El Sahili proved that $g(T) \leq 3n - 3$ for every oriented tree T on n vertices, and this remains the best known upper bound on $g(T)$ for small n. Kühn, Mycroft and Osthus [11] also gave a stronger bound for large oriented trees of bounded maximum degree, proving that for every $\alpha, \Delta > 0$, if n is sufficiently large then every oriented tree T on n vertices with $\Delta(T) \leq \Delta$ has $g(T) \leq (1 + \alpha)n$. In other words, bounded degree oriented trees are close to being unavoidable, in that they are contained in every tournament of slightly larger order.

Our proof of Theorem 1.4 makes use of the aforementioned random embedding algorithm of Kühn, Mycroft and Osthus, using somewhat sharper estimates on certain quantities associated with the random embedding. In particular, using these stronger estimates we are able to establish the same bound on $g(T)$ for oriented trees whose maximum degree is at most polylogarithmic in n (rather than bounded by a constant as above). This is the following theorem, which we use repeatedly in the proof of Theorem 1.4, and which may be of independent interest.

Theorem 1.7. *For every $\alpha, C > 0$ there exists n_0 such that if T is an oriented tree on $n \geq n_0$ vertices with $\Delta(T) \leq (\log n)^C$ and G is a tournament on at least $(1 + \alpha)n$ vertices, then G contains a copy of T.*

Observe that, under the assumption that Theorem 1.4 holds, we can deduce Theorem 1.7 immediately by appending a linear number of pendant stars to T. However, since Theorem 1.7 plays a crucial role in the proof of Theorem 1.4, we can not use this deduction.

1.1 Proof outline for Theorem 1.4

Our proof of Theorem 1.4 uses a structural characterisation of large tournaments (Lemma 2.3) which is obtained by combining results of Kühn, Mycroft and Osthus [11]. Loosely speaking, this shows that every large tournament G has one of the following two possible structures. The first possibility is that $V(G)$ can be partitioned into two sets U and W such that almost all edges of G between U and W are directed from U to W. We refer to such a structure as an 'almost-directed pair'. The second possibility is that $V(G)$ contains disjoint subsets $V_1, \ldots, V_k$ of equal size called 'clusters' whose union includes almost all vertices of G and such that the edges of G directed from V_i to V_{i+1} (with addition taken modulo k) are 'randomlike'. We refer to this structure as a 'cycle of cluster tournaments'. Given a tournament G on n vertices and a nice oriented tree T on n vertices with polylogarithmic maximum degree we consider separately these two cases for the structure of G.

Almost-directed pairs. Suppose first that G admits an almost-directed pair (U, W). In this case we begin by identifying the set Z of 'atypical' vertices of G, namely those which lie too many edges directed 'the wrong way', that is, from W to U. Since (U, W) is an almost-directed pair Z must be small. We then choose a set S of $|Z|$ distinct vertices of T, each of which lies in an out-star of T and is adjacent to both an in-leaf and an out-leaf of T. We also choose a small set S^- of vertices of T, each of which lies in an in-star of T and is adjacent to an out-leaf of T, and a small set S^+ of vertices of T, each of which lies in an out-star of T and is adjacent to an in-leaf of T. The fact that T is nice ensures that we can choose such sets. Having done so, we form a subtree T' of T by removing one out-leaf adjacent to each vertex in S^-, one in-leaf adjacent to each vertex in S^+, and one in-leaf and one out-leaf adjacent to each vertex in S. We then embed T' in G; this can be achieved by *ad hoc* methods (Lemma 4.2) using the fact that G has slightly more vertices than T' to give us a little 'room to spare'. Moreover, we can insist that the image P^- of S^- under this embedding has $P^- \subseteq U$, and likewise that the image P^+ of S^+ has $P^+ \subseteq W$.

It then suffices to embed the removed leaves into the set $Q \subseteq V(G)$ of vertices of G not covered by the embedding of T'. To do this, we first embed the removed leaves adjacent to vertices of S so as to cover the set Z of atypical vertices of G. This is achieved as follows. Let b be an atypical vertex of G, choose a vertex $s \in S$, and let s^+ and s^- be the removed out-leaf and in-leaf (respectively) adjacent to s. Since s is a vertex of T', s has already been embedded in G, say to a vertex x. Let x^+ be an outneighbour of x in Q, and let x^- be an inneighbour of x in Q (our embedding of T' in G will ensure that such vertices exist). Since G is a tournament, we must have either an edge $b \to x$ or $x \to b$ in G. In the former case we embed s^- to b and s^+ to x^+, and in the latter case we embed s^+ to b and s^- to x^-; either way we have extended our embedding to cover the atypical vertex b.

Having dealt with all atypical vertices in this manner, we let $Q^- \subseteq U$ and $Q^+ \subseteq W$ be the sets of vertices in U and W respectively which remain uncovered. The only vertices of T not yet embedded are the removed neighbours of vertices in $S^- \cup S^+$. We now use the fact that all vertices of Q^- and Q^+ are typical to find perfect matchings in the graphs $G[P^- \to Q^+]$ and $G[Q^- \to P^+]$ (our embedding of T' in G will ensure for this that we have $|P^-| = |Q^+| = |P^+| = |Q^-|$). Recall that each $s \in S^-$ was embedded to some vertex $p \in P^-$, which is matched to some $q \in Q^+$; we embed the removed outneighbour of s to q. Likewise, each $s \in S^+$ was embedded to some vertex $p \in P^+$, which is matched to some $q \in Q^-$; we embed the removed inneighbour of s to q. This completes the embedding of T in G.

Cycles of cluster tournaments. Now suppose that G contains an almost spanning cycle of cluster tournaments with clusters $V_1, \ldots, V_k$ of equal size. Again we begin by identifying the small set B of atypical vertices, which in this case are those vertices in some cluster V_i which have atypically small inneighbourhood in V_{i-1} or atypically small outneighbourhood in V_{i+1}, as well as those vertices not contained in any cluster V_i. We also choose a small set L of vertices of T each of which is adjacent to at least one in-leaf and at least one out-leaf of T (this is possible since T is nice). Following this we split T into subtrees T_1 and T_2 which partition the edge-set of T and have precisely one vertex in common, so that T_1 and T_2 each contain many vertices of L. Next we form subtrees T_1' and T_2' of T_1 and T_2 respectively by removing one in-leaf and one out-leaf adjacent to each vertex of L. Finally, we embed T into G by the following two steps.

First, we embed T_1 in G so that all atypical vertices are covered and also so that the number of vertices of T_1 embedded in each cluster V_i is approximately equal (more specifically, with an additive error on the order of $\frac{n}{\log \log n}$). To do this, we apply a 'random embedding algorithm' of Kühn, Mycroft and Osthus [11] to embed T_1' into G so that approximately the same number of vertices of each cluster are covered and also so that roughly the same number of vertices of L are embedded to each cluster. (In fact, at this point we use slightly sharper estimates on the numbers of vertices embedded in each cluster than those given in [11]; these arise from the same proofs). Then, by a similar argument to that used for covering atypical vertices in the previous case, for each $i \in [k]$ and each vertex $x \in L$ which was embedded in the cluster V_i we may use the fact that $G[V_i]$ is a tournament to choose an atypical vertex b and an uncovered vertex $y \in V_i$

so that the removed inneighbour and outneighbour of x can be embedded to b and y. This gives the desired embedding of T_1 in G.

Secondly, to complete the embedding of T in G we embed T_2 into the uncovered vertices of G (except for the single common vertex of T_1 and T_2 which is already embedded). For this we again apply the random embedding algorithm to embed T_2' in G with approximately the same number of vertices embedded within each cluster. We then carefully embed the removed inneighbours and outneighbours of a small number of vertices of L to achieve the following property. Let $U_i \subseteq V_i$ be the set of vertices of V_i which remain uncovered, and let $P_i \subseteq V_i$ be the image of vertices of L embedded to V_i whose removed inneighbour and outneighbour have not yet been embedded. We ensure that $2|P_1| = \cdots = 2|P_k| = |U_1| = \cdots = |U_k|$. Having done so, we partition each set U_i into two equal-size parts U_i^- and U_i^+, and use the fact that all vertices which remain uncovered are typical to find perfect matchings in $G[U_{i-1}^- \to P_i]$ and $G[P_i \to U_{i+1}^+]$ for each $i \in [k]$. Then, for each vertex x in L whose removed inneighbour and outneighbour have not yet been embedded, let $p \in P_i$ be the vertex to which x was embedded, and let q^- and q^+ be the vertices to which p is matched in U_{i-1} and U_{i+1} respectively. We may then embed the removed inneighbour and outneighbour of x to q^- and q^+ respectively; doing so for every $x \in L$ completes the embedding of T in G.

1.2 Structure of this paper

This paper is organised as follows. In Section 2 we give definitions and preliminary results which we will use later on in the paper. These include structural results for tournaments and probabilistic estimates. Next, in Section 3 we consider the 'random embedding algorithm' of Kühn, Mycroft and Osthus [11] and explain how to modify the proofs of the associated results to obtain slightly sharper bounds. In particular this includes Theorem 1.7; we also use these sharper bounds when considering cycles of cluster tournaments (as described in the proof sketch above). In Section 4 we consider tournaments G whose vertex set can be partitioned into two large sets which form an almost-directed pair in G, proceeding as outlined in the proof sketch above to show that every such tournament contains a copy of every nice oriented tree of polylogarithmic maximum degree (this is Lemma 4.3, which can be interpreted as proving Theorem 1.4 for such tournaments). Then, in Section 5 we do the same for tournaments G which contain an almost-spanning cycle of cluster tournaments (Lemma 5.9), making use of the sharper estimates established in Section 3. In Section 6 we prove Theorem 1.4 by using the structural results of Section 2 to show that every tournament must have one of the two structures described above, and then applying the results of Sections 4 and 5. We also give the proof of Theorem 1.6. Finally, in Section 7 we conclude by discussing related results and possible areas for future research.

2 Notation and auxiliary results

A *directed graph*, or *digraph* for short, consists of a vertex set V and edge set E, where each edge is an ordered pair of distinct vertices. We think of the edge (u, v) as being directed from u to v, and write $x \to y$ or $y \leftarrow x$ to denote the edge (x, y). In a digraph G, the *outneighbourhood* $N_G^+(x)$ of a vertex x is the set $\{ y : x \to y \in E(G) \}$. Similarly, the *inneighbourhood* $N_G^-(x)$ of x is the set $\{ y : x \leftarrow y \in E(G) \}$. The *outdegree* and *indegree* of x in G are respectively $\deg_G^+(x) \coloneqq |N_G^+(x)|$ and $\deg_G^-(x) \coloneqq |N_G^-(x)|$, and the *semidegree* $\deg_G^0(x)$ of x is the minimum of the outdegree and indegree of x. The *minimum semidegree* of G is $\delta^0(G) \coloneqq \min_{x \in V(G)} \deg_G^0(x)$. For any subset $Y \subseteq V(G)$, we write $\deg_G^-(x, Y)$ for $|N_G^-(x) \cap Y|$, the *indegree of x in Y*; the *outdegree of x in Y*, denoted by $\deg_G^+(x, Y)$, is defined similarly. The *semidegree of x in Y*, denoted by $\deg_G^0(x, Y)$, is the minimum of those two values. We drop the subscript when there is no danger of confusion, writing $N^-(x)$, $\deg^0(x)$, and so forth. Also, we write $|G|$ and $e(G)$ for the number of vertices and edges of G respectively. For digraphs G and H we say that H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For any set $X \subseteq V(G)$, we write $G[X]$ for the subgraph of G *induced* by X, which has vertex set X and whose edges are all edges of G with both

endvertices in X. If H is a subgraph of G then we write $G - H$ for $G[V(G) \setminus V(H)]$. Likewise, for a vertex v or set of vertices S, we write $G - v$ or $G - S$ for $G[V(G) \setminus \{v\}]$ or $G[V(G) \setminus S]$ respectively.

Every digraph considered in this paper will be an *oriented graph*, meaning that there is at most one edge between each pair of vertices (and there are no loops). Equivalently, an oriented graph G can be formed by orienting each edge of some (undirected) graph H; in this case we refer to H as the *underlying graph* of G, and say that G is an *orientation* of H. We refer to the *maximum degree* of an oriented graph G, denoted $\Delta(G)$, to mean the maximum degree of the underlying graph H. A *tournament* is an orientation of a complete graph, and a *subtournament* of a tournament G is a subgraph of G which is a tournament. A *regular tournament* is a tournament in which every vertex has equal indegree and outdegree; it is easily checked that regular tournaments of order n exist for every odd $n \in \mathbb{N}$. A *transitive tournament* is a tournament whose vertices can be ordered $v_1, \ldots, v_n$ such that $v_i \to v_j$ is an edge for each $i < j$. A *directed path* of length k is an oriented graph with vertex set $v_0, \ldots, v_k$ and edges $v_{i-1} \to v_i$ for each $1 \le i \le k$, and an *antidirected path* of length k is an oriented graph with vertex set $v_0, \ldots, v_k$ and edges $v_{i-1} \to v_i$ for odd $i \le k$ and $v_{i-1} \leftarrow v_i$ for even $i \le k$ (or *vice versa*). A *directed cycle* of length k is an oriented graph with vertex set $v_1, \ldots, v_k$ and edges $v_i \to v_{i+1}$ for each $1 \le i \le k$ with addition taken modulo k.

A *tree* is an acyclic connected graph, and an *oriented tree* or *directed tree* is an orientation of a tree. A *leaf* in a tree or oriented tree is a vertex of degree one. A *star* is a tree in which at most one vertex (the *centre*) is not a leaf. A *subtree* T' of a tree T is a subgraph of T which is also a tree, and we define subtrees of oriented trees similarly. For oriented trees T and T' we say that T' is an *out-subtree* (respectively an *in-subtree*) of T if both T' and $T - T'$ are subtrees of T, and the unique edge of T between T' and $T - T'$ is directed towards T' (respectively away from T'). In a similar way we say that a vertex is an *in-leaf* or *out-leaf* of T. Now let T be a tree or oriented tree. It is often helpful to nominate a vertex r of T as the *root* of T; to emphasise this fact we sometimes refer to T as a *rooted tree*. If so, then every vertex x other than r has a unique *parent*; this is defined to be the neighbour p of x in the unique path in T from x to r, and x is said to be a *child* of p. An *ancestral ordering* of the vertices of a rooted tree T is an ordering of $V(T)$ in which the root vertex appears first and every non-root vertex appears later than its parent. Where it is clear from the context that a tree is oriented, we may refer to it simply as a tree.

We say that a sequence of events $A_1, A_2, \ldots$ holds *asymptotically almost surely* if $\mathbb{P}(A_n) \to 1$ as $n \to \infty$. Likewise, in this paper all occurrences of the standard asymptotic notation $o(f)$ refer to sequences $f(n)$ with parameter n as $n \to \infty$. We will often have sets indexed by $\{1, 2, \ldots, k\}$ (*e.g.* $V_1, \ldots, V_k$), and addition of indices will always be performed modulo k. Also, if $\varphi : A \to B$ is a function from A to B and $A' \subseteq A$, then we write $\varphi(A')$ for the image of A' under φ. We omit floors and ceilings whenever they do not affect the argument, and write $a = b \pm c$ to indicate that $b - c \le a \le b + c$. For $k \in \mathbb{N}$ we denote by $[k]$ the set $\{1, 2, \ldots, k\}$, and write $\binom{S}{k}$ to denote the set of all k-element subsets of a set S. We use the notation $x \ll y$ to indicate that for every positive y there exists a positive number x_0 such that for every $0 < x < x_0$ the subsequent statements hold. Such statements with more variables are defined similarly. We always write $\log x$ to mean the natural logarithm of x.

2.1 Structural results for tournaments

Let G be a bipartite graph with vertex classes A and B. Loosely speaking, G is 'regular' if the edges of G are 'randomlike' in the sense that they are distributed roughly uniformly. More formally, for any sets $X \subseteq A$ and $Y \subseteq B$, we write $G[X, Y]$ for the bipartite subgraph of G with vertex classes X and Y and whose edges are the edges of G with one endvertex in each of the sets X and Y, and define the *density* $d_G(X, Y)$ of edges between X and Y to be $d_G(X, Y) := e(G[X, Y])/|X||Y|$. Then, for any $d, \varepsilon > 0$, we say that G is (d, ε)-*regular* if for every $X \subseteq A$ and every $Y \subseteq B$ such that $|X| \ge \varepsilon|A|$ and $|Y| \ge \varepsilon|B|$ we have $d_G(X, Y) = d \pm \varepsilon$. The following well-known proposition is immediate from this definition.

Lemma 2.1 (Slicing lemma)**.** *Fix $\alpha, \varepsilon, d > 0$ and let G be a (d, ε)-regular bipartite graph with vertex classes A and B. If $A' \subseteq A$ and $B' \subseteq B$ have sizes $|A'| \geq \alpha|A|$ and $|B'| \geq \alpha|B|$, then $G[A', B']$ is $(d, \varepsilon/\alpha)$-regular.*

We say that G is $(d_{\geq}, \varepsilon)$-*regular* if G is (d', ε)-regular for some $d' \geq d$. Another immediate consequence of the definition of regularity is that, for small ε, if G is (d, ε)-regular then almost all vertices of A have degree close to $d|B|$ in B and almost all vertices of B have degree close to $d|A|$ in A. We say that G is 'super-regular' if no vertex has degree much lower than this. More precisely, G is (d, ε)-*super-regular* if (A, B) is $(d_{\geq}, \varepsilon)$-regular and also for every $a \in A$ and $b \in B$ we have $\deg(a, B) \geq (d - \varepsilon)|B|$ and $\deg(b, A) \geq (d - \varepsilon)|A|$.

To complete the embedding of a spanning oriented tree in a tournament, we will make use of the following well-known lemma, which states that every balanced super-regular bipartite graph contains a perfect matching (a bipartite graph is *balanced* if its vertex classes have equal size).

Lemma 2.2. *For every $\varepsilon, d > 0$ with $d \geq 2\varepsilon$, if G is a (d, ε)-super-regular balanced bipartite graph, then G contains a perfect matching.*

Proof. Let A and B be the vertex classes of G, and let m denote their common size. Consider an arbitrary set $S \subseteq A$, and let $N(S) \subseteq B$ denote the set of vertices of G with a neighbour in S. If $|S| < \varepsilon m$, then for each $a \in S$ we have $\deg(a, B) \geq (d - \varepsilon)m \geq \varepsilon m > |S|$, so certainly $|N(S)| \geq |S|$. Alternatively, if $\varepsilon m \leq |S| \leq (1 - \varepsilon)m$, then, since G is $(d_{\geq}, \varepsilon)$-regular, at most εm vertices of B have no neighbours in S, so $|N(S)| \geq (1 - \varepsilon)m \geq |S|$. Finally, if $|S| > (1 - \varepsilon)m$ then every vertex $b \in B$ has a neighbour in S, since $\deg(b, A) \geq (d - \varepsilon)m \geq \varepsilon m > |A \setminus S|$, so $|N(S)| = m \geq |S|$. In each case Hall's criterion holds, that is, we have $|N(S)| \geq |S|$ for every subset $S \subseteq A$. So G contains a perfect matching. $\qquad\square$

Now let G be a digraph. For disjoint subsets $X, Y \subseteq V(G)$ we denote by $G[X \to Y]$, or equivalently by $G[Y \leftarrow X]$, the subdigraph of G with vertex set $X \cup Y$ and edge set

$$E(X \to Y) \coloneqq \{ x \to y \in E(G) : x \in X, y \in Y \}.$$

We call the ordered pair (X, Y) a *directed pair* in G if there are no edges in $G[X \leftarrow Y]$, that is, if every edge between X and Y is directed towards Y. Similarly, for any $\mu \geq 0$ we call (X, Y) a *μ-almost-directed pair* if $e\big(G(X \leftarrow Y)\big) \leq \mu|X||Y|$, so any directed pair is a 0-almost-directed pair. These structures will play a key role in our proof.

Observe that the underlying graph of $G[X \to Y]$ is a bipartite graph with vertex classes X and Y. We say that $G[X \to Y]$ is (d, ε)-regular (respectively (d, ε)-super-regular) to mean that this underlying graph is (d, ε)-regular (respectively (d, ε)-super-regular). In this way we may apply the previous results of this subsection to directed graphs.

We now define another structure which is crucial for our proof. Let d and ε be positive real numbers, and let G be a digraph whose vertex set is the disjoint union of sets $V_1, \ldots, V_k$. We say that G is a *(d, ε)-regular cycle of cluster tournaments* if for each $i \in [k]$ the induced subgraph $G[V_i]$ is a tournament and the digraph $G[V_i \to V_{i+1}]$ is $(d_{\geq}, \varepsilon)$-regular (where addition on the subscript is taken modulo k). Likewise, we say that G is a *(d, ε)-super-regular cycle of cluster tournaments* if for each $i \in [k]$ the induced subgraph $G[V_i]$ is a tournament and the digraph $G[V_i \to V_{i+1}]$ is (d, ε)-super-regular. In either case we refer to the sets $V_1, \ldots, V_k$ as the *clusters* of G.

The following lemma, a combination of two lemmas of Kühn, Mycroft and Osthus [11] about so-called 'robust outexpanders', shows that every tournament with large minimum semidegree either admits a partition $\{S, S'\}$ where S and S' are not too small and (S, S') is an almost-directed pair, or contains an almost-spanning cycle of cluster tournaments.

Lemma 2.3 ([11, Lemmas 2.7 and 2.8])**.** *Suppose that $1/n \ll 1/k_1 \ll 1/k_0 \ll \varepsilon \ll d \ll \mu \ll \nu \ll \eta$, and let G be a tournament on n vertices. Then either*

 (a) $\delta^0(G) < \eta n$,

 (b) there is a partition of $V(G)$ into sets S and S' with $\nu n < |S|, |S'| < (1 - \nu)n$ and such that (S, S') is a μ-almost-directed pair in G, or

(c) there is an integer k with $k_0 \leq k \leq k_1$ for which G contains a (d, ε)-regular cycle of cluster tournaments with clusters $V_1, \ldots, V_k$ of equal size such that $|\bigcup_{i=1}^{k} V_i| > (1 - \varepsilon)n$.

2.2 Useful estimates and bounds

In this section we present various useful estimates. The first is the following lemma which is used in Section 3 to show that our random allocation of vertices of an oriented tree T to the clusters of a cycle of cluster tournaments G gives a roughly uniform distribution. We write $\mathcal{B}(n, p)$ to denote the binomial distribution (the result of n independent Bernoulli experiments, each with success probability p).

Lemma 2.4. *Suppose that $1/n \ll 1/k$. If $X := \mathcal{B}(n, \frac{1}{2})$, then for every $r \in [k]$ we have*

$$\mathbb{P}(\, X \equiv r \mod k \,) = \frac{1}{k} \pm \frac{4}{\sqrt{n}}.$$

Proof. Define $p_\mu := \max_{x \in \{0, \ldots, n\}} \mathbb{P}(X = x)$. Kühn, Mycroft and Osthus [11, proof of Lemma 2.1] gave a straightforward argument to show that $\mathbb{P}(\, X \equiv r \mod k \,) = \frac{1}{k} \pm 2p_\mu$, and the result then follows from a standard estimate on the binomial distribution (see, for example, [2, Section 1.2]) which states that $p_\mu \sim 1/\sqrt{\pi n/2}$. $\square$

The following straightforward lemma shows that a tournament can only have a few vertices of small in- or outdegree.

Lemma 2.5. *For each $d \in \mathbb{N}$, every tournament contains at most $4d - 2$ vertices with semidegree less than d.*

Proof. Let G be a tournament, and let X be the set of vertices $x \in V(G)$ with $\deg^+(x) \leq d - 1$. Then

$$\binom{|X|}{2} = e(G[X]) \leq \sum_{x \in X} \deg^+(x) \leq (d - 1)|X|,$$

where the central inequality holds because every edge of $G[X]$ contributes one to the given sum. It follows that $|X| \leq 2d - 1$, that is, there are at most $2d - 1$ vertices with outdegree less than d. Essentially the same argument shows that there are at most $2d - 1$ vertices with indegree less than d, so in total at most $4d - 2$ vertices have semidegree less than d. $\square$

Suppose N is an n-element set, and let M be a subset of N with m elements. If we choose a subset $S \in \binom{N}{k}$ uniformly at random, then the random variable $X = |S \cap M|$ is said to have *hypergeometric distribution* with parameters n, m and k. Note that the expectation of X is then $\mathbb{E}X = km/n$.

Theorem 2.6 ([9], Corollary 2.3 and Theorem 2.10)**.** *For every $0 < a < 3/2$, if X has binomial or hypergeometric distribution, then $\mathbb{P}(\, |X - \mathbb{E}X| \geq a\mathbb{E}X \,) \leq 2\exp(-a^2\mathbb{E}X/3)$.*

We also use an Azuma-type concentration result for martingales due to Mc Diarmid [15], in the form stated by Sudakov and Vondrák [21].

Lemma 2.7. *Fix $n \in \mathbb{N}$ and let $X_1, \ldots, X_n$ be random variables taking values in $[0, 1]$ such that for each $i \in [n]$ we have $\mathbb{E}(\, X_i \mid X_1, \ldots, X_{i-1} \,) \leq a_i$. If $\mu \geq \sum_{i=1}^{n} a_i$, then for every δ with $0 < \delta < 1$ we have*

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i > (1 + \delta)\mu \Big) \leq e^{-\delta^2\mu/3}.$$

3 Allocating and embedding

In this section we show how to obtain somewhat sharper estimates from the random allocation and embedding algorithms used by Kühn, Mycroft and Osthus [11] to embed oriented trees in slightly larger tournaments. We begin with the following lemma, which is a slightly modified version of [11, Lemma 2.10].

Lemma 3.1. *For every $C > 0$ there exists n_0 such that for every rooted tree T on $n \geq n_0$ vertices with $\Delta(T) \leq (\log n)^C$ and root r, there exist $s \in \mathbb{N}$, pairwise-disjoint subsets $F_1 \ldots, F_s \subseteq V(T)$, and not-necessarily-distinct vertices $v_1, \ldots, v_s$ of T with the following properties.*

(1) $\left| \bigcup_{i \in [s]} F_i \right| \geq n - n^{5/12}$.

(2) $|F_i| \leq n^{2/3}$ for each $i \in [s]$.

(3) For any $i \in [s]$, any $x \in \{r\} \cup \bigcup_{j < i} F_j$, and any $y \in F_i$, the path from x to y in T includes v_i.

(4) For any $i \in [s]$ and $y \in F_i$ we have $\mathrm{dist}_T(v_i, y) \geq (\log \log n)^3$.

The original version of this lemma had constants $\Delta, \varepsilon, k > 0$ rather than $C > 0$, assumed additionally that $\Delta(T) \leq \Delta$, had $n - \varepsilon n$ in place of $n - n^{5/12}$ in (1) and had k in place of $\log \log n$ in (4). However, the form of the lemma given above can be established by an essentially identical proof, replacing each instance of k by $\log \log n$ and each instance of Δ by $(\log n)^C$. The crucial point is that we then replace the bound $3n^{1/3}\Delta^{k^3} \leq \varepsilon n$ by the bound $3n^{1/3}(\log n)^{C(\log \log n)^3} \leq n^{5/12}$. These changes yield (1) and (4) above, whilst (2) and (3) are unchanged.

We now consider the random allocation algorithm of Kühn, Mycroft and Osthus [11, Vertex Allocation Algorithm], which is presented below as Algorithm 1. Given a rooted oriented tree T and a cycle of cluster tournaments G with clusters $V_1, V_2, \ldots, V_k$, this assigns each vertex of T to a cluster of G. We allocate vertices of T one at a time in an ancestral ordering. This ensures that whenever we allocate a vertex x other than the root, the parent p of x has previously been allocated to some cluster V_i. We then say that x is allocated *canonically* if either $p \to x \in E(T)$ and x is allocated to the cluster V_{i+1}, or $p \leftarrow x \in E(T)$ and x is allocated to the cluster V_{i-1}. Moreover, we say that an allocation of the vertices of T to the clusters of G is *semi-canonical* if every vertex of T is either allocated canonically or allocated to the same cluster as its parent, every vertex adjacent to the root of T is allocated canonically, and for each $i \in [k]$ the set U_i of vertices allocated to V_i induces a forest $F = T[U_i]$ in which no connected component has more than $\Delta(T)$ vertices.

Algorithm 1: The Vertex Allocation Algorithm [11]

> **Input**: an oriented tree T on n vertices, a root vertex r of T, and clusters $V_1, \ldots, V_k$.
>
> Choose an ancestral ordering $t_1, \ldots, t_n$ of $V(T)$ (so in particular $t_1 = r$).
>
> **for** $\tau = 1$ **to** n **do**
>> **if** $\tau = 1$ **then** allocate r to V_1.
>> **else**
>>> Let t_σ be the parent of t_τ.
>>> **if** $\mathrm{dist}_T(t_\tau, r)$ *is odd* **then** allocate t_τ canonically.
>>> **else** Allocate t_τ to the same cluster as t_σ with probability $1/2$ and
>>> allocate t_τ canonically with probability $1/2$, independently of all previous choices.

The following lemma, a slightly modified version of [11, Lemma 3.3], states that Algorithm 1 will always return a semi-canonical allocation, and moreover that if T is sufficiently large then the allocation of vertices to clusters will be approximately uniform.

Lemma 3.2. *Let T be an oriented tree on n vertices rooted at r. If we allocate the vertices of T to clusters $V_1, \ldots, V_k$ by applying the Vertex Allocation Algorithm, then the following properties hold.*

(a) The allocation obtained will be semi-canonical.

(b) *Let u and v be distinct vertices of T such that u lies on the path from r to v, let P be the path between u and v, and let $E \subseteq V(T)$ consist of all vertices $x \in V(P) \setminus \{u\}$ for which $\mathrm{dist}(r, x)$ is even. If we condition on the event that u is allocated to some cluster V_j, then v is allocated to cluster V_{j+R+F} (taking addition in the subscript modulo k) where $R := \mathcal{B}(|E|, \frac{1}{2})$ and F is a deterministic variable depending only on $\mathrm{dist}(r, u)$ and the orientations of edges of P (that is, F is unaffected by the random choices made by the Vertex Allocation Algorithm).*

(c) *Suppose that $1/n \ll 1/k$. Let u and v be vertices of T such that u lies on the path from r to v, and $\mathrm{dist}_T(u, v) \geq (\log \log n)^3$. Then for any $i, j \in [k]$,*

$$\mathbb{P}(\, v \text{ is allocated to } V_i \mid u \text{ is allocated to } V_j \,) = \frac{1}{k}\left(1 \pm \frac{1}{4 \log \log n}\right).$$

(d) *Suppose that $1/n \ll 1/k, \alpha, 1/C$ and that $\Delta(T) \leq (\log n)^C$. Let S be a subset of $V(T)$ with at least αn vertices. Then with probability $1 - \mathrm{o}(1)$ each of the k clusters V_i has $|S|(\frac{1}{k} \pm \frac{1}{\log \log n})$ vertices of S allocated to it.*

The statement above differs from the original version of the lemma in the following ways. Firstly, (b) was not stated explicitly, but was established in the original proof. Secondly, the original version of (c) instead had constants $1/k \ll \delta$, assumed that $\mathrm{dist}_T(u, v) \geq k^3$ instead of $\mathrm{dist}_T(u, v) \geq (\log \log n)^3$, and had δ in place of $\frac{1}{\log \log n}$ in the displayed equation. Finally, the original version of (d) had constants $1/n \ll 1/\Delta, 1/k \ll \delta$, assumed instead that $\Delta(T) \leq \Delta$, had δ in place of $\frac{1}{\log \log n}$, was only stated for the special case $S = V(T)$, and only provided an upper bound on the number of vertices allocated to each cluster. So our version of the lemma allows the bounds in (c) and (d) to decrease with n, and $\Delta(T)$ to grow with n, rather than being fixed constants. We now show how the original proof can be modified to establish our altered versions of (c) and (d).

Proof. To prove (c), let $\ell := \mathrm{dist}_T(u, v)$, and define E as in (b), so $|E| = \lfloor \frac{\ell}{2} \rfloor$ or $|E| = \lceil \frac{\ell}{2} \rceil$. By (b) it suffices to show that $\mathbb{P}(\mathcal{B}(|E|, \frac{1}{2}) = r \mod k) = \frac{1}{k} \pm \frac{1}{4k \log \log n}$ for each $r \in [k]$, and since $|E| \geq \frac{1}{3}(\log \log n)^3$ this holds by Lemma 2.4.

We now prove (d). By Lemma 3.1, there exist an integer $s \leq 3n^{1/3}$, vertices $v_1, \ldots, v_s \in V(T)$ and pairwise-disjoint subsets $F_1, \ldots, F_s$ of $V(T)$ such that $|\bigcup_{i=1}^{s} F_i| \geq n - n^{5/12}$ and $|F_i| \leq n^{2/3}$ for each $i \in [k]$, such that if $j < i$, then any path from r or any vertex of F_j to any vertex of F_i passes through the vertex v_i, and also such that $\mathrm{dist}(v_i, F_i) \geq (\log \log n)^3$. Write $\delta := \frac{1}{\log \log n}$; we shall prove that

(†) with probability $1 - \mathrm{o}(1)$, for any $j \in [k]$ the total number of vertices from $\bigcup_{i \in [s]} F_i \cap S$ allocated to cluster V_j is at most $|S|(\frac{1}{k} + \frac{\delta}{2k})$.

Note that (†) implies (d). Indeed, since the number of vertices of T not contained in any of the sets F_i is at most $n^{5/12} \leq \alpha n / 2k \log \log n \leq \delta |S| / 2k$, if (†) holds then for any $j \in [k]$ in total at most $|S|(1+\delta)/k$ vertices of S are allocated to V_j. It follows that at least $|S| - (k-1)|S|(1+\delta)/k \geq |S|(1/k - \delta)$ vertices of S are allocated to V_j, so (d) holds.

To prove (†), define random variables X_i^j for each $i \in [s]$ and $j \in [k]$ by

$$X_i^j := \frac{\# \text{ of vertices of } F_i \cap S \text{ allocated to cluster } V_j}{n^{2/3}},$$

so each X_i^j lies in the range $[0, 1]$. Then since the cluster to which a vertex x of T is allocated is dependent only on the cluster to which the parent of x is allocated and on the outcome of the random choice made when allocating x, we have for each $q \in [k]$ that $\mathbb{E}(X_i^j \mid X_{i-1}^j, \ldots, X_1^j, v_i \in V_q) = \mathbb{E}(X_i^j \mid v_i \in V_q)$, where we write $x \in V_q$ to denote the event that x is allocated to V_q. So for any $i \in [s]$ and $j \in [k]$ we have

$$\mathbb{E}(X_i^j \mid X_{i-1}^j, \ldots, X_1^j) \leq \max_{q \in [k]} \mathbb{E}(X_i^j \mid X_{i-1}^j, \ldots, X_1^j, v_i \in V_q) = \max_{q \in [k]} \mathbb{E}(X_i^j \mid v_i \in V_q)$$

$$= \max_{q \in [k]} \frac{\sum_{x \in F_i \cap S} \mathbb{P}(x \in V_j \mid v_i \in V_q)}{n^{2/3}} \leq \left(\frac{1}{k} + \frac{\delta}{4k}\right)\frac{|F_i \cap S|}{n^{2/3}},$$

using (c). We apply Lemma 2.7 with

$$\mu := \left(\frac{1}{k} + \frac{\delta}{4k}\right) \frac{|S|}{n^{2/3}} \geq \left(\frac{1}{k} + \frac{\delta}{4k}\right) \sum_{i \in [s]} \frac{|F_i \cap S|}{n^{2/3}},$$

to obtain

$$\mathbb{P}\left(\sum_{i \in [s]} X_i^j > (1 + \delta/8)\mu\right) \leq \exp\left(\frac{-(\delta/8)^2 \mu}{3}\right) = \exp\left(-\frac{\delta^2(1 + \delta/4)|S|}{192kn^{2/3}}\right) \leq \exp(-n^{1/4})$$

where the second inequality holds since we assumed that $1/n \ll 1/k, \alpha$. Taking a union bound, we find that with probability $1 - o(1)$ we have for each $j \in [k]$ that

$$n^{2/3} \sum_{i \in [s]} X_i^j \leq n^{2/3}(1 + \delta/8)\mu \leq |S| \left(\frac{1}{k} + \frac{\delta}{2k}\right).$$

In other words, for each $j \in [k]$ there are at most $|S|(\frac{1}{k} + \frac{\delta}{2k})$ vertices from $\bigcup_{i=1}^{r} F_i \cap S$ allocated to V_j, so (†) holds. $\qquad\square$

Having applied the random allocation algorithm to allocate the vertices of an oriented tree T to the clusters of a slightly larger cycle of cluster tournaments G, Kühn, Mycroft and Osthus proceeded to embed T in G using a vertex embedding algorithm which successively embedded vertices of T in G following an ancestral ordering of the vertices of T, with each vertex being embedded in the cluster to which is was allocated. Studying this algorithm yields the following lemma, which is a modified form of [11, Lemma 3.4].

Lemma 3.3. *Suppose that $1/n \ll 1/C$ and that $1/n \ll 1/k \ll \varepsilon \ll \gamma \ll d \ll \alpha$, and let $m := n/k$.*
  (1) Let T be an oriented tree on at most n vertices with root r and $\Delta(T) \leq (\log n)^C$.
  (2) Let G be a (d, ε)-regular cycle of cluster tournaments on clusters $V_1, \ldots, V_k$, each of size at least $(1 + \alpha)m$ and at most $3m$, and let v be a vertex of V_1 with at least γm inneighbours in V_k and at least γm outneighbours in V_2.
  (3) Let the vertices of T be allocated to the clusters $V_1, \ldots, V_k$ so that at most $(1 + \alpha/2)m$ vertices are allocated to each cluster V_i, and so that the allocation is semi-canonical.
Then G contains a copy of T in which r is embedded to v, and such that each vertex is embedded in the cluster to which it was allocated.

The differences between Lemma 3.3 as stated above and the original version in [11] are twofold. Firstly, the original assumption that $\Delta(T) \leq \Delta$ for some (fixed) Δ with $1/n \ll 1/\Delta \ll \varepsilon$ has been replaced by our assumption that $\Delta(T) \leq (\log n)^C$. Secondly, we allow the cluster sizes to vary between the bounds in (2), whereas the original form insisted that all clusters have size exactly $(1 + \alpha)n$. Neither of these changes materially affects the original proof given in [11].

Combining Lemma 3.2 and Lemma 3.3 immediately yields the following corollary, a modified version of [11, Lemma 3.2], in which the original constant bound on $\Delta(T)$ has been replaced by a polylogarithmic bound.

Corollary 3.4. *Suppose that $1/n \ll 1/C$ and that $1/n \ll 1/k \ll \varepsilon \ll d \ll \alpha \leq 2$, and let $m := n/k$. Let T be an oriented tree on at most n vertices with $\Delta(T) \leq (\log n)^C$ and with root r. Also let G be a (d, ε)-regular cycle of cluster tournaments on clusters $V_1, \ldots, V_k$, each of size $(1 + \alpha)m$, and let v be a vertex of V_1 with at least $d^2 m$ inneighbours in V_k and at least $d^2 m$ outneighbours in V_2. Then G contains a copy of T in which r is embedded to v.*

Recall that Theorem 1.7 of this paper is a sharpened version of [11, Theorem 1.4(2)]. The proof of Theorem 1.7 is identical to the proof of [11, Theorem 1.4(2)] given in [11] from this point onwards, using Corollary 3.4 above in place of [11, Lemma 3.2]. More precisely, we first derive an analogous statement to [11, Lemma 3.1], in which the bound $\Delta(T) \leq \Delta$ for constant Δ is

replaced by $\Delta(T) \leq (\log n)^C$. The (short) derivation of this statement is identical to that given in [11], except that Corollary 3.4 is used in place of [11, Lemma 3.2]. We then follow the proof of [11, Theorem 1.4(2)] in Section 6 of [11], with the only changes being that we now use this modified version of [11, Lemma 3.1]. The other results used in the proof (from Sections 2 and 5 of [11]) are applied exactly as they are. Note that the results of Section 4 of [11] are not relevant to this argument, since they address the case in which we have no bound on the maximum degree of T.

In the proof of Theorem 1.4 we use the following corollary. This is a consequence of Theorem 1.7 and El Sahili's theorem [19] that, for every $m \in \mathbb{N}$, every tournament on at least $3m - 3$ vertices contains every oriented tree on m vertices. Indeed, this corollary is simpler to apply since it holds for both small and large trees.

Corollary 3.5. *Suppose that $1/n \ll \alpha, 1/C$. Let T be an oriented tree on $n' \leq n$ vertices with $\Delta(T) \leq (\log n)^C$, and let G be a tournament on at least $n' + \alpha n$ vertices. Then G contains a copy of T.*

Proof. Fix $\alpha, C > 0$ and choose n_0 sufficiently large to apply Theorem 1.7 with $2C$ in place of C, and also so that $\log n_0 \geq \left(1 + \log(2/\alpha)\right)^2$. Then we may assume that $n \geq 2n_0/\alpha$. If $n' > \alpha n/2$, then $n' > n_0$, so G contains a copy of T by Theorem 1.7, since G has at least $n' + \alpha n \geq (1 + \alpha)n'$ vertices and $\Delta(T) \leq (\log n)^C \leq (\log n')^{2C}$. On the other hand, if $n' \leq \alpha n/2$, then $|G| \geq n' + \alpha n \geq 3n'$, and thus G contains a copy of T by the aforementioned theorem of El Sahili. $\qquad\square$

The modified proofs of Lemmas 3.1 and 3.3 and Theorem 1.7 are presented in full in [17].

4   Almost-directed pairs

Our aim in this section is to prove Lemma 4.3, which states that every nice oriented tree T of polylogarithmic maximum degree is contained in every tournament whose vertex set admits a partition $\{U, W\}$ into not-too-small sets U and W such that the pair (U, W) is almost-directed.

We begin with a definition and two lemmas. If (X, Y) is a μ-almost-directed pair in a digraph G, we say that an edge $e \in E(G)$ is a *reverse edge* if $e \in E(X \leftarrow Y)$ (so, by definition, an almost-directed pair has at most $\mu|X||Y|$ reverse edges). Our first lemma guarantees that we may partition the vertex set of an oriented tree T into sets A and B so that (A, B) is a directed pair in T and so that specific in-subtrees of T have all their vertices in A and specific out-subtrees of T have all their vertices in B. Moreover, we may specify the sizes of A and B (subject to the trivial necessary conditions).

Lemma 4.1. *Let T be an oriented tree on n vertices. Let $\mathcal{T}^-$ be a collection of in-subtrees of T, and let $\mathcal{T}^+$ be a collection of out-subtrees of T, such that the trees in $\mathcal{T}^- \cup \mathcal{T}^+$ are pairwise vertex-disjoint. If a and b are integers with*

$$a \geq \Big| \bigcup_{S \in \mathcal{T}^-} V(S) \Big|, \quad b \geq \Big| \bigcup_{S \in \mathcal{T}^+} V(S) \Big| \quad and \quad a + b = n,$$

then there exists a partition $\{A, B\}$ of $V(T)$ with $|A| = a$ and $|B| = b$ such that (A, B) is a directed pair in T and

$$\bigcup_{S \in \mathcal{T}^-} V(S) \subseteq A \quad and \quad \bigcup_{S \in \mathcal{T}^+} V(S) \subseteq B.$$

Proof. The key observation is that in every oriented forest there is a vertex with no inneighbours (since a forest has more vertices than edges). Define $V^- := \bigcup_{S \in \mathcal{T}^-} V(S)$ and $V^+ := \bigcup_{S \in \mathcal{T}^+} V(S)$, and let $k := a - |V^-|$, so $0 \leq k \leq n - |V^-| - |V^+|$. Greedily choose distinct vertices $v_1, v_2, \ldots, v_k$ of $V(T) \setminus (V^- \cup V^+)$ such that v_i has no inneighbours in $T - (V^- \cup V^+ \cup \{v_1, \ldots, v_{i-1}\})$ for each $i \in [k]$. The desired partition is then $A := V^- \cup \{v_1, \ldots, v_k\}$ and $B := V(T) \setminus A$. Indeed, we have $V^- \subseteq A$, $V^+ \subseteq B$, $|A| = |V^-| + k = a$ and $|B| = n - |A| = b$. It remains to show

that (A, B) is a directed pair in T. So suppose that $u \to v$ is an edge of T and $v \in A$. It then suffices to show that we must have $u \in A$ as well. For this, observe that since $V^+ \subseteq B$ consists of outstars of T, and $v \in A$, we cannot have $u \in V^+$. So if $v \notin V^-$, then $v = v_i$ for some $i \in [k]$, and by choice of v_i we then have $u \in V^- \cup \{v_1, \dots, v_{i-1}\} \subseteq A$. On the other hand, if $v \in V^-$ then v is a vertex of some in-subtree of T, so u must be a vertex of the same in-subtree; it follows that $u \in A$. $\qquad \square$

Suppose now that T is an oriented tree of polylogarithmic maximum degree whose vertex set is partitioned into sets A and B which form a directed pair (A, B) in T, and also that G is a tournament whose vertex set admits a partition into sets U and W such that (U, W) is an almost-directed pair in G. The next lemma shows that if U and W are slightly larger than A and B respectively, then under the additional assumption that every vertex of G lies in few reverse edges, we may embed T in G so that vertices of A are embedded in U and vertices of B are embedded in W. (Recalling the proof outline of Theorem 1.4, we will use this lemma to embed the subtree T' in G.)

Lemma 4.2. *Suppose that $1/n \ll 1/C$ and that $1/n \ll \mu \ll \alpha$. Let T be an oriented tree with $\Delta(T) \leq (\log n)^C$ and let $\{A, B\}$ be a partition of $V(T)$ such that (A, B) is a directed pair in T. Also let G be a tournament on n vertices. If $V(G)$ admits a partition $\{U, W\}$ such that*
  (i) $|U| \geq |A| + \alpha n$,
  (ii) $|W| \geq |B| + \alpha n$,
  (iii) for each $u \in U$ we have $\deg^-(u, W) \leq \mu n$, and
  (iv) for each $w \in W$ we have $\deg^+(w, U) \leq \mu n$,
then there exists a copy of T in G such that every vertex in A is embedded in U and every vertex in B is embedded in W.

Proof. Consider the oriented forest F with $F := T[A] \cup T[B]$ (in other words, $V(F) = V(T)$ and the edges of F are the edges of T with both endvertices in A or both endvertices in B). Let $C_1, \dots, C_s$ be the components of F, and let T' be the minor of T that we obtain by contracting $V(C_j)$ to a single vertex v_j, for each $j \in [s]$. We may assume the components are labelled so that $v_1, \dots, v_s$ is an ancestral ordering of $V(T')$. We will greedily embed $C_1, \dots, C_s$ in G in that order, defining a mapping $\varphi \colon V(T) \to U \cup W$. For each $j \in [s]$, let U_j (respectively W_j) be the set of vertices of U (respectively W) which have not been covered by the embedding of $C_1, \dots, C_{j-1}$.

If $V(C_1) \subseteq A$, then by (i) we have $|U_1| = |U| \geq |A| + \alpha n \geq |C_1| + \alpha n$, so there exists a copy of C_1 in $G[U_1]$ by Corollary 3.5. By a similar argument using (ii) we may embed C_1 in $G[W_1]$ if $V(C_1) \subseteq B$. Now suppose that we have already embedded components $C_1, \dots, C_{j-1}$ for some $1 < j \leq n$, so $\varphi(v)$ is defined for every $v \in \bigcup_{i=1}^{j-1} V(C_i)$. Since we assumed that $v_1, \dots, v_s$ was an ancestral ordering of $V(T')$, there exists a unique integer $i \in [j-1]$ for which some vertex $u \in V(C_i)$ is adjacent to some vertex $v \in C_j$. Suppose first that $u \to v \in E(T)$. Then C_i has been embedded in U and C_j is a component of $T[B]$, and we want to embed C_j in $W_j \cap N^+(\varphi(u))$. Note that $\varphi(u)$ has at most μn inneighbours in W by (iii), so by (ii) the number of outneighbours of $\varphi(u)$ in W which are not in the image of φ (that is, which are not covered by the embedding so far) is at least $|W_j| - \mu n \geq (|W| - |B| + |C_j|) - \mu n \geq |C_j| + \alpha n - \mu n \geq |C_j| + \alpha n/2$. We may therefore embed C_j in $G[W_j]$ by Corollary 3.5. If instead $u \leftarrow v \in E(T)$ then C_j is a component of $T[A]$ and we may embed C_j in $G[U_j]$ by a similar argument using (i) and (iv). In either case we have extended φ as desired, and so proceeding in this manner gives a copy of T in G. $\qquad \square$

We are now ready to state and prove Lemma 4.3, the main result of this section, following the approach sketched in the proof outline of Theorem 1.4.

Lemma 4.3. *Suppose that $1/n \ll 1/C$ and that $1/n \ll \mu \ll \alpha, \nu$. Let G be a tournament on n vertices, and let T be an α-nice oriented tree on n vertices with $\Delta(T) \leq (\log n)^C$. If there is a partition $\{U, W\}$ of $V(G)$ with $|U|, |W| \geq \nu n$ such that (U, W) is a μ-almost-directed pair in G, then G contains a (spanning) copy of T.*

Proof. Introduce new constants ψ and β so that $1/n \ll \mu \ll \psi \ll \beta \ll \alpha, \nu$. Since (U, W) is a μ-almost directed pair in G, there are at most $\mu|U||V|$ reverse edges, so at most $\sqrt{\mu}|U|$ vertices of U are incident to at least $\sqrt{\mu}|W|$ reverse edges, and at most $\sqrt{\mu}|W|$ vertices of W are incident to at least $\sqrt{\mu}|U|$ reverse edges. Let Z be the set of all such vertices, so $z := |Z| \leq \sqrt{\mu}(|U| + |W|) = \sqrt{\mu}n$. Now let $W_0 := W \setminus Z$, and let X be the set of all vertices $w \in W_0$ with $\deg^0(w, W_0) < \psi n$. Then by Lemma 2.5 we have $|X| < 4\psi n$. Choose a subset $Y \subseteq W_0$ of size ψn uniformly at random. Note that for each $w \in W_0 \setminus X$ the values of $\deg^-(w, Y)$ and of $\deg^+(w, Y)$ then have a hypergeometric distribution with expectation at least $\psi n |Y|/|W_0| \geq \psi^2 n$, so $\mathbb{P}(\deg^0(w, Y) < \psi^2 n/2)$ decreases exponentially with n by Theorem 2.6. Taking a union bound over the at most n vertices $w \in W_0 \setminus X$ we find that with positive probability every $w \in W_0 \setminus X$ has $\deg^0(w, Y) \geq \psi^2 n/2 \geq 2z$. Fix a choice of Y for which this event occurs and define $U' := U \setminus Z$ and $W' := W_0 \setminus (Y \cup X)$. Also let $n' := |U' \cup W'|$, so $n' \geq n - |X| - |Y| - |Z| \geq (1 - 6\psi)n$. Observe that we then have the following properties.

(a) Every vertex $u \in U \setminus Z$ has $\deg^-(u, W') \leq \sqrt{\mu}|W| \leq \psi n'$.
(b) Every vertex $w \in W \setminus Z$ has $\deg^+(w, U') \leq \sqrt{\mu}|U| \leq \psi n'$.
(c) Every vertex $w \in W'$ has $\deg^0(w, Y) \geq 2z$.
(d) $|U'| \geq |U| - |Z| \geq |U| - \sqrt{\mu}n$ and $|W'| \geq |W| - |X| - |Y| - |Z| \geq |W| - 6\psi n$.
(e) $\Delta(T) \leq (\log n)^C \leq (\log n')^{2C}$.

Define $t := \lceil \beta n \rceil$. Let $\mathcal{S}^-$ be the set of pendant instars of T which contain an out-leaf of T, and let $\mathcal{S}^+$ be the set of pendant outstars of T which contain both an in-leaf of T and an out-leaf of T. Observe that $\mathcal{S}^- \cup \mathcal{S}^+$ is then a set of vertex-disjoint subtrees of T. Moreover, since T is α-nice, we have $|\mathcal{S}^-|, |\mathcal{S}^+| \geq \alpha n$. We define $S_1^-, \ldots, S_t^-$ to be the smallest t members of $\mathcal{S}^-$ and $S_1^+, \ldots, S_{t+z}^+$ to be the smallest $t + z$ members of $\mathcal{S}^+$. Since $t + z \leq 2\beta n$ we must then have $|\bigcup_{i \in [t]} V(S_i^-)|, |\bigcup_{i \in [t+z]} V(S_i^+)| \leq 2\beta n/\alpha$. For each $i \in [t]$ let ℓ_i^+ be an out-leaf of T in S_i^- and let c_i^- be the centre of the star S_i^-, and for each $i \in [z]$ let ℓ_{t+i}^+ be an out-leaf of T in S_{t+i}^+. Similarly, for each $i \in [t + z]$ let ℓ_i^- be an in-leaf of T in S_i^+ and let c_i^+ be the centre of the star S_i^+. We can be sure that these leaves exist by definition of $\mathcal{S}^+$ and $\mathcal{S}^-$.

We now define T' to be the subtree of T obtained by deleting the leaves ℓ_i^+ and ℓ_i^- from T for each $i \in [t + z]$. So $L_i^- := S_i^- - \ell_i^+$ (respectively $L_i^+ := S_i^+ - \ell_i^-$) is an in-subtree (respectively out-subtree) of T' for each $i \in [t]$, and $L_{t+j}^+ := S_{t+j}^+ - \{\ell_{t+j}^-, \ell_{t+j}^+\}$ is an out-subtree of T' for each $j \in [z]$. Also define $a := |U| - t$ and $b := |W| - t - 2z$. Then we have $a \geq \nu n - t \geq 2\beta n/\alpha \geq |\bigcup_{i \in [t]} V(L_i^-)|$ and $b \geq \nu n - t - 2z \geq 2\beta n/\alpha \geq |\bigcup_{i \in [t+z]} V(L_i^+)|$, and also $a + b = |U| + |W| - 2t - 2z = |T'|$, so we may apply Lemma 4.1 to obtain a partition $\{A, B\}$ of $V(T')$ with $|A| = a$ and $|B| = b$ such that (A, B) is a directed pair in T' and so that $V(L_i^-) \subseteq A$ for each $i \in [t]$ and $V(L_i^+) \subseteq B$ for each $i \in [t + z]$. Next, since by (d) we have $|U'| \geq a + \beta n'/2$ and $|W'| \geq b + \beta n'/2$, by (a), (b) and (e) we may apply Lemma 4.2 (with $n', \psi, 2C$ and $\beta/2$ in place of n, μ, C and α respectively) to obtain an embedding φ of T' in G so that $\varphi(A) \subseteq U'$ and $\varphi(B) \subseteq W'$.

We next embed the vertices ℓ_{t+j}^+ and ℓ_{t+j}^- for $j \in [z]$ so that all vertices of Z are covered. Note that our embedding of T' in G ensured that for each $j \in [z]$ the centre c_{t+j}^+ of S_{t+j}^+ was embedded to a vertex $w_{t+j} := \varphi(c_{t+j}^+)$ in W', so in particular we have $\deg^0(w_{t+j}, Y) \geq 2z$ by (c). This means that we can greedily choose distinct vertices $y_1^-, y_1^+, \ldots, y_z^-, y_z^+ \in Y$ so that for each $j \in [z]$ the vertex y_j^- is an inneighbour of w_{t+j} and y_j^+ is an outneighbour of w_{t+j}. Write $Z := \{q_1, \ldots, q_z\}$, and for each $j \in [z]$ consider the orientation of the edge of G between q_j and w_{t+j}. If $q_j \to w_{t+j} \in E(G)$, then we set $\varphi(\ell_{t+j}^-) := q_j$ and $\varphi(\ell_{t+j}^+) := y_j^+$. Similarly, if $q_j \leftarrow w_{t+j} \in E(G)$, then we set $\varphi(\ell_{t+j}^-) := y_j^-$ and $\varphi(\ell_{t+j}^+) := q_j$.

Observe that we have now embedded all of the vertices of T except for the leaves $\ell_1^+, \ldots, \ell_t^+$ and $\ell_1^-, \ldots, \ell_t^-$. Let $P^- := \{\varphi(c_i^-) : i \in [t]\}$ and $P^+ := \{\varphi(c_i^+) : i \in [t]\}$, so $P^- \subseteq U'$ and $P^+ \subseteq W'$. Also, let Q^- be the set of uncovered vertices of U and let Q^+ be the set of uncovered vertices of W. Then $|Q^-| = |U| - a = t$, and $|Q^+| = |W| - b - 2z = t$, so we have $|P^-| = |P^+| = |Q^-| = |Q^+| = t$. Observe that since we already covered all vertices of Z, we also have $Q^- \subseteq U \setminus Z$ and $Q^+ \subseteq W \setminus Z$. Together with the fact that $t = \lceil \beta n \rceil$, by (a) and (b) it follows that $G[P^- \to Q^+]$ and $G[Q^- \to P^+]$ are both $(1, \frac{1}{2})$-super-regular, so the balanced bipartite underlying graph of each contains a perfect matching by Lemma 2.2. For

each $j \in [t]$ let $\varphi(\ell_j^+) \in Q^+$ (respectively $\varphi(\ell_j^-) \in Q^-$) be the vertex matched to $\varphi(c_j^-) \in P^-$ (respectively $\varphi(c_j^+) \in P^+$); this completes the embedding φ of T in G. $\qquad\square$

5 Cycles of cluster tournaments

Our goal in this section is to prove Lemma 5.9, which states that every sufficiently large tournament containing an almost-spanning regular cycle of cluster tournaments contains a spanning copy of every nice oriented tree T with polylogarithmic maximum degree. Recall from the proof sketch of Theorem 1.4 that for this we split T into two subtrees T_1 and T_2. We then embed T_1 so that all 'atypical' vertices are covered and so that roughly the same number of vertices from each cluster are covered. Since T_1 covered all atypical vertices, the vertices which remain uncovered then form a super-regular cycle of cluster tournaments, and we use this fact to embed T_2 to cover all vertices which remain uncovered and so complete the embedding of T in G. In Section 5.1 we focus on the embedding of T_1, showing that can find an embedding with the desired properties (Lemma 5.1). Likewise, in Section 5.2 we consider the embedding of T_2, and prove that we can indeed embed T_2 so as to cover all remaining vertices, as desired (Lemma 5.2). Finally, in Section 5.3 we combine these results to prove Lemma 5.9 by first splitting T into subtrees T_1 and T_2 and then successively embedding these subtrees using Lemmas 5.1 and 5.2.

5.1 Embedding the first subtree

The subtree T_1 will have polylogarithmic maximum degree and will contain many vertices which are adjacent to at least one in-leaf and at least one out-leaf of T, and we wish to embed T_1 into a tournament G which contains an almost-spanning cycle of cluster tournaments so that approximately the same number of vertices of T_1 are embedded in each cluster. The following lemma states that we can indeed do this.

Lemma 5.1. *Suppose that $1/n \ll 1/C$ and that $1/n \ll 1/k \ll \varepsilon \ll d \ll \psi \ll \beta \ll \alpha$. Let T be an oriented tree on n vertices with root r, with maximum degree $\Delta(T) \le (\log n)^C$, and which contains at least βn distinct vertices that are each adjacent to at least one in-leaf and at least one out-leaf of T. Let G be a tournament which contains a (d, ε)-regular cycle of cluster tournaments whose clusters $V_1, \ldots, V_k$ have size $(1+\alpha)\frac{n}{k} \le |V_i| \le \frac{3n}{k}$ for each $i \in [k]$, and assume additionally that $B := V(G) \setminus \bigcup_{i \in [k]} V_i$ has size $|B| \le \psi n$. Then there exists an embedding φ of T in G covering B, such that r is embedded in V_1 and such that for each $i \in [k]$ we have*

$$|\varphi(V(T)) \cap V_i| = (n - |B|) \left(\frac{1}{k} \pm \frac{2}{\log \log n} \right).$$

Loosely speaking the proof proceeds as follows. We begin by selecting from each cluster V_i a large subset V_i' of vertices which each have large semidegree in $V_i \setminus V_i'$. Then $V_1', \ldots, V_k'$ are the clusters of a regular cycle of cluster tournaments in $G' := G[\bigcup_{i \in [k]} V_i']$. We remove a small number of leaves from T to obtain a subtree T', and embed T' in G' by using the Vertex Allocation Algorithm (Algorithm 1) and Lemma 3.3. Lemma 3.2 then ensures that approximately the same number of vertices are embedded in each cluster. Finally, we extend the embedding of T' in G to an embedding of T in G by embedding the removed leaves so as to cover all vertices of B.

Proof. Define $m := \frac{n}{k}$, so $(1+\alpha)m \le |V_i| \le 3m$ for each $i \in [k]$, and let $\delta := \frac{1}{\log \log n}$. Let B_i be the set of all vertices $x \in V_i$ such that $\deg^0(x, V_i) < \alpha m/20$. By Lemma 2.5 we have $|B_i| < \alpha m/4$. For each $i \in [k]$, pick a subset $Y_i \subseteq V_i$ of size $|Y_i| = \alpha m/4$ uniformly at random with choices made independently for each i. Note that for each $i \in [k]$ and each $x \in V_i \setminus B_i$, the random variables $\deg^-(x, Y_i)$ and $\deg^+(x, Y_i)$ then have hypergeometric distributions with expected value at least $(\alpha m/20)|Y_i|/|V_i| > 5\beta m$, and thus $\mathbb{P}(\deg^0(x, Y_i) < 4\beta m)$ decreases exponentially with n by Theorem 2.6. Taking a union bound, we find that there is a positive probability that for

every $i \in [k]$ and every $x \in V_i \setminus B_i$ we have $\deg^0(x, Y_i) \geq 4\beta m$. Fix a choice of sets $Y_1, \ldots, Y_k$ such that this event occurs, and for each $i \in [k]$ let $V_i' := V_i \setminus (Y_i \cup B_i)$, so

$$3m \geq |V_i| \geq |V_i'| \geq |V_i| - |B_i| - |Y_i| > (1 + \alpha)m - \frac{\alpha m}{4} - \frac{\alpha m}{4} = \left(1 + \frac{\alpha}{2}\right)m.$$

Furthermore, every vertex $x \in V_i'$ has $\deg^0(x, Y_i) \geq 4\beta m$. Now define $G' := G[V_1' \cup \cdots \cup V_k']$, and observe that since $V_1, \ldots, V_k$ were the clusters of a (d, ε)-regular cycle of cluster tournaments in G, by Lemma 2.1 the sets $V_1', \ldots, V_k'$ are the clusters of a spanning $(d, 3\varepsilon)$-regular cycle of cluster tournaments in G'. In particular we may choose a vertex $v \in V_1'$ with at least $(d - 3\varepsilon)|V_k'|$ inneighbours in V_k' and at least $(d - 3\varepsilon)|V_2'|$ outneighbours in V_2'. The tournament G', the clusters $V_1', \ldots, V_k'$ and the vertex v then meet the conditions of Lemma 3.3 with $\alpha/2$ and 3ε in place of α and ε respectively (and with n playing the same role there as here).

Let $t := \lceil \beta n \rceil - 1$, and choose a set $W := \{w_1, \ldots, w_t\}$ of t distinct vertices in T so that each w_i is adjacent to at least one in-leaf and at least one out-leaf of T and so that r is not a leaf of T which is adjacent to a vertex of W (such a set exists by the assumptions of the lemma). For each $j \in [t]$, let w_j^- and w_j^+ be respectively an in-leaf and an out-leaf adjacent to w_j. Let T' be the oriented tree we obtain by deleting from T the vertices w_j^- and w_j^+ for each $j \in [t]$, so $|T'| = n - 2t$ and $\Delta(T') \leq \Delta(T) \leq (\log n)^C \leq (\log(n - 2t))^{2C}$. Also take r to be the root of T', and apply the Vertex Allocation Algorithm (Algorithm 1) to allocate the vertices of T' to the clusters $V_1', \ldots, V_k'$. By Lemma 3.2(a) the obtained allocation will be semi-canonical. Moreover, by two applications of Lemma 3.2(d) (with $\beta/2$ and $2C$ in place of α and C respectively) we have with probability $1 - o(1)$ that for each $i \in [k]$ the number of vertices of T' allocated to the cluster V_i' is

$$(n - 2t)\left(\frac{1}{k} \pm \frac{1}{\log\log(n - 2t)}\right) = \frac{n - 2t}{k} \pm \frac{3\delta n}{2}, \tag{1}$$

and the number of vertices of W allocated to the cluster V_i' is

$$t\left(\frac{1}{k} \pm \frac{1}{\log\log(n - 2t)}\right) = \frac{t}{k} \pm \frac{3\delta t}{2}. \tag{2}$$

Fix an outcome of the Vertex Allocation Algorithm for which each of these events occurs, and apply Lemma 3.3 to obtain an embedding φ of T' in G' so that r is embedded to v and each vertex of T' is embedded in the cluster V_i' to which it is allocated. In particular r is embedded in V_1, as required.

We now extend φ to an embedding of T in G which covers B. Let $b := |B| \leq \psi n$, and let $q_1, \ldots, q_b$ be the vertices of B. Also let $p \in [k]$ be such that $b \equiv p \mod k$, and for each $i \in [k]$ choose $W_i \subseteq W$ such that $\varphi(W_i) \subseteq \varphi(W) \cap V_i'$ and so that $|W_i| = \lceil b/k \rceil$ if $i \in [p]$ and $|W_i| = \lfloor b/k \rfloor$ if $i \in [k] \setminus [p]$. (Since $b/k \leq \psi n/k$ and $\psi \ll \beta$, (2) ensures that we can indeed choose such sets.) The sets $W_1, \ldots, W_k$ are then vertex-disjoint and $|\bigcup_{i \in [k]} W_i| = b$, so by relabelling if necessary we may assume that $\bigcup_{i \in [k]} W_i = \{w_1, \ldots, w_b\}$. For each $j \in [t]$ set $p_j := \varphi(w_j)$ and write i_j to denote the index such that $p_j \in V_{i_j}$. Greedily choose $2t$ distinct vertices $c_1^-, c_1^+, \ldots, c_t^-, c_t^+$ so that for each $j \in [t]$ we have that $c_j^-, c_j^+ \in Y_{i_j}$, that c_j^- is an inneighbour of p_j and that c_j^+ is an outneighbour of p_j. It is possible to make such choices since for each $i \in [k]$ there are at most $2t/k$ vertices w_j with $i_j = i$ by (2), and because for each $j \in [t]$ we have $p_j \in V_{i_j}'$ (since w_j is a vertex of T'), so the semidegree of p_j in Y_{i_j} is at least $4\beta m \geq 2 \cdot (2t/k)$ by our choice of the sets Y_i.

Recall that each vertex in W is adjacent to precisely one removed in-leaf w_j^- of T and one removed out-leaf w_j^+ of T, and that these leaves have not yet been embedded. For each $s \in [b]$ we embed one of these leaves to the vertex q_s and the other to either c_s^- or c_s^+ according to the direction of the edge between q_s and p_s. For each $b + 1 \leq s \leq t$ we then embed the in-leaf of w_s to c_s^- and the out-leaf of w_s to c_s^+. More precisely, for all integers s with $1 \leq s \leq b$ we set $\varphi(w_s^-) := q_s$ and $\varphi(w_s^+) := c_s^+$ if $q_s \to p_s \in E(G)$, and set $\varphi(w_s^+) := q_s$ and $\varphi(w_s^-) := c_s^-$ if $q_s \leftarrow p_s \in E(G)$. Then, for all integers s with $b < s \leq t$ we set $\varphi(w_s^-) := c_s^-$ and $\varphi(w_s^+) := c_s^+$.

Following this extension φ is an embedding of T in G which covers every vertex in B. Moreover, for each $i \in [k]$ the number of vertices embedded in the cluster V_i is

$$|\varphi(V(T)) \cap V_i| = \left(\frac{n - 2t}{k} \pm \frac{3\delta n}{2} \right) + 2 \left(\frac{t}{k} \pm \frac{3\delta t}{2} \right) - \left(\frac{b}{k} \pm 1 \right) = (n - |B|) \left(\frac{1}{k} \pm 2\delta \right)$$

where the first term counts the number of vertices of T' embedded in V_i (see (1)), and the second and third terms count the number of removed leaves embedded in V_i. Indeed, by (2) there are $t/k \pm 3\delta t/2$ vertices of W embedded in V_i, each of which is adjacent to two removed leaves, and these removed leaves are each embedded in V_i except for the $\lfloor b/k \rfloor$ or $\lceil b/k \rceil$ leaves embedded in B. $\qquad \square$

5.2 Embedding the second subtree

Recall from the outline at the beginning of this section that, following the embedding of the first subtree T_1, the vertices which remain uncovered form a super-regular cycle of cluster tournaments. We wish to embed the second subtree T_2 so that all of these vertices are covered. The following lemma demonstrates that this is possible.

Lemma 5.2. *Suppose that $1/n \ll 1/C$ and that $1/n \ll 1/k \ll \varepsilon \ll d \ll \beta$. Let T be an oriented tree on n vertices with root r, with maximum degree $\Delta(T) \leq (\log n)^C$, and which contains at least βn distinct vertices that are each adjacent to at least one in-leaf and at least one out-leaf of T. Let G be a (d, ε)-super-regular cycle of cluster tournaments on n vertices whose clusters $V_1, \ldots, V_k$ each have size $\frac{n}{k} \pm \frac{2n}{\log \log n}$, and let v be a vertex of V_1. Then G contains a (spanning) copy of T in which r is embedded to v.*

Loosely speaking, the proof of Lemma 5.2 begins by removing a small number of in-leaves and out-leaves of T to obtain a subtree T'. We then select small disjoint subsets X_i and Y_i of V_i for each $i \in [k]$ with the property that each vertex in V_i has many inneighbours in each of X_{i-1} and Y_{i-1} and many outneighbours in each of X_{i+1} and Y_{i+1}, and so that most vertices in V_i have large semidegree in X_i. Removing these sets from G yields a subgraph G' of G which is a regular cycle of cluster tournaments, and we embed T' in G' using Lemmas 3.2 and 3.3. It remains to embed the removed leaves of T so as to cover all vertices of G which remain uncovered. We first use the fact that the image of each vertex of T' embedded in V_i has large semidegree in X_i to embed a small number of removed leaves to equalise the numbers of uncovered vertices in each cluster and the numbers of removed leaves needing to be embedded in that cluster, before completing the embedding by using the super-regularity of G to find perfect matchings in appropriate auxiliary bipartite graphs.

Proof. Introduce new constants η and γ such that $\varepsilon \ll \eta \ll \gamma \ll d$. Also define $\delta := \frac{2}{\log \log n}$ and $m := \frac{n}{k}$, so each cluster has size $m \pm \delta n$, assume without loss of generality that $\beta \leq \frac{1}{4}$, and let $t := \lceil \beta n \rceil - 1$. Choose a set W of t distinct vertices of T so that each $w \in W$ is adjacent to at least one in-leaf of T and at least one out-leaf of T and so that r is neither in W nor a leaf of T which is adjacent to a vertex of W (our assumption on T ensures that we can choose such a set W). Let T' be the oriented tree formed by deleting from T precisely one in-leaf and one out-leaf adjacent to each vertex of W, and take r to be the root of T'. Observe that T' then has precisely $n - 2t$ vertices and maximum degree $\Delta(T') \leq \Delta(T) \leq (\log n)^C \leq (\log(n - 2t))^{2C}$; in other words, T' meets the conditions of Lemma 3.3 with $n - 2t$ and $2C$ in place of n and C respectively. We will embed T' in an appropriate subgraph of G, which we find by using the following claim.

Claim 5.3. *For each $i \in [k]$ there exist sets $F_i, X_i, Y_i \subseteq V_i$ with $X_i, Y_i \subseteq F_i$ such that, writing $V_i' := V_i \setminus F_i$, we have*
   (i) $|F_i| \leq 3\gamma m$,
   (ii) X_i and Y_i are disjoint, and $v \in V_1'$,
   (iii) for each $x \in V_i' \setminus \{v\}$ we have $\deg^0(x, X_i) \geq \eta m$, and

(iv) for each $x \in V_i$ we have $\deg^-(x, X_{i-1}), \deg^-(x, Y_{i-1}), \deg^+(x, X_{i+1}), \deg^+(x, Y_{i+1}) \geq \eta m$.

Proof. For each $i \in [k]$ let $D_i \subseteq V_i$ consist of all vertices $x \in V_i$ with $\deg^0(x, V_i) < \gamma m/5$, so $|D_i| \leq \gamma m$ by Lemma 2.5. Then every vertex $x \in V_i \setminus D_i$ has $\deg^0(x, V_i) \geq \gamma m/5$. Also, since $V_1, \ldots, V_k$ are the clusters of a (d, ε)-super-regular cycle of cluster tournaments, every vertex $x \in V_i$ has at least $(d - \varepsilon)|V_{i-1}| \geq dm/2$ inneighbours in V_{i-1} and at least $(d - \varepsilon)|V_{i+1}| \geq dm/2$ outneighbours in V_{i+1}. For each $i \in [k]$ choose disjoint subsets $X_i, Y_i \subseteq V_i$ with $|X_i| = |Y_i| = \lfloor \gamma m \rfloor$ uniformly at random and independently of all other choices. Then for each $i \in [k]$ and each $x \in V_i$ the random variables $\deg^-(x, X_{i-1}), \deg^-(x, Y_{i-1}), \deg^+(x, X_{i+1})$ and $\deg^+(x, Y_{i+1})$ each have hypergeometric distribution with expectation at least $(dm/2)\lfloor \gamma m \rfloor/(m + \delta n) \geq d\gamma m/3 \geq 2\eta m$; if additionally $x \in V_i \setminus D_i$, then the random variables $\deg^+(x, X_i)$ and $\deg^-(x, X_i)$ each have hypergeometric distribution with expectation at least $(\gamma m/5)\lfloor \gamma m \rfloor/(m + \delta n) \geq \gamma^2 m/6 \geq 2\eta m$. The probability that any given one of these random variables is less than $\eta m + 1$ therefore decreases exponentially with n by Theorem 2.6, so by taking a union bound over all of these at most $6n$ events we find that with positive probability none of these random variables is less than $\eta m + 1$. Fix a choice of the sets X_i and Y_i with this property, then removing v from the sets X_1 and Y_1 if necessary and taking $F_1 := (X_1 \cup Y_1 \cup D_1) \setminus \{v\}$ and $F_i := X_i \cup Y_i \cup D_i$ for each $2 \leq i \leq k$ gives the desired sets. □

Fix sets F_i, X_i, Y_i and V_i' as in Claim 5.3, and observe that

$$|V_i'| = |V_i| - |F_i| \geq m - \delta n - 3\gamma m \geq \left(1 - \frac{\beta}{4}\right) m \geq (1 + \beta) \frac{n - 2t}{k}.$$

Let $G' := G[V_1' \cup \cdots \cup V_k']$, and note that by Lemma 2.1 G' is then a $(d, 2\varepsilon)$-regular cycle of cluster tournaments with clusters $V_1', \ldots, V_k'$. Observe also that v has at least $(d - \varepsilon)|V_2| - |F_2| \geq \gamma m$ outneighbours in V_2' and at least $(d - \varepsilon)|V_k| - |F_k| \geq \gamma m$ inneighbours in V_k'. In other words, G' meets the conditions of Lemma 3.3 with $n - 2t$, β and 2ε in place of n, α and ε respectively (so, in particular, m there corresponds to $m - 2t/k$ here).

Apply the Vertex Allocation Algorithm (Algorithm 1) to allocate the vertices of T' to the clusters $V_1', \ldots, V_k'$ of G'. For each $i \in [k]$ let T_i' consist of all vertices of T' allocated to the cluster V_i', and likewise let $W_i \subseteq W$ consist of all vertices of W allocated to the cluster V_i'. By Lemma 3.2(a) the allocation we obtain from the Vertex Allocation Algorithm will be semi-canonical. Furthermore, by two applications of Lemma 3.2(d) (with $n - 2t$, $2C$ and $\beta/2$ in place of n, C and α respectively) we find with probability $1 - o(1)$ that for every $i \in [k]$ we have

$$|W_i| := |W| \left(\frac{1}{k} \pm \frac{1}{\log\log(n - 2t)}\right) = \frac{t}{k} \pm \delta n \tag{3}$$

and

$$|T_i'| := (n - 2t) \left(\frac{1}{k} \pm \frac{1}{\log\log(n - 2t)}\right) = m - \frac{2t}{k} \pm \delta n. \tag{4}$$

Fix an allocation with these properties, and observe that this allocation then meets the conditions of Lemma 3.3 with $n - 2t$ and β in place of n and α respectively. So we may apply Lemma 3.3 to obtain an embedding φ of T' in G' such that each vertex of T' is embedded in the cluster to which it was allocated and so that $\varphi(r) = v$.

For each $i \in [k]$, let $U_i \subseteq V_i$ be the set of vertices of V_i not covered by φ and let $U := \bigcup_{i \in [k]} U_i$. Then, since every vertex was embedded in the cluster to which it was allocated, by (4) we have for each $i \in [k]$ that

$$|U_i| = |V_i| - |T_i'| = (m \pm \delta n) - \left(m - \frac{2t}{k} \pm \delta n\right) = \frac{2t}{k} \pm 2\delta n, \tag{5}$$

and since $|G| = n$ and $|T'| = n - 2t$ we have $|U| = 2t$. Also, for each $i \in [k]$, let $P_i := \varphi(W_i)$ and write $P := \bigcup_{i \in [k]} P_i$. In other words, P_i (respectively P) is the set of vertices of G to which vertices of W_i (respectively W) were embedded. So $P_i \subseteq V_i'$ and $|P_i| = |W_i|$, and similarly $|P| = |W| = t$.

Our goal for the remaining part of the proof is to choose, for each $x \in P$, an inneighbour x^- of x in U and an outneighbour x^+ of x in U such that the chosen inneighbours and outneighbours are all distinct. Indeed, for each vertex $w \in W$ there is a unique vertex $x \in P$ with $\varphi(w) = x$. Let w^+ and w^- denote the out-leaf and in-leaf adjacent to w which we removed when forming T'; we could then embed w^+ to x^+ and w^- to x^-, and doing so for each $w \in W$ would extend φ to an embedding of T in G, completing the proof. If for every $i \in [k]$ both $G[U_{i-1} \to P_i]$ and $G[P_i \to U_{i+1}]$ are super-regular and $|U_i| = |P_{i-1}| + |P_{i+1}|$, then (after appropriately partitioning the sets U_i) we could apply Lemma 2.2 to find, for each $i \in [k]$ and each $x \in P_i$, vertices $x^- \in U_{i-1}$ and $x^+ \in U_{i+1}$ satisfying the above properties. However, neither of these assumptions is necessarily valid. Over the following steps of the proof we embed the removed leaves adjacent to a small number of vertices of W so that these assumptions do indeed hold for the remaining vertices; we then complete the embedding of T in G in the manner described above.

Step 1: Balancing the sets W_i and ensuring super-regularity. The first step of this process is to embed the removed leaves adjacent to a small number of vertices of W so that equally many vertices in each set W_i have not had their adjacent removed leaves embedded. We also cover all vertices in each set U_i which have too few inneighbours in P_{i-1} or too few outneighbours in P_{i+1}; this will ensure that the auxiliary bipartite graphs which we consider at the end of the proof are super-regular.

For each $i \in [k]$ define $s_i := \lfloor 4\varepsilon m \rfloor + |W_i| - \min_{i \in [k]} |W_i|$, so by (3) we have $\lfloor 4\varepsilon m \rfloor \leq s_i \leq 4\varepsilon m + 2\delta n$. Also, for each $i \in [k]$, let B_i^- be the set of vertices in U_i with fewer than ηm inneighbours in P_{i-1}, and let B_i^+ be the set of vertices in U_i with fewer than ηm outneighbours in P_{i+1}. Since $G[V_{i-1} \to V_i]$ is $(d_\geq, \varepsilon)$-regular, and $|P_{i-1}| = |W_{i-1}| \geq t/k - \delta n > \varepsilon |V_{i-1}|$ by (3), we must have $|B_i^-| \leq \varepsilon |V_i| \leq 2\varepsilon m$; likewise, since $G[V_i \to V_{i+1}]$ is $(d_\geq, \varepsilon)$-regular and $|P_{i+1}| > \varepsilon |V_{i+1}|$, we must have $|B_i^+| \leq \varepsilon |V_i| \leq 2\varepsilon m$. So we may choose for each $i \in [k]$ a subset $B_i \subseteq U_i$ of size $|B_i| = s_i$ with $B_i^- \cup B_i^+ \subseteq B_i$.

Next, for each $i \in [k]$ we proceed as follows. Let $\{b_1, \ldots, b_{s_i}\}$ be the vertices in B_i, arbitrarily choose distinct vertices $w_1, \ldots, w_{s_i} \in W_i$, and for each $j \in [s_i]$ let $p_j := \varphi(w_j)$, so $p_j \in P_i$. Since $W \subseteq V(T') \setminus \{r\}$, for each $j \in [s_i]$ the vertex p_j was embedded in $V_i' \setminus \{v\}$, so by Claim 5.3(iii) we have $\deg^0(p_j, X_i \setminus B_i) \geq \deg^0(p_j, X_i) - |B_i| = \eta m - s_i \geq s_i$. We may therefore choose distinct vertices $x_1, \ldots, x_{s_i}$ in $X_i \setminus B_i$ such that for each $j \in [s_i]$, the vertex x_j is an inneighbour of p_j if $b_j \in N^+(p_j)$, whilst x_j is an outneighbour of p_j if $b_j \in N^-(p_j)$. For each $j \in [s_i]$ let w_j^+ be the removed out-leaf of T adjacent to w_j and let w_j^- be the removed in-leaf of T adjacent to w_j. If $b_j \in N^+(p_j)$ then we set $\varphi(w_j^+) = b_j$ and $\varphi(w_j^-) = x_j$, whilst if $b_j \in N^-(p_j)$ then we set $\varphi(w_j^-) = b_j$ and $\varphi(w_j^+) = x_j$. Observe that our choice of vertices $x_1, \ldots, x_{s_i}$ ensures that these embeddings are consistent with the directions of the edges $w_j^- \to w_j$ and $w_j \to w_j^+$.

Having carried out these steps for each $i \in [k]$ we have extended the embedding φ to cover all vertices in $B_1 \cup \cdots \cup B_k$. For each $i \in [k]$ we now define $W_i^0 := W_i \setminus \{w_1, \ldots w_{s_i}\}$ and $P_i^0 := P_i \setminus \{p_1, \ldots, p_{s_i}\}$. In other words, W_i^0 is the set of vertices of W which were embedded in V_i' and whose adjacent removed leaves have not yet been embedded, and P_i^0 is the set of vertices of G to which vertices of W_i^0 have been embedded. By (3) we then have

$$|P_i^0| = |W_i^0| = |W_i| - s_i = \min_{i \in [k]} |W_i| - \lfloor 4\varepsilon m \rfloor = \frac{t}{k} - 4\varepsilon m \pm \delta n, \tag{6}$$

so in particular we have $|W_1^0| = \cdots = |W_k^0| = |P_1^0| = \cdots = |P_k^0|$. Similarly, for each $i \in [k]$ we define $U_i^0 := U_i \setminus \{b_1, x_1, \ldots, b_{s_i}, x_{s_i}\}$. In other words, U_i^0 is the set of vertices of V_i which have not yet been covered by φ. By (5) we then have

$$|U_i^0| = |U_i| - 2s_i = \frac{2t}{k} - 8\varepsilon m \pm 6\delta n. \tag{7}$$

Write $W^0 := \bigcup_{i \in [k]} W_i^0$, $P^0 := \bigcup_{i \in [k]} P_i^0$, and $U^0 := \bigcup_{i \in [k]} U_i^0$. So in particular U^0 is the set of vertices of G which remain uncovered. Since there are two such vertices for each vertex of W^0, and $|W_1^0| = |W_2^0| = \cdots = |W_k^0|$, it follows that $|U^0|$ is divisible by $2k$.
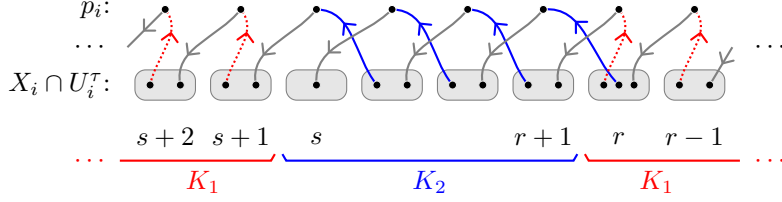
Figure 2: This diagram illustrates how the embedding φ is extended at each step of the balancing algorithm. The vertices at the top are the vertices $p_1, \ldots, p_k$, which lie in the sets $P_1^\tau, \ldots, P_k^\tau$ respectively, and the shaded areas represent the sets $X_i \cap U_i^\tau$ for $i \in [k]$ (that is, the vertices of X_i not yet covered by φ). The extension of φ at step τ then covers the vertices appearing in the shaded areas, so three extra vertices are covered from V_r, one from V_s, and two from each other cluster.

Step 2: Balancing the numbers of uncovered vertices. Our next step is to embed the removed leaves adjacent to a small number of vertices of W so that, following these embeddings, there are equally many uncovered vertices within each cluster (we also preserve the properties ensured in Step 1). We achieve this by applying the following 'balancing algorithm'. Each iteration of this algorithm will extend φ by embedding, for each $i \in [k]$, the removed in-leaf and out-leaf adjacent to some vertex in W_i.

More precisely, the balancing algorithm proceeds as follows. For each time $\tau \geq 0$ and for each $i \in [k]$, we let $W_i^\tau \subseteq W_i$ be the set of vertices of T whose adjacent removed leaves have not yet been embedded, we let $P_i^\tau \subseteq P_i$ be the set of vertices of G to which vertices of W_i^τ have been embedded, and we let $U_i^\tau \subseteq V_i$ be the set of uncovered vertices in V_i at time τ. Observe that these definitions of W_i^0, P_i^0 and U_i^0 coincide with those given above. We also define the quantity $M^\tau := \frac{1}{k} \sum_{i \in [k]} |U_i^\tau|$, so M^τ is the average number of uncovered vertices per cluster at time τ. Our observation above that $|U^0|$ is divisible by $2k$ ensures that M^0 is an even integer, and in fact the algorithm will ensure that M^τ is an even integer at each time $\tau \geq 0$. At time step τ, if $|U_i^\tau| = M^\tau$ for all $i \in [k]$, then we stop with success. Otherwise, since M^τ is an integer, we may choose $r, s \in [k]$ with $|U_r^\tau| \geq M^\tau + 1$ and $|U_s^\tau| \leq M^\tau - 1$. Define $K_1 := \{s+1, s+2, \ldots, r-1, r\}$ and $K_2 := \{r+1, \ldots, s\} = [k] \setminus K_1$, with addition taken modulo k. For each $i \in [k]$, we choose a vertex $w_i \in W_i^\tau$, and let $p_i \in P_i^\tau$ be the vertex to which w_i was embedded. We also choose a vertex $x_i^+ \in N^+(p_i) \cap X_{i+1} \cap U_{i+1}^\tau$ and, if $i \in K_1$ then we choose a vertex $x_i^- \in N^-(p_i) \cap X_i \cap U_i^\tau$, whilst if $i \in K_2$ then we choose a vertex $x_i^- \in N^-(p_i) \cap X_{i-1} \cap U_{i-1}^\tau$. We make these choices so that the $2k$ vertices $\{x_1^-, x_1^+, \ldots, x_k^-, x_k^+\}$ are all distinct (if it is not possible to make such choices then we terminate with failure, but we shall see shortly that this will not happen). For each $i \in [k]$ let w_i^+ be the removed out-leaf of T adjacent to w_i and let w_i^- be the removed in-leaf of T adjacent to w_i; we then set $\varphi(w_i^-) := x_i^-$ and $\varphi(w_i^+) := x_i^+$ (see Figure 2 for an illustration of this embedding). To conclude this iteration of the algorithm, for each $i \in [k]$ we update the sets W_i^τ, P_i^τ and U_i^τ by setting $W_i^{\tau+1} := W_i^\tau \setminus \bigcup_{i \in [k]} \{w_i\}$, $P_i^{\tau+1} := P_i^\tau \setminus \bigcup_{i \in [k]} \{p_i\}$, and $U_i^{\tau+1} := U_i^\tau \setminus \bigcup_{i \in [k]} \{x_i^+, x_i^-\}$. Observe that we then have

$$|U_i^{\tau+1}| = \begin{cases} |U_i^\tau| - 3 & \text{if } i = r, \\ |U_i^\tau| - 1 & \text{if } i = s, \text{ and} \\ |U_i^\tau| - 2 & \text{otherwise.} \end{cases} \tag{8}$$

In particular it follows that $M^{\tau+1} = M^\tau - 2$; since M^τ was an even integer it follows that $M^{\tau+1}$ is an even integer, as required.

Claim 5.4. *The balancing algorithm described above stops with success after at most $3k\delta n$ iterations.*

Proof. We first check that we can choose vertices w_i, p_i, x_i^- and x_i^+ as described whenever $\tau \leq 3k\delta n$. First observe that, for each $i \in [k]$, since $|W_i^0| \geq t/2k > 3k\delta n$ by (6), and at most

one vertex is removed from W_i^τ at each step τ of the balancing algorithm (and its image is removed from P_i^τ), there are at least $|W_i^0| - \tau \geq 1$ possible choices for w_i at step $\tau \leq 3k\delta n$ of the balancing algorithm. So we may choose the vertices w_i and p_i for $i \in [k]$ as claimed. Next observe that for each $i \in [k]$ at most $2s_i \leq 8\varepsilon m + 4\delta n$ vertices were embedded in X_i in Step 1. Also, each iteration of the balancing algorithm embeds at most three vertices in X_i, so at time $\tau \leq 3k\delta n$ the total number of vertices which have so far been embedded in X_i is at most $3\tau + 8\varepsilon m + 4\delta n \leq 9k\delta n + 9\varepsilon m \leq \eta m/2$. Since $p_i \in V_i'$, it follows by Claim 5.3(iii) and (iv) that $\deg^-(p_i, X_{i-1} \cap U_{i-1}^\tau) \geq \eta m/2$, that $\deg^+(p_i, X_{i+1} \cap U_{i+1}^\tau) \geq \eta m/2$ and that $\deg^-(p_i, X_i \cap U_i^\tau) \geq \eta m/2$. So we may greedily choose the vertices x_i^- and x_i^+ for each $i \in [k]$ as desired.

It therefore suffices to prove that the algorithm stops after at most $3k\delta n$ iterations and thus, because it cannot fail in these early steps, it always stops successfully. For each $\tau \geq 0$ let $\Upsilon^\tau := \sum_{i \in [k]} \big| |U_i^\tau| - M^\tau \big|$, so Υ^τ is a non-negative integer. In particular by (7) we have

$$\Upsilon^0 = \sum_{i \in [k]} \big| |U_i^0| - M^0 \big| \leq 6k\delta n, \tag{9}$$

Also, by (8) we have $|U_r^{\tau+1}| = |U_r^\tau| - 3$ and $M^{\tau+1} = M^\tau - 2$; by our choice of r it follows that $\big| |U_r^{\tau+1}| - M^{\tau+1} \big| = \big| |U_r^\tau| - M^\tau \big| - 1$. Similarly we find that $\big| |U_s^{\tau+1}| - M^{\tau+1} \big| = \big| |U_s^\tau| - M^\tau \big| - 1$ and that $\big| |U_j^{\tau+1}| - M^{\tau+1} \big| = \big| |U_j^\tau| - M^\tau \big|$ for each $j \in [k] \setminus \{r, s\}$. Together these equalities imply that $\Upsilon^{\tau+1} = \Upsilon^\tau - 2$. Since Υ^τ is always non-negative, we conclude that for some $\tau \leq 3k\delta n$ we must have $\Upsilon^\tau = 0$. It follows that $|U_j^\tau| = M^\tau$ for all $j \in [k]$, and so the algorithm will stop at step τ. $\qquad\square$

Returning to the proof of Lemma 5.2, we conclude that the balancing algorithm will stop with success at some time τ_{end} with $\tau_{\text{end}} \leq 3k\delta n$. For each $i \in [k]$, let $W_i^* := W_i^{\tau_{\text{end}}}$, $P_i^* := P_i^{\tau_{\text{end}}}$, and $U_i^* := U_i^{\tau_{\text{end}}}$, and write $W^* := \bigcup_{i \in [k]} W_i^*$, $P^* := \bigcup_{i \in [k]} P_i^*$, and $U^* := \bigcup_{i \in [k]} U_i^*$. So the embedding φ now covers all vertices of $V(G)$ except for those in U^*, and the only vertices of T which remain to be embedded are one in-leaf and one out-leaf of each vertex of W^*. In particular we have $|U^*| = 2|P^*| = 2|W^*|$. Observe that in the execution of the balancing algorithm, at each time τ and for each $i \in [k]$ precisely one vertex was removed from W_i^τ. Therefore, since we initially had $|W_1^0| = \cdots = |W_k^0|$ by (6), we now have $|W_1^*| = \cdots = |W_k^*|$. We denote this common size by L, and note that by (6) we then have $L \geq t/k - 4\varepsilon m - \delta n - \tau_{\text{end}} \geq 2t/3k$. Also, since $\Upsilon^{\tau_{\text{end}}} = 0$, we must have $|U_1^*| = \cdots = |U_k^*| = M^{\tau_{\text{end}}}$, so

$$L = |W_1^*| = \cdots = |W_k^*| = |P_1^*| = \cdots = |P_k^*| = \frac{1}{2}|U_1^*| = \cdots = \frac{1}{2}|U_k^*| \geq \frac{2t}{3k} \geq \frac{\beta m}{2}. \tag{10}$$

Step 3: Completing the embedding. We are now ready to complete the embedding of T in G as described previously, beginning with the following claim.

Claim 5.5. *For each $i \in [k]$ each vertex in U_i^* has at least $\eta m/2$ inneighbours in P_{i-1}^* and at least $\eta m/2$ outneighbours in P_{i+1}^*, and each vertex in P_i^* has at least ηm inneighbours in U_{i-1}^* and at least ηm outneighbours in U_{i+1}^*.*

Proof. Recall that the set B_i chosen in Step 1 contained all vertices of U_i with fewer than ηm inneighbours in P_{i-1} or fewer than ηm outneighbours in P_{i+1}. All vertices of B_i were covered in Step 1, so no vertex of B_i is contained in U_i^*. The first statement then follows from the fact that for each $j \in [k]$ we have

$$|P_j \setminus P_j^*| = |P_j \setminus P_j^0| + |P_j^0 \setminus P_j^*| \leq s_j + \tau_{\text{end}} \leq 4\varepsilon m + 2\delta n + 3k\delta n \leq \frac{\eta m}{2}.$$

For the second statement observe that no vertices have yet been embedded in any set Y_j, so $Y_{i-1} \subseteq U_{i-1}^*$ and $Y_{i+1} \subseteq U_{i+1}^*$. Moreover, since $P_i^* \subseteq P_i \subseteq V_i$, by Claim 5.3(iv) every vertex of P_i^* has at least ηm inneighbours in Y_{i-1} and at least ηm outneighbours in Y_{i+1}. $\qquad\square$

For each $i \in [k]$ we now partition U_i^* into disjoint sets U_i^- and U_i^+ each of size L uniformly at random and independently of all other choices. Since $G[V_i \to V_{i+1}]$ is $(d_\geq, \varepsilon)$-regular for each $i \in [k]$, by (10) and Lemma 2.1 both $G[U_{i-1}^- \to P_i^*]$ and $G[P_i^* \to U_{i+1}^+]$ are then $(d_\geq, \varepsilon')$-regular, where $\varepsilon' := 3\varepsilon/\beta$. Also, by Claim 5.5, each $u \in U_{i-1}^-$ has $\deg^+(u, P_i^*) \geq \eta m/2 \geq \eta L/2$ and each $u \in U_{i+1}^+$ has $\deg^-(u, P_i^*) \geq \eta m/2 \geq \eta L/2$. Furthermore, for each $p \in P_i^*$ the random variables $\deg^-(p, U_{i-1}^-)$ and $\deg^+(p, U_{i+1}^+)$ each have hypergeometric distributions with expectation at least $\eta m L/2L \geq \eta L/2$. Applying Theorem 2.6 and taking a union bound we find that with positive probability we have for every $i \in [k]$ and every $p \in P_i^*$ that $\deg^-(p, U_{i-1}^-) \geq \eta L/4$ and $\deg^+(p, U_{i+1}^+) \geq \eta L/4$. Fix such an outcome of our random selection; then for each $i \in [k]$ the underlying graphs of both $G[U_{i-1}^- \to P_i^*]$ and $G[P_i^* \to U_{i+1}^+]$ are $(\eta/4, \varepsilon')$-super-regular balanced bipartite graphs with vertex classes of size L.

We may therefore apply Lemma 2.2 to obtain, for each $i \in [k]$, a perfect matching M_i^- in $G[U_{i-1}^- \to P_i^*]$ and a perfect matching M_i^+ in $G[P_i^* \to U_{i+1}^+]$. For each $i \in [k]$ and each $w \in W_i^*$ let w^- be the removed in-leaf of T adjacent to w and let w^+ be the removed out-leaf of T adjacent to w. Also let $p = \varphi(w)$ and let $q^- \in U_{i-1}^-$ and $q^+ \in U_{i+1}^+$ be the vertices matched to p in M_i^- and M_i^+ respectively, and set $\varphi(w^-) := q^-$ and $\varphi(w^+) := q^+$. Since each $p \in P_i^*$ is matched to precisely one inneighbour in U_{i-1}^- and precisely one outneighbour in U_{i+1}^+, this extends φ to an embedding of T in G. $\qquad\square$

5.3 Joining the pieces

As outlined at the start of this section, we will 'split' our tree T into two subtrees T_1 and T_2, which we embed successively in G using Lemmas 5.1 and 5.2. Definition 5.6 makes this notion precise, following which Lemma 5.7 shows that every oriented tree admits such a split.

Definition 5.6. Let T be a tree or oriented tree. A *tree-partition* of T is a pair $\{T_1, T_2\}$ of edge-disjoint subtrees of T such that $V(T_1) \cup V(T_2) = V(T)$ and $E(T_1) \cup E(T_2) = E(T)$ (so in particular T_1 and T_2 have precisely one vertex in common).

Lemma 5.7. *Let T be a tree or oriented tree. For every set $L \subseteq V(T)$ there exists a tree-partition $\{T_1, T_2\}$ of T such that T_1 and T_2 each contain at least $|L|/3$ vertices of L.*

We prove Lemma 5.7 using the following simple fact.

Fact 5.8. *Let $x_1, \ldots, x_s$ be non-negative integers. If $x_1, \ldots, x_s \leq c$ and $x_1 + \cdots + x_s \geq 3c$, then there exists $i \in [s]$ such that $c \leq x_1 + \cdots + x_i \leq 2c$.*

For a tree or oriented tree T, an edge $e \in E(T)$ and a vertex $v \in e$, we write $T - e$ for the oriented forest we obtain by deleting e from T, and write C_v^e for the vertex set of the component of $T - e$ which contains v.

Proof of Lemma 5.7. Since edge orientations do not affect the validity of a tree-partition, we may assume that T is an undirected tree. Define $\ell := |L|$. For each edge $e = \{u, v\} \in E(T)$ we say that v is a *heavy neighbour* of u if $|C_v^e \cap L| \geq \ell/3$. Observe that if u and v are both heavy neighbours of each other, then $\{T[C_u^e], T[C_v^e \cup \{u\}]\}$ is the desired tree-partition. We may therefore assume that for each edge $e = \{u, v\} \in E(T)$ either u is a heavy neighbour of v or v is a heavy neighbour of u, but not both. It follows that some vertex v has no heavy neighbours (to see this, form an auxiliary orientation of $E(T)$ with each edge directed $u \to v$ where v is a heavy neighbour of u, and choose v to be a vertex with no outneighbours). Let $C_1, \ldots, C_s$ be the vertex sets of the components of $T - v$. For each $i \in [s]$ let $\ell_i := |C_i \cap L|$, and observe that since v has no heavy neighbours we have $\ell_i < \ell/3$; since ℓ_i is an integer, we then have $\ell_i \leq (\ell - 1)/3$.

If $v \in L$, then $\ell_1 + \cdots + \ell_s = \ell - 1$ and by Fact 5.8 there exists $j \in [s]$ with $(\ell - 1)/3 \leq \ell_1 + \cdots + \ell_j \leq 2(\ell - 1)/3$. In this case the desired tree-partition is $\{T[\{v\} \cup \bigcup_{1 \leq i \leq j} C_i], T[\{v\} \cup \bigcup_{j < i \leq s} C_i]\}$, because each of these subtrees contains v and hence contains at least $(\ell - 1)/3 + 1 > \ell/3$ vertices of L. On the other hand, if $v \notin L$, then $\ell_1 + \cdots + \ell_s = \ell$, so by Fact 5.8 above there exist $j \in [s]$ with $\ell/3 \leq \ell_1 + \cdots + \ell_j \leq 2\ell/3$, and again $\{T[\{v\} \cup \bigcup_{1 \leq i \leq j} C_i], T[\{v\} \cup \bigcup_{j < i \leq s} C_i]\}$ is the desired tree-partition. $\qquad\square$

We are now ready to state and prove Lemma 5.9, the main result of this section.

Lemma 5.9. *Suppose that $1/n \ll 1/C$ and that $1/n \ll 1/k \ll \varepsilon \ll d \ll \psi \ll \alpha$. Let T be an α-nice oriented tree on n vertices with maximum degree $\Delta(T) \leq (\log n)^C$. Also let G be a tournament on n vertices which contains a (d, ε)-regular cycle of cluster tournaments whose clusters $V_1, \ldots, V_k$ have equal size such that $B := V(G) \setminus \bigcup_{i \in [k]} V_i$ has size $|B| \leq \psi n$. Then G contains a (spanning) copy of T.*

Proof of Lemma 5.9. Introduce a new constant β with $\psi \ll \beta \ll \alpha$, and define $m := |V_1| = \cdots = |V_k| = (n - |B|)/k$ and $s := \lceil \alpha n \rceil$. Since T is α-nice we may choose a set L of s distinct vertices of T such that each vertex in L is adjacent to at least one in-leaf and at least one out-leaf of T. Apply Lemma 5.7 to obtain a tree-partition $\{T_1, T_2\}$ of T such that the subtrees T_1 and T_2 each contain at least $s/3$ vertices of L. Let r be the unique common vertex of T_1 and T_2, which we take as the root of each subtree, and observe that for each vertex $x \neq r$ every neighbour of x is contained in the same subtree as x. So in particular T_1 contains at least $s/3 - 1 \geq \alpha n/4 \geq \beta n$ vertices each adjacent to at least one in-leaf and at least one out-leaf of T_1, and likewise T_2 contains at least $\alpha n/4 \geq \beta n$ vertices each adjacent to at least one in-leaf and at least one out-leaf of T_2. Write $n_1 := |T_1|$ and $n_2 := |T_2|$, so $3\alpha n/4 \leq n_1, n_2$ and $n_1 + n_2 = n + 1$. By relabelling if necessary we may assume that $n_1 \leq n_2$. Observe also that $\Delta(T_1) \leq \Delta(T) \leq (\log n)^C \leq (\log n_1)^{2C}$ and likewise that $\Delta(T_2) \leq (\log n_2)^{2C}$. So T_1 meets the conditions of Lemma 5.1 with $2C$ and n_1 in place of C and n respectively, and likewise T_2 meets the conditions of Lemma 5.2 with $2C$ and n_2 in place of C and n respectively.

Next, proceed as follows for each $i \in [k]$. Define $B_i^+ := \{\, v \in V_i : \deg^+(v, V_{i+1}) < (d - \varepsilon)m \,\}$, and $B_i^- := \{\, v \in V_i : \deg^-(v, V_{i-1}) < (d - \varepsilon)m \,\}$. Since $G[V_{i-1} \to V_i]$ and $G[V_i \to V_{i+1}]$ are each $(d_{\geq}, \varepsilon)$-regular, we must then have $|B_i^-|, |B_i^+| < \varepsilon m$. Let B_i be a set of $2\varepsilon m$ vertices such that $B_i^- \cup B_i^+ \subseteq B_i \subseteq V_i$ and define $V_i' := V_i \setminus B_i$. It follows that for every vertex $x \in V_i'$ we have $\deg^-(v, V_{i-1}'), \deg^+(v, V_{i+1}') \geq (d - \varepsilon)m - 2\varepsilon m = (d - 3\varepsilon)m$. Choose a subset $Z_i \subseteq V_i'$ of size $|Z_i| = \alpha m/5$ uniformly at random and independently of all other choices. So for each $x \in V_i'$ the random variables $\deg^-(x, Z_{i-1})$ and $\deg^+(x, Z_{i+1})$ have hypergeometric distributions with expectation at least $(d - 3\varepsilon)\alpha m/5$. Applying Theorem 2.6 and taking a union bound we find that with positive probability we have for every $x \in V_i'$ that $\deg^-(x, Z_{i-1}), \deg^+(x, Z_{i+1}) \geq d\alpha m/10$. Fix an outcome of the random selections for which this event occurs.

Define $B' := B \cup \bigcup_{i \in [k]} B_i$, so $|B'| = |B| + 2k\varepsilon m \leq 2\psi n$. Next choose arbitrarily a set $X_i \subseteq V_i' \setminus Z_i$ of size $(1 + \alpha/4)n_1/k$ for each $i \in [k]$; this is possible since for each $i \in [k]$ we have

$$|V_i' \setminus Z_i| = (1 - 2\varepsilon)m - \frac{\alpha m}{5} \geq \left(1 - \frac{\alpha}{4}\right) m \geq \left(1 - \frac{\alpha}{4}\right)(1 - \psi)\frac{n}{k} \geq \left(1 - \frac{\alpha}{3}\right)\frac{n}{k} \geq \left(1 + \frac{\alpha}{3}\right)\frac{n_1}{k},$$

where the final inequality uses the fact that $n_1 \leq n + 1 - n_2 \leq n + 1 - 3\alpha n/4 \leq (1 - 2\alpha/3)n$. Define $G_1 := G[B' \cup \bigcup_{i \in [k]} X_i]$. Since $G[V_i \to V_{i+1}]$ is $(d_{\geq}, \varepsilon)$-regular for each $i \in [k]$, and $n_1 \geq 3\alpha n/4$, it follows by Lemma 2.1 that the sets $X_1, \ldots, X_k$ are the clusters of a (d, ε')-regular cycle of cluster tournaments in G_1, where $\varepsilon' := 4\varepsilon/3\alpha$. The tournament G_1, the clusters X_i and the set B' therefore meet the conditions of Lemma 5.1 with $n_1, \alpha/3, \varepsilon'$ and 2ψ in place of n, α, ε and ψ respectively. So we may apply Lemma 5.1 to obtain an embedding φ of T_1 in G_1 so that r is embedded in X_1, so that every vertex of B' is covered, and so that for each $i \in [k]$ we have

$$|\varphi(V(T)) \cap X_i| = (n_1 - |B'|)\left(\frac{1}{k} \pm \frac{2}{\log\log n_1}\right) = \frac{n_1 - |B'|}{k} \pm \left(\frac{2n_1}{\log\log n_1} - 2\right). \tag{11}$$

For convenience of notation write $E := \frac{2n_2}{\log\log n_2} \geq \frac{2n_1}{\log\log n_1}$. For each $i \in [k]$ define $U_i := V_i \setminus \varphi(V(T))$, so U_i contains all vertices of V_i not covered by our embedding of T_1. Then by (11) we have for each $i \in [k]$ that

$$|U_i| = |V_i \setminus B_i| - \frac{n_1 - |B'|}{k} \pm (E - 2) = m - 2\varepsilon m - \frac{n_1}{k} + \frac{|B| + 2k\varepsilon m}{k} \pm (E - 2)$$
$$= \frac{n - |B|}{k} - \frac{n_1}{k} + \frac{|B|}{k} \pm (E - 2) = \frac{n_2}{k} \pm (E - 1),$$

where the second equality uses the fact that $|B'| = |B| + 2k\varepsilon m$, and the final equality uses the fact that $n_2 = n + 1 - n_1$. Let $v = \varphi(r)$, so $v \in X_1$, and set $U_1^* := U_1 \cup \{v\}$ and $U_i^* := U_i$ for $2 \leq i \leq k$, so $|U_i^*| = \frac{n_2}{k} \pm E$ for each $i \in [k]$. In particular, we have $|U_i^*| \geq \alpha n / 2k \geq \alpha |V_i| / 2$ for each $i \in [k]$, so by Lemma 2.1 the sets $U_1^*, \ldots, U_k^*$ are the clusters of a spanning $(d, 2\varepsilon/\alpha)$-regular cycle of cluster tournaments in the tournament $G_2 := G[U_1^* \cup \cdots \cup U_k^*]$. Furthermore, for each $i \in [k]$ we have $Z_i \subseteq U_i^* \subseteq V_i'$ (since we chose X_i to be disjoint from Z_i, and every vertex of B was covered by the embedding of T_1), so every vertex $u \in U_i^*$ has $\deg^-(x, U_{i-1}^*), \deg^+(x, U_{i+1}^*) \geq d\alpha m / 10$. So in fact the clusters $U_1^*, \ldots, U_k^*$ form a spanning $(d\alpha/10, 2\varepsilon/\alpha)$-super-regular cycle of cluster tournaments in G_2. In other words, the tournament G_2, the clusters $U_1^*, \ldots, U_k^*$ and the vertex v meet the conditions of Lemma 5.2 with $d\alpha/10, 2\varepsilon/\alpha$ and n_2 in place of d, ε and n respectively. Since $|G_2| = |G| - |T_1| + 1 = n - n_1 + 1 = n_2 = |T_2|$ we may therefore apply Lemma 5.2 to find a spanning copy of T_2 in G_2 in which r is embedded to v, and then the embeddings of T_1 and T_2 together form a spanning copy of T in G. $\qquad\square$

6 Proofs of main theorems

In this section we give the proofs of Theorem 1.4 (that every large nice oriented tree of polylogarithmic maximum degree is unavoidable) and Theorem 1.6 (that a random labelled oriented tree is nice asymptotically almost surely).

6.1 A class of unavoidable oriented trees

We begin by combining the results of the previous two sections to prove Theorem 1.4. The main task is to use Lemma 2.3 to show that we can find either an almost-directed pair in G which partitions $V(G)$ or an almost-spanning cycle of cluster tournaments in G. In the former case we then embed T in G using Lemma 4.3, whilst in the latter case we embed T in G using Lemma 5.9.

Proof of Theorem 1.4. Introduce new constants $k_0, k_1, \varepsilon, d, \mu, \eta, \omega$ and γ such that

$$\frac{1}{n} \ll \frac{1}{k_1} \ll \frac{1}{k_0} \ll \varepsilon \ll d \ll \mu \ll \eta \ll \omega \ll \gamma \ll \alpha.$$

We may also assume that $1/n \ll 1/C$. Let G be a tournament on n vertices, and let T be an α-nice tree on n vertices such that $\Delta(T) \leq (\log n)^C$. We choose vertex-disjoint subsets $X, Y, Z \subseteq V(G)$ such that
 (a) $X \cup Y \cup Z = V(G)$,
 (b) $|Y| \geq n/3$, and
 (c) $e(G(Y \to X)) + e(G(Z \to X)) + e(G(Z \to Y)) \leq \min(\eta(|X| + |Z|)n, 3\gamma\eta n^2)$.
Moreover, we make this choice so that $|Y|$ is minimal among all choices of X, Y and Z which satisfy (a)–(c) above (taking $Y = V(G)$ and $X = Z = \varnothing$ shows that such subsets do exist).

Suppose first that $|Y| \leq (1 - 2\gamma)n$. Then we have either $|X| \geq \gamma n$ or $|Z| \geq \gamma n$. If $|X| \geq \gamma n$ then, taking $A := X$ and $B := Y \cup Z$, we have a partition $\{A, B\}$ of $V(G)$ into sets $|A|, |B| \geq \gamma n$ such that the number of edges directed from B to A is $e(G(Y \to X)) + e(G(Z \to X)) \leq 3\gamma\eta n^2 \leq \omega|A||B|$ by (c), so (A, B) is an ω-almost-directed pair in G. If instead $|Z| \geq \gamma n$ then a similar argument shows that taking $A := X \cup Y$ and $B = Z$ gives a partition $\{A, B\}$ of $V(G)$ into sets $|A|, |B| \geq \gamma n$ such that (A, B) is an ω-almost-directed pair in G. Either way, we may then apply Lemma 4.3 (with ω and γ in place of μ and ν respectively) to find a copy of T in G.

Now suppose instead that $|Y| > (1 - 2\gamma)n$, and write $G' := G[Y]$. Observe in particular that we then have $|X| + |Z| = n - |Y| < 2\gamma n$, so (c) states that $e(G(Y \to X)) + e(G(Z \to X)) + e(G(Z \to Y)) \leq \eta(|X| + |Z|)n$. If there exists a vertex $y \in Y$ with $\deg_{G'}^-(y) < \eta n$, then moving y from Y to X would increase $e(G(Y \to X))$ by less than ηn whilst increasing $|X|$ by one and leaving $e(G(Z \to X)) + e(G(Z \to Y))$ and $|Z|$ unchanged. The resulting sets would then satisfy (a), (b) and (c) with a smaller value of $|Y|$, contradicting the minimality of $|Y|$ in our choice of X, Y and Z. So every vertex $y \in Y$ must have $\deg_{G'}^-(y) \geq \eta n$. Likewise, if there exists a vertex $y \in Y$ with $\deg_{G'}^+(y) < \eta n$, then we obtain a similar contradiction

by moving y from Y to Z. We conclude that every vertex $y \in Y$ must have $\deg^+_{G'}(y) \geq \eta n$, so $\delta^0(G') \geq \eta n \geq \eta|Y|$. Now suppose that there exists a partition $\{S, S'\}$ of Y such that (S, S') is a μ-almost-directed pair in G'. Observe that moving all vertices of S from Y to X would increase $e(Y \to X)$ by at most $e(S' \to S) \leq \mu|S||S'| \leq \gamma\eta|S|n$ whilst increasing $|X|$ by $|S|$ and leaving $e(G(Z \to X)) + e(G(Z \to Y))$ and $|Z|$ unchanged. So if $|S| \leq n/2$, then at least $|Y| - n/2 \geq n/3$ vertices would remain in Y, and so the resulting sets would satisfy (a), (b) and (c) with a smaller value of $|Y|$, again contradicting the minimality of $|Y|$. On the other hand, if $|S| > n/2$ then $|S'| \leq n/2$, and we obtain a similar contradiction by moving all vertices of S' from Y to Z. We conclude that no such partition $\{S, S'\}$ of Y exists. Therefore by Lemma 2.3 there is an integer k with $k_0 \leq k \leq k_1$ such that G' contains a (d, ε)-regular cycle of cluster tournaments with clusters $V_1, \ldots, V_k$ of equal size such that $|\bigcup_{i \in [k]} V_i| > (1 - \varepsilon)|Y| \geq (1 - 3\gamma)n$. We may therefore apply Lemma 5.9 (with 3γ in place of ψ) to obtain a copy of T in G. $\qquad\square$

6.2 Most oriented trees are nice

We now turn to the proof of Theorem 1.6, for which we use the following classical result, known as Cayley's theorem.

Theorem 6.1 (Borchardt 1860; Cayley 1889)**.** *There are n^{n-2} labelled undirected trees with vertex set $[n]$.*

A *cherry* is a path of length two, and its *centre* is the vertex of degree two. In an oriented tree T we refer to an in-subtree (respectively out-subtree) which is an (oriented) cherry as an *in-cherry* (respectively *out-cherry*). Our next lemma states that most labelled undirected trees have many pendant cherries. This is a special case of a much more general result for simply generated trees due to Janson [8]. For completeness, we include a proof of the particular statement that suffices for our purposes.

Lemma 6.2. *Fix $\varepsilon > 0$, and let T be a tree chosen uniformly at random from the set of all labelled undirected trees with vertex set $[n]$. Then asymptotically almost surely T contains $(1 \pm \varepsilon)\frac{e^{-3}}{2}n$ pendant cherries.*

Proof. For each set $S \in \binom{[n]}{3}$, let $\widehat{S}$ be the indicator random variable which has value 1 if S spans a pendant cherry in T and 0 otherwise. We first note that

$$\mathbb{P}(\widehat{S} = 1) = \frac{3(n-3)(n-3)^{n-5}}{n^{n-2}} = \frac{3}{n^2}\left(1 - \frac{3}{n}\right)^{n-4}.$$

Indeed, there are three possible choices for the centre of the cherry, this centre is adjacent to one of the $n - 3$ vertices in $[n] \setminus S$, and by Theorem 6.1 there are $(n-3)^{n-5}$ distinct possibilities for the undirected labelled tree spanned by $[n] \setminus S$, giving the numerator, whilst the denominator is simply the total number of labelled undirected trees on n vertices (again by Theorem 6.1). The number of pendant cherries in T is $X := \sum_{S \in \binom{[n]}{3}} \widehat{S}$, so by linearity of expectation it follows that

$$\mathbb{E}(X) = \sum_{S \in \binom{[n]}{3}} \mathbb{P}(\widehat{S} = 1) = \binom{n}{3}\frac{3}{n^2}\left(1 - \frac{3}{n}\right)^{n-4} = (1 + o(1))\frac{e^{-3}}{2}n. \qquad (12)$$

It therefore suffices to show that X is concentrated around $\mathbb{E}(X)$. Consider any distinct $S, S' \in \binom{[n]}{3}$, and note that if S intersects S' then we must have $\widehat{S} \cdot \widehat{S'} = 0$. On the other hand, if S and S' are disjoint then by a similar argument as above we have

$$\mathbb{E}(\widehat{S} \cdot \widehat{S'}) = \mathbb{P}(\widehat{S} = \widehat{S'} = 1) = \frac{[3(n-6)]^2(n-6)^{n-8}}{n^{n-2}} = \frac{9}{n^4}\left(1 - \frac{6}{n}\right)^{n-6},$$

so

$$\mathbb{E}(X^2) = \mathbb{E}\Big(\sum_{S \in \binom{[n]}{3}} \widehat{S}^2 + \sum_{\substack{S,S' \in \binom{[n]}{3} \\ S \neq S'}} \widehat{S} \cdot \widehat{S'} \Big) = \sum_{S \in \binom{[n]}{3}} \mathbb{E}(\widehat{S}) + \sum_{\substack{S,S' \in \binom{[n]}{3} \\ S \cap S' = \varnothing}} \mathbb{E}\big(\widehat{S} \cdot \widehat{S'} \big)$$

$$= \binom{n}{3}\frac{3}{n^2}\left(1 - \frac{3}{n}\right)^{n-4} + \binom{n}{3}\binom{n-3}{3}\frac{9}{n^4}\left(1 - \frac{6}{n}\right)^{n-6}. \tag{13}$$

Combining (12) and (13) we find that

$$\mathrm{Var}(X) = \binom{n}{3}\frac{3}{n^2}\left(1 - \frac{3}{n}\right)^{n-4} + \binom{n}{3}\binom{n-3}{3}\frac{9}{n^4}\left(1 - \frac{6}{n}\right)^{n-6} - \left[\binom{n}{3}\frac{3}{n^2}\left(1 - \frac{3}{n}\right)^{n-4}\right]^2$$

$$= (1 + \mathrm{o}(1))\frac{\mathrm{e}^{-3}}{2}n + (1 + \mathrm{o}(1))\frac{\mathrm{e}^{-6}}{4}n^2 - \left((1 + \mathrm{o}(1))\frac{\mathrm{e}^{-3}}{2}n\right)^2 = \mathrm{o}(n^2). \tag{14}$$

By Chebyshev's inequality, (12) and (14) it follows that

$$\mathbb{P}\left(|X - \mathbb{E}(X)| > \frac{\varepsilon}{2} \cdot \mathbb{E}(X)\right) \leq \frac{\mathrm{Var}(X)}{\left(\varepsilon\mathbb{E}(X)/2\right)^2} = \mathrm{o}(1),$$

which together with (12) proves the lemma. □

We are now ready to prove Theorem 1.6, that almost all labelled oriented trees are $\frac{1}{250}$-nice.

Proof of Theorem 1.6. Let $\mathcal{T}_n$ be the set of all labelled oriented trees with vertex set $[n]$. Note that we can select an oriented tree T uniformly at random from $\mathcal{T}_n$ using the following two-step random procedure: first select a tree T_0 uniformly at random from the set of all labelled undirected trees with vertex set $[n]$, then form a labelled oriented tree T by orienting each edge e of T_0 uniformly at random and independently of all other choices. Indeed, since there are n^{n-2} possibilities for T_0 by Theorem 6.1, and every tree in $\mathcal{T}_n$ has $n-1$ edges, the probability that a given labelled oriented tree T is selected by this two-step procedure is $n^{2-n}2^{1-n}$.

Let C be the number of pendant cherries of T_0, let X be the number of pendant in-cherries of T which contain an out-leaf of T, and let Y be the number of pendant out-cherries of T which contain both an in-leaf and out-leaf of T. Observe that the probability that a fixed pendant cherry of T_0 contributes to X is $3/8$, and likewise the probability that a fixed pendant cherry of T_0 contributes to Y is $1/4$. So $X \sim \mathcal{B}(C, 3/8)$ and $Y \sim \mathcal{B}(C, 1/4)$. Since by Lemma 6.2 we have $C \geq n/50$ asymptotically almost surely (where we use the fact that $\mathrm{e}^{-3}/2 > 1/50$), it follows by Theorem 2.6 that we also have $|X|, |Y| \geq C/5 \geq n/250$ asymptotically almost surely. Since no pendant cherry of T can be counted by both X and Y, it follows that T is $\frac{1}{250}$-nice. □

7   Concluding remarks

Recall that Theorem 1.4 states that all large nice oriented trees of polylogarithmic maximum degree are unavoidable. Together with Moon's theorem on the maximum degree of a random labelled tree (Theorem 1.5) and our proof that almost all labelled oriented trees are nice (Theorem 1.6) this established Theorem 1.2, that almost all labelled oriented trees are unavoidable.

The same method can be used to show that other classes of random oriented trees are asymptotically almost surely unavoidable. More precisely, let $\mathcal{T}$ be a class of undirected trees, let $\mathcal{T}_n$ consist of all members of $\mathcal{T}$ with n vertices, and let T be a tree selected uniformly at random from $\mathcal{T}_n$. If we can show, for some constants C and ξ, that

  (a) $\Delta(T) \leq (\log n)^C$ asymptotically almost surely, and

  (b) T has at least ξn pendant stars asymptotically almost surely,

then by a similar argument to the proof of Theorem 1.6 it follows that a uniformly-random orientation T^* of T is asymptotically almost surely α-nice (where $\alpha \ll \xi$), and therefore by Theorem 1.4 that T^* is asymptotically almost surely unavoidable. Following the methods of Janson [8] it is not hard to show that (a) and (b) hold for many classes $\mathcal{T}$ of simply-generated random trees, such as uniformly-random ordered trees (see [8, Example 10.1]), binary trees (see [8, Example 10.3]) and d-ary trees for a fixed integer $d \geq 3$ (see [8, Example 10.6]) In the same way Theorem 1.4 directly shows that for many fixed trees T, such as not-too-unbalanced d-ary trees for a fixed integer $d \geq 3$, a random orientation of T is unavoidable asymptotically almost surely. Finally we note that for many oriented trees it is straightforward to check directly that the conditions of Theorem 1.4 are satisfied, for instance in the case of balanced antidirected binary trees, in which every non-leaf vertex has one child as an inneighbour and one child as an outneighbour.

However, there do exist oriented trees which are not nice but which are unavoidable, such as the paths and claws discussed in Section 1. In this context it is natural to ask whether the property of being unavoidable can be succinctly characterised or easily tested.

Question 7.1.
(i) Is there a concise characterisation of unavoidable oriented trees?
(ii) Given an oriented tree T, can we determine in polynomial time if T is unavoidable?

We suspect that it would be very difficult to establish such a characterisation. As a more attainable goal, it would be interesting to establish further classes of unavoidable oriented trees. For example, say that an oriented tree T with root r is *outbranching* if for every vertex $v \in V(T)$ the path in T from r to v is directed from r to v. In particular, if the root of T is not a leaf then T then has no in-leaves at all, so T is not α-nice for any $\alpha > 0$.

Problem 7.2. *What conditions are sufficient to ensure that an outbranching oriented tree T is unavoidable?*

To shed some light on this problem it may help to consider the outbranching balanced binary trees B_d on $2^{d+1} - 1$ vertices, in which every non-leaf vertex has two children as outneighbours and every leaf is at distance precisely d from the root.

Conjecture 7.3. *B_d is unavoidable for d sufficiently large (possibly $d > 1$ is sufficient).*

It seems that further new ideas and techniques would be necessary to prove Conjecture 7.3, since the existence of both many in-leaves and many out-leaves of T is crucial to the approach we use in this paper.

Finally, recall that in Section 1 we defined $g(T)$ for an oriented tree T to be the smallest integer such that every tournament on $g(T)$ vertices contains a copy of T. So T is unavoidable if and only if $g(T) = |T|$. As noted earlier, if T is an out-directed star on n vertices then $g(T) \geq 2n - 2$, and Kühn, Mycroft and Osthus's proof of Sumner's conjecture for large trees shows that this is the maximum possible value of $g(T)$ for large n. That is, every oriented tree T on n vertices, where n is large, has $g(T) \leq 2n - 2$. The following 'double-star' construction due to Allen and Cooley (see [11]) also yields an oriented tree T for which $g(T)$ is significantly larger than $|T|$. Fix $a, b, c \in \mathbb{N}$ with $a + b + c = n$, and let T be the oriented tree on n vertices formed from a directed path P on b vertices by adding a new vertices as inneighbours of the initial vertex of P and adding c new vertices as outneighbours of the terminal vertex of P. Now take disjoint sets of vertices A, B and C of sizes $2a - 1, b - 1$ and $2c - 1$ respectively, and let G be the tournament in which $G[A]$ and $G[C]$ are regular tournaments, $G[B]$ is an arbitrary tournament, and all remaining edges of G are directed from A to B, from B to C or from A to C. So G has $2a + b + 2c - 3 = 2n - b - 3$ vertices, but G does not contain a copy of T, since then (as $|B| < b$) either the initial vertex of P would be in A, which cannot occur since each vertex of A has only $a - 1$ inneighbours, or the terminal vertex of P would be in C, which cannot occur since each vertex of C has only $c - 1$ outneighbours. So $g(T) \geq 2n - b - 2$ (and it is not too hard to check that in fact $g(T) = 2n - b - 2$).

For any $\Delta, n \in \mathbb{N}$, taking $a = c = \Delta - 1$ and $b = n - 2\Delta + 2$ in the above construction yields an oriented tree T on n vertices with $\Delta(T) = \Delta$ and $g(T) = n + 2\Delta - 4$. In other words, for any $n \in \mathbb{N}$ and any $\Delta \geq 3$ there exist oriented trees on n vertices with maximum degree at most Δ which are not contained in some tournament on $n + 2\Delta - 5$ vertices. On the other hand, Theorem 1.7 shows that every oriented tree whose maximum degree is at most polylogarithmic in n is contained in every tournament on $n + \mathrm{o}(n)$ vertices. Kühn, Mycroft and Osthus [11] asked whether this $\mathrm{o}(n)$ term can be replaced by a constant for oriented trees whose maximum degree is at most a constant Δ, and the previous construction shows that a constant of $2\Delta - 4$ would be best possible. More generally it would be interesting to know whether the previous construction is extremal for any bound on $\Delta(T)$ (as a function of n), with the exception of the antidirected paths $\check{P}_3, \check{P}_5$ and $\check{P}_7$ on 3, 5 and 7 vertices respectively — as described in the introduction, these three paths are avoidable and so are not contained in any tournament on the same number of vertices.

Question 7.4. *With the exception of $\check{P}_3, \check{P}_5$ and $\check{P}_7$, is every oriented tree T on n vertices contained in every tournament on $n + 2\Delta(T) - 4$ vertices?*

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