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MIYAMOTO INVOLUTIONS IN AXIAL ALGEBRAS OF JORDAN TYPE HALF

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Abstract. Nonassociative commutative algebras A generated by idempotents e whose adjoint operators $\mathrm{ad}_e\colon A \to A$, given by $x \mapsto xe$, are diagonalizable and have few eigenvalues are of recent interest. When certain fusion (multiplication) rules between the associated eigenspaces are imposed, the structure of these algebras remains rich yet rather rigid. For example vertex operator algebras give rise to such algebras. The connection between the Monster algebra and Monster group extends to many axial algebras which then have interesting groups of automorphisms.

Axial algebras of Jordan type η are commutative algebras generated by idempotents whose adjoint operators have a minimal polynomial dividing $(x-1)x(x-\eta)$, where $\eta \notin \{0,1\}$ is fixed, with well-defined and restrictive fusion rules. The case of $\eta \neq \frac{1}{2}$ was thoroughly analyzed by Hall, Rehren, and Shpectorov in a recent paper, in which axial algebras were introduced. Here we focus on the case where $\eta = \frac{1}{2}$, which is less well understood and is of a different nature.

1. Introduction

Axial algebras, introduced in [HRS1, HRS2], are certain commutative and nonassociative algebras. Their definition was motivated by constructions from the theory of vertex operator algebras. The second degree piece $V_2^\natural$ of the Monster vertex operator algebra $V^\natural$ [Bo] has the structure of a commutative nonassociative algebra. Within this algebra certain idempotents, called axes by Conway [C], give rise to $V^\natural$ and $V_2^\natural$ automorphisms of order two. Miyamoto [M] observed that this is a special case of a more general phenomenon. He called the commutative algebras V_2 of the appropriate VOAs V Griess algebras. Ivanov [I] took certain of the properties of these as axioms for his Majorana algebras. The axial algebras of [HRS1, HRS2] are then a further abstraction, with the corresponding automorphisms of order two being called Miyamoto involutions.

The historical development is discussed at greater length in the introduction to [HRS2].

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That earlier paper is focused upon those axial algebras said to have Jordan type η. Specifically, all primitive axial algebras of Jordan type $\eta \neq \frac{1}{2}$ are essentially classified in [HRS2]. The purpose of this paper is to understand better the structure of primitive axial algebras of Jordan type $\frac{1}{2}$. These remain unclassified and are more varied than those with $\eta \neq \frac{1}{2}$. Especially, many Jordan algebras occur, hence the name.

Throughout this paper $\mathbb{F}$ is a field of characteristic not 2.

1.1. **The definition of an axial algebra.**

Let A be a commutative algebra over $\mathbb{F}$ that is not necessarily associative. For $a \in A$ and $\lambda \in \mathbb{F}$ let

$$A_\lambda(a) := \{x \in A \mid xa = \lambda x\}\,.$$

That is, $A_\lambda(a)$ is the λ-eigenspace of the adjoint operator

$$\mathrm{ad}_a\colon A \to A,\ x \mapsto xa.$$

(We allow $A_\lambda(a) = 0$.)

An *axis* in A is an idempotent element $a = a^2$ of A such that the minimal polynomial of ad_a is a product of distinct linear factors. The $\mathbb{F}$-algebra A is an *axial algebra* if it is generated by axes. An axis $a \in A$ is *absolutely primitive* if $A_1(a) = \mathbb{F}a$ (note that this is stronger than being a primitive idempotent). The structure of axial algebras can be very loose. This is typically remedied by specifying *fusion rules* (that is, *multiplication rules*) which restrict how eigenspaces are allowed to multiply.

Fix $\eta \in \mathbb{F}$ with $\eta \notin \{0,1\}$. In this paper we shall be concerned with axial algebras generated by a set $\mathcal{A}$ of absolutely primitive axes with the following eigenvalue and fusion requirements:

(1) *For each $a \in \mathcal{A}$, the minimal polynomial of ad_a divides $(x-1)x(x-\eta)$.*
(2) *We have*

$$A_\delta(a)A_\epsilon(a) \subseteq A_{\delta\epsilon}(a)\,,$$

for $\delta, \epsilon \in \{+,-\}$, and

$$A_0(a)A_0(a) \subseteq A_0(a)\,,$$

for all $a \in \mathcal{A}$. Here $\delta\epsilon$ has the obvious meaning, and

$$A_+(a) = A_1(a) \oplus A_0(a) \quad \textit{and} \quad A_-(a) = A_\eta(a)\,.$$

An absolutely primitive axis a having these two properties will be called an *η-axis.* A *primitive axial algebra of Jordan type η* is a commutative algebra generated by η-axes. The terminology arises from the fact that Jordan algebras generated by absolutely primitive axes are primitive axial algebras of Jordan type $\frac{1}{2}$. That particular choice for η will be of the greatest interest to us.

1.2. **Main results.**

The current paper has its origins in the paper [HRS2] of Hall, Rehren, and Shpectorov. However, we observed that Theorem 5.4 of that paper is false in two senses—the actual result is false for primitive axial algebras of Jordan type $\eta = \frac{1}{2}$ and, while the result is true for the cases $\eta \neq \frac{1}{2}$, the proof given there is not sufficient. This paper began as an effort to resolve these difficulties. It should be noted that the results from [HRS2] used in this paper come from Sections 1–4 of that paper, which were free of error.[1] The current Theorem 7.1 provides a brief proof of Theorem 5.4 from [HRS2] in the important, generic case $\eta \neq \frac{1}{2}$. The problematic case $\eta = \frac{1}{2}$ is then the main focus of this paper.

To state our main results, let A be a primitive axial algebra of Jordan type η. We need to recall certain definitions. The *Miyamoto involution* $\tau(a)$ corresponding to an η-axis $a \in A$ is the automorphism of A defined by $\tau(a)\colon x \mapsto x_+ - x_-$, where $x = x_+ + x_-$, with $x_+ \in A_+(a)$ and $x_- \in A_-(a)$. It is easy to check that $\tau(a)$ is an automorphism of A of order at most 2 (see Definition 2.1 below).

1.2.1. *Results concerning the structure of A.*

Consider the (undirected) graph Δ on the set of all η-axes of A, where distinct a, b form an edge if and only if $ab \neq 0$. In §6 we show the existence of a certain decomposition of our axial algebras.

Theorem A. *Let A be a primitive axial algebra of Jordan type η. Let $\{\Delta_i \mid i \in I\}$ be the connected components of Δ and let A_i be the subalgebra of A generated by the axes in Δ_i. Then*

(1) $A = \sum_{i \in I} A_i$ *is the sum of its ideals* A_i;
(2) $A_i A_j = 0$, *for distinct* $i, j \in I$;
(3) *for each* $i \in I$ *exactly one of the following holds:*
   (a) *the map* $a \mapsto \tau(a),\ a \in \Delta_i$ *is injective.*
   (b) A_i *is a Jordan algebra of Clifford type, in which case the map* $a \mapsto \tau(a),\ a \in \Delta_i$ *is two-to-one.*

Jordan algebras of Clifford type are discussed in §5. Theorem 6.10 below gives a more detailed (and refined) version of Theorem A.

To prove part (3b) of Theorem A we prove:

Theorem B. *Let A be a primitive axial algebra of Jordan type η. Assume that Δ is connected and that there are two distinct η-axes $a, b \in A$ such that $\tau(a) = \tau(b)$. Then $\eta = \frac{1}{2}$, $a + b = \mathbb{1}$ is the identity of A and A is a Jordan algebra of Clifford type.*

The proof of Theorem B uses Theorem 5.4 and Proposition 6.6.

[1] Indeed, other than Theorem 5.4, the only result from [HRS2] whose statement needs change is Theorem 6.3 to which the hypothesis $\eta \neq \frac{1}{2}$ should be added. Corollary 5.5 is true as stated, but its proof must be augmented with an appeal to the current Proposition 6.6.

Remark. Let A be a primitive axial algebra of Jordan type η, let a be an η-axis in A and let Δ_i be the connected component of Δ with $a \in \Delta_i$. Then exactly one of the following holds.

(1) $\tau(b) \neq \tau(a)$, for all η-axes $b \neq a$.
(2) $\tau(a) = \mathrm{id}$ (which is equivalent to $\Delta_i = \{a\}$). In this case for an η-axis b of A we have $\tau(b) = \tau(a)$ if and only if $\{b\}$ is a connected component of Δ.
(3) $\eta = \frac{1}{2}$, the corresponding algebra A_i of Theorem A is a Jordan algebra of Clifford type, and there is a unique η-axis $b \neq a$ in A such that $\tau(a) = \tau(b)$. Indeed $b = \mathbb{1}_i - a$, where $\mathbb{1}_i$ is the identity element of A_i.

This is a consequence of Theorems A and B and Lemma 3.2.1.

1.2.2. *Results concerning* 3*-transpositions.*

Let G be a group generated by a normal set of involutions D. Recall that D is called *a set of* 3*-transpositions in* G if $|st| \in \{1, 2, 3\}$, for all $s, t \in D$. The group G is then called a 3*-transposition group*.

Let $\mathcal{A}$ be a generating set of η-axes in A. Suppose that $a^{\tau(b)} \in \mathcal{A}$, for all $a, b \in \mathcal{A}$, where $a^{\tau(b)}$ is the image of a under the Miyamoto involution $\tau(b)$ (and this and similar notation will prevail throughout this paper). In other words, assume that $\mathcal{A}$ is *closed*.

As we will see, the set $D := \{\tau(a) \mid a \in \mathcal{A} \text{ and } \tau(a) \neq \mathrm{id}\}$ is a normal set of involutions in $G = \langle D \rangle$. Suppose D is a set of 3-transpositions in G, then we call A *a* 3*-transposition algebra with respect to* $\mathcal{A}$.

In [HRS2] it is shown that *every* primitive axial algebra of Jordan type $\eta \neq \frac{1}{2}$ is a 3-transposition algebra with respect to *any* closed generating set of η-axes (see Theorem 7.1). The case $\eta = \frac{1}{2}$ is very different. However, the following theorem holds in the case where A is a Jordan algebra of Clifford type (see §7).

Theorem C. *Assume* $\mathrm{char}(\mathbb{F}) \neq 3$ *and that* A *is a Jordan algebra of Clifford type that is additionally a* 3*-transposition algebra with respect to the closed set* $\mathcal{A}$ *of* $\frac{1}{2}$*-axes. Assume further that* $D := \{\tau(a) \mid a \in \mathcal{A} \text{ and } \tau(a) \neq \mathrm{id}\}$ *is a conjugacy class in* $G = \langle D \rangle$*. Then* G *is of symplectic type and no subgroup* $H = \langle D \cap H \rangle$ *is isomorphic to a central quotient of* $\mathrm{W}_2(\tilde{D}_4)$.

For more detail about the terminology in Theorem C, see §4. In that section we prove a result about 3-transposition groups which is of independent interest.

Theorem D. *Let* G *be a finite* 3*-transposition group of symplectic type generated by the conjugacy class* D *of* 3*-transpositions, such that there is no subgroup* $H = \langle D \cap H \rangle$ *isomorphic to a central quotient of* $\mathrm{W}_2(\tilde{D}_4)$.

Then there is an $n \in \mathbb{Z}^+$ *with* G *a central quotient of* $\mathrm{W}(A_n)$ *for* $n \geq 2$, $\mathrm{W}(D_n)$ *for* $n \geq 4$*, or* $\mathrm{W}(E_n)$ *for* $n \in \{6, 7, 8\}$.

Of course Theorem D can be used in conjunction with Theorem C to impose severe restrictions on the groups G that can occur in Theorem C.

1.3. Some open problems.

Here we list certain remaining open problems.

Problem 1. Let A be a primitive axial algebra of Jordan type $\frac{1}{2}$ and assume that the graph Δ above is connected and that the map $a \mapsto \tau(a)$ is bijective on the set of $\frac{1}{2}$-axes.

(i) What is the structure of subalgebras of A generated by three $\frac{1}{2}$-axes?
(ii) Suppose A is generated by a finite set of $\frac{1}{2}$-axes. Is A finite dimensional over $\mathbb{F}$?
(iii) What else can be said about the structure of A?

Problem 2. Let A be a Jordan algebra of Clifford type and assume that the graph Δ above is connected. Let $\mathcal{A}$ be a closed subset (see above) of generating $\frac{1}{2}$-axes in A such that the map $a \mapsto \tau(a)$ is injective on $\mathcal{A}$. Classify these sets $\mathcal{A}$.

Problem 3. Are there restriction on G and D if we assume in Theorem C that $\mathrm{char}(\mathbb{F}) = 3$?

Contents

2. Notation and some definitions

In this section we assemble the notation and definitions that will prevail throughout this paper. Other, more specific ones, will be given in the beginning of each of the following sections.

As mentioned above, throughout this paper $\mathbb{F}$ is a field of *characteristic not* 2. Also, A is a primitive axial algebra of Jordan type η.

Definition 2.1 (Miyamoto involution)**.** Let B be an algebra over the field $\mathbb{F}$ (not necessarily associative).

(1) Suppose that B is a direct sum $B = B_+ \oplus B_-$, such that B_+ and B_- are subspaces of B and such that $B_\delta B_\epsilon = B_{\delta\epsilon}$, for $\delta, \epsilon \in \{+, -\}$. For $x \in B$, write $x = x_+ + x_-$, with $x_\epsilon \in B_\epsilon$. The *Miyamoto involution* corresponding to the above decomposition of B is the map $\tau\colon B \to B$ defined by $x^\tau = x_+ - x_-$, for all $x \in B$. It is easy to check that τ is an automorphism of B of order at most 2. (It has order 2 if and only if $B_- \neq 0$.)

(2) Let $x \in B$, and assume that B decomposes into a direct sum $B_+(x) \oplus B_-(x)$ of ad_x-invariant subspaces of the adjoint operator $\mathrm{ad}_x\colon B \to B$, $b \mapsto bx$. Suppose further that $B_+(x)$ and $B_-(x)$ satisfy the rules as in (1) above. Then we denote the corresponding Miyamoto involution by $\tau(x)$ (where B is understood from the context) and call $\tau(x)$ the *Miyamoto involution corresponding to* x.

(3) When A is a primitive axial algebra of Jordan type η and $a \in A$ is an η-axis, then $\tau(a)$ will denote the Miyamoto involution corresponding to a (as in (2) above, recall the definition of $A_+(a)$ and $A_-(a)$ from subsection 1.1). It is easy to check that if ρ is an automorphism of A, then $\tau(a^\rho) = \tau(a)^\rho$. This fact will be used throughout this paper without further mention.

Notation 2.2 (Notation and definitions related to axes)**.**

(1) We denote by $\mathcal{X}$ the set of all η-axes in A.

(2) For a subalgebras $N \subseteq A$ and $x \in N$, we denote by $N_\lambda(x)$ the λ-eigenspace of the adjoint endomorphism $\mathrm{ad}_x\colon b \mapsto bx$ of N. (Recall that we allow the possibility $N_\lambda(x) = 0$.)

(3) We let $\mathcal{X}^1 := \{a \in \mathcal{X} \mid \tau(a) = \mathrm{id}\}$ and $\mathcal{X}^\eta := \mathcal{X} \smallsetminus \mathcal{X}^1$.

(4) For a subset $\mathcal{B} \subseteq \mathcal{X}$, we let $\mathcal{B}^1 = \mathcal{B} \cap \mathcal{X}^1$ and $\mathcal{B}^\eta = \mathcal{B} \cap \mathcal{X}^\eta$.

(5) A subset $\mathcal{B} \subseteq \mathcal{X}$ is *closed* if $\mathcal{B}^{\tau(b)} \subseteq \mathcal{B}$, for all $b \in \mathcal{B}$.

(6) For a set $\mathcal{B} \subseteq \mathcal{X}$, the *closure* of $\mathcal{B}$ in $\mathcal{X}$ is the intersection of all closed subsets of $\mathcal{X}$ containing $\mathcal{B}$. We denote it by $[\mathcal{B}]$.

(7) For a subset $\mathcal{B} \subseteq \mathcal{X}$, we denote by $D_{\mathcal{B}} := \{\tau(b) \mid b \in \mathcal{B}\}$ and $G_{\mathcal{B}} = \langle D_{\mathcal{B}} \rangle$.

Notation 2.3 (General notation for algebras and subalgebras)**.** Let $a, b \in \mathcal{X}$ with $a \neq b$.

(1) For a subset $\mathcal{B} \subseteq \mathcal{X}$, we denote by $N_{\mathcal{B}}$ the subalgebra of A generated by $\mathcal{B}$. If $\mathcal{B} = \{a, b\}$, we sometimes write $N_{\mathcal{B}} = N_{a,b}$.

(2) Note that $N_{a,b}$ satisfies the multiplication rules of Proposition 3.1.1 below. We use the notation $\varphi_{a,b}$, $\pi_{a,b}$ and $\sigma_{a,b}$ as in Proposition 3.1.1.

(3) We denote by $1_{a,b}$ the identity element of $N_{a,b}$ if $N_{a,b}$ is 3-dimensional and has an identity element.

Note that by Theorem 3.1.3(3), $N_{a,b}$ contains an identity element if and only if $\sigma_{a,b} \neq 0$ and $\pi_{a,b} \neq 0$. In this case the identity element of $N_{a,b}$ is $1_{a,b} = \frac{1}{\pi_{a,b}}\sigma_{a,b}$.

Notation 2.4 (Some specific two generated algebras)**.** The following 2-generated algebras were defined in [HRS2]. In several cases though we changed notation. Let $a, b \in \mathcal{X}$ with $a \neq b$.

(1) If $ab = 0$, we denote $N_{a,b} = 2B_{a,b}$, thus $N_{a,b}$ is 2-dimensional.
(2) If $ab = -a-b$, we denote $N_{a,b} = 3C(-1)^{\times}_{a,b}$, thus $N_{a,b}$ is 2-dimensional (see Lemma 3.1.8).
(3) If $ab = \frac{1}{2}a + \frac{1}{2}b$ we denote $N_{a,b} = J_{a,b}$, thus $N_{a,b}$ is 2-dimensional (see Lemma 3.1.9). Our notation here for this algebra differs from the notation in [HRS2].
(4) If $N_{a,b}$ is 3-dimensional, we denote $N_{a,b} = B(\eta, \varphi_{a,b})_{a,b}$, where $\varphi_{a,b} \in \mathbb{F}$ is defined in Proposition 3.1.1 (see Theorem 3.1.3).
(5) We denote $3C(\eta)_{a,b} = B(\eta, \frac{1}{2}\eta)_{a,b}$ (see Lemma 3.1.4 and Remark 3.1.5).

3. Preliminaries

In this section we give some preliminary properties of the various algebras from Notation 2.4. In addition, we assemble some preliminary results.

3.1. Details about the algebras in Notation 2.4.

Proposition 3.1.1 (Proposition 4.6 [HRS2])**.** *Let $a, b \in \mathcal{X}$ with $a \neq b$. Let $\sigma = \sigma_{a,b} = ab - \eta a - \eta b \in N_{a,b}$. Then there exists a scalar $\varphi = \varphi_{a,b} \in \mathbb{F}$ such that if we set $\pi = \pi_{a,b} = (1-\eta)\varphi - \eta$, then*

(1) $ab = \sigma + \eta a + \eta b$;
(2) $\sigma v = \pi v$, for all $v \in \{a, b, \sigma\}$.

Lemma 3.1.2. *Let a, b be two distinct η-axes in A and suppose that $N_{a,b}$ is 2-dimensional. Then*

(1) one of the following three statements holds:
 (a) (i) $\varphi_{a,b} = 0, \pi_{a,b} = -\eta$ and $\sigma_{a,b} = -\eta a - \eta b$, also
 (ii) $ab = 0$ and $N_{a,b} = 2B_{a,b}$.
 (b) (i) $\eta = -1,\ \varphi_{a,b} = -\frac{1}{2},\ \pi_{a,b} = 0$ and $\sigma_{a,b} = 0$, also
 (ii) $ab = -a - b$ and $N_{a,b} = 3C(-1)^{\times}_{a,b}$.
 (c) (i) $\eta = \frac{1}{2},\ \varphi_{a,b} = 1, \pi_{a,b} = 0$ and $\sigma_{a,b} = 0$, also
 (ii) $ab = \frac{1}{2}a + \frac{1}{2}b$, and $N_{a,b} = J_{a,b}$.

(2) If $ab \neq 0$, then $N_{a,b}$ does not have an identity element.

Proof. Suppose first that $\sigma_{a,b} \neq 0$. Let $\sigma = \sigma_{a,b}$ and $\pi = \pi_{a,b}$. Write $\sigma = \alpha a + \beta b$. Multiplying by a we get that $\pi a = \alpha a + \beta ab$. Suppose $\beta = 0$.

Then $\pi a = \alpha a$, thus $\pi \neq 0$, and $\alpha = \pi$. Multiplying by b we get that $\pi b = \pi ab$, so $ab = b$. But this contradicts the absolute primitivity of a. Hence $\beta \neq 0$. Similarly, $\alpha \neq 0$.

Thus we have $ab = \frac{\pi-\alpha}{\beta}a$ and similarly multiplying by b we get $ab = \frac{\pi-\beta}{\alpha}b$. Hence we must have $\alpha = \beta = \pi \neq 0$. It follows that $ab = 0$ and $N_{a,b} = 2B_{a,b}$. Further $\sigma = -\eta a - \eta b$. Now $\pi a = \sigma a = -\eta a$. Hence $\pi = -\eta$ and then $\varphi_{a,b} = 0$. This shows part (1a).

Suppose now that $\sigma = 0$. Then $ab = \eta a + \eta b$. Thus, if $a(\alpha a + \beta b) = 0$, for some vector $\alpha a + \beta b \in N_{a,b}$, then $\alpha a + \beta(\eta a + \eta b) = 0$. Hence $\beta\eta = 0$, so $\beta = 0$, and then also $\alpha = 0$, and we see that the 0-eigenspace of a is $\{0\}$ and so is the 0-eigenspace of b.

We now compute the η-eigenspace of a. Clearly, it is a 1-dimensional space spanned by some $\alpha a + \beta b$, with $\alpha \neq 0 \neq \beta$.

Now since A is of Jordan type η, we must have $(\alpha a + \beta b)^2 \in \mathbb{F}a$. Thus

$$\alpha^2 a + \beta^2 b + 2\alpha\beta(\eta a + \eta b) \in \mathbb{F}a.$$

It follows that $\beta^2 + 2\alpha\beta\eta = 0$, or

$$\beta = -2\alpha\eta.$$

Canceling α, we may assume that the η-eigenspace of a is spanned by $a - 2\eta b$.

Next we have $a(a - 2\eta b) = \eta(a - 2\eta b)$. Hence

$$a - 2\eta(\eta a + \eta b) = \eta a - 2\eta^2 b.$$

It follows that $2\eta^2 + \eta - 1 = 0$ so, $\eta = -1$ or $\eta = \frac{1}{2}$. If $\eta = -1$, then $ab = -a - b$, and this is the definition of $3C(-1)^{\times}_{a,b}$. This shows part (1bii). Since $0 = \sigma a = \pi a$, it follows that $\pi = 0$ and then $\varphi = -\frac{1}{2}$ and (1bi) holds.

If $\eta = \frac{1}{2}$, then $ab = \frac{1}{2}a + \frac{1}{2}b$. This is the algebra $J_{a,b}$. This shows part (1cii). As above, $\pi = 0$ and then $\varphi = 1$.

Part (2) is an easy calculation and we omit the details. □

Theorem 3.1.3. *Let a, b be two distinct η-axes in A and assume that $N := N_{a,b}$ is 3-dimensional. Set $\sigma = \sigma_{a,b}$, $\varphi = \varphi_{a,b}$ and $\pi = \pi_{a,b}$. Then*

(1) $\sigma \neq 0$;
(2) $\sigma z = \pi z$, *for* $z \in \{a, b, \sigma\}$;
(3) $\pi \neq 0$ *if and only if* N *contains an identity element* $1_{a,b} = \frac{1}{\pi}\sigma$;
(4) $ab = \sigma + \eta a + \eta b$; *in particular, if* $\pi \neq 0$ *then* $ab = \pi 1_{a,b} + \eta a + \eta b$;
(5) $N_1(x) = \mathbb{F}x$, $N_0(x) = \mathbb{F}(\pi x - \sigma)$ *and* $N_\eta(x) = \mathbb{F}((\eta - \varphi)x + \eta y + \sigma)$;
(6) $x^{\tau(y)} = -\frac{2}{\eta}\sigma - \frac{2(\eta-\varphi)}{\eta}y - x$, *for* $\{x, y\} = \{a, b\}$;
(7) $xx^{\tau(y)} = -\frac{2(\eta-\varphi)}{\eta}\sigma - \left(2(\eta - \varphi) + \frac{2}{\eta}\pi + 1\right)x - 2(\eta - \varphi)y$, *for* $\{x, y\} = \{a, b\}$.

Proof. Note that $N_{a,b}$ is isomorphic to the algebra $B(\eta, \varphi)$ defined in [HRS2, Theorem 4.7, p. 98], with (a, b) in place of (c, d) and $\sigma_{a,b}$ in place of ρ. Hence, parts (1)–(5) are [HRS2, Theorem 4.7(a), p. 99]. Note that if $\pi = 0$ and N contains an identity element $1_{a,b}$, then $\sigma = 1_{a,b}\sigma = 0$, a contradiction.

For (6) we have

$$\begin{aligned} x &= -\tfrac{1}{\eta}\sigma - \tfrac{\eta-\varphi}{\eta}y + \tfrac{1}{\eta}\left((\eta-\varphi)y + \eta x + \sigma\right) \iff \\ x^{\tau(y)} &= -\tfrac{1}{\eta}\sigma - \tfrac{\eta-\varphi}{\eta}y - \tfrac{1}{\eta}\left((\eta-\varphi)y + \eta x + \sigma\right) \iff \\ x^{\tau(y)} &= -\tfrac{2}{\eta}\sigma - \tfrac{2(\eta-\varphi)}{\eta}y - x. \end{aligned}$$

Finally, for (7) we have

$$\begin{aligned} xx^{\tau(y)} &= x(-\tfrac{2}{\eta}\sigma - \tfrac{2(\eta-\varphi)}{\eta}y - x) \\ &= -\tfrac{2}{\eta}\pi x - \tfrac{2(\eta-\varphi)}{\eta}xy - x \\ &= -\tfrac{2}{\eta}\pi x - \tfrac{2(\eta-\varphi)}{\eta}(\sigma + \eta x + \eta y) - x \\ &= -\tfrac{2}{\eta}\pi x - \tfrac{2(\eta-\varphi)}{\eta}\sigma - 2(\eta-\varphi)x - 2(\eta-\varphi)y - x \\ &= -\tfrac{2(\eta-\varphi)}{\eta}\sigma - \left(2(\eta-\varphi) + \tfrac{2}{\eta}\pi + 1\right)x - 2(\eta-\varphi)y. \qquad \square \end{aligned}$$

Lemma 3.1.4. *Let a, b be two distinct η-axes in A and suppose that $N_{a,b} = B(\eta, \frac{1}{2}\eta)_{a,b}$ (in particular, $N_{a,b}$ is 3-dimensional). Set $N = N_{a,b}$, $\pi = \pi_{a,b}$ and $\sigma = \sigma_{a,b}$. Then $\pi = -\frac{1}{2}(\eta + \eta^2)$, and*

(1) (a) *$ab = \sigma + \eta a + \eta b$;*
  (b) *$N_\eta(x) = \mathbb{F}(\sigma + \frac{1}{2}\eta x + \eta y)$, for $\{x, y\} = \{a, b\}$;*
  (c) *$x^{\tau(y)} = -\frac{2}{\eta}\sigma - x - y$, for $\{x, y\} = \{a, b\}$;*
  (d) *$a^{\tau(b)} = b^{\tau(a)}$, so $|\tau(a)\tau(b)| = 3$.*

(2) *If $\eta \neq -1$, then $1_{a,b} = \frac{1}{\pi}\sigma$ is the identity of N while if $\eta = -1$, then N has no identity element.*

Proof. Part (1a) follows from Theorem 3.1.3(4). Also

$$\pi = (1-\eta)\varphi - \eta = (1-\eta)\tfrac{\eta}{2} - \eta = -\tfrac{1}{2}(\eta + \eta^2).$$

Next, by Theorem 3.1.3(5), for $x \in \{a, b\}$, $N_\eta(x)$ is spanned by $(\eta - \frac{1}{2}\eta)x + \eta y + \sigma$, so (1b) holds. By 3.1.3(6), $x^{\tau(y)} = -\frac{2}{\eta}\sigma - \frac{2(\eta - \frac{1}{2}\eta)}{\eta}y - x$, for $\{x, y\} = \{a, b\}$, this shows (1c) and (1d). Part (2) follows from Theorem 3.1.3(3). $\square$

Remark 3.1.5. Let a, b be two distinct η-axes in A and suppose that $N_{a,b} = B(\eta, \frac{1}{2}\eta)_{a,b}$. Set $c_0 = a, c_1 = b$ and $c_2 = -\frac{2}{\eta}\sigma_{a,b} - a - b$. Then it is readily verified that $\{c_0, c_1, c_2\}$ is a basis of $N_{a,b}$, and for $\{i, j, k\} = \{0, 1, 2\}$ we have $c_i^2 = c_i$ and $c_ic_j = \frac{1}{2}\eta(c_i + c_j - c_k)$. Thus $N_{a,b}$ is the algebra denoted $3C(\eta)$ in [HRS2, Example 3.3, p. 90]. See also [HRS2], p. 91, line 9 (with (a, b) in place of (c_0, c_1)). This explains our notation $3C(\eta)_{a,b}$ (see Notation 2.4(5)).

Lemma 3.1.6. *Let $a, b \in A$ be two distinct η-axes. Suppose that $\eta = \varphi_{a,b} = \frac{1}{2}$. Then $N := N_{a,b}$ is 3-dimensional, so $N = B(\frac{1}{2}, \frac{1}{2})_{a,b}$ and*

(1) *$\pi_{a,b} = -\frac{1}{4}$;*
(2) *$x^{\tau(y)} = 1_{a,b} - x$, for $\{x, y\} = \{a, b\}$;*
(3) *$N_{\frac{1}{2}}(x) = \mathbb{F}(1_{a,b} - 2y)$, for $\{x, y\} = \{a, b\}$;*

(4) $|\tau(a)\tau(b)| \in \{2,4\}$;
(5) $|\tau(a)\tau(b)| = 2$ *if and only if* $\tau(x) = \tau(1_{a,b} - x)$, *for* $x \in \{a,b\}$;
(6) *if* $|\tau(a)\tau(b)| = 4$ *then* $\tau(a)\tau(1_{a,b} - a) = \tau(b)\tau(1_{a,b} - b) =: t$ *and* $Z(\langle \tau(a), \tau(b)\rangle) = \langle t \rangle$;
(7) $N = B(\frac{1}{2}, \frac{1}{2})_{x,y}$, *for all* $x \in \{a, 1_{a,b} - a\}$ *and* $y \in \{b, 1_{a,b} - b\}$.

Proof. By Lemma 3.1.2, N is 3-dimensional. Let $\varphi = \varphi_{a,b}$, $\pi = \pi_{a,b}$, $\sigma = \sigma_{a,b}$ and $\mathbb{1} = 1_{a,b}$. Since $\eta = \frac{1}{2}$, we have $\pi = (1-\eta)\varphi - \eta = (1-\frac{1}{2})\frac{1}{2} - \frac{1}{2} = -\frac{1}{4}$. By Theorem 3.1.3(6), $x^{\tau(y)} = -4\sigma - x = \mathbb{1} - x$. By Theorem 3.1.3(5) part (3) holds. Since $a^{\tau(a)} = a$ and $b^{\tau(a)} = \mathbb{1} - b$, it follows that $N = B(\frac{1}{2}, \frac{1}{2})_{a,\,(\mathbb{1}-b)}$. Similarly we see that (7) holds.

From (7) it follows that $x^{\tau(y)\tau(\mathbb{1}-y)} = x$, and clearly $y^{\tau(y)\tau(\mathbb{1}-y)} = y$, for $\{x,y\} = \{a,b\}$. Hence $\tau(y)\tau(\mathbb{1} - y) \in Z(H)$, where $H = \langle \tau(a), \tau(b)\rangle$. We have

$$\begin{aligned}\tau(\mathbb{1} - b)\tau(b) &= \tau(b^{\tau(a)})\tau(b) = (\tau(a)\tau(b))^2 = \tau(a)\tau(a^{\tau(b)}) \\ &= \tau(a)\tau(\mathbb{1} - a) \in Z(H).\end{aligned}$$

If $\tau(x) = \tau(\mathbb{1} - x)$ for $x = a$ or b, then $|\tau(a)\tau(b)| = 2$ and clearly if $|\tau(a)\tau(b)| = 2$, then $\tau(x) = \tau(\mathbb{1} - x)$. This completes the proof of the lemma. □

Remark 3.1.7. Suppose that $\eta = \frac{1}{2}$. Let $a, b \in \mathcal{X}$. Note that $N_{a,b} = B(\frac{1}{2}, \frac{1}{2})_{a,b}$ if and only if $N_{a,b}$ contains an identity $1_{a,b}$ and $ab = -\frac{1}{4}1_{a,b} + \frac{1}{2}a + \frac{1}{2}b$.

Lemma 3.1.8. *Let* $C := 3C(-1)^{\times}_{a,b}$. *Then*

(1) $ab = -a - b$, $\sigma_{a,b} = 0$, $\pi_{a,b} = 0$ *and* $\varphi_{a,b} = -\frac{1}{2}$;
(2) *C has no identity element;*
(3) $C_{-1}(a) = a + 2b$ *and* $C_{-1}(b) = b + 2a$;
(4) $ab = a^{\tau(b)} = b^{\tau(a)} = -a - b$, *so* $(\tau(a)\tau(b))^3 = \mathrm{id}$;
(5) *if* $\mathrm{char}(\mathbb{F}) \neq 3$, *the only non-zero idempotents in* C *are* $a, b, -a - b$, *and they are all* -1*-axis with* $C_{-1}(-a - b) = \mathbb{F}(a - b)$*;*
(6) *if* $\mathrm{char}(\mathbb{F}) = 3$ *then the non-zero idempotents in* C *are*

$$\mathcal{E} := \{\alpha a + (1 - \alpha)b \mid \alpha \in \mathbb{F}\},$$

and each $e \in \mathcal{E}$ *is a* -1*-axis in* C *with* $C_{-1}(e) = \mathbb{F}(a - b)$.

Proof. Part (1) is by definition, and part (2) is Lemma 3.1.2(2).

(3): $a(a + 2b) = a + 2ab = a - (2a + 2b) = -a - 2b$, and similarly for b.

(4): We have $a = -\frac{1}{2}b + \frac{1}{2}(b+2a)$, so, by definition, $a^{\tau(b)} = -\frac{1}{2}b - \frac{1}{2}(b+2a) = -a - b$. By symmetry, $b^{\tau(a)} = -a - b$. It follows that $a^{\tau(b)} = b^{\tau(a)}$, and then $|\tau(b)\tau(a)| = 3$.

(5&6): Assume that $e := \alpha a + \beta b$ is a non-zero idempotent. Then

$$\alpha a + \beta b = (\alpha a + \beta b)^2 = \alpha^2 a + \beta^2 b + 2\alpha\beta ab = (\alpha^2 - 2\alpha\beta)a + (\beta^2 - 2\alpha\beta)b.$$

If $\alpha = 0$, then $\beta = 1$, and $e = b$. Similarly if $\beta = 0$, then $\alpha = 1$ and $e = a$. Suppose $\alpha \neq 0 \neq \beta$. Then $\alpha^2 - 2\alpha\beta = \alpha$, and so $\alpha - 2\beta = 1$. Similarly $\beta - 2\alpha = 1$, and we see that if $\mathrm{char}(\mathbb{F}) \neq 3$, then $\alpha = \beta = -1$. If $\mathrm{char}(\mathbb{F}) = 3$, then we get $\alpha + \beta = 1$.

If $\mathrm{char}(\mathbb{F}) \neq 3$, then $(-a-b)(a-b) = -(a+b)(a-b) = -(a-b)$, so $-a-b$ is a -1-axis.

Next if $\mathrm{char}(\mathbb{F}) = 3$, then

$$\begin{aligned}(\alpha a + (1-\alpha)b)(a-b) &= \alpha a - \alpha ab + (1-\alpha)ab - (1-\alpha)b \\ &= \alpha a + \alpha a + \alpha b - a - b + \alpha a + \alpha b - b + \alpha b = \\ &-a - 2b = -a + b = -(a-b).\end{aligned}$$ □

Lemma 3.1.9. *Let $a, b \in \mathcal{X}$ such that $N := N_{a,b} = J_{a,b}$ (so that $\eta = \frac{1}{2}$). Let $\{x, y\} = \{a, b\}$, then*

(1) $\sigma_{a,b} = 0$, $\pi_{a,b} = 0$ *and* $\varphi_{a,b} = 1$;
(2) $ab = \frac{1}{2}a + \frac{1}{2}b$;
(3) $N_{\frac{1}{2}}(x) = \mathbb{F}(x-y)$;
(4) $x^{\tau(y)} = 2y - x$;
(5) $x^{(\tau(y)\tau(x))^k} = (2k+1)x - 2ky$, *for all* $k \geq 0$.

Proof. Parts (1) and (2) are by definition. We have $a(a-b) = a - ab = a - (\frac{1}{2}a + \frac{1}{2}b) = \frac{1}{2}a - \frac{1}{2}b$ and similarly for b, so (3) holds. Since $a = b + (a-b)$, then, by definition, $a^{\tau(b)} = b - (a-b) = 2b - a$, and similarly for b. This shows (4). We leave the calculations of (5) to the reader. □

Lemma 3.1.10. *Suppose that $\eta = \frac{1}{2}$, and let $a, b \in \mathcal{X}$ with $a \neq b$. Set $\varphi = \varphi_{a,b}$ and $\sigma = \sigma_{a,b}$, then*

$$b^{\tau(a)} = -4\sigma - (2 - 4\varphi)a - b.$$

Proof. Suppose $\sigma \neq 0$. If $N_{a,b} \neq 2B_{a,b}$, this follows from Lemma 3.1.2 and Theorem 3.1.3(6). If $N_{a,b} = 2B_{a,b}$, then $\varphi = 0$ and $\sigma = -\frac{1}{2}(a+b)$, so $b^{\tau(a)} = 2(a+b) - 2a - b = b$. Suppose $\sigma = 0$, then this follows from Lemma 3.1.2, Lemma 3.1.9 and from Lemma 3.1.8 when $\mathrm{char}(\mathbb{F}) = 3$. □

3.2. **Some further consequences.**

In this subsection we derive further properties of the algebras discussed in subsection 3.1.

Lemma 3.2.1. *Let a, b be two distinct η-axes in A. Then $N_{a,b} = 2B_{a,b}$ if and only if $a^{\tau(b)} = a$. In particular, if $\tau(a) = \tau(b)$, then $N_{a,b} = 2B_{a,b}$.*

Proof. We have $a^{\tau(b)} = a$ if and only if

$(*)$ the projection of a into the η-eigenspace of ad_b is 0.

Now if $N_{a,b} = 2B_{a,b}$, then $ab = 0$, so clearly $(*)$ holds. If $(*)$ holds then, by [HRS2, Proposition 2.8, p. 88], $ab = 0$, so $N_{a,b} = 2B_{a,b}$.

Next, if $\tau(a) = \tau(b)$, then $a^{\tau(b)} = a^{\tau(a)} = a$, so $N_{a,b} = 2B_{a,b}$. □

Lemma 3.2.2. *Let a, b be two distinct η-axes in A and assume that $N := N_{a,b} = B(\eta, \varphi_{a,b})_{a,b}$ is 3-dimensional. Set $\sigma = \sigma_{a,b}$, $\varphi = \varphi_{a,b}$ and $\pi = \pi_{a,b}$. Set $V := N_{b,b^{\tau(a)}}$. Then either $V = N$ or N has an identity $1_{a,b}$ and exactly one of the following holds:*

(i) $N = B(\frac{1}{2}, 0)_{a,b}$, $V = J_{b,b^{\tau(a)}}$ *and then* $\mathrm{Span}\{b, b^{\tau(a)}\} \cap \mathrm{Span}\{1_{a,b}, a\} = \mathbb{F}(1_{a,b} - a)$.

(ii) $N = B(\frac{1}{2}, \frac{1}{2})_{a,b}$, $V = 2B_{b,b^{\tau(a)}}$ *and then* $\mathrm{Span}\{b, b^{\tau(a)}\} \cap \mathrm{Span}\{1_{a,b}, a\} = \mathbb{F}1_{a,b}$.

Proof. Assume that V is 2-dimensional. By Lemma 3.1.2 we must consider 2 cases.

Case 1. $\eta \in \{-1, \frac{1}{2}\}$ and $bb^{\tau(a)} = \eta b^{\tau(a)} + \eta b$.

In this case, by Theorem 3.1.3(6)

$$\begin{aligned} bb^{\tau(a)} = \eta b^{\tau(a)} + \eta b &= -\tfrac{2\eta}{\eta}\sigma - \tfrac{2\eta(\eta-\varphi)}{\eta} a - \eta b + \eta b \\ &= -2\sigma - 2(\eta - \varphi)a. \end{aligned}$$

Hence by Theorem 3.1.3(7),

$$\begin{aligned} & -2\sigma - 2(\eta - \varphi)a \\ &= -\tfrac{2(\eta-\varphi)}{\eta}\sigma - \Big(2(\eta - \varphi) + \tfrac{2}{\eta}\pi + 1\Big) b - 2(\eta - \varphi)a. \end{aligned}$$

We conclude that $2 = \frac{2(\eta-\varphi)}{\eta}$. This implies

$$\varphi = 0 \quad \text{and} \quad \pi = -\eta.$$

But also $2\eta + \frac{2}{\eta}\pi + 1 = 0$, thus

$$\pi = -\tfrac{2\eta^2+\eta}{2}, \qquad \text{so for } \eta \in \{-1, \tfrac{1}{2}\},\ \pi = -\tfrac{1}{2}.$$

Note that $-1 = \frac{1}{2}$ if $\mathrm{char}(\mathbb{F}) = 3$, hence the only case that can occur is (i). Also by Lemma 3.1.10 we have

$$-\tfrac{1}{2}b + \tfrac{1}{2}b^{\tau(a)} = -\tfrac{1}{2}b + \tfrac{1}{2}(-4\sigma - 2a - b) = -2\sigma - a = 1_{a,b} - a.$$

Case 2. $bb^{\tau(a)} = 0$.

In this case, by Theorem 3.1.3(7),

$$-\tfrac{2(\eta-\varphi)}{\eta}\sigma - \Big(2(\eta - \varphi) + \tfrac{2}{\eta}\pi + 1\Big) b - 2(\eta - \varphi)a = 0.$$

Hence $\eta = \varphi$ and $\pi = -\frac{\eta}{2}$. Since $\pi = (1 - \eta)\varphi - \eta$ we get $(1 - \eta)\eta = \frac{\eta}{2}$, and $\eta = \frac{1}{2}$. This is case (ii). Since $b^{\tau(a)} = 1_{a,b} - b$ the last claim of (ii) holds. □

Lemma 3.2.3. *Suppose there exists an element $\mathbb{1} \in A$ such that $\mathbb{1}^2 = \mathbb{1}$ and $\mathbb{1} \cdot a = a$, for all $a \in \mathcal{A}$. Then $\mathbb{1}$ is the identity element of A.*

Proof. Since $\mathbb{1} = a + (\mathbb{1} - a) \in A_1(a) + A_0(a)$, by the definition of $\tau(a)$ we have, $\mathbb{1}^{\tau(a)} = \mathbb{1}$ for all $a \in \mathcal{A}$. Let $G := \langle \tau(a) \mid a \in \mathcal{A}\rangle$. Then $\mathbb{1}^g = \mathbb{1}$, for all $g \in G$. Since any $x \in [\mathcal{A}]$ has the form a^g, for some $g \in G$ and $a \in \mathcal{A}$, we see that $\mathbb{1} \cdot x = \mathbb{1}$, for all $x \in [\mathcal{A}]$. By [HRS2, Corollary 1.2], A is spanned by $[\mathcal{A}]$, so $\mathbb{1}$ is the identity of A. □

Lemma 3.2.4. *Assume that $\eta = \frac{1}{2}$ and that $a, b, c \in \mathcal{X}$ are distinct. Suppose there exists an element $\mathbb{1} \in A$ such that $\mathbb{1}^2 = \mathbb{1}$ and $\mathbb{1}x = x$, for all $x \in \{a, b, c\}$. Suppose further that*

$$xy = \alpha_{x,y}\mathbb{1} + \tfrac{1}{2}x + \tfrac{1}{2}y,$$

where $\alpha_{x,y} \in \mathbb{F}$, for all distinct $x, y \in \{a, b, c\}$. Then

(1) $\alpha_{x,y} = \pi_{x,y}$, *for all distinct* $x, y \in \{a, b, c\}$;
(2) $ab^{\tau(c)} = \big(8\pi_{a,c}\pi_{b,c} + 2\pi_{a,c} - \pi_{a,b} + 2\pi_{b,c}\big)\mathbb{1} + \frac{1}{2}a + \frac{1}{2}b^{\tau(c)}$.

Proof. (1): Let $x, y \in \{a, b, c\}$ with $x \neq y$. Set $N := N_{x,y}$. Suppose first that $\dim(N) = 2$. If $xy = 0$, then $xy = -\frac{1}{2}(x+y) + \frac{1}{2}x + \frac{1}{2}y$, and by Lemma 3.1.2(1), $\pi_{x,y} = -\frac{1}{2}$. Also $x+y$ is the identity element of N. Hence $\mathbb{1} = x+y$ and $\alpha_{x,y} = \pi_{x,y} = -\frac{1}{2}$. Otherwise, since N contains no identity element, $\alpha_{x,y} = \pi_{x,y} = 0$, by Lemma 3.1.2(2&3).

Suppose $\dim(N) = 3$. By Theorem 3.1.3(4), $xy = \sigma_{x,y} + \frac{1}{2}x + \frac{1}{2}y$ and $\sigma_{x,y} \neq 0$. Hence $\sigma_{x,y} = \alpha_{x,y}\mathbb{1} \neq 0$. Thus $\mathbb{1} \in N$ so by Lemma 3.2.3, N contains an identity element $\mathbb{1}$. But if $\pi_{x,y} = 0$, then N contains no identity element (Theorem 3.1.3(3)). Hence $\pi_{x,y} \neq 0$, and then $1_{x,y} = \mathbb{1}$. But $\sigma_{x,y} = \pi_{x,y}1_{x,y} = \pi_{x,y}\mathbb{1}$. Hence $\alpha_{x,y} = \pi_{x,y}$.

(2): Set

$$\alpha := \pi_{a,b}, \quad \beta := \pi_{b,c} \quad \gamma := \pi_{a,c} \quad \text{and } \varphi := \varphi_{b,c}$$

By Lemma 3.1.10 (and since here $\sigma_{b,c} = \pi_{b,c}\mathbb{1}$),

$$b^{\tau(c)} = -4\beta\mathbb{1} - (2 - 4\varphi)c - b.$$

Hence (recalling that $\beta = \pi_{b,c} = \frac{1}{2}\varphi - \frac{1}{2}$) we have

$$\begin{aligned}
ab^{\tau(c)} &= a(-4\beta\mathbb{1} - (2 - 4\varphi)c - b) = -4\beta a - (2 - 4\varphi)ac - ab \\
&= -4\beta a + (4\varphi - 2)(\gamma\mathbb{1} + \tfrac{1}{2}a + \tfrac{1}{2}c) - (\alpha\mathbb{1} + \tfrac{1}{2}a + \tfrac{1}{2}b) \\
&= -4\beta a + (4\varphi - 2)\gamma\mathbb{1} + (2\varphi - 1)a + (2\varphi - 1)c - \alpha\mathbb{1} - \tfrac{1}{2}a - \tfrac{1}{2}b \\
&= ((4\varphi - 2)\gamma - \alpha)\mathbb{1} + (2\varphi - \tfrac{3}{2} - 4\beta)a + (2\varphi - 1)c - \tfrac{1}{2}b \\
&= ((4\varphi - 2)\gamma - \alpha + 2\beta)\mathbb{1} + (2\varphi - \tfrac{3}{2} - 4\beta)a - 2\beta\mathbb{1} + (2\varphi - 1)c - \tfrac{1}{2}b \\
&= ((4\varphi - 2)\gamma - \alpha + 2\beta)\mathbb{1} + \tfrac{1}{2}a + \tfrac{1}{2}b^{\tau(c)}.
\end{aligned}$$

Since $\beta = \frac{1}{2}\varphi - \frac{1}{2}$ we get that $\varphi = 2\beta + 1$. □

Lemma 3.2.5. *Let a, c be two distince η-axes in A and assume that there exists an η-axis $b \in N_{a,c}$ such that $ab = 0$. Then either $N_{a,c} = 2B_{a,c}$ and $b = c$, or $\eta = \frac{1}{2}$ and $N_{a,c}$ contains an identity $1_{a,c} = a + b$.*

Proof. If $b = c$, then this is clear. So assume $b \neq c$. Suppose that $N_{a,c}$ is 2-dimensional. Then $N_{a,c} = N_{a,b} = 2B_{a,b}$, so $b = c$, a contradiction.

Hence $N := N_{a,c}$ is 3-dimensional. Set $B := N_{a,b} = 2B_{a,b}$. Let $\sigma = \sigma_{a,c} \neq 0$. If $\pi_{a,c} = 0$, then $\sigma \notin B$ because σ is in the annihilator of N and B has $\{0\}$ annihilator. But this implies that $N = B \oplus \mathbb{F}\sigma$ is associative, contradicting the fact that η is an eigenvalue of ad_a (and ad_c). (The eigenvalues of idempotents in an associative algebra are 0 and 1.)

Thus $\pi_{a,c} \neq 0$ and then $1_{a,c} = \frac{1}{\pi_{a,c}}\sigma$ is the identity element of N. Now if $1_{a,c} \notin B$, then again we get that $N = B \oplus \mathbb{F} \cdot 1_{a,c}$ is associative a contradiction. Hence $1_{a,c} \in B$, so $1_{a,c} = a + b$. Also, since $a = 1_{a,c} - b$ is an η-axis in A and it has eigenvalues $0, 1, 1 - \eta$, we must have $\eta = \frac{1}{2}$. □

The following properties of dihedral groups are well-known and easy to check:

Lemma 3.2.6. *Let $D := \langle t, s \rangle$ be a dihedral group such that t, s are involutions and such that $|ts| = k \geq 2$. Then*

(1) *if k is odd then $t^{(st)^{\frac{k-1}{2}}} = s$ and $s^{(ts)^{\frac{k-1}{2}}} = t$;*
(2) *if k is even then*
  (a) $t^{(st)^{\frac{k-2}{2}}s} = t$;
  (b)
$$(st)^{\frac{k-2}{2}}s = \begin{cases} s^{(ts)^{\frac{k-2}{4}}}, & \text{if } k \equiv 2 \pmod 4 \\ t^{(st)^{\frac{k-4}{4}}s}, & \text{if } k \equiv 0 \pmod 4). \end{cases}$$

Lemma 3.2.7. *Let a, b be two distinct η-axes in A and assume that $|\tau(a)\tau(b)| = k < \infty$. Set $t = \tau(a)$ and $s = \tau(b)$ and assume that $s \neq t$. Let $N = N_{a,b}$. Then,*

(1) *if k is odd then either $a = b^{(ts)^{\frac{k-1}{2}}}$, or $\eta = \frac{1}{2}$ and $N_{a,b}$ contains an identity $1_{a,b}$. Further $1_{a,b} - a = b^{(ts)^{\frac{k-1}{2}}}$, $\tau(1_{a,b} - a) = \tau(a)$ and $\tau(1_{a,b} - b) = \tau(b)$.*
(2) *If $k = 2$ then either $N = 2B_{a,b}$ or $N = B(\frac{1}{2}, \frac{1}{2})_{a,b}$ and $\tau(1_{a,b} - x) = \tau(x)$ for $x \in \{a, b\}$.*
(3) *If $k \geq 4$ is even then $\eta = \frac{1}{2}$, N contains an identity $1_{a,b}$, and $(1_{a,b} - x) \in \mathcal{X}$ for $x \in \{a, b\}$. Furthermore*
  (a) *If $k \equiv 2 \bmod(4)$ then either $\tau(1_{a,b} - x) \neq \tau(x)$, for $x \in \{a, b\}$ and*
$$a + b^{(ts)^{\frac{k-2}{2}}} = 1_{a,b} = b + a^{(st)^{\frac{k-2}{2}}}$$
  or for $x \in \{a, b\}$ there exists $c_x \in \mathcal{X} \cap N$ such that $N = B(\frac{1}{2}, \frac{1}{2})_{x,c_x}$, $\tau(x) = \tau(1_{a,b} - x)$ and
$$a + a^{(st)^{\frac{k-2}{2}}s} = 1_{a,b} = b + b^{(ts)^{\frac{k-2}{2}}t}.$$

(b) *If* $k \equiv 0 \bmod(4)$, *then* $\eta = \frac{1}{2}$ *and for* $x \in \{a, b\}$ *there exists* $c_x \in \mathcal{X} \cap N$ *such that* $N = B(\frac{1}{2}, \frac{1}{2})_{x,c_x}$. *Also*

$$a + a^{(st)^{\frac{k-2}{2}}s} = 1_{a,b} = b + b^{(ts)^{\frac{k-2}{2}}t},$$

$$\tau(1_{a,b} - a) \neq \tau(a) \text{ and } \tau(1_{a,b} - b) \neq \tau(b).$$

Proof. (1): Assume that k is odd. Set $g = (st)^{\frac{k-1}{2}}$. By Lemma 3.2.6

$$\tau(a^g) = \tau(b).$$

By Lemma 3.2.1, either $a^g = b$ or $a^g b = 0$. Suppose $a^g b = 0$. Since $a^g \neq a$ (because $ab \neq 0$), Lemma 3.2.5 implies that $\eta = \frac{1}{2}$ and $N_{a,b}$ contains an identity $1_{a,b} = a^g + b$. Hence $1_{a,b} - b = a^g$, so $\tau(1_{a,b} - b) = \tau(a^g) = \tau(b)$. Similarly $\tau(1_{a,b} - a) = \tau(a)$.

(2): Assume that k is even. Let

$$g = (st)^{\frac{k-2}{2}}s.$$

By Lemma 3.2.6, $t^g = t$, so

$$\tau(a^g) = \tau(a),$$

and

$$g = \begin{cases} \tau(b^h), & \text{if } k \equiv 2 \pmod 4 \\ \tau(a^h), & \text{if } k \equiv 0 \pmod 4, \end{cases}$$

where

$$h = \begin{cases} (ts)^{\frac{k-2}{4}}, & \text{if } k \equiv 2 \pmod 4 \\ (st)^{\frac{k-4}{4}}s, & \text{if } k \equiv 0 \pmod 4. \end{cases}$$

By Lemma 3.2.1, either $a^g = a$ or $aa^g = 0$. Assume first that $a^g = a$. If $k = 2$, then $a^{\tau(b)} = a$ so by Lemma 3.2.1, $N_{a,b} = 2B_{a,b}$. So let $k \geq 4$. We have

$$a = \begin{cases} a^{\tau(b^h)}, & \text{if } k \equiv 2 \pmod 4 \\ a^{\tau(a^h)}, & \text{if } k \equiv 0 \pmod 4. \end{cases}$$

Since $a \notin \{a^h, b^h\}$ in the respective cases (because $t \notin \{t^h, s^h\}$), Lemma 3.2.1 and Lemma 3.2.5 imply that

$$1_{a,b} = \begin{cases} a + b^h, & \text{if } k \equiv 2 \pmod 4 \\ a + a^h, & \text{if } k \equiv 0 \pmod 4 \end{cases}$$

If $k \equiv 2 \bmod(4)$, then $b^h = 1_{a,b} - a$ and $\tau(1_{a,b} - a) \neq \tau(a)$. Similarly $a^{h^{-1}} = 1_{a,b} - b$ so $\tau(1_{a,b} - b) \neq \tau(b)$.

If $k \equiv 0 \bmod(4)$, then $a^h = 1_{a,b} - a$. Since $h = \tau(c)$ for some $c \in \mathcal{X} \cap N_{a,b}$, we see that $N = B(\frac{1}{2}, \frac{1}{2})_{a,c}$. Further $\tau(a) \neq \tau(1_{a,b} - a)$. As we will see later (see Theorem 6.7) this also yields $\tau(1_{a,b} - b) \neq \tau(b)$

Assume next that $aa^g = 0$. Note that $a^g \neq b$ since otherwise $ab = 0$, and then $a^s = a$ and it would follow that $a^g = a$, a contradiction. By Lemma 3.2.5 we get that $N_{a,b}$ contains an identity $1_{a,b} = a + a^g$. Then

$\tau(1_{a,b} - a) = \tau(a^g) = \tau(a)$. Since $g = \tau(b^h)$ or $g = \tau(a^h)$, and since $a^g = 1_{a,b} - a$, Lemma 3.1.10 implies that $N = B(\frac{1}{2}, \frac{1}{2})_{a,c_a}$, with $c_a \in \{a^h, b^h\}$. The argument above (i.e. the case $a^g = a$) shows that necessarily the roles of a and b can be interchanged (since $b^g = b$ implies that $\tau(x) \neq \tau(1_{a,b} - x)$, for $x \in \{a, b\}$). So Parts (1) and (2) of the Lemma hold in case $aa^g = 0$, and the proof of the lemma is complete. □

Lemma 3.2.8. *Let $\mathcal{B} \subseteq \mathcal{X}$. Then*

(1) $[\mathcal{B}] = \bigcup_{g \in G_{\mathcal{B}}} \mathcal{B}^g$;
(2) *for each $g \in G_{\mathcal{B}}$ there are $g_1, g_2, \dots, g_k \in G_{\mathcal{B}}$ $(k \geq 2)$, with $g_1 = \mathrm{id}$ and $g_k = g$, such that $\mathcal{B}^{g_i} \cap \mathcal{B}^{g_{i+1}} \neq \emptyset$, for all $i = 1, \dots, k-1$, in particular;*
(3) *for each $x \in [\mathcal{B}]$ there are $g_1, \cdots, g_k$ as in (2), such that $x \in \mathcal{B}^{g_k}$;*
(4) *$G_{\mathcal{B}} = G_{[\mathcal{B}]}$ and $G_{\mathcal{B}} = G_{\mathcal{B}^\eta}$;*
(5) $[B^\eta] = [\mathcal{B}]^\eta$;
(6) *if $a \in \mathcal{X}^1$, then $ax = 0$ and $a^{\tau(x)} = a$, for all $x \in \mathcal{X} \smallsetminus \{a\}$;*
(7) *$[\mathcal{B}] = \mathcal{B}^1 \cup [\mathcal{B}^\eta]$ a disjoint union and $[\mathcal{B}]^1 = \mathcal{B}^1$;*
(8) $N_{\mathcal{B}} = N_{[\mathcal{B}]}$.

Proof. (1): Set $G := G_{\mathcal{B}}$. Let $\mathcal{C} := \bigcup_{g \in G} \mathcal{B}^g$. Clearly $\mathcal{C} \subseteq [\mathcal{B}]$. Let now $c \in \mathcal{C}$. Then $c \in \mathcal{B}^g$ for some $g \in G$. Thus $\tau(c) = \tau(b^g) = \tau(b)^g$, for some $b \in \mathcal{B}$. But then $\tau(c) \in G$, so $\mathcal{C}^{\tau(c)} \subset \mathcal{C}$. Thus $\mathcal{C}$ is closed, so $\mathcal{C} = [\mathcal{B}]$.

(2): Let $g \in G$ and write $g = \tau(b_1)\tau(b_2)\cdots\tau(b_m)$, with $b_i \in \mathcal{B}$ for all i. We prove (2) by induction on m. If $m = 1$, then $b_1 \in \mathcal{B} \cap \mathcal{B}^{\tau(b_1)}$ so (2) holds. Next let $h := \tau(b_1)\cdots\tau(b_{m-1})$, and let $\mathrm{id} = h_1, \dots, h_k = h$ with $\mathcal{B}^{h_i} \cap \mathcal{B}^{h_{i+1}} \neq \emptyset$, for all $i = 1, \dots, k-1$. Then $b_m \in \mathcal{B} \cap \mathcal{B}^{\tau(b_m)}$ and letting $g_{i+1} = h_i\tau(b_m)$, $i = 1, \dots, k$ and $g_1 = \mathrm{id}$ we have $g_{k+1} = g$, and clearly (2) hold for g (and $k+1$).

(3): This is immediate from (1) and (2).

(4): Let $x \in [\mathcal{B}]$. By (1), $x = b^g$, for some $b \in \mathcal{B}$ and some $g \in G_{\mathcal{B}}$, so $\tau(x) = \tau(b)^g \in G_{\mathcal{B}}$. Hence $G_{[\mathcal{B}]} \leq G_{\mathcal{B}}$. Also, since (by definition) $\tau(a) = \mathrm{id}$, for $a \in \mathcal{B}^1$, it is clear that $G_{\mathcal{B}} = G_{\mathcal{B}^\eta}$.

(5): Since $\mathcal{B}^\eta \subseteq \mathcal{B}$, we have $[\mathcal{B}^\eta] \subseteq [\mathcal{B}]$. Let $a \in [\mathcal{B}^\eta]$. By (1), there is $b \in \mathcal{B}^\eta$ and $g \in G_{\mathcal{B}^\eta}$ such that $a = b^g$. Since $\tau(b) \neq \mathrm{id}$, also $\tau(a) = \tau(b)^g \neq \mathrm{id}$. Hence $a \in [\mathcal{B}]^\eta$.

Let $a \in [\mathcal{B}]^\eta$, by (1) there exists $b \in \mathcal{B}$ and $g \in G_{\mathcal{B}} = G_{\mathcal{B}^\eta}$ (by (4)) such that $a = b^g$. Since $\tau(a) \neq \mathrm{id}$ it follows that $\tau(b) \neq \mathrm{id}$ so $b \in \mathcal{B}^\eta$ and we see that $a \in [\mathcal{B}^\eta]$.

(6): By definition $\tau(a) = \mathrm{id}$, so $x^{\tau(a)} = x$, for all $x \in \mathcal{X}$. Hence (6) follows from Lemma 3.2.1.

(7): Clearly $\mathcal{B}^1 \cup [\mathcal{B}^\eta] \subseteq [\mathcal{B}]$. Let $a \in [\mathcal{B}]$. Using (1) and (4) write $a = b^g$, with $b \in \mathcal{B}$ and $g \in G_{\mathcal{B}^\eta}$. If $b \in \mathcal{B}^1$, then by (6), $b^g = b$. Otherwise $b \in B^\eta$ and then $b^g \in [\mathcal{B}^\eta]$. By (5) the union is disjoint and $[\mathcal{B}]^1 = \mathcal{B}^1$.

(8): Clearly $N_{\mathcal{B}} \subseteq N_{[\mathcal{B}]}$. Let $b \in \mathcal{B}$. Since $N_{\mathcal{B}}$ is a subalgebra of A it is invariant under the adjoint action ad_b; that is ad_b is a linear transformation of $N_{\mathcal{B}}$. Since ad_b is semi-simple on A, it is semi-simple on $N_{\mathcal{B}}$. By the definition of $\tau(b)$ it follows that $N_{\mathcal{B}}$ is $\tau(b)$-invariant. As this holds for all $b \in \mathcal{B}$ we see that $N_{\mathcal{B}}$ is $G_{\mathcal{B}}$-invariant. By (1), since $\mathcal{B} \subseteq N_{\mathcal{B}}$ also $[\mathcal{B}] \subseteq N_{\mathcal{B}}$ so $N_{[\mathcal{B}]} \subseteq N_{\mathcal{B}}$. □

3.3. Properties related to 3-transpositions.

This subsection is devoted to results related to the question of when an axial algebra is a 3-transposition algebra with respect to a generating set of η-axes (see subsection 1.2.2 of the introduction for a definition). These results will be applied in §7.

Lemma 3.3.1. *Let a, b be two distinct η-axes in A and suppose that $N_{a,b}$ is 2-dimensional. Assume further that $|\tau(a)\tau(b)| \in \{2, 3\}$. Then either $N_{a,b} = 3C(-1)^{\times}_{a,b}$, $|\tau(a)\tau(b)| = 3$, and $a^{\tau(b)} = b^{\tau(a)}$. Or $|\tau(a)\tau(b)| = 2$ and $N_{a,b} = 2B_{a,b}$.*

Proof. We use Lemma 3.1.2. Set $N = N_{a,b}$. If N is as in Lemma 3.1.2(1c), then by Lemma 3.1.9, $a^{(\tau(a)\tau(b))^2} = 5a - 4b$ and $a^{(\tau(a)\tau(b))^3} = 7a - 6b$. Hence $|\tau(a)\tau(b)| \neq 2$. If $\mathrm{char}(\mathbb{F}) \neq 3$, then it follows that $|\tau(a)\tau(b)| \neq 3$, while if $\mathrm{char}(\mathbb{F}) = 3$, then $N = 3C(-1)^{\times}_{a,b}$ and then by Lemma 3.1.2, $a^{\tau(b)} = b^{\tau(a)}$.

If $N_{a,b} = 3C(-1)^{\times}_{a,b}$, then again $a^{\tau(b)} = b^{\tau(a)}$. Finally if $N_{a,b} = 2B_{a,b}$, then, by Lemma 3.2.1, $|\tau(a)\tau(b)| = 2$. □

Corollary 3.3.2. *Let a, b be two distinct η-axes in A, and assume that $a^{\tau(b)} = b^{\tau(a)}$. Then $\varphi_{a,b} = \frac{1}{2}\eta$ and one of the following holds:*

(i) *$N_{a,b}$ is 2-dimensional and $N_{a,b} = 3C(-1)^{\times}_{a,b}$.*
(ii) *$N_{a,b}$ is 3-dimensional and $N_{a,b} = 3C(\eta)_{a,b}$.*

Proof. Set $\varphi := \varphi_{a,b}$. By Lemma 4.1 and Lemma 4.4 in [HRS2],

$$\varphi b + \tfrac{\eta}{2}(a - a^{\tau(b)}) = \varphi a + \tfrac{\eta}{2}(b - b^{\tau(a)}),$$

Hence $0 = \varphi(a - b) + \frac{\eta}{2}(b - a)$, so since $a \neq b$, we have $\varphi = \frac{\eta}{2}$. Notice that $|\tau(a)\tau(b)| = 3$. If $N_{a,b}$ is 2-dimensional, then (i) follows from Lemma 3.3.1. If $N_{a,b}$ is 3-dimensional, then $N_{a,b} = B(\eta, \frac{1}{2}\eta)_{a,b}$, so, by definition, $N_{a,b} = 3C(\eta)_{a,b}$. □

Lemma 3.3.3. *Let $\eta = \frac{1}{2}$ and let $a, b \in A$ be two distinct $\frac{1}{2}$-axes. Set $N := N_{a,b}$ and assume that N is contained in a 3-dimensional subalgebra M of A such that M contains an identity element $\mathbb{1}$. Then the following are equivalent:*

(i) $\mathrm{char}(\mathbb{F}) \neq 3$ *and* $N = 3C(\frac{1}{2})_{a,b}$, *or* $\mathrm{char}(\mathbb{F}) = 3$ *and* $N = 3C(-1)^{\times}_{a,b}$.
(ii) $ab = -\frac{3}{8} \cdot \mathbb{1} + \frac{1}{2}a + \frac{1}{2}b$.
(iii) $a^{\tau(b)} = b^{\tau(a)}$.

If these conditions hold then

(1) $\varphi_{a,b} = \frac{1}{4}$, $\pi_{a,b} = -\frac{3}{8}$ *and* $\sigma_{a,b} = -\frac{3}{8}\mathbb{1}$;
(2) $|\tau(a)\tau(b)| = 3$;
(3) *if* $\mathrm{char}(\mathbb{F}) \neq 3$, *then* $\mathbb{1} = 1_{a,b}$;
(4) *if* $c := \mathbb{1} - b$ *is a* $\frac{1}{2}$*-axis in* A *then* $N_{a,c}$ *is* 3*-dimensional and*

$$ac = -\tfrac{1}{8}\mathbb{1} + \tfrac{1}{2}a + \tfrac{1}{2}c.$$

Also if $\mathrm{char}(\mathbb{F}) \neq 3$, *then* $N_{a,b} = N_{a,c}$, *while if* $\mathrm{char}(\mathbb{F}) = 3$ *then* $N_{a,b} \subsetneq N_{a,c}$ *and* $N_{a,c} = B(-1, 0) = B(\frac{1}{2}, 0)$.

Proof. (i) $\iff$ (ii): If (i) holds and $\mathrm{char}(\mathbb{F}) \neq 3$, then $N = M$ is 3-dimensional, $1_{a,b} \neq 0$, and by Lemma 3.1.4, $\pi_{a,b} = -\frac{3}{8}$ and of course $1_{a,b} = \mathbb{1}$, so (ii) holds. If $\mathrm{char}(\mathbb{F}) = 3$, then (ii) holds by the definition of $3C(-1)^{\times}_{a,b}$.

Assume that (ii) holds. If $\mathrm{char}(\mathbb{F}) = 3$ then clearly (i) holds. Suppose $\mathrm{char}(\mathbb{F}) \neq 3$. Then $\mathbb{1} \in N_{a,b}$ so $\mathbb{1} = 1_{a,b}$. Now Theorem 3.1.3(4) shows that $\pi_{a,b} = -\frac{3}{8}$ and that $\pi_{a,b} = \frac{1}{2}\varphi_{a,b} - \frac{1}{2}$. Hence $\varphi_{a,b} = \frac{1}{4}$ and $N = B(\frac{1}{2}, \frac{1}{4})_{a,b} = 3C(\frac{1}{2})_{a,b}$, by Remark 3.1.5. Hence (i) holds.

(i) $\iff$ (iii): Suppose (i) holds. If $\mathrm{char}(\mathbb{F}) \neq 3$, then by Lemma 3.1.4(1d) (iii) holds. If $\mathrm{char}(\mathbb{F}) = 3$ then (iii) holds by Lemma 3.1.8. If (iii) holds then by Corollary 3.3.2, (i) holds. Note that when $\mathrm{char}(\mathbb{F}) = 3$, N cannot be 3-dimensional since $3C(-1)_{a,b}$ does not contain an identity element (Lemma 3.1.4(2)).

Part (1) follows from Lemma 3.1.4 and part (2) is an immediate consequence of (iii). We already saw that (3) holds. To see (4) we have

$$\begin{aligned} ac = a(\mathbb{1} - b) = a - ab &= a - (-\tfrac{3}{8} \cdot \mathbb{1} + \tfrac{1}{2}a + \tfrac{1}{2}b) = \tfrac{3}{8} \cdot \mathbb{1} + \tfrac{1}{2}a - \tfrac{1}{2}b \\ &= \tfrac{3}{8} \cdot \mathbb{1} + \tfrac{1}{2}a - \tfrac{1}{2}\mathbb{1} + \tfrac{1}{2}\mathbb{1} - \tfrac{1}{2}b = -\tfrac{1}{8} \cdot \mathbb{1} + \tfrac{1}{2}a + \tfrac{1}{2}c. \end{aligned}$$

This shows that $\mathbb{1} \in N_{a,c}$. By Lemma 3.1.2 and (iii), $N_{a,c}$ is 3-dimensional and $\mathbb{1} = 1_{a,c}$. Hence if $\mathrm{char}(\mathbb{F}) \neq 3$ then $N_{a,c} = N_{a,b}$. If $\mathrm{char}(\mathbb{F}) = 3$ then $\pi_{a,c} = -\frac{1}{8} = 1 = -\frac{1}{2}$ and then, by the definition of $\pi_{a,c}$, we have $\varphi_{a,c} = 0$. This shows (4). □

Lemma 3.3.4. *Let* $\eta = \frac{1}{2}$ *and let* $a, b \in A$ *be two distinct* $\frac{1}{2}$*-axes. Set* $N := N_{a,b}$. *Assume that* $\dim(N) = 3$ *and that* N *contains an identity element* $1_{a,b}$. *Then the following are equivalent*

(i) $1_{a,b} - x$ *is a* $\frac{1}{2}$*-axis in* A *and either* $\mathrm{char}(\mathbb{F}) \neq 3$ *and* $N = 3C(\frac{1}{2})_{x,\,(1_{a,b}-y)}$, *or* $\mathrm{char}(\mathbb{F}) = 3$ *and* $N_{x,\,(1_{a,b}-y)} = 3C(-1)^{\times}_{x,\,(1_{a,b}-y)}$, *for* $\{x, y\} = \{a, b\}$.
(ii) $ab = -\frac{1}{8} \cdot 1_{a,b} + \frac{1}{2}a + \frac{1}{2}b$;
(iii) $a^{\tau(b)} + b^{\tau(a)} = 1_{a,b}$.

If these conditions hold then

(1) $\varphi_{a,b} = \frac{3}{4}$ *(so* $\varphi_{a,b} = 0$ *if* $\mathrm{char}(\mathbb{F}) = 3$*),* $\pi_{a,b} = -\frac{1}{8}$ *(so* $\pi_{a,b} = 1$ *if* $\mathrm{char}(\mathbb{F}) = 3$*) and* $\sigma_{a,b} = -\frac{1}{8} \cdot 1_{a,b}$ *(so* $\sigma_{a,b} = 1_{a,b}$ *if* $\mathrm{char}(\mathbb{F}) = 3$*);*
(2) $|\tau(1_{a,b} - x)\tau(y)| = 3$, *for* $\{x, y\} = \{a, b\}$;

(3) $|\tau(a)\tau(b)| \in \{3, 6\}$;
(4) $|\tau(a)\tau(b)| = 3$ *if and only if* $\tau(x) = \tau(1_{a,b} - x)$ *for* $x \in \{a, b\}$;
(5) *if* $|\tau(a)\tau(b)| = 6$ *then* $\tau(1_{a,b} - x)\tau(x)$ *is an involution in the center of* $\langle\tau(a), \tau(b)\rangle$, *for* $x \in \{a, b\}$.

Proof. Set $\pi = \pi_{a,b}$, $\varphi = \varphi_{a,b}$ and $\sigma = \sigma_{a,b}$.

(i) $\Longrightarrow$ (ii): Assume that (i) holds. By Lemma 3.3.3, and by the definition of $3C(-1)^{\times}_{a,b}$ (when $\mathrm{char}(\mathbb{F}) = 3$),

$$\begin{aligned} a - ab = a(1_{a,b} - b) &= -\tfrac{3}{8} \cdot 1_{a,b} + \tfrac{1}{2}a + \tfrac{1}{2}(1_{a,b} - b) \\ &= \tfrac{1}{8} \cdot 1_{a,b} + \tfrac{1}{2}a - \tfrac{1}{2}b, \end{aligned}$$

so (ii) holds. Also, by (i) and Lemma 3.3.3(iii) (respectively Lemma 3.1.8), part (2) holds.

(ii) $\Longrightarrow$ (iii): By Lemma 3.1.10,

$$x^{\tau(y)} = -4\sigma + y - x \quad \text{for } \{x, y\} = \{a, b\}.$$

Adding we see that $a^{\tau(b)} + b^{\tau(a)} = -8\sigma = 1_{a,b}$.

(iii) $\Longrightarrow$ (ii): We know that $N = B(\frac{1}{2}, \varphi)$, so by Lemma 3.1.10,

$$x^{\tau(y)} = -4\sigma - (2 - 4\varphi)y - x \quad \text{for } \{x, y\} = \{a, b\}.$$

Adding we get $a^{\tau(b)} + b^{\tau(a)} = -8\sigma - (3 - 4\varphi)a - (3 - 4\varphi)b$. But this expression equals $1_{a,b}$. Hence $\varphi = \frac{3}{4}$ and $\pi = -\frac{1}{8}$. Now Theorem 3.1.3(4) yields (ii).

(ii) $\Longrightarrow$ (i): Assume that (ii) holds. By (ii) $\Longrightarrow$ (iii) we already know that $1_{a,b} - x$ is a $\frac{1}{2}$-axis, for $x \in \{a, b\}$. By Theorem 3.1.3, $\pi = -\frac{1}{8}$ and $\pi = \frac{1}{2}\varphi - \frac{1}{2}$, so $\varphi = \frac{3}{4}$. Now

$$\begin{aligned} a(1_{a,b} - b) = a - ab = a - (-\tfrac{1}{8} \cdot 1_{a,b} + \tfrac{1}{2}a + \tfrac{1}{2}b) = \\ \tfrac{1}{8} \cdot 1_{a,b} + \tfrac{1}{2}a - \tfrac{1}{2}b = \tfrac{1}{8} \cdot 1_{a,b} + \tfrac{1}{2}a - \tfrac{1}{2}1_{a,b} + \tfrac{1}{2}1_{a,b} - \tfrac{1}{2}b \\ = -\tfrac{3}{8} \cdot 1_{a,b} + \tfrac{1}{2}a + \tfrac{1}{2}(1_{a,b} - b). \end{aligned}$$

Hence if $\mathrm{char}(\mathbb{F}) \neq 3$, this show that $1_{a,b}$ is in the subalgebra of N generated by a and $1_{a,b} - b$ and hence a and $1_{a,b} - b$ generate N. Now Lemma 3.3.3 shows that that $N = 3C(\frac{1}{2})_{a,(1_{a,b} - b)}$. If $\mathrm{char}(\mathbb{F}) = 3$ then $N_{a,(1_{a,b} - a)} = 3C(-1)^{\times}_{a,(1_{a,b} - b)}$. By symmetry the same holds for b.

We already saw that (1) and (2) hold. Since $1_{a,b} - b = a^{\tau(b)\tau(a)}$ we have $\tau(a)\tau(1_{a,b} - b) = (\tau(b)\tau(a))^2$. Hence, by (2), $|\tau(a)\tau(b)| \in \{3, 6\}$. Also $\tau(1_{a,b} - b)\tau(b) = (\tau(a)\tau(b))^3$ so (3), (4) and (5) hold for b, and by symmetry they also holds for a. □

4. 3-transposition groups of ADE-type

The purpose of this section is to characterize central quotients of finite simply-laced Weyl/Coxeter groups of type A, D, and E (see Proposition 4.4 for a precise description of these groups). Thus we define 3-transpositions groups of ADE-type (see Definition 4.2), and Theorem 4.3 is the main theorem of this section. In §7 we will see how these groups are related to primitive axial algebras of Jordan type half.

We start with a short discussion. In the 3-transposition group G, the normal set of generating 3-transpositions D is said to be of *symplectic type* if for every $d, e, f \in D$ with $\langle e, f\rangle$ isomorphic to S_3, the transposition d commutes with at least one of $\{e, f, efe = fef\} = D \cap \langle e, f\rangle$. Equivalently (see [CH, H1, HSo]) G has no subgroup $H = \langle D \cap H\rangle$ with $|D \cap H| = 9$; that is, $|H| \neq 18, 54$.

The name comes from the fact that (see Theorem 4.7 below) every group of symplectic type arises from a subgroup of a symplectic group over $\mathbb{F}_2$ that is generated by transvections (a generating 3-transposition class in the full symplectic group).

Let us recall the notion of the *diagram*: Given a subset $Y \subseteq D$, the *diagram* of Y is the graph whose vertex set is Y and $a, b \in Y$ form an edge if and only if $|ab| = 3$.

It is well-known and easy to see [CH, H2] that the finite simply-laced Weyl/Coxeter groups of type A, D, and E are 3-transposition groups with the Weyl generators contained in a 3-transposition class of symplectic type. These facts were of great help in the classification [CH] of 3-transposition groups with trivial center. For instance, the diagram A_3 $(= D_3)$ is complete bipartite $K_{1,2}$, and the isomorphism $\mathrm{W}(A_3) \cong S_4$ leads directly to a result that is often used without reference:

Lemma 4.1. *Let G be a group generated by the conjugacy class D of* 3-*transpositions. Then $D \cap dZ(G) = \{d\}$ for each $d \in D$ and $Z(G) = Z_2(G)$.*

Proof. This is due to Fischer and can be found as [CH, Lemma (3.16)] and [H1, (4.3)] (where the assumption of symplectic type is not used).

If $G = \langle D\rangle \cong S_2$, then this is certainly true. Otherwise there is an $e \in D$ with $\langle d, e\rangle \cong S_3$. Were there to be an $f \in dZ(G)$ with $d \neq f$, then $\{d, e, f\}$ would have diagram A_3 and so generate a subgroup $H \cong S_4$. But then $1 \neq df \in Z(G)$ while $Z(H) = 1$. The contradiction shows that no such f exists.

The subgroup $Z(G)$ is clearly the kernel of the action of $G = \langle D\rangle$ on D by conjugation. But the previous paragraph implies that $Z_2(G)$ is also in this kernel. Thus $Z(G) = Z_2(G)$. □

Let us now define groups of ADE-type.

Definition 4.2. In the 3-transposition group G, the normal set of generating 3-transpositions D is said to be of *ADE-type* provided it is of symplectic type and there is no subgroup $H = \langle D \cap H \rangle$ isomorphic to a central quotient of $\mathrm{W}_2(\tilde{D}_4)$. The group G is then called a group of ADE-type.

Recall that $\tilde{D}_4$ is the complete bipartite graph $K_{1,4}$, and see Proposition 4.5(4) for $\mathrm{W}_2(\tilde{D}_4)$.

In this section we will prove:

Theorem 4.3. *Let G be a finite 3-transposition group generated by the conjugacy class D of 3-transpositions having ADE-type. Then there is an $n \in \mathbb{Z}^+$ with G a central quotient of $\mathrm{W}(A_n)$ for $n \geq 2$, $\mathrm{W}(D_n)$ for $n \geq 4$, or $\mathrm{W}(E_n)$ for $n \in \{6, 7, 8\}$. All of these groups are of ADE-type.*

Given the appropriate definitions, Theorem 4.3 remains true for infinite 3-transposition groups of ADE-type. In this paper we are only concerned with the finite case.

Proposition 4.4. *Let X be a subset of D, a normal set of 3-transpositions in the group G. Set $H = \langle X \rangle$.*

(1) *If X has diagram (isomorphic to) A_n then H is isomorphic to the Weyl/Coxeter group $\mathrm{W}(A_n) \cong S_{n+1}$.*
(2) *If X has diagram D_n then H is isomorphic to a central quotient of $\mathrm{W}(D_n)$. That is, either $H \cong \mathrm{W}(D_n) \cong 2^{n-1}{:}\, S_n$ or $H \cong \mathrm{W}(D_{2k})/Z(\mathrm{W}(D_{2k})) \cong 2^{2k-2}{:}\, S_{2k}$.*
(3) *If X has diagram E_6 then H is isomorphic to $\mathrm{W}(E_6) \cong O_6^-(2)$.*
(4) *If X has diagram E_7 then H is isomorphic to a central quotient of $\mathrm{W}(E_7)$. That is, either $H \cong \mathrm{W}(E_7) \cong 2 \times Sp_6(2)$ or $H \cong \mathrm{W}(E_7)/Z(\mathrm{W}(E_7)) \cong Sp_6(2)$.*
(5) *If X has diagram E_8 then H is isomorphic to a central quotient of $\mathrm{W}(E_8)$. That is, either $H \cong \mathrm{W}(E_8) \cong 2 \cdot O_8^+(2)$ or the group $H \cong \mathrm{W}(E_8)/Z(\mathrm{W}(E_8)) \cong O_8^+(2)$.*

Proof. In each case, H must be a quotient of the related Weyl/Coxeter group. As the elements of X are distinct, the only possible kernels for this quotient are central. □

As is noted in [CH], in each of these 3-transposition groups the 3-transposition class is uniquely determined except for $\mathrm{W}(A_5) \cong S_6$, $\mathrm{W}(D_{2k})$, and $\mathrm{W}(E_8)$ where there are two classes of 3-transpositions, exchanged by an outer automorphism (a central automorphism except in the case of S_6).

The simply-laced affine Weyl group $\mathrm{W}(\tilde{X})$ for $X \in \{A_n, D_n, E_n\}$ is the split extension of the corresponding rank n root lattice Λ_X by the finite Weyl group $\mathrm{W}(X)$ [B81, p. 173]. These are not 3-transposition groups but become such if we factor by $2\Lambda_X$ or $3\Lambda_X$; see again [CH]. Indeed the factor group $\mathrm{W}_2(\tilde{X}) = \mathrm{W}(\tilde{X})/2\Lambda_X$ is a finite 3-transposition group of symplectic type. For instance, S_4 is $\mathrm{W}(A_3) = \mathrm{W}(D_3)$ but it is also $\mathrm{W}_2(\tilde{A}_2) = 2^2{:}\, S_3$. (The diagram $\tilde{A}_2$ is a triangle.) Here the normal elementary abelian 2^2

is the mod 2 root lattice $V_4 = \Lambda_{A_2}/2\Lambda_{A_2}$ of type A_2, naturally admitting $\mathrm{W}(A_2) \cong S_3$.

As already mentioned, the diagram $A_3 = D_3$ is complete bipartite $K_{1,2}$. Additionally D_4 is $K_{1,3}$ and $\tilde{D}_4$ is $K_{1,4}$.

Proposition 4.5. *Let $H = (\bigoplus_{i=1}^m V(i))\colon S_3$ be the split extension by S_3 of $V = \bigoplus_{i=1}^m V(i)$, a direct sum of copies $V(i)$ of the S_3-module V_4.*

(1) *H is a 3-transposition group, generated by the class $E = d^H = e^H$ for $\langle d, e\rangle \cong S_3$, a complement to V. The diagram of E is a complete tripartite graph $K_{2^m,2^m,2^m}$ with parts d^V, e^V, and $(ede)^V$. The group H is generated by d together with a basis of the elementary abelian subgroup $\langle e^V\rangle$, this generating set having diagram $K_{1,m}$.*
(2) *For $m = 1$, the group H is isomorphic to $\mathrm{W}(A_3) = \mathrm{W}(D_3) \cong \mathrm{W}_2(\tilde{A}_2)$ and is isomorphic to S_4.*
(3) *For $m = 2$, the group H is isomorphic to the quotient of $\mathrm{W}(D_4) \cong 2^{1+(2\oplus 2)}\colon S_3$ by its center of order 2.*
(4) *For $m = 3$, the group H is isomorphic to the quotient of*

$$\mathrm{W}_2(\tilde{D}_4) \cong 2^4\colon \mathrm{W}(D_4) \cong 2^4\colon (2^{1+(2\oplus 2)}\colon S_3) = (2^3 \cdot (2^2 \oplus 2^2 \oplus 2^2))\colon S_3$$

by its elementary center of order 2^3.

Proof. The first part is a direct computation. The rest then come from expanding $\{d, e\}$ to a generating set X from E of size $2 + m$ and having the appropriate diagram. The group $\mathrm{W}_2(\tilde{D}_4)$ is the group $F(5, 24)$ of [HSo]. □

Proposition 4.6. *Let X be a subset of D, a normal set of 3-transpositions of symplectic type in the group G. If X has diagram $\tilde{D}_4$ and $H = \langle X\rangle$ is not a central quotient of $\mathrm{W}(D_4)$, then H is a central quotient of $\mathrm{W}_2(\tilde{D}_4)$.*

Proof. The group H must be a quotient of the affine Weyl/Coxeter group $\mathrm{W}(\tilde{D}_4) \cong \mathbb{Z}^4\colon \mathrm{W}(D_4)$. As it is a 3-transposition group of symplectic type, it is in fact a quotient of $\mathrm{W}_2(\tilde{D}_4)$. Since it is not a central quotient of $\mathrm{W}(D_4)$, the only possible kernels are central. □

Theorem 4.7. *Let G be a finite group generated by a conjugacy class D of 3-transpositions of symplectic type.*

(1) *There is a normal subgroup N of G such that $\bar{G} = G/N$ is isomorphic to one of the groups S_n, $O^\epsilon_{2m}(2)$, or $Sp_{2m}(2)$ for $4 \neq n \geq 2$ and $m \geq 3$ with $(m, \epsilon) \neq (3, +)$. This isomorphism can be chosen to map D to the collection of symplectic transvections in $\bar{G}$. For $x, y \in D$, $\bar{x} = \bar{y}$ if and only if $\mathrm{C}_D(x) = \mathrm{C}_D(y)$.*
(2) *The normal subgroup $[G, N]$ is a 2-group, generated by its normal elementary abelian 2-subgroups $[x, N] = \langle\, xy \mid y \in D\,, \bar{x} = \bar{y}\,\rangle$ for $x \in D$.*

Proof. The first part of this theorem is the finite part of [H1, Theorem 5]. The second part of the theorem then follows directly from the last sentence of the first part. □

The restrictions on m in the theorem arise from isomorphisms of the smaller groups with certain symmetric groups.

The papers [H1, H2] provide a full classification (up to a central quotient) of all 3-transposition groups of symplectic type, and the paper [CH] describes the near-complete classification of all 3-transposition groups with trivial center. In our proof of Theorem 4.3 we only need the elementary [H1], as detailed in Theorem 4.7; in particular the cohomological arguments of [H2] are not necessary.

Proposition 4.8. *Let G be a finite group generated by a conjugacy class D of 3-transpositions of symplectic type. Assume additionally there is no subgroup $H = \langle D \cap H \rangle$ isomorphic to a central quotient of $\mathrm{W}(D_4)$. Then there is an $n \in \mathbb{Z}^+$ with G isomorphic to $\mathrm{W}(A_n) \cong S_{n+1}$ for $n \geq 1$.*

Proof. This is nearly equivalent to the finite version of [H1, (2.17)], which is a step in the proof of [H1, Theorem 5] (the finite version of which is the first part of Theorem 4.7). Here we prove it as a consequence of Theorem 4.7.

Let G, D, N, and $\bar{G} = G/N$ be as in the previous theorem.

As D is a conjugacy class, if $\bar{G} \cong S_2$ then $G = \bar{G} \cong S_2 \cong \mathrm{W}(A_1)$, and we are done. So we may assume that there are a, b in D with $\langle a, b \rangle \cong \langle \bar{a}, \bar{b} \rangle \cong S_3$.

First suppose that there is a $c \in D$ for which $\{a, b, c\}$ has diagram A_3 (with $|ac| = 2$ and $|bc| = 3$) and additionally that $\langle \bar{a}, \bar{b}, \bar{c} \rangle$ is isomorphic to $\langle a, b, c \rangle$ and hence to $\mathrm{W}(A_3) \cong S_4$.

As G is generated by $D = a^G$, if $[G, N] \neq 1$ then there is an $x \in N$ with $[a, x] \neq 1$. In that case $a \neq d = x^{-1}ax \in D$ with $\bar{a} = \bar{d}$, so that $\{a, b, c, d\}$ has diagram D_4 and generates a central quotient of $\mathrm{W}(D_4)$. This contradicts the hypothesis, so $[G, N] = 1$ and G is a central extension of one of the groups of Theorem 4.7.

The groups $O^{\epsilon}_{2m}(2)$ and $Sp_{2m}(2)$, for $m \geq 3$ with $(m, \epsilon) \neq (3, +)$ all contain $\mathrm{W}(E_6) \cong O^-_6(2)$ as a transvection generated subgroup. As E_6 has D_4 as a subdiagram, these have subgroups $H = \langle D \cap H \rangle$ that are isomorphic to a central quotient of $\mathrm{W}(D_4)$ by Proposition 4.4. (Indeed this subgroup is actually $\mathrm{W}(D_4)$.) Thus the only possibilities for the quotient $\bar{G} = G/Z(G)$ are S_n for $n \geq 3$. Such a group G will be generated by a subset of D with diagram A_{n-1} (by Lemma 4.1), and so by Proposition 4.4 we have $G \cong S_n$.

The only groups $\bar{G}$ of the theorem that contain no S_4 are S_2 and S_3. We have already dealt with the first case, so we may assume now that $\bar{G} = G/N$ is $S_3 = \langle \bar{a}, \bar{b} \rangle$ for $a, b \in D$.

If $[G, N] = 1$, then $G = \bar{G} \cong \mathrm{W}(A_2) \cong S_3$ by Proposition 4.4. If $|[G, N]| = 2$, then as above there is a $d (\neq a)$ with $\bar{a} = \bar{d}$ and $G = \langle a, b, d \rangle \cong \mathrm{W}(A_3) \cong S_4$. Finally, if $|[G, N]| > 2$ then there are distinct $d, e (\neq a)$ with $\bar{a} = \bar{d} = \bar{e}$ and $\langle a, b, d, e \rangle$ a central quotient of $\mathrm{W}(D_4)$, against hypothesis. □

Lemma 4.9. (1) *The 3-transpositions of $O_8^-(2)$ are not of ADE-type.*
(2) *The 3-transpositions of $O_8^+(2)$ are of ADE-type.*

Proof. (1) $O_8^-(2)$ contains a parabolic subgroup $2^6\colon O_6^-(2)$. As $O_6^-(2) \cong \mathrm{W}(E_6)$ contains a (central quotient of) $\mathrm{W}(D_4)$ (as mentioned before), the noncentral extension $2^6\colon O_6^-(2)$ contains a central quotient of $\mathrm{W}_2(\tilde{D}_4)$ by Proposition 4.6.

(2) $O_8^+(2)$ is of ADE-type if and only if $\mathrm{W}(E_8)$ is (by Lemma 4.1). Suppose $\mathrm{W}(E_8)$ is not. Then it has a subset S of five reflections with diagram $\tilde{D}_4$ that generate H, a central quotient of $\mathrm{W}_2(\tilde{D}_4)$. In the action of $\mathrm{W}(E_8)$ on $V = \mathbb{Q}^8$ we have $W = [V,H] = [V,S]$ of dimension at most 5 and positive definite, as V is. But by Proposition 4.5(4) the reflection group H contains eight pairwise commuting reflections. These cannot act on the positive definite space W of dimension less than 8, a contradiction.

We conclude that $\mathrm{W}(E_8)$ and $O_8^+(2)$ are both of ADE-type. □

We are now in a position to prove Theorem 4.3.

Proof of Theorem 4.3. Let G, N, and $\bar{G} = G/N$ be as in Theorem 4.7, and assume that the conjugacy class D of 3-transpositions is of ADE-type.

As D is a conjugacy class, if $\bar{G} \cong S_2$, then $G = \bar{G} \cong S_2 \cong \mathrm{W}(A_1)$; and we are done. So we may assume that in G there are $a, b \in \bar{D}$ with $\langle \bar{a}, \bar{b}\rangle \cong \langle a, b\rangle \cong S_3$.

First suppose the normal 2-group $[G,N]$ is nontrivial. If $\bar{G}$ had a subgroup $\bar{H} = \langle \bar{H} \cap \bar{D}\rangle$ that was a central quotient $\mathrm{W}(D_4)$, then as $[N,H] \neq 1$ there would be a new, fifth generator that together with four lifted from $\bar{H}$ would provide a $\tilde{D}_4$ diagram and a central quotient of $\mathrm{W}_2(\tilde{D}_4)$ by Proposition 4.6. But this is not the case. Therefore by Proposition 4.8 the group $\bar{G}$ is S_n for some $n \geq 3$. For $a \in D$, if $|[a,N]| > 2$, then within a^N there are enough elements of D to produce together with b a diagram $\tilde{D}_4$ as in Proposition 4.6. But then G must contain a subgroup $H = \langle H \cap D\rangle$ that is a central quotient of $\mathrm{W}_2(\tilde{D}_4)$, against hypothesis. Therefore $|[a,N]| = 2$. Let $\{a, d\} = a^N = D \cap aN$. A generating set for $\bar{G}$ containing $\bar{a}$ and having diagram A_{n-1} can then be lifted to an $(n-1)$-subset A of G with the same diagram and $H = \langle A\rangle \cong S_n$. The set $\{d, A\}$ then has diagram D_n and generates $G = \langle D\rangle$ since $D = a^H \cup d^H$. Therefore G is a central quotient of $\mathrm{W}(D_n)$ by Proposition 4.4.

Now we may assume $[G,N] = 1$ so that G is an extension of central N by $\bar{G}$, which is one of the groups of Theorem 4.7. If $\bar{G}$ is S_n for some $n \geq 3$, then $G = \langle D\rangle = \bar{G} \cong S_n \cong \mathrm{W}(A_{n-1})$ by Lemma 4.1 and Proposition 4.4.

If $\bar{G}$ is $O_{2m}^{\epsilon}(2)$, or $Sp_{2m}(2)$ with $m \geq 4$ and $(m,\epsilon) \neq (4,+)$, then $\bar{G}$ has a $\bar{D}$-subgroup $O_8^-(2)$. By Lemma 4.9 the groups $\bar{G}$ and G are not of ADE-type, against hypothesis. We are left with three possible examples:

$$\bar{G} \in \{O_6^-(2), Sp_6(2), O_8^+(2)\}\,.$$

Thus by Proposition 4.4, the group G is a central quotient of $\mathrm{W}(E_n)$ for $n \in \{6,7,8\}$. Each of these is a genuine example by Lemma 4.9, the groups $\mathrm{W}(E_6)$ and $\mathrm{W}(E_7)$ being subgroups of $\mathrm{W}(E_8)$ generated by reflections. □

We conclude this section with a lemma that will enable us to apply Theorem 4.3 in our primitive axial algebras setup.

Lemma 4.10. *Let G be a group generated by the normal set D of 3-transpositions of symplectic type.*

(1) *A subgroup $H = \langle H \cap D\rangle$ is generated by a subset $Y \subseteq D$ with diagram the complete bipartite graph $K_{3,2}$ if and only if H is a central quotient of $\mathrm{W}(D_4)$ or $\mathrm{W}_2(\tilde{D}_4)$.*
(2) *For the subgroup $H = \langle Y\rangle$ of the previous part, the following are equivalent:*
   (a) *some 4-subset of Y generates a subgroup isomorphic to S_4;*
   (b) *H is generated by a 4-subset of Y with diagram D_4;*
   (c) *H is a central quotient of $\mathrm{W}(D_4)$.*

Proof. Let $H = \langle H \cap D\rangle$ be a central quotient of $\mathrm{W}(D_4)$ or $\mathrm{W}_2(\tilde{D}_4)$. We show that H is generated by a subset $Y \subset D$ with diagram $K_{3,2}$. By Proposition 4.5(1) we can choose $\{a,b,c,d\} \subset H \cap D$ having diagram $D_4 = K_{3,1}$, with parts $\{a,b,c\}$ and $\{d\}$. Further, $K = \langle a,b,c,d\rangle$ is a central quotient of $\mathrm{W}(D_4)$ and $|d^{O_2(K)}| = 4$. In the case $H = K$, choose any $e \in d^{O_2(H)} = d^{O_2(K)}$ with $e \neq d$. Then $H = \langle a,b,c,d\rangle = \langle a,b,c,d,e\rangle$ and $\{a,b,c,d,e\}$ has diagram $K_{3,2}$ with parts $\{a,b,c\}$ and $\{d,e\}$, by Proposition 4.5(1).

Suppose next that H is a central quotient of $\mathrm{W}_2(\tilde{D}_4)$. Then $|d^{O_2(H)}| = 8$. Choose $e \in d^{O_2(H)} \backslash d^{O_2(K)}$. Then $Y = \{a,b,c,d,e\}$ has diagram $K_{3,2}$. Here $\{d,a,e\}$ and $\{a,e,a^{de}\}$ both have diagram A_3 with $\langle d,a,e\rangle = \langle a,e,a^{de} = d^{ae}\rangle$, a copy of S_4. Thus $W = \{a,b,c,a^{de},e\}$ has diagram $K_{4,1} = \tilde{D}_4$ and $\langle Y\rangle = \langle W\rangle$. By Proposition 4.6, the full group H is $\langle Y\rangle = \langle W\rangle$ as $e \notin KZ(H)$. This gives one direction of (1).

For the remainder of the proof of (1) and the proof of (2), let the subset Y of D have diagram $K_{3,2}$ and generate H. Specifically, let $Y = X \cup Z$ with $X = \{a,b,c\}$ and $Z = \{d,e\}$ such that $|xz| = 3$ for all $x \in X$ and $z \in Z$ and $(wy)^2 = 1$ for $w,y \in X$ or $w,y \in Z$. Let

$$K = \langle a,b,c,d\rangle.$$

By Proposition 4.4(2), K is a central quotient of $\mathrm{W}(D_4)$.

(1) If $e \in K$, then $K = H$ is a central quotient of $\mathrm{W}(D_4)$, as claimed. So we may assume $e \notin K$. As such, the result follows directly by checking the list of [HSo, Theorem 6.6]; but we provide a direct proof here.

By Proposition 4.5 the set $D \cap K$ consists of 12 transpositions with diagram the complete tripartite graph $K_{4,4,4}$, and every S_3 subgroup $S = \langle D \cap S\rangle \leq K$ meets each of the parts exactly once. As H is symplectic, $C_K(e)$ must meet each such S in at least one element of $D \cap S$. The only proper subgroups $J = \langle J \cap D\rangle$ of K with this property are those isomorphic to S_4 and the three elementary abelian 2-groups generated by one of the parts. Again by Proposition 4.5 any $J\,(\leq K)$ isomorphic to S_4 would contain three pairs of commuting 3-transpositions, and so at least two members of $\{a,b,c,x\}$, the part of $D \cap K$ containing $\{a,b,c\}$. But then $J = C_K(e)$ would contain at least one of $\{a,b,c\}$, which is not the case by hypothesis.

Therefore $D \cap C_K(e) = \{d,f,g,h\}$, the part of $D \cap K$ that contains d. Here $\{a,d,f,g\}$ has diagram D_4 and so $\langle a,d,f,g\rangle = K$ and $H = \langle K,e\rangle = \langle a,d,f,g,e\rangle$ with $\{a,d,f,g,e\}$ having diagram $\tilde{D}_4 = K_{4,1}$. As K is a central quotient of $\mathrm{W}(D_4)$ and $e \notin K$, the group H is a central quotient of $\mathrm{W}_2(\tilde{D}_4)$ by Proposition 4.6.

(2) $(a) \implies (b)$: A 4-subset of Y not containing both d and e generates an abelian group or a central quotient of $\mathrm{W}(D_4)$ by Proposition 4.4. Suppose instead that $\langle a,b,d,e\rangle$ is a copy of S_4. Then $\langle a,b,d,e\rangle = \langle a,b,d\rangle$, so $H = \langle a,b,c,d\rangle = K$.

$(b) \implies (c)$: This follows from Proposition 4.4(2).

$(c) \implies (a)$: Let $E = a^{O_2(H)} = \{a,b,c,x\}$. For each $y \in E$ the set $\{y,d,e\}$ has diagram $A_3 = D_3 = K_{1,2}$. Therefore $\{d,e\}$ is in exactly two subgroups K_1 and K_2 of H isomorphic to $S_4\,(\cong \mathrm{W}(A_3))$. For these we have $|K_i \cap E| = 2$ and $|K_1 \cap K_2 \cap E| = 0$. Therefore there is an $i \in \{1,2\}$ with $|K_i \cap \{a,b,c\}| = 2$, hence $|K_i \cap \{a,b,c,d,e\}| = 4$ and $K_i = \langle K_i \cap \{a,b,c,d,e\}\rangle$ is a copy of S_4. □

5. Jordan algebras of Clifford type

In this section we discuss a class of Jordan algebras that appear as subalgebras of the Jordan algebra $\mathrm{Cl}(V,q)^+$, which comes from the Clifford algebra $\mathrm{Cl}(V,q)$ of the quadratic space (V,q). These appear in [HRS2, Example (3.5)] where they are denoted $V^J(b)$ (for b equal to half the form B defined below). In [Mc, 3.6, p. 74] these algebras are called *Jordan spin factors* and are denoted $\mathcal{JS}pin(V,B)$.

We also prove a result connecting primitive axial algebras of Jordan type $\frac{1}{2}$ and these Jordan algebras of Clifford type (see Theorem 5.4). This result will be used in §6.

As is well known, if M is an associative algebra over of field $\mathbb{F}$ of characteristic not two then the same M taken with the product $x * y = \frac{1}{2}(xy+yx)$ is a Jordan algebra. This Jordan algebra is denoted M^+.

Let V be a vector space over $\mathbb{F}$ endowed with a quadratic form q. Let $B(u,v) = \frac{1}{2}(q(u+v) - q(u) - q(v))$ be the associated symmetric bilinear form (and so $q(u) = B(u,u)$).

Consider the Clifford algebra $\mathrm{Cl}(V,q)$. This is an associative unital algebra (having the identity $\mathbb{1}$) which is generated by V and satisfies the relations $u^2 = q(u)\mathbb{1}, u \in V$. Equivalently we have relations $uv + vu = 2B(u,v)\mathbb{1}$, for all $u,v \in V$. Thus $\mathrm{Cl}(V,q)$ with the product $x * y = \frac{1}{2}(xy+yx) = B(u,v)\mathbb{1}$ is a Jordan algebra.

It is easy to see that $\mathbb{1} * \mathbb{1} = \mathbb{1}$ and $\mathbb{1} * u = u$ for $u \in V$. Therefore, the subspace $\mathbb{F}\mathbb{1} \oplus V$ of $\mathrm{Cl}(V, q)$ is a subalgebra of the Jordan algebra $\mathrm{Cl}(V, q)^+$, hence itself a Jordan algebra. We say that this Jordan algebra is of *Clifford type* and denote it by $J(V, B)$.

Here are some relevant properties of $J(V, B)$. (Many of these can be found in [HRS2], sometimes with different notation.) Recall the notion of a Miyamoto involution from Notation 2.1, and the Notation in 2.2(2).

Lemma 5.1. *Let $J = J(V, B)$.*

(1) *For $u \in V$ and $\alpha \in \mathbb{F}$, the vector $a := \alpha\mathbb{1} + u$ is an idempotent if and only if either (i) $a \in \{0, \mathbb{1}\}$ or (ii) $\alpha = \frac{1}{2}$ and $q(u) = \frac{1}{4}$.*

(2) *Assume that $a = \frac{1}{2}\mathbb{1} + u$ is an idempotent in J. Then $J_1(a) = \mathbb{F}a$ (and so a is a $\frac{1}{2}$-axis), $J_0(a) = \mathbb{F}(\frac{1}{2}\mathbb{1} - u)$, and $J_{\frac{1}{2}}(a) = u^\perp = \{v \in V \mid B(u, v) = 0\}$.*

(3) *For a as in (2), J decomposes into a directs sum $J_+ \oplus J_-$, where $J_+ = J_1(a) \oplus J_0(a)$ and $J_- = J_{\frac{1}{2}}(a)$, with $J_\delta J_\epsilon = J_{\delta\epsilon}$. The Miyamoto involution $\tau(a)$ fixes $\mathbb{1}$ and acts on V as minus the reflection through $u^\perp$ (that is $v^{\tau(a)} = -v$, for $v \in u^\perp$ and $u^{\tau(a)} = u$). (We recall that $\tau(a)$ is sometimes called the Peirce reflection of a.)*

Proof. We have that $(\alpha\mathbb{1} + u) * (\alpha\mathbb{1} + u) = \alpha^2\mathbb{1} + \alpha\mathbb{1} * u + u * \alpha\mathbb{1} + u * u = (\alpha^2 + q(u))\mathbb{1} + 2\alpha u$. Hence $\alpha\mathbb{1} + u$ is an idempotent if and only if $2\alpha u = u$, and $\alpha^2 + q(u) = \alpha$. This shows (1).

Now $a \in J_1(a)$ because $a^2 = a$, and then $\mathbb{1} - a = \frac{1}{2}\mathbb{1} - u \in J_0(a)$. Next, $a * v = (\frac{1}{2}\mathbb{1} + u) * v = \frac{1}{2}v + B(u, v) = \frac{1}{2}v$, for $v \in u^\perp$, so $u^\perp \subseteq J_{\frac{1}{2}}(a)$. Since a, $\mathbb{1} - a$, and $u^\perp$ together span all of J, (2) holds. Part (3) is immediate from (2). □

Remark 5.2. Let $J(V, B) = \mathbb{F}\mathbb{1} \oplus V$ be a Jordan algebra of Clifford type. Since $v * w = B(v, w)\mathbb{1}$ for all $v, w \in V$, Lemma 5.1 implies that

$J(V, B)$ is a primitive axial algebra of type $\frac{1}{2}$ if and only if
V is linearly spanned by vectors $u \in V$ with $q(u) = \frac{1}{4}$.

In this case it will be appropriate to refer to $J(V, B)$ as an *axial algebra of Clifford type*. Then the $\frac{1}{2}$-axes of J have the form $\frac{1}{2}\mathbb{1} + u$, with $q(u) = \frac{1}{4}$. Furthermore $\frac{1}{2}\mathbb{1} - u$ is also an absolutely primitive idempotent, as $q(-u) = \frac{1}{4}$. Finally Lemma 5.1 implies that $\tau(a) = \tau(b)$, where $a = \frac{1}{2}\mathbb{1} + u$ and $b = \mathbb{1} - a = \frac{1}{2}\mathbb{1} - u$ are distinct axes.

Next we prove a result that enables us to identify a primitive axial algebra of Jordan type $\frac{1}{2}$ as a Jordan algebra of Clifford type. Throughout the rest of this section A is a primitive axial algebra of type $\frac{1}{2}$ generated by a set of $\frac{1}{2}$-axes $\mathcal{A}$.

Lemma 5.3. *Assume that A contains an identity element $\mathbb{1}$. For $a \in \mathcal{A}$ set $v_a := a - \frac{1}{2}\mathbb{1}$. Suppose further that*

$$(*) \qquad v_a v_b \in \mathbb{F}\mathbb{1} \quad \text{for all } a, b \in \mathcal{A}.$$

Then $A = J(V, B)$ for some vector space V and a symmetric bilinear form B on V.

Proof. If $A = \mathbb{F}\mathbb{1}$ then the claim holds with $V = 0$. Let us assume that $A \neq \mathbb{F}\mathbb{1}$. In particular, $\mathbb{1}$ is not an axis.

We set V to be the $\mathbb{F}$-linear span of v_a for all $\frac{1}{2}$-axes $a \in \mathcal{A}$. It follows from $(*)$ that $uv \in \mathbb{F}\mathbb{1}$ for all $u, v \in V$.

Note that $V + \mathbb{F}\mathbb{1}$ is closed under multiplication and contains $\mathcal{A}$. Hence $A = V + \mathbb{F}\mathbb{1}$. Let $a \in \mathcal{A}$. Note that if $\mathbb{1} \in V$ then $a - \frac{1}{2}\mathbb{1} = (a - \frac{1}{2}\mathbb{1})\mathbb{1} = v_a\mathbb{1} \in \mathbb{F}\mathbb{1}$. This yields that $a \in \mathbb{F}\mathbb{1}$, and so $a = \mathbb{1}$, a contradiction. Therefore, $A = V \oplus \mathbb{F}\mathbb{1}$.

Let us define the bilinear form B on V by $uv = B(u, v)\mathbb{1}$. Clearly, B is symmetric since A is commutative. Also B is bilinear, since the algebra product is bilinear. Hence, by definition, $A = J(V, B)$. □

Theorem 5.4. *Assume that A contains two $\frac{1}{2}$-axes $a, b \in \mathcal{A}$ such that $a+b = \mathbb{1}$ is the identity element of A and such that $v_a v_c \in \mathbb{F}\mathbb{1}$, for all $c \in \mathcal{A}$, where $v_c = c - \frac{1}{2}\mathbb{1}$. Then $A = J(V, B)$ for some vector space V and a symmetric bilinear form B on V.*

Proof. We show that $(*)$ of Lemma 5.3 holds. Let $c, d \in \mathcal{A}$. Note that

$$v_b = b - \tfrac{1}{2}\mathbb{1} = (\mathbb{1} - a) - \tfrac{1}{2}\mathbb{1} = \tfrac{1}{2}\mathbb{1} - a = -v_a.$$

Hence also $v_b v_c \in \mathbb{F}\mathbb{1}$, for all $c \in \mathcal{A}$. Also (recall the notation $\sigma_{c,d}$ from Notation 2.3(2)),

$$(5.1) \qquad v_c v_d = (c - \tfrac{1}{2}\mathbb{1})(d - \tfrac{1}{2}\mathbb{1}) = cd - \tfrac{1}{2}c - \tfrac{1}{2}d + \tfrac{1}{4}\mathbb{1} = \sigma_{c,d} + \tfrac{1}{4}\mathbb{1}.$$

Set $\sigma = \sigma_{c,d}$. We show that

$$(5.2) \qquad v_c v_d \in \mathbb{F}\mathbb{1}.$$

If $c = a$ or b then equation (5.2) holds by hypothesis. So assume now that $\{c, d\}$ is disjoint from $\{a, b\}$. Notice that

$$v_a v_a = (a - \tfrac{1}{2}\mathbb{1})(a - \tfrac{1}{2}\mathbb{1}) = a^2 - \tfrac{1}{2}a - \tfrac{1}{2}a + \tfrac{1}{4}\mathbb{1} = \tfrac{1}{4}\mathbb{1}.$$

Let λ be defined by $v_a v_c = \lambda\mathbb{1}$. Then $v_c - 4\lambda v_a$ is a $\frac{1}{2}$-eigenvector for ad_a. Indeed, $(a - \frac{1}{2}\mathbb{1})(v_c - 4\lambda v_a) = v_a(v_c - 4\lambda v_a) = v_a v_c - 4\lambda v_a v_a = \lambda\mathbb{1} - 4\lambda\frac{1}{4}\mathbb{1} = 0$. Similarly, $v_d - 4\delta v_a$ is also a $\frac{1}{2}$-eigenvector for ad_a, where δ is defined by $v_a v_d = \delta\mathbb{1}$.

In view of the fusion rules in A, $(v_c - 4\lambda v_a)(v_d - 4\delta v_a)$ lies in $A_1(a) + A_0(a)$. On the other hand,

$$\begin{aligned}(v_c - 4\lambda v_a)(v_d - 4\delta v_a) &= v_c v_d - 4\delta v_c v_a - 4\lambda v_a v_d + 16\lambda\delta v_a v_a \\ &= v_c v_d - 4\delta\lambda\mathbb{1} - 4\lambda\delta\mathbb{1} + 4\lambda\delta\mathbb{1} = v_c v_d - 4\lambda\delta\mathbb{1}.\end{aligned}$$

Since $\mathbb{1} \in A_1(a) + A_0(a)$, we conclude that $v_c v_d \in A_1(a) + A_0(a)$. Now since a and b are absolutely primitive, $A_1(a) = \mathbb{F}a$. Also $\mathbb{F}b = A_1(b) = A_0(a)$, since $b = \mathbb{1} - a$. It follows that $v_c v_d \in \mathbb{F}a \oplus \mathbb{F}b$. Recall that $v_c v_d = \sigma + \frac{1}{4}\mathbb{1}$. Hence σ is contained in $\mathbb{F}a \oplus \mathbb{F}b$.

We now note that $ab = 0$, which means that $\mathbb{F}a \oplus \mathbb{F}b$ is isomorphic to the associative algebra $\mathbb{F} \oplus \mathbb{F}$. This algebra does not have nilpotent elements. Since $\sigma^2 = \pi\sigma$, where $\pi = \pi_{c,d}$, either $\sigma = 0$ or $\pi \neq 0$ and $\frac{1}{\pi}\sigma$ is an idempotent, namely $\frac{1}{\pi}\sigma$ is one of a, b, or $\mathbb{1} = a + b$. Let us look at these possibilities in turn. If $\sigma = 0$ then $v_c v_d = \frac{1}{4}\mathbb{1}$, a multiple of $\mathbb{1}$. If $\frac{1}{\pi}\sigma = a$ then a is the identity in the subalgebra generated by c and d. However, this means that a is not absolutely primitive, a contradiction. Symmetrically, we also rule out the possibility that $\frac{1}{\pi}\sigma = b$. Finally, if $\frac{1}{\pi}\sigma = \mathbb{1}$ then $v_c v_d = \sigma + \frac{1}{4}\mathbb{1} = (\pi + \frac{1}{4})\mathbb{1}$, again a multiple of $\mathbb{1}$. Hence equation (5.2) holds and the proof is complete. □

6. The graph Δ and some consequences

The purpose of this section is to discuss the graph Δ given in Notation 6.1(1) below. Our main result in this section is Theorem 6.7. Throughout this section $\mathcal{A}$ is a generating set of η-axes of the axial algebra A.

Notation 6.1. (1) We define the graph Δ as follows. The vertex set of this graph is the set $\mathcal{X}$ of all the η-axes in A. Two distinct axes $x, y \in \mathcal{X}$ form an edge if and only if $xy \neq 0$.
(2) For a subset $\mathcal{B} \subseteq \mathcal{X}$ we denote by $\Delta_{\mathcal{B}}$ the full subgraph of Δ on the vertex set $\mathcal{B}$.
(3) Recall the notation $\mathcal{B}^1$ and $\mathcal{B}^\eta$ from Notation 2.2(4).
(4) For a subset $\mathcal{B} \subseteq \mathcal{X}$ we denote by $\mathcal{B}^u$ (u for unique) the set $\{x \in \mathcal{B}^\eta \mid \tau(x) \neq \tau(y) \text{ for all } x \neq y \in \mathcal{B}\}$.
(5) $\mathcal{B}^{nu}$ (nu for not unique) is the set $\mathcal{B}^\eta \smallsetminus \mathcal{B}^u$.

Remark 6.2. By Lemma 3.2.8(6), if $a \in \mathcal{X}^1$, then $\{a\}$ is a connected component of Δ and hence $\{a\}$ is a connected component of $\Delta_{\mathcal{B}}$ for all $a \in \mathcal{B} \subseteq \mathcal{X}$. Hence from now on we may assume that

$$\mathcal{A} = \mathcal{A}^\eta.$$

Lemma 6.3. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two distinct connected components of $\Delta_{\mathcal{A}}$, and let $\{\mathcal{A}_i \mid i \in I\}$ be the set of connected components of $\Delta_{\mathcal{A}}$. Then*

(1) $[G_{\mathcal{A}_1}, G_{\mathcal{A}_2}] = 1$;
(2) $b^{g_2} = b$, *for all* $b \in [\mathcal{A}_1]$ *and* $g_2 \in G_{\mathcal{A}_2}$;
(3) $b_1 b_2 = 0$, *for all* $b_1 \in [\mathcal{A}_1]$ *and* $b_2 \in [\mathcal{A}_2]$;
(4) *the graph* $\Delta_{[\mathcal{A}_1]}$ *is connected;*
(5) $[\mathcal{A}] = \dot{\bigcup}_{i \in I} [\mathcal{A}_i]$ *is a disjoint union;*

(6) *there is a bijection* $\mathcal{A}_i \mapsto [\mathcal{A}_i]$ *between the connected components* $\{\mathcal{A}_i \mid i \in I\}$ *of* $\Delta_{\mathcal{A}}$ *and the connected components* $\{[\mathcal{A}_i] \mid i \in I\}$ *of* $\Delta_{[\mathcal{A}]}$.

Proof. (1): By definition $a_1a_2 = 0$, for all $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$. By Lemma 3.2.1 we see that $\tau(a_1)$ commutes with $\tau(a_2)$. Since $G_{\mathcal{A}_i} = \langle \tau(a_i) \mid a_i \in \mathcal{A}_i\rangle$, $i = 1, 2$, part (1) follows.

(2): Assume first that $b \in \mathcal{A}_1$. Then $b^{\tau(a_2)} = b$, for all $a_2 \in \mathcal{A}_2$, because $ba_2 = 0$. It follows that $b^{g_2} = b$. Next write $b = a_1^{g_1}$ with $a_1 \in \mathcal{A}_1$ and $g_1 \in G_{\mathcal{A}_1}$ (see Lemma 3.2.8(1)). Then using (1) we get $b^{g_2} = a_1^{g_1g_2} = a_1^{g_2g_1} = a_1^{g_1} = b$.

(3): Write $b_i = a_i^{g_i}$, with $a_i \in \mathcal{A}_i$ and $g_i \in G_{\mathcal{A}_i}$, $i = 1, 2$. Then, by (1) and (2), $b_1b_2 = a_1^{g_1g_2}a_2^{g_1g_2} = (a_1a_2)^{g_1g_2} = 0$.

(4): This follows from Lemma 3.2.8(3).

(5): We first show that $[\mathcal{A}] = \bigcup_{i\in I}[\mathcal{A}_i]$. Clearly it suffices to show that $[\mathcal{A}]$ is contained in the union. Let $b \in [\mathcal{A}]$, then, by Lemma 3.2.8(1), there exists $a \in \mathcal{A}$ and $g \in G_{\mathcal{A}}$ such that $b = a^g$. Let $i \in I$ so that $a \in \mathcal{A}_i$. By (1) and (2) it follows that $a^g = a^{g_i}$ for some $g_i \in G_i$. Hence $b \in [\mathcal{A}_i]$. Since by (4), $\Delta_{[\mathcal{A}_i]}$ is connected, for each $i \in I$, the fact that the union is disjoint is immediate from (3).

(6): This follows from (4) and (5). □

Lemma 6.4. *Let* $\{\mathcal{A}_i \mid i \in I\}$ *be the set of connected components of* $\Delta_{\mathcal{A}}$. *For each* $i \in I$, *let* $A_i = N_{\mathcal{A}_i}$ *(see Notation 2.3) and let* $\mathcal{X}_i$ *be the set of all* η*-axes in* A_i. *Then*

(1) $A_iA_j = \{0\}$ *for all* $i \neq j$, *so* A *is the sum of its ideals* $\{A_i \mid i \in I\}$;
(2) *the connected components of* Δ *are* $\{\mathcal{X}_i \mid i \in I\}$.

Proof. (1): By Lemma 3.2.8(8), $A_i = N_{[\mathcal{A}_i]}$ and by [HRS2, Corollary 1.2], A_i is spanned over $\mathbb{F}$ by $[\mathcal{A}_i]$. By Lemma 6.3(3), $A_iA_j = \{0\}$ for $i \neq j$. Hence A_i is an ideal of A. Since the sum of A_i contains $\mathcal{A}$ we see that it equals A.

(2): Let $u \in \mathcal{X}_i$. Suppose that $ua = 0$ for all $a \in [\mathcal{A}_i]$. Since A_i is spanned by $[\mathcal{A}_i]$, there is a basis $\mathcal{B}_i$ of A_i such that $\mathcal{B}_i \subseteq [\mathcal{A}_i]$. Hence there is $b \in \mathcal{B}_i$ and $0 \neq \alpha \in \mathbb{F}$ such that $b = \alpha u + x$, with $x \in \mathrm{Span}(\mathcal{B}_i \smallsetminus \{b\})$. But then $0 = ub = \alpha u$, a contradiction. It follows that u is in the same connected component of $[\mathcal{A}_i]$ of Δ (see Lemma 6.3(6)). By (1), $\mathcal{X}_i$ is a connected component of Δ. □

Proposition 6.5. *Let* $a, b \in \mathcal{X}^\eta$ *be two distinct* η*-axes in* A. *Assume that* $\tau(a) = \tau(b)$. *Then* $ab = 0$ *and*

(1) *for any* η*-axis* $c \in A$ *exactly one the following holds:*
  (i) $ac = bc = 0$.

(ii) $\eta = \frac{1}{2}$, *and for some* $x \in \{a, b\} = \{x, y\}$, *we have* $N_{x,c} = B(\frac{1}{2}, 0)_{x,c}$ *is* 3*-dimensional,* $N_{y,c} = J_{y,c}$ *(see Lemma 3.1.2(1c)) and* $N_{y,c} \subset N_{x,c}$. *Further* $N_{x,c}$ *contains an identity* $1_{x,c} = a+b$.

(iii) $\eta = \frac{1}{2}$, $N_{a,c} = N_{b,c}$ *is* 3*-dimensional and contains an identity* $1_{a,c} = a + b$.

(2) *If* d *is an* η*-axis in* A *such that* $\tau(d) = \tau(a)$, *then* $d \in \{a, b\}$.

Proof. By Lemma 3.2.1, $N_{a,b} = 2B_{a,b}$. If $N_{a,c} = 2B_{a,c}$, then $c = c^{\tau(a)} = c^{\tau(b)}$, so by Lemma 3.2.1 again, $N_{b,c} = 2B_{b,c}$. Hence part (1i) holds.

So we may assume that both $N_{a,c}$ and $N_{b,c}$ are not of type $2B$. If $\eta \neq \frac{1}{2}$, then by [HRS2, Prop. 4.8], $N_{x,c} = B(\eta, \frac{1}{2}\eta)_{x,c}$, for $x \in \{a, b\}$. By Lemma 3.1.4, $a^{\tau(c)} = c^{\tau(a)} = c^{\tau(b)} = b^{\tau(c)}$, and then $a = b$, a contradiction.

Hence $\eta = \frac{1}{2}$. Let

$$V := N_{c,c^{\tau(a)}} \subseteq N_{a,c} \cap N_{b,c}.$$

Suppose that $N_{a,c}$ is 2-dimensional. By Lemma 3.2.1, $c \neq c^{\tau(a)}$ so $N_{a,c} = V$. If $N_{b,c}$ is 2-dimensional, then $N_{a,c} = V = N_{b,c}$ is spanned by a and b. So $N_{a,c} = N_{a,b} = N_{b,c}$ is of type $2B$, a contradiction. Hence $N_{a,c} = V \subset N_{b,c}$. Since $a, b \in N_{b,c}$ and $ab = 0$, Lemma 3.2.5 implies that $1_{b,c} = a + b$, and the last part of (1ii) holds. Also, by Lemma 3.2.2, if $N_{b,c} = B(\frac{1}{2}, \frac{1}{2})_{b,c}$, then $1_{b,c} \in N_{c,c^{\tau(b)}} = V = N_{a,c}$. But then $b \in N_{c,c^{\tau(b)}}$, a contradiction. Hence, by Lemma 3.2.2 the first part of (1ii) holds.

If V is 3-dimensional, then since $V = N_{c,c^{\tau(b)}}$ we see that $N_{a,c} = V = N_{b,c}$ and as above $1_{a,c} = a + b$ and the proposition holds.

Hence we may assume that $N_{a,c}$, $N_{b,c}$ are 3-dimensional and V is 2-dimensional. We use Lemma 3.2.2. We must consider 2 cases.

Assume first that $V = 2B_{c,c^{\tau(a)}}$. Then $N_{x,c} = B(\frac{1}{2}, \frac{1}{2})_{x,c}$, for $x \in \{a, b\}$. By Lemma 3.1.6(2), $c^{\tau(x)} = 1_{x,c} - c$, for $x \in \{a, b\}$. Since $\tau(a) = \tau(b)$ we see that

$$\mathbb{1} := 1_{a,c} = 1_{b,c}.$$

Also $x^{\tau(c)} = \mathbb{1} - x$ for $x \in \{a, b\}$. We have $(\mathbb{1} - a)b = b$. Since $\mathbb{1} - a$ is an axis A this forces $\mathbb{1} - a = b$. So we see that $N_{a,c} = N_{b,c}$ and $a + b$ is its identity element.

Assume next that $V = J_{c,c^{\tau(a)}}$. Then, by Lemma 3.2.2(i), $N_{a,c} = B(\frac{1}{2}, 0)_{a,c}$ and $N_{b,c} = B(\frac{1}{2}, 0)_{b,c}$. We claim that $A_{\frac{1}{2}}(a) = A_{\frac{1}{2}}(b)$. Indeed, for any $v \in A$ we have $v \in A_{\frac{1}{2}}(a)$ if and only if $v^{\tau(a)} = -v$. and the same holds for b. Since $\tau(a) = \tau(b)$ the claim follows.

Now by Theorem 3.1.3(5) (since $\varphi_{x,c} = 0$, for $x \in \{a, b\}$),

$$v_x := \tfrac{1}{2}x + \tfrac{1}{2}c + \sigma_{x,c} \in (N_{x,c})_{\frac{1}{2}}(x), \text{ for } x \in \{a, b\}.$$

Hence

$$v_a - v_b = (\sigma_{a,c} + \tfrac{1}{2}a) - (\sigma_{b,c} + \tfrac{1}{2}b) \in A_{\frac{1}{2}}(a).$$

Note now that $\tau(a) = \tau(b)$ fixes $v_a - v_b$. But it also negates it. Hence

$$-\sigma_{a,c} - \tfrac{1}{2}a = -\sigma_{b,c} - \tfrac{1}{2}b.$$

Since $\pi_{x,c} = -\frac{1}{2}$ for $x \in \{a, b\}$, we get: $\frac{1}{2}1_{a,c} - \frac{1}{2}a = \frac{1}{2}1_{b,c} - \frac{1}{2}b$, or

$$1_{a,c} - a = 1_{b,c} - b.$$

It follows that $1_{a,c}b = (1_{a,c} - a)b = (1_{b,c} - b)b = 0$. So $1_{a,c}b = 0$. Applying $\tau(c)$ we see that $1_{a,c}b^{\tau(c)} = 0$. But

$$\dim\big(\mathrm{Span}\big(\{b, b^{\tau(c)}\}\big)\big) = \dim\big(\mathrm{Span}\big(\{c, c^{\tau(b)}\}\big)\big) = \dim(V) = 2,$$

and both spaces live in the algebra $N_{b,c}$ of dimension 3. Hence $W := V \cap \mathrm{Span}(\{b, b^{\tau(c)}\}) \neq 0$ and $1_{a,c}$ annihilates W. However $W \subset N_{a,c}$ and $1_{a,c}$ is the identity of this algebra, and we finally reached a contradiction. This proves (1).

Let d be an η-axis in A such that $\tau(d) = \tau(a)$ and $d \neq a$. Since $a \in \mathcal{X}^\eta$ also $d \in \mathcal{X}^\eta$. Let $c \in \mathcal{X}$ such that $c^{\tau(a)} \neq c$. This is possible since $\tau(a) \neq \mathrm{id}$. Then $N_{a,c} \neq 2B_{a,c}$. So either (1ii) or (1iii) holds and we may assume that $N_{a,c}$ is 3-dimensional and contains an identity $1_{a,c} = a + b$. By (1ii) and (1iii) applied to d in place of b we see that also $a + d$ is the identity of $N_{a,c}$. Hence $a + d = 1_{a,c} = a + b$, and $d = b$. □

Proposition 6.6. *Let $\Delta_{\mathcal{A}}$ be as in Notation 6.1(3). Assume there are distinct $a, b \in \mathcal{A}^\eta$ such that $\tau(a) = \tau(b)$. Then $\eta = \frac{1}{2}$ and a, b are contained in a connected component $\mathcal{B}$ of $\Delta_{\mathcal{A}}$.*

Let $B = N_{\mathcal{B}}$ be the subalgebra of A generated by $\mathcal{B}$, and let $\mathcal{C}$ be the set of all $\frac{1}{2}$-axes in B. Then

1. *$\mathcal{C}$ is a connected component of Δ;*
2. *$xa \neq 0 \neq xb$, for all $x \in \mathcal{C} \smallsetminus \{a, b\}$;*
3. *B contains an identity element $\mathbb{1} = a + b$;*
4. *for any $x \in \mathcal{C}$ such that $N_{a,x}$ is 3-dimensional we have $\mathbb{1} = 1_{a,x}$.*

Proof. Part (1) is Lemma 6.4(2). Since $a, b \in \mathcal{A}^\eta$, there exists $c \in \mathcal{A}$ such that $c^{\tau(a)} \neq c$ (indeed if $c^{\tau(a)} = c$, for all $c \in \mathcal{A}$, it would follow that $\tau(a) = \mathrm{id}$ as A is generated by $\mathcal{A}$). Hence $ac \neq 0$ and $bc \neq 0$. Thus a, b are in the same connected component $\Delta_{\mathcal{A}}$.

By Proposition 6.5, $N_{a,b} = 2B_{a,b}$. Let $d(\ ,\ )$ be the distance function on Δ. Since $\mathcal{C}$ is connected there exists $c \in \mathcal{C}$ with $d(a, c) = 1$. Thus $c \neq b$, and by Proposition 6.5, $d(b, c) = 1$ and we may assume without loss that $N_{a,c}$ is 3-dimensional and contains $1_{a,c} = a + b$. Also $\eta = \frac{1}{2}$. Set

$$\mathbb{1} = 1_{a,c} = a + b.$$

Consider the set

$$\mathcal{C}_1(a) := \{x \in \mathcal{C} \mid d(a, x) = 1\}.$$

Replacing c with $x \in \mathcal{C}_1(a)$ in the above argument shows that there exists $u \in \{a, b\}$ such that $N_{u,x}$ is 3-dimensional and contains an identity $1_{u,x} =$

$a + b = \mathbb{1}$. Hence $\mathbb{1}x = x$, for all $x \in \mathcal{C}_1(a)$. Note also that by Proposition 6.5, $\mathcal{C}_1(a) = \mathcal{C}_1(b)$.

Let $y \in \mathcal{C} \smallsetminus \mathcal{C}_1(a)$ be at distance 2 from a in Δ. Then $ay = 0 = by$ and we can find $x \in \mathcal{C}_1(a)$ such that $d(x, y) = 1$. Thus by Proposition 6.6 (without loss after perhaps interchanging a and b), we have $\mathbb{1} = 1_{a,x}$ and $ay = 0$.

Notice that by Proposition 6.5(1i), $(a + b)^{\tau(y)} = a + b$, that is $\mathbb{1}^{\tau(y)} = \mathbb{1}$. Also $N_{x,y} \neq 2B_{x,y}$.

Now $\mathbb{1}y = 0$ and hence $\mathbb{1}y^{\tau(x)} = 0$ (because $\mathbb{1} = 1_{a,x}$ so $\mathbb{1}^{\tau(x)} = \mathbb{1}$). Also $\mathbb{1}x = x$, so $\mathbb{1}x^{\tau(y)} = x^{\tau(y)}$ (because $\mathbb{1}^{\tau(y)} = \mathbb{1}$). Let $W := \mathrm{Span}\big(\{y,\ y^{\tau(x)}\}\big) \cap \mathrm{Span}\big(\{x,\ x^{\tau(y)}\}\big)$. Since $N_{x,y}$ is at most 3-dimensional, $W \neq 0$. But the above shows that multiplication by $\mathbb{1}$ both annihilates W and acts as the identity map on W, a contradiction.

Hence $\mathcal{C}_1(a) = \mathcal{C} \smallsetminus \{a, b\}$ and clearly $d(a, b) = 2$ in $\mathcal{C}$. But now, as we saw above $\mathbb{1}x = x$ for all $x \in \mathcal{C}$. By Lemma 3.2.3 (with $\mathcal{B}$ in place of $\mathcal{A}$ and $N_{\mathcal{B}}$ in place of A) we see that $\mathbb{1}$ is the identity of B. □

We can now prove the main result of this section

Theorem 6.7. *Assume that $\Delta_{\mathcal{A}}$ is connected and that there are distinct $a, b \in \mathcal{A}^\eta$ such that $\tau(a) = \tau(b)$. Then $A = J(V, B)$ is a Jordan algebra of Clifford type.*

Proof. We show that the hypotheses of Theorem 5.4 are satisfied. By Proposition 6.6(3), $a + b = \mathbb{1}$ is the identity element of A. Let $c \in \mathcal{A}$. Then clearly $v_a v_c \in \mathbb{F}\mathbb{1}$, for $c \in \{a, b\}$. Otherwise, by Proposition 6.6(2), $N_{a,c}$ is not $2B_{a,c}$. Also, as in equation (5.1) (in the proof of Theorem 5.4), $v_a v_c = \sigma_{a,c} + \frac{1}{4}\mathbb{1}$. If $N_{a,c}$ is 2-dimensional, then by Lemma 3.1.2, $\sigma_{a,c} = 0$ (because $ac \neq 0$), and so $v_a v_c \in \mathbb{F}\mathbb{1}$. If $N_{a,c}$ is 3-dimensional, then by Proposition 6.6(4), $\mathbb{1} = 1_{a,c}$, so by Theorem 3.1.3(3), $\sigma_{a,c} = \pi_{a,c}\mathbb{1}$ and again $v_a v_c \in \mathbb{F}\mathbb{1}$. □

Definition 6.8. (1) Let the ideals A_i, $i \in I$ be as in Lemma 6.4. We call A_i the *components* of the algebra A. Note that by Lemma 6.4(2), the components A_i are independent of $\mathcal{A}$.

(2) Let A_i be a componet of the algebra A. If A_i is a Jordan algebra of Clifford type we call A_i a component of *Clifford type*. Otherwise (in the case where $\tau(x) \neq \tau(y)$ for distinct $x, y \in \mathcal{X} \cap A_i$) we call A_i a component of *Unique type*.

(3) Let $\mathcal{B} \subseteq \mathcal{X}^\eta$ be a closed set of η-axes ($\mathcal{B} = [\mathcal{B}]$). Suppose that $\mathcal{B}$ is contained in a connected component $\mathcal{X}_i$ of Δ.
   - (i) We say that $\mathcal{B}$ is of *Non-unique type* if there exists distinct $a, b \in \mathcal{B}$ such that $\tau(a) = \tau(b)$ (and then $\eta = \frac{1}{2}$ and A_i is a Jordan algebra of Clifford type).
   - (ii) We say that $\mathcal{B}$ is of *C-unique type* if A_i is a Jordan algebra of Clifford type and the map $b \mapsto \tau(b)$ is bijective on $\mathcal{B}$.
   - (iii) We say that $\mathcal{B}$ is of *NC-unique type* if A_i is a **not** a Jordan algebra of Clifford type. (And then the map $b \mapsto \tau(b)$ is necessarily bijective on $\mathcal{B}$.)

Remark 6.9. Let A_i be a component of A of Clifford type. Then $A_i \cap A_j = \{0\}$ for any component A_j of A with $A_j \neq A_i$. Indeed, by Lemma 6.4(1), $A_iA_j = 0$, so since A_i contains an identity element we must have $A_i \cap A_j = \{0\}$. If A_i and A_j are distinct components of A of Unique type, then it may happen that $A_i \cap A_j \neq \{0\}$.

We close this section with a theorem that summarizes some of the results in this section.

Theorem 6.10. *Let $\{\mathcal{A}_i \mid i \in I\}$ be the set of connected components of $\Delta_{\mathcal{A}}$. For each $i \in I$, let $A_i = N_{\mathcal{A}_i}$ and let $\mathcal{X}_i = A_i \cap \mathcal{X}$. Then*

(1) *$A = \sum_{i\in I} A_i$ is the sum of its ideals A_i;*
(2) *$A_iA_j = 0$, for distinct $i, j \in I$;*
(3) *$\{A_i \mid i \in I\}$ are the components of A and $\{\mathcal{X}_i \mid i \in I\}$ are the connected components of Δ.*
(4) *If $i \in I$, then exactly one of the following holds:*
   (i) *$[\mathcal{A}_i]$ is of Non-unique type, so A_i is a Jordan algebra of Clifford type.*
   (ii) *$[\mathcal{A}_i]$ is of C-unique type.*
   (iii) *$[\mathcal{A}_i]$ is of NC-unique type.*

Proof. Parts (1), (2) and (3) are Lemma 6.4. Part (4) follows from Theorem 6.7. □

7. The case where A is a 3-transposition algebra

Recall from subsection 1.2.2 of the introduction the notion of a 3-transposition algebra with respect to a generating set of η-axes. The following theorem is taken from [HRS2]:

Theorem 7.1 (Theorem 5.4 in [HRS2])**.** *Assume $\eta \neq \frac{1}{2}$. Then A is a 3-transposition algebra with respect to any subset $\mathcal{B} \subseteq \mathcal{X}$ that generates A.*

Proof. It suffices to show that $|\tau(a)\tau(b)| \in \{2, 3\}$ for any $a, b \in \mathcal{X}$, with $a \neq b$ (note that by Proposition 6.6, $\tau(a) \neq \tau(b)$). If $N_{a,b}$ is 2-dimensional then by Lemma 3.1.2, $N_{a,b}$ is either $3C^{\times}(-1)$ (and $\eta = -1$ so $\mathrm{char}(\mathbb{F}) \neq 3$) and so by Lemma 3.1.8, $|\tau(a)\tau(b)| = 3$, or $N_{a,b} = 2B_{a,b}$ and $|\tau(a)\tau(b)| = 2$. If N is 3-dimensional then, by [HRS2, Proposition 4.8], $N = B(\eta, \frac{1}{2}\eta)_{a,b}$, so by Lemma 3.1.4, $|\tau(a)\tau(b)| = 3$. □

What then about the case $\eta = \frac{1}{2}$?

Most of this section is devoted to the case where A is a 3-transposition algebra with respect to a generating set of η-axes $\mathcal{A} \subseteq \mathcal{X}$. By Theorem 7.1 we may (and we will) assume that

$$\eta = \tfrac{1}{2}.$$

Thus by an axis in A we mean a $\frac{1}{2}$-axis. Note that if $\mathrm{char}(\mathbb{F}) = 3$ then $\frac{1}{2} = -1$.

The Miyamoto involution set $D := D_{[\mathcal{A}]}$ is a normal subset of 3-transpositions generating the group $G := G_{[\mathcal{A}]}$. Further, we assume that D is a conjugacy class in G. This implies that the graph $\Delta_{\mathcal{A}}$ is connected (see Notation 6.1(2)). In particular A is either of Clifford type or of Unique type (see Definition 6.8(2)).

Our main result in this section is:

Theorem 7.2. *Assume that* $\mathrm{char}(\mathbb{F}) \neq 3$, *that A is of Clifford type, and that A is a 3-transposition algebra with respect to $\mathcal{A}$. Then $G_{[\mathcal{A}]}$ is a 3-transposition group of ADE-type.* (*See Definition 4.2.*)

By Remark 6.2 we may ignore the axes in $\mathcal{A} \cap \mathcal{X}^1$.

Lemma 7.3. *Let G be a 3-transposition group generated by a conjugacy class of 3-transpositions D. Let $r, s, t \in D$ be three distinct involutions such that $|uv| = 3$, for all distinct $u, v \in \{r, s, t\}$. Set $H = \langle r, s, t\rangle$, then*

(1) *if $r^s = t$, then $H \cong S_3$;*
(2) *if $|r^s t| = 3$, then $H \cong 3^2 : 2$ or $3^{1+2} : 2$;*
(3) *if $|r^s t| = 2$, then $H \cong S_4$.*

Proof. See, e.g., [HSo, 4.1, p. 2526]. □

Lemma 7.4. *Let a, b be two axes in A and assume that $|\tau(a)\tau(b)| = 2$. Set $N = N_{a,b}$. Then either*

(i) *A is of Unique type and $N = 2B_{a,b}$.*
(ii) *A is of Clifford type and $N = B(\frac{1}{2}, \frac{1}{2})_{a,b}$.*

Proof. This is Lemma 3.2.7(2). Note that if $N = B(\frac{1}{2}, \frac{1}{2})_{a,b}$, then we must have $\tau(1_{a,b} - x) = \tau(x)$, for $x \in \{a, b\}$, so A is of Clifford type. Note further that if A is of Clifford type and $ab = 0$, then $(\mathbb{1} - a)b = b$. But since $\mathbb{1} - a$ is an axis in A we must have $b = \mathbb{1} - a$, so $\tau(a) = \tau(b)$. □

Lemma 7.5. *Let a, b be two distinct axes in A and set $N = N_{a,b}$. Then*

$$(*) \qquad |\tau(a)\tau(b)| = 3,$$

if and only if one of the following holds

(i) $\dim(N) = 2$ *and* $N = 3C(-1)^{\times}_{a,b}$ (*so* $\mathrm{char}(\mathbb{F}) = 3$).
(ii) $\dim(N) = 3$, *N contains no identity element and* $N = 3C(-1)_{a,b}$ (*so* $\mathrm{char}(\mathbb{F}) = 3$).
(iii) $\dim(N) = 3$, *N contains an identity element* $1_{a,b}$ *and* $N = 3C(\frac{1}{2})_{a,b} = B(\frac{1}{2}, \frac{1}{4})$ (*so* $\mathrm{char}(\mathbb{F}) \neq 3$). *We then have*
$$ab = -\tfrac{3}{8}1_{a,b} + \tfrac{1}{2}a + \tfrac{1}{2}b.$$
(iv) *A is of Clifford type and* $N_{a,b} = B(\frac{1}{2}, \frac{3}{4})$ (*so when* $\mathrm{char}(\mathbb{F}) = 3$, $N_{a,b} = B(-1, 0)$). *We then have*
$$ab = -\tfrac{1}{8}1_{a,b} + \tfrac{1}{2}a + \tfrac{1}{2}b.$$

Further, if $\mathrm{char}(\mathbb{F}) \neq 3$, *then* $N = 3C(\frac{1}{2})_{x,\,(1_{a,b}-y)}$, *while if* $\mathrm{char}(\mathbb{F}) = 3$, *then* $N_{x,(1_{a,b}-y)} = 3C(-1)^{\times}_{x,(1_{a,b}-y)}$, *for* $\{x, y\} = \{a, b\}$.

Proof. Suppose that $(*)$ holds. Then by Lemma 3.2.7 with $k = 3$ we either have:

(1) $a^{\tau(b)} = b^{\tau(a)}$, or
(2) $\dim(N) = 3$, N contains an identity element $1_{a,b}$, $1_{a,b} - x$ is an axis in A, $\tau(1_{a,b} - x) = \tau(x)$ and $1_{a,b} - x = y^{\tau(x)\tau(y)}$, for $\{x, y\} = \{a, b\}$.

Suppose that (1) holds. Then (i), (ii) or (iii) hold by Corollary 3.3.2, and Lemma 3.3.3. Also use Lemma 3.1.4(2) if $\mathrm{char}(\mathbb{F}) = 3$ (because $\frac{1}{2} = -1$ if $\mathrm{char}(\mathbb{F}) = 3$).

Suppose that (2) holds. Then (iv) holds by Lemma 3.3.4. Finally if (i) holds then $(*)$ holds by Lemma 3.1.8, if (ii) or (iii) hold then $(*)$ holds by Lemma 3.1.4 and Lemma 3.3.3. Finally, if (iv) holds, then $(*)$ holds by Lemma 3.3.4. □

Corollary 7.6. *Let* $a, b \in A$ *be two distinct axes and suppose* $|\tau(a)\tau(b)| = 3$. *Then either*

(i) *A is of Unique type and* $N_{a,b} = 3C(\frac{1}{2})_{a,b}$.
(ii) $\mathrm{char}(\mathbb{F}) = 3$ *and* $N_{a,b} = 3C(-1)^{\times}_{a,b}$.
(iii) *A is of Clifford type,* $\mathrm{char}(\mathbb{F}) \neq 3$ *and* $N_{a,b} = 3C(\frac{1}{2})_{a,b}$.
(iv) *A is of Clifford type and* $N_{a,b} = B(\frac{1}{2}, \frac{3}{4})_{a,b}$.

Proof. Assume that A is of Clifford type. Let $\mathbb{1}$ be the identity element of A. Since $\tau(a) = \tau(\mathbb{1} - a)$, Proposition 6.6 implies that we may assume (after perhaps interchanging a and $\mathbb{1} - a$) that $\mathbb{1}$ is the identity element of $N_{a,b}$. Since $3C(-1)_{x,y}$ does not contain an identity element we must have $\mathrm{char}(\mathbb{F}) \neq 3$ in (iii). The rest of the lemma follows from Lemma 7.5. □

Definition 7.7. Assume $\eta = \frac{1}{2}$. Let $a, b \in \mathcal{X}$. If $ab = -\frac{3}{8}1_{a,b} + \frac{1}{2}a + \frac{1}{2}b$ we will say that $N_{a,b}$ is of type $-\frac{3}{8}$ while if $ab = -\frac{1}{8}1_{a,b} + \frac{1}{2}a + \frac{1}{2}b$ we will say that $N_{a,b}$ is of type $-\frac{1}{8}$. Notice that if $|\tau(a)\tau(b)| = 3$ and A is of Clifford type, then, by Corollary 7.6, $N_{a,b}$ is necessarily of type $-\frac{3}{8}$ or of type $-\frac{1}{8}$. (Indeed when $\mathrm{char}(\mathbb{F}) = 3$ and $N_{a,b}$ is of type $-\frac{3}{8}$, then $N_{a,b} = 3C(-1)^{\times}_{a,b}$.)

Lemma 7.8. *Assume that A is of Clifford type, and let* $a, b, c \in \mathcal{X}$ *be three distinct axes. Then we have*

(1) *if* $N_{x,y}$ *is of type* $-\frac{3}{8}$ *for all distinct* $x, y \in \{a, b, c\}$ *and* $\langle\tau(a), \tau(b), \tau(c)\rangle$ *is* not *isomorphic to* S_3, *then* $N_{a,b^{\tau(c)}} = J_{a,b^{\tau(c)}}$. *In particular, if* $\mathrm{char}(\mathbb{F}) \neq 3$, *then* $|\tau(a)\tau(b^{\tau(c)})| \notin \{2, 3\}$;
(2) *if* $N_{a,b}$ *and* $N_{a,c}$ *are of type* $-\frac{3}{8}$ *and* $N_{b,c}$ *is of type* $-\frac{1}{8}$, *then* $N_{a,b^{\tau(c)}} = B(\frac{1}{2}, \frac{1}{2})_{a,b^{\tau(c)}}$. *In particular,* $|\tau(a)\tau(b^{\tau(c)})| = 2$.

Proof. By Lemma 3.2.4 we have

$$\pi_{a,b^{\tau(c)}} = 8\pi_{a,c}\pi_{b,c} + 2\pi_{a,c} - \pi_{a,b} + 2\pi_{b,c}.$$

(1): In this case $\pi_{x,y} = -\frac{3}{8}$, for all distinct $x, y \in \{a, b, c\}$, so $\pi_{a,b^{\tau(c)}} = 0$. Hence, by Lemma 3.2.4(2), the first part of (1) holds. The second part follows from Lemma 3.3.1.

(2): In this case $\pi_{a,b} = \pi_{a,c} = -\frac{3}{8}$ and $\pi_{b,c} = -\frac{1}{8}$ so $\pi_{a,b^{\tau(c)}} = -\frac{1}{4}$. Now (2) follows from Remark 3.1.7. □

Lemma 7.9. *Suppose that* $\mathrm{char}(\mathbb{F}) \neq 3$, *that A is of Clifford type, and that A is a 3-transposition algebra with respect to $\mathcal{A}$. Let $\mathbb{1}$ be the identity element of A. Then*

(1) *If $\mathcal{B} = \{a, b, c\} \subset [\mathcal{A}]$ is a set of size 3 such that$\langle \tau(a), \tau(b), \tau(c)\rangle$ is* not *isomorphic to S_3 and $|\tau(x)\tau(y)| = 3$, for all distinct $x, y \in \mathcal{B}$. Then either $N_{x,y}$ is of type $-\frac{1}{8}$ for all distinct $x, y \in \mathcal{B}$, or there exists distinct $x, y \in \mathcal{B}$ such that $N_{x,y}$ is of type $-\frac{1}{8}$ and both $N_{x,z}$ and $N_{y,z}$ are type $-\frac{3}{8}$, where $\{x, y, z\} = \{a, b, c\}$.*

(2) *If $\mathcal{B} = \{a, b, c, d\} \subset [\mathcal{A}]$ is such that*
   (a) $\big\langle \tau(x) \mid x \in \{a, b, c, d\}\big\rangle$ *is* not *isomorphic to S_4.*
   (b) $|\tau(a)\tau(c)| = 2 = |\tau(b)\tau(d)|$.
   (c) $|\tau(x)\tau(b)| = 3 = |\tau(x)\tau(d)|$, *for $x \in \{a, c\}$.*

   (Thus $\mathcal{B}$ has diagram $\tilde{A}_3$ (see the definition of a diagram in §4)). Then the following holds
   (∗) *either $N_{a,b}$ and $N_{a,d}$ have different type, or $N_{c,b}$ and $N_{c,d}$ have different type.*

Proof. (1): Assume (1) does not hold. If $N_{x,y}$ is of type $-\frac{3}{8}$ for all distinct $x, y \in \{a, b, c\}$, then Lemma 7.8(1) applies and so $|\tau(a)\tau(b^{\tau(c)})| \notin \{2, 3\}$, a contradiction.

Otherwise, for some distinct $x, y \in \mathcal{B}$ we have $N_{x,y}$ is of type $-\frac{3}{8}$ and both $N_{x,z}$ and $N_{y,z}$ are of type $-\frac{1}{8}$, where $\{x, y, z\} = \{a, b, c\}$. But by Lemma 7.5(iv) we see that both $N_{x,\mathbb{1}-z}$ and $N_{y,\mathbb{1}-z}$ are of type $-\frac{3}{8}$. By Lemma 7.8, $N_{(\mathbb{1}-z),x^{\tau(y)}} = J_{(\mathbb{1}-z),x^{\tau(y)}}$. Since $\tau(z) = \tau(\mathbb{1} - z)$, we get from Lemma 3.3.1 that $|\tau(z)\tau(x^{\tau(y)})| \notin \{2, 3\}$, a contradiction.

(2): Suppose there exists $\mathcal{B} \subset [\mathcal{A}]$ satisfying (a), (b) and (c) of (2), and (∗) does not hold. Notice that by hypothesis (b), and by Lemma 7.4, $N_{a,c} = B(\frac{1}{2}, \frac{1}{2})_{a,c}$ so $\mathbb{1} - x \in \mathcal{A}$, for all $x \in \{a, b, c, d\}$. Interchanging a with $\mathbb{1} - a$ and c with $\mathbb{1} - c$ if necessary, we may assume that $N_{a,b},\ N_{a,d},\ N_{c,b},\ N_{c,d}$ are all of type $-\frac{3}{8}$. Note that $\langle \tau(x), \tau(y), \tau(z)\rangle \cong S_4$, for all distinct $x, y, z \in \{a, b, c, d\}$. Hence $|\tau(x)\tau(c)^{\tau(d)}| = 3$, for $x \in \{a, b\}$. Notice that $N_{c,c^{\tau(d)}}$ is of type $-\frac{3}{8}$ (because $N_{c,d} = 3C(\frac{1}{2})_{c,d}$). Hence applying (1) with $\{b, c, c^{\tau(d)}\}$ in place of $\{a, b, c\}$ and using Corollary 7.6 shows that $N_{a,c^{\tau(d)}}$ is of type $-\frac{1}{8}$. Similarly, $N_{b,c^{\tau(d)}}$ is of type $-\frac{1}{8}$. By hypothesis (a), $\langle \tau(a), \tau(b), \tau(c^d)\rangle$ is not isomorphic to S_3. Hence we apply part (1) with $c^{\tau(d)}$ in place of c to get a contradiction. □

Observe that

Theorem 7.10. *Assume that* $\mathrm{char}(\mathbb{F}) \neq 3$, *that* A *is of Clifford type, and that* A *is a* 3*-transposition algebra with respect to* $\mathcal{A}$. *Then* $G_{[\mathcal{A}]}$ *is a* 3*-transposition group of symplectic type.*

Proof. Let $\mathbb{1}$ be the identity element of A. Assume that $G_{[\mathcal{A}]}$ is not of symplectic type. Then there are $a, b, c \in [\mathcal{A}]$ such that $\langle \tau(a), \tau(b), \tau(c) \rangle$ is *not* isomorphic to S_3, and such that $|\tau(x)\tau(y)| = 3$ for all $x, y \in \{a, b, c, b^{\tau(c)}\}$. By Lemma 7.8(1) there are distinct $x, y \in \{a, b, c\}$ such that $N_{x,y}$ is of type $-\frac{1}{8}$. Hence by Lemma 3.3.4, $\mathbb{1} - x \in [\mathcal{A}]$, so $\mathbb{1} - z \in \mathcal{A}$, for all $z \in \{a, b, c\}$.

By Lemma 3.3.4, after perhaps interchanging x with $\mathbb{1} - x$ for $x \in \{b, c\}$ we may assume that

$$N_{a,x} = 3C(\tfrac{1}{2})_{a,x} \text{ for } x \in \{b, c\}.$$

Thus $\pi_{a,b} = \pi_{a,c} = -\frac{3}{8}$. By Lemma 7.8 we have $|\tau(a)\tau(b^{\tau(c)})| = 2$, a contradiction. □

We are now in a position to prove Theorem 7.2.

Proof of Theorem 7.2. Assume that D is not of ADE-type. Then, by Lemma 4.10, there exists a subset $\mathcal{Y} \subset [\mathcal{A}]$ of size 5 such that if we let $Y := \{\tau(x) \mid x \in \mathcal{Y}\}$, then (1) $H = \langle Y \rangle$ is isomorphic to a central quotient of $\mathrm{W}_2(\tilde{D}_4)$. (2) The diagram of Y is the complete bipartite graph $K_{3,2}$ and (3) no 4-subset of Y generates a subgroup isomorphic to S_4.

Let the parts of Y be $\{a_1, a_2, a_3\}$ and $\{b_1, b_2\}$. As we already noted, Lemma 7.4 and Lemma 3.1.6 imply that $\mathbb{1} - c \in [\mathcal{A}]$, for all $c \in \mathcal{Y}$.

Hence, after perhaps interchanging c with $\mathbb{1} - c$ for $c \in \{a_1, a_2, a_3\}$, using Corollary 7.6 and Lemma 3.3.4, we may assume that

$$N_{b_1,a_i} = 3C(\tfrac{1}{2})_{b_1,a_i}, \quad i = 1, 2, 3.$$

But then interchanging b_2 with $\mathbb{1} - b_2$ if necessary we see that for at least two of $\{a_1, a_2, a_3\}$ say a_1 and a_2 we have

$$N_{b_2,a_i} = 3C(\tfrac{1}{2})_{b_2,a_i}, \quad i = 1, 2.$$

But now taking b_1, a_1, b_2, a_2 in place of a, b, c, d in Lemma 7.9(2), we get a contradiction. This completes the proof of the Theorem. □

Examples 7.11. Finally we observe that, in a certain sense, the converse to Theorem 7.2 holds. More precisely, for any field $\mathbb{F}$ of characteristic not 2 and for any ADE-type X_n, there is a Jordan-axial $\mathbb{F}$-algebra A of Clifford type such that

(1) *A is a* 3*-transposition algebra with respect to* $\mathcal{A}$*;*
(2) $D_{[\mathcal{A}]}$ *is a conjugacy class of* 3*-transpositions of type* X_n*;*
(3) *the Miyamoto group* $G_{[\mathcal{A}]}$ *is isomorphic to one of the groups* $\mathrm{W}(X_n)$ *or* $\mathrm{W}(X_n)/Z(\mathrm{W}(X_n))$. *(The possible groups being listed in Proposition 4.4.)*

Consider a root system Φ of type X_n. Let E be the Euclidean space containing (and spanned by) Φ and $E_{\mathbb{Z}}$ the root lattice in E, the $\mathbb{Z}$-span of Φ. We assume that each root in Φ is of length 1. Then the values of the inner product on $E_{\mathbb{Z}}$ belong to $\frac{1}{2}\mathbb{Z}$. (For instance, in the standard action of $\mathrm{W}(A_m) = S_{m+1}$ on its permutation module $\mathbb{R}^{m+1}$ equipped with the dot product, the roots corresponding to transpositions have square length 2 and inner-products ± 1.) Hence $V = E_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}$ is a vector space over $\mathbb{F}$ of dimension n endowed with a symmetric bilinear form B such that $q(\bar{r}) := B(\bar{r}, \bar{r}) = 1$ for all $r \in \Phi$. Here we use the notation $\bar{e} = e \otimes 1_{\mathbb{F}} \in V$ for $e \in E_{\mathbb{Z}}$.

The Weyl group $W = \mathrm{W}(X_n)$ of Φ generated by the reflections in all $r \in \Phi$ acts naturally on Φ and $E_{\mathbb{Z}}$ and hence on V. Namely, the reflection in a root r acts on V as the reflection in the corresponding vector $\bar{r}$. Let $\widehat{W}$ be the (isomorphic) image of W in $GL(V)$.

Consider $A = J(V, B)$ and take $\mathcal{A} = \{a = \frac{1}{2}(\mathbb{1} + \bar{r}) \mid r \in \Phi\}$. It follows from §5 and the discussion above that $\mathcal{A}$ is a set of $\frac{1}{2}$-axes generating A. The Miyamoto involution $\tau(a)$, for $\frac{1}{2}(\mathbb{1} + \bar{r}) = a \in \mathcal{A}$, fixes $\mathbb{1} \in A$ and acts as the negative of the reflection in $\bar{r}$ on V. Therefore the group G generated by the Miyamoto involutions for $\mathcal{A}$ is a subgroup of index at most 2 of the group $\langle -\mathrm{id}_V \rangle \widehat{W}$.

The order of the product of two Miyamoto involutions is the same as the order of the product of the corresponding reflections. Hence G is a group of 3-transpositions isomorphic to $\widehat{W}$ or $\widehat{W}/\langle -\mathrm{id}_V \rangle$. The second case occurs only if $\langle -\mathrm{id}_V \rangle$ is in $\widehat{W}$ but not in its subgroup generated by negative reflections. This in turn happens if and only if $-\mathrm{id}_V \in \widehat{W} \smallsetminus \widehat{W}'$. The only such example is $\mathrm{W}(E_7)$ with $\widehat{W}$ isomorphic to $\langle -\mathrm{id}_V \rangle \times Sp_6(2)$ but G isomorphic to $Sp_6(2)$.

The space (V, B) may have a nontrivial radical (depending upon the type X_n and the characteristic of $\mathbb{F}$), in which case there is a further example $J(\tilde{V}, \tilde{B})$ corresponding to $\tilde{V} = V/\mathrm{Rad}(V, B)$.

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