Testing for panel cointegration using common correlated effects estimators

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Testing for Panel Cointegration using Common Correlated Effects Estimators

Supplementary material

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A Supplementary material

Lemma 1 *Define the vector of stochastic processes* $V_{i,t} = \left(e'_{i,t}, v'_t\right)'$ *that satisfies the panel functional central limit theorem (CLT):*

$$T^{-1/2} \sum_{t=1}^{[Tr]} V_{i,t} \Rightarrow C_i(1) W_i(r) \qquad \text{as } T \to \infty \text{ for all } i,$$

where $C_i(1)$ *is a* $((1+k+r) \times (1+k+r))$*-matrix of conditional long-run standard deviations.*

Proof: see Lemma 3 in Phillips and Moon (1999).

Lemma 2 *Define the* $(1+k+r)$*-vector* $W_i(r)$ *of standard Brownian motions. The expected value of the cross-product matrix for the demeaned –* $W_i^*(r) = W_i(r) - \int_0^1 W_i(s)\,ds$ *– and detrended –* $W_i^*(r) = W_i(r) - (4-6r)\int_0^1 W_i(s)\,ds - (-6+12r)\int_0^1 sW_i(s)\,ds$ *– vectors of Brownian motions is given by:*
a) Demeaned Brownian motions:

$$E\left(W^*(r) W^*(s)\right) = (r \wedge s) - \frac{2r - r^2}{2} - \frac{2s + s^2}{2} + \frac{1}{3}.$$

b) Detrended Brownian motions:

$$\begin{aligned} E\left(W^*(r) W^*(s)\right) &= (r \wedge s) + 2r^3 s - r^3 - 3r^2 s + 2r^2 + 2rs^3 - 3rs^2 \\ &\quad + \frac{6}{5} rs - \frac{11}{10} r - s^3 + 2s^2 - \frac{11}{10} s + \frac{2}{15}. \end{aligned}$$

Proof. Constant term. Define a vector of demeaned Brownian motions $W_i^*(r) = W_i(r) - \int_0^1 W_i(s)\,ds$ for which we want to compute $E\left(W_i^*(r) W_i^{*\prime}(s)\right)$. Since the Brownian motions are independent across i, consider the cross-product for one element of the vector, and remove the subscript to simplify notation:

$$\begin{aligned} E\left(W^*(r) W^*(s)\right) &= E\left[\left(W(r) - \int_0^1 W(u)\,du\right)\left(W(s) - \int_0^1 W(u)\,du\right)\right] \\ &= E\left[W(r) W(s) - W(r) \int_0^1 W(u)\,du - W(s) \int_0^1 W(u)\,du + \left(\int_0^1 W(u)\,du\right)^2\right] \\ &= E\left[A1 - A2 - A3 + A4\right]. \end{aligned}$$

The expected value of $A1$ is:

$$E(A1) = E\left(W(r) W(s)\right) = (r \wedge s).$$

For the second element we have:

$$
\begin{aligned}
E(A2) &= E\left[W(r)\int_0^1 W(u)\,du\right] \\
&= \int_0^r E(W(r)W(u))\,du + \int_r^1 E(W(r)W(u))\,du \\
&= \left(\int_0^r udu + \int_r^1 rdu\right) = \left(\frac{r^2}{2} + r(1-r)\right).
\end{aligned}
$$

The computation of the expected value for the third element is similar:

$$
\begin{aligned}
E(A3) &= E\left[W(s)\int_0^1 W(u)\,du\right] \\
&= \int_0^s E(W(s)W(u))\,du + \int_s^1 E(W(s)W(u))\,du \\
&= \left(\int_0^s udu + \int_s^1 sdu\right) = \left(\frac{s^2}{2} + s(1-s)\right).
\end{aligned}
$$

Finally, for the fourth element:

$$
\begin{aligned}
E(A4) &= E\left[\left(\int_0^1 W(u)\,du\right)^2\right] \\
&= E\left[\int_0^1\int_0^1 W(u)\,W(v)\,dudv\right] = 2\int_0^1\int_0^u E(W(u)\,W(v))\,dudv \\
&= 2\int_0^1\int_0^u vdudv = \frac{1}{3}.
\end{aligned}
$$

Taken all these elements together, we obtain:

$$
E(W^*(r)\,W^*(s)) = (r \wedge s) - \frac{2r - r^2}{2} - \frac{2s + s^2}{2} + \frac{1}{3}.
$$

Time trend case. Define a vector of detrended Brownian motions $W_i^*(r) = W_i(r) - (4 - 6r)$ $\int_0^1 W_i(s)\,ds - (-6 + 12r)\int_0^1 sW_i(s)\,ds$ for which we want to compute $E(W_i^*(r)\,W_i^{*\prime}(s))$. Since the Brownian motions are independent across i, consider the cross-product for one element of the vector, and remove the subscript to simplify notation:

$$
\begin{aligned}
E(W^*(r)\,W^*(s)) = \; & E\Big[(W(r) - (4-6r)\int_0^1 W(u)\,du - (-6+12r)\int_0^1 uW(u)\,du) \\
& (W(s) - (4-6s)\int_0^1 W(u)\,du - (-6+12s)\int_0^1 uW(u)\,du)\Big]
\end{aligned}
$$

so that

$$\begin{aligned}
E\left(W^*(r)\,W^*(s)\right) &= E\Big[W(r)\,W(s) - (4-6s)\,W(r)\int_0^1 W(u)\,du - W(r)\,(-6+12s)\int_0^1 uW(u)\,du \\
&\quad - (4-6r)\,W(s)\int_0^1 W(u)\,du + (4-6r)(4-6s)\left(\int_0^1 W(u)\,du\right)^2 \\
&\quad + (4-6r)(-6+12s)\int_0^1 W(u)\,du\int_0^1 uW(u)\,du \\
&\quad - (-6+12r)\,W(s)\int_0^1 uW(u)\,du \\
&\quad + (-6+12r)(4-6s)\int_0^1 W(u)\,du\int_0^1 uW(u)\,du \\
&\quad + (-6+12r)(-6+12s)\left(\int_0^1 uW(u)\,du\right)^2\Big] \\
&= E\left[B1 - B2 - B3 - B4 + B5 + B6 - B7 + B8 + B9\right].
\end{aligned}$$

Let us focus on the expected value of each of these nine elements. For the first element we have:

$$E(B1) = E\left(W(r)\,W(s)\right) = (r \wedge s).$$

The second element:

$$\begin{aligned}
E(B2) &= E\left[(4-6s)\,W(r)\int_0^1 W(u)\,du\right] \\
&= (4-6s)\left[\int_0^r E\left(W(r)\,W(u)\right)du + \int_r^1 E\left(W(r)\,W(u)\right)du\right] \\
&= (4-6s)\left(\int_0^r udu + \int_r^1 rdu\right) = (4-6s)\left(\frac{r^2}{2} + r(1-r)\right).
\end{aligned}$$

The third element:

$$\begin{aligned}
E(B3) &= E\left[(-6+12s)\,W(r)\int_0^1 uW(u)\,du\right] \\
&= (-6+12s)\left[\int_0^r uE\left(W(r)\,W(u)\right)du + \int_r^1 uE\left(W(r)\,W(u)\right)du\right] \\
&= (-6+12s)\left(\frac{r^3}{3} + r\left(\frac{1}{2} - \frac{r^2}{2}\right)\right).
\end{aligned}$$

The fourth element – note that is similar to $B2$:

$$\begin{aligned}
E(B4) &= E\left[(4-6r)\,W(s)\int_0^1 W(u)\,du\right] \\
&= (4-6r)\left[\int_0^s E\left(W(s)\,W(u)\right)du + \int_s^1 E\left(W(s)\,W(u)\right)du\right] \\
&= (4-6r)\left(\int_0^s udu + \int_s^1 sdu\right) = (4-6r)\left(\frac{s^2}{2} + s(1-s)\right).
\end{aligned}$$

The fifth element:

$$\begin{aligned}
E(B5) &= E\left[(4-6r)(4-6s)\left(\int_0^1 W(u)\,du\right)^2\right] \\
&= (4-6r)(4-6s)\,E\left[\left(\int_0^1 W(u)\,du\right)^2\right] \\
&= (4-6r)(4-6s)\,E\left[\left(\int_0^1\int_0^1 W(u)\,W(v)\,dudv\right)\right] \\
&= (4-6r)(4-6s)\left[2\left(\int_0^1\int_0^u E(W(u)\,W(v))\,dudv\right)\right] \\
&= (4-6r)(4-6s)\left[2\left(\int_0^1\int_0^u vdudv\right)\right] = \frac{1}{3}(4-6r)(4-6s)\,.
\end{aligned}$$

The sixth element:

$$\begin{aligned}
E(B6) &= E\left[(4-6r)(-6+12s)\int_0^1 W(u)\,du\int_0^1 uW(u)\,du\right] \\
&= (4-6r)(-6+12s)\,E\left[\int_0^1 W(u)\,du\int_0^1 uW(u)\,du\right] \\
&= (4-6r)(-6+12s)\,E\left[\int_0^1\int_0^1 vW(u)\,W(v)\,dudv\right] \\
&= (4-6r)(-6+12s)\left[\int_0^1\left(\int_0^u vE(W(u)\,W(v)) + \int_u^1 vE(W(u)\,W(v))\right)dvdu\right] \\
&= (4-6r)(-6+12s)\left[\int_0^1\left(\int_0^u v^2 + \int_u^1 uv\right)dvdu\right] = \frac{5}{24}(4-6r)(-6+12s)\,.
\end{aligned}$$

The seventh element – similar to $B3$:

$$\begin{aligned}
E(B7) &= E\left[(-6+12r)\,W(s)\int_0^1 uW(u)\,du\right] \\
&= (-6+12r)\left[\int_0^s uE(W(s)\,W(u))\,du + \int_s^1 uE(W(s)\,W(u))\,du\right] \\
&= (-6+12r)\left(\frac{s^3}{3} + s\left(\frac{1}{2} - \frac{s^2}{2}\right)\right).
\end{aligned}$$

The eight element – similar to $B6$:

$$\begin{aligned}
E(B8) &= E\left[(4-6s)(-6+12r)\int_0^1 W(u)\,du\int_0^1 uW(u)\,du\right] \\
&= \frac{5}{24}(4-6s)(-6+12r)\,.
\end{aligned}$$

Finally, the ninth element:

$$\begin{aligned}
E(B9) &= E\left[(-6+12r)(-6+12s)\left(\int_0^1 uW(u)\,du\right)^2\right] \\
&= (-6+12r)(-6+12s)\,E\left[\left(\int_0^1 uW(u)\,du\right)^2\right] \\
&= (-6+12r)(-6+12s)\,E\left[\int_0^1\int_0^1 uvW(u)\,W(v)\,dudv\right] \\
&= (-6+12r)(-6+12s)\left[2\int_0^1\int_0^u uvE(W(u)\,W(v))\,dudv\right] \\
&= (-6+12r)(-6+12s)\left[2\int_0^1\int_0^u uv^2 dudv\right] = \frac{2}{15}(-6+12r)(-6+12s)\,.
\end{aligned}$$

Taking all these elements together, we obtain:

$$\begin{aligned}
E(W^*(r)\,W^*(s)) &= (r\wedge s) - (4-6s)\left(\frac{r^2}{2} + r(1-r)\right) - (-6+12s)\left(\frac{r^3}{3} + r\left(\frac{1}{2} - \frac{r^2}{2}\right)\right) \\
&\quad - (4-6r)\left(\frac{s^2}{2} + s(1-s)\right) + \frac{1}{3}(4-6r)(4-6s) \\
&\quad + \frac{5}{24}(4-6r)(-6+12s) \\
&\quad - (-6+12r)\left(\frac{s^3}{3} + s\left(\frac{1}{2} - \frac{s^2}{2}\right)\right) \\
&\quad + \frac{5}{24}(4-6s)(-6+12r) \\
&\quad + \frac{2}{15}(-6+12r)(-6+12s) \\
&= (r\wedge s) + 2r^3s - r^3 - 3r^2s + 2r^2 + 2rs^3 - 3rs^2 \\
&\quad + \frac{6}{5}rs - \frac{11}{10}r - s^3 + 2s^2 - \frac{11}{10}s + \frac{2}{15}
\end{aligned}$$

■

A.1 Proof of Theorem 1

A.1.1 No deterministic component

In this section we analyze the model specification that does not include any deterministic component, i.e., $D_{i,t} = 0\ \forall i$ in (1). For ease of exposition, we start considering that all common factors in the model are I(1), but the derivations also apply if there is a mixture of I(0) and I(1) common factors, or all common factors are I(0) – see below.

Let $M_i(r) = (M_{y_i}(r)', M_{x_i}(r)', M_F(r)')' = (M_{U_i}(r)', M_F(r)')' = C_i(1)\,W_i(r)$, where $M_i(r)$ is a randomly scaled Brownian motion with a conditional covariance matrix $C_i(1)\,C_i(1)'$ that has a well defined expectation provided that $\|EC_i(1)\,C_i(1)'\| < \infty$ as shown in Lemma 1(d) in Phillips and Moon (1999). Let us define $Z_{i,t} = \left(U_{i,t}', F_t'\right)'$, by the continuous mapping theorem we have that as $T \to \infty$ for a fixed N

$$T^{-2} \sum_{t=1}^{T} Z_{i,t} Z_{i,t}' \Rightarrow C_i(1) \int_0^1 W_i(r) W_i'(r)\, dr \; C_i(1)' = \int_0^1 M_i(r) M_i'(r)\, dr.$$

Further, we define the long-run conditional covariance matrix of $Z_{i,t} = \left(U_{i,t}', F_t'\right)' = (U_{y_i,t}, U_{x_{i,1},t}, \ldots, U_{x_{i,k},t}, F_t')'$ as

$$\begin{aligned}
\Omega_i &= \begin{bmatrix} \Omega_{U_{y_i}U_{y_i}} & \Omega_{U_{y_i}U_{x_i}} & \Omega_{U_{y_i}F} \\ \Omega_{U_{x_i}U_{y_i}} & \Omega_{U_{x_i}U_{x_i}} & \Omega_{U_{x_i}F} \\ \Omega_{FU_{y_i}} & \Omega_{FU_{x_i}} & \Omega_{FF} \end{bmatrix} = \begin{bmatrix} \Omega_{U_i} & \Omega_{U_iF} \\ \Omega_{FU_i} & \Omega_{FF} \end{bmatrix} \\
&= C_i(1) C_i(1)' = \begin{bmatrix} C_{U_{y_i}}(1) C_{U_{y_i}}(1)' & C_{U_{y_i}}(1) C_{x_i}(1)' & C_{U_{y_i}}(1) C_F(1)' \\ C_{U_{x_i}}(1) C_{U_{y_i}}(1)' & C_{U_{x_i}}(1) C_{U_{x_i}}(1)' & C_{U_{x_i}}(1) C_F(1)' \\ C_F(1) C_{U_{y_i}}(1)' & C_F(1) C_{U_{x_i}}(1)' & C_F(1) C_F(1)' \end{bmatrix},
\end{aligned}$$

with $C_i(1) = (C_{y_i}(1)', C_{x_i}(1)', C_F(1)')' = (C_{U_i}(1)', C_F(1)')'$ and the long-run average covariance matrix of $Z_{i,t}$ as:

$$\Omega = \begin{bmatrix} \Omega_{U_yU_y} & \Omega_{U_yU_x} & \Omega_{U_yF} \\ \Omega_{U_xU_y} & \Omega_{U_xU_x} & \Omega_{U_xF} \\ \Omega_{FU_y} & \Omega_{FU_x} & \Omega_{FF} \end{bmatrix} = \begin{bmatrix} \Omega_U & \Omega_{UF} \\ \Omega_{FU} & \Omega_{FF} \end{bmatrix} = E\left(C_i(1) C_i(1)'\right).$$

Let K be the invariant σ-field generated by F_t, so that $U_{i,t}$ are independent across i conditional on K. Then, we define the expected value of the cross product matrix as

$$\begin{aligned}
E\left(\int_0^1 M_i(r) M_i'(r)\, dr\right) &= E\left(C_i(1) E\left(\int_0^1 W_i(r) W_i'(r)\, dr\right) C_i(1)'\right) \\
&= E\left(C_i(1) \frac{1}{2} I_{(1+k+r)} C_i(1)'\right) \\
&= \frac{1}{2}\Omega.
\end{aligned}$$

Note that averaging across i the cross-products involving $U_{i,t}$ we have, conditional on K,

$$N^{-1} \sum_{i=1}^{N} T^{-2} \sum_{t=1}^{T} U_{i,t} U_{i,t}' \Rightarrow N^{-1} \sum_{i=1}^{N} \left(\int_0^1 M_{U_i}(r) M_{U_i}'(r)\, dr \middle| K \right).$$

Using Lemma 4 in Phillips and Moon (1999) and Theorem 9 in Kao, Trapani and Urga (2011), we have that $E\left\| \int_0^1 M_{U_i}(r) M_{U_i}'(r)\, dr \middle| K \right\|^2 < \infty$, so that as $N \to \infty$ the law of strong numbers gives

$$N^{-1} \sum_{i=1}^{N} \left(\int_0^1 M_{U_i}(r) M_{U_i}'(r)\, dr \middle| K \right) \overset{a.s.}{\to} E\left(\int_0^1 M_{U_i}(r) M_{U_i}'(r)\, dr \middle| K \right) = \frac{1}{2}\Omega_U.$$

We have defined the pooled estimator as

$$\hat{\beta} = \left[\frac{1}{N}\sum_{i=1}^{N} T^{-2}\left(x_i^{*\prime} x_i^*\right)\right]^{-1} \frac{1}{N}\sum_{i=1}^{N} T^{-2}\left(x_i^{*\prime} y_i^*\right),$$

where, in this case, $x_i^* = M_F x_i$. Note that

$$\begin{aligned} T^{-2} x_i^{*\prime} x_i^* &= T^{-2} x_i' M_F x_i = T^{-2} U_{x_i}' M_F U_{x_i} \\ &= T^{-2} U_{x_i}' U_{x_i} - T^{-2} U_{x_i}' F \left(T^{-2} F' F\right)^{-1} T^{-2} F' U_{x_i}, \end{aligned}$$

so that, in the limit,

$$\begin{aligned} T^{-2} x_i^{*\prime} x_i^* \Rightarrow\ & C_{U_{x_i}}(1) \left(\int_0^1 W_{U_{x_i}}(r) W_{U_{x_i}}'(r)\, dr \middle| K\right) C_{U_{x_i}}(1)' \\ & - \left[\left(C_{U_{x_i}}(1) \left(\int_0^1 W_{U_{x_i}}(r) W_F'(r)\, dr \middle| K\right) C_F(1)'\right)\right. \\ & \left(C_F(1) \left(\int_0^1 W_F(r) W_F'(r)\, dr \middle| K\right) C_F(1)'\right)^{-1} \\ & \left.\left(C_F(1) \left(\int_0^1 W_F(r) W_{U_{x_i}}'(r)\, dr \middle| K\right) C_{U_{x_i}}(1)'\right)\right]. \end{aligned}$$

Using the fact that $E\left(W_{U_i}(r) W_{U_i}'(s)\right) = (r \wedge s) I_{1+k}$, with $(r \wedge s) = \min\{r, s\}$, we have, conditional on K,

$$\begin{aligned} E\left(T^{-2} x_i^{*\prime} x_i^*\right) \overset{a.s.}{\to}\ & E\left[C_{U_{x_i}}(1) \left(\int_0^1 W_{U_{x_i}}(r) W_{U_{x_i}}'(r)\, dr \middle| K\right) C_{U_{x_i}}(1)'\right] \\ & - E\left[\int_0^1 \int_0^1 C_{U_{x_i}}(1) W_{U_{x_i}}(r) h(r,s) W_{U_{x_i}}'(s) C_{U_{x_i}}(1)'\, dr ds \middle| K\right], \end{aligned}$$

with $h(r,s) = W_F'(r) C_F(1)' \left(C_F(1) \left(\int_0^1 W_F(r) W_F'(r)\, dr\right) C_F(1)'\right)^{-1} C_F(1) W_F(s)$. Note that

$$E\left[C_{U_{x_i}}(1) \left(\int_0^1 W_{U_{x_i}}(r) W_{U_{x_i}}'(r)\, dr \middle| K\right) C_{U_{x_i}}(1)'\right] = \left(\int_0^1 r dr\right) \Omega_{U_x U_x} = \frac{1}{2} \Omega_{U_x U_x},$$

and

$$E\left[\int_0^1 \int_0^1 C_{U_{x_i}}(1) W_{U_{x_i}}(r) h(r,s) W_{U_{x_i}}'(s) C_{U_{x_i}}(1)'\, dr ds \middle| K\right] = \left(\int_0^1 \int_0^1 (r \wedge s) h(r,s)\, ds dr\right) \Omega_{U_x U_x},$$

so that, conditional on K, we have

$$N^{-1} \sum_{i=1}^{N} T^{-2} x_i^{*\prime} x_i^* \overset{a.s.}{\to} \left(\frac{1}{2} - \int_0^1 \int_0^1 (r \wedge s) h(r,s)\, ds dr\right) \Omega_{U_x U_x}.$$

Similarly, for the numerator of the pooled estimator

$$\begin{aligned} T^{-2}x_i^{*\prime}y_i^* &= T^{-2}x_i^{\prime}M_F y_i = T^{-2}U_{x_i}^{\prime}M_F U_{y_i} \\ &= T^{-2}U_{x_i}^{\prime}U_{y_i} - T^{-2}U_{x_i}^{\prime}F\left(T^{-2}F^{\prime}F\right)^{-1}T^{-2}F^{\prime}U_{y_i}, \end{aligned}$$

so that, conditional on K,

$$N^{-1}\sum_{i=1}^{N}T^{-2}x_i^{*\prime}y_i^* \overset{a.s.}{\rightarrow} \left(\frac{1}{2} - \int_0^1\int_0^1 (r\wedge s)\,h(r,s)\,dsdr\right)\Omega_{U_xU_y}.$$

Finally,

$$\begin{aligned} \hat{\beta} \overset{p}{\rightarrow} & \left[\left(\frac{1}{2} - \int_0^1\int_0^1 (r\wedge s)\,h(r,s)\,dsdr\right)\Omega_{U_xU_x}\right]^{-1} \\ & \left[\left(\frac{1}{2} - \int_0^1\int_0^1 (r\wedge s)\,h(r,s)\,dsdr\right)\Omega_{U_xU_y}\right] \\ = & \ \Omega_{U_xU_x}^{-1}\Omega_{U_xU_y} = \beta. \end{aligned}$$

Note that the conditioning variables that appear in the numerator and denominator of the estimator cancel out so that the conditional limit of the estimator is also the unconditional limit.

So far, the proof has used sequential limits to show the consistency of the pooled estimator. However and following Phillips and Moon (1999), the same result is achieved if we base our analysis on joint limit theory. By the Beveridge-Nelson (BN) decomposition:

$$Z_{i,t} \overset{a.s.}{=} C_i(1)\,P_{i,t} + \tilde{V}_{i,0} - \tilde{V}_{i,t} + Z_{i,0},$$

with $P_{i,t} = \sum_{t=1}^{[Tr]} S_{i,t}$, $S_{i,t} = \left(w_t^{\prime}, \varepsilon_{i,t}^{\prime}\right)^{\prime}$. Then, define

$$N^{-1}\sum_{i=1}^{N}T^{-2}\sum_{t=1}^{T}Z_{i,t}Z_{i,t}^{\prime} \overset{a.s.}{=} N^{-1}\sum_{i=1}^{N}\left(Q_{i,t}+R_{i,t}\right),$$

where

$$\begin{aligned} Q_{i,t} &= T^{-2}\sum_{t=1}^{T}C_i(1)\,P_{i,t}P_{i,t}^{\prime}C_i(1) \\ R_{i,t} &= R_{1,i,t} + R_{1,i,t}^{\prime} + R_{2,i,t} \\ R_{1,i,t} &= T^{-2}\sum_{t=1}^{T}C_i(1)\,P_{i,t}\left(\tilde{V}_{i,0} - \tilde{V}_{i,t} + Z_{i,0}\right)^{\prime} \\ R_{2,i,t} &= T^{-2}\sum_{t=1}^{T}\left(\tilde{V}_{i,0} - \tilde{V}_{i,t} + Z_{i,0}\right)\left(\tilde{V}_{i,0} - \tilde{V}_{i,t} + Z_{i,0}\right)^{\prime}. \end{aligned}$$

We need to show that $\|Q_{i,t}\|$ is uniformly integrable in T, provided that then

$$N^{-1}\sum_{i=1}^{N}Q_{i,t}\xrightarrow{p}\frac{1}{2}\Omega,$$

as $(T,N)\to\infty$ jointly. By $\|AB\|\leq\|A\|\,\|B\|$ and the triangle inequality

$$\|Q_{i,t}\|\leq\|C_i(1)\|^2\,T^{-2}\sum_{t=1}^{T}\|P_{i,t}\|^2.$$

Note that as $T\to\infty$

$$T^{-2}\sum_{t=1}^{T}\|P_{i,t}\|^2\Rightarrow\int_0^1\|W_{i,t}\|^2\,dr,$$

and that

$$E\left(T^{-2}\sum_{t=1}^{T}\|P_{i,t}\|^2\right)=tr\left(T^{-2}\sum_{t=1}^{T}E\left(P_{i,t},P_{i,t}'\right)\right)\to E\left(\int_0^1\|W_{i,t}\|^2\,dr\right)=\frac{1}{2}tr\left(I_{1+k+r}\right).$$

Then, it follows from Billingsley (1968) that $T^{-2}\sum_{t=1}^{T}\|P_{i,t}\|^2$ is uniformly integrable in T. Since $E\|C_i(1)\|^2<\infty$, we can conclude that $\|C_i(1)\|^2\,T^{-2}\sum_{t=1}^{T}\|P_{i,t}\|^2$ is uniformly integrable in T and, hence, $\|Q_{i,t}\|$ is uniformly integrable in T. Consequently, $N^{-1}\sum_{i=1}^{N}Q_{i,t}\xrightarrow{p}\frac{1}{2}\Omega$ as stated above.

So far, we have assumed that all r common factors are I(1), but it would be the case that there is a subset of r_0 I(0) common factors and a subset of r_1 I(1) common factors, $r=r_0+r_1$. Let us define $F_t=\left(F_{1,t}',F_{0,t}'\right)'$, with $F_{1,t}$ the $(r_1\times 1)$-vector of I(1) common factors and $F_{0,t}$ the $(r_0\times 1)$-vector of I(0) common factors. In this case,

$$\begin{aligned}T^{-2}x_i^{*\prime}x_i^{*}&=T^{-2}U_{x_i}'M_FU_{x_i}\\&=T^{-2}U_{x_i}'U_{x_i}-T^{-2}U_{x_i}'F\Psi\left(\Psi F'F\Psi\right)^{-1}\Psi F'U_{x_i},\end{aligned}$$

with $\Psi=diag\left(\Psi_1,\Psi_0\right)$ a rescaling diagonal matrix defined by the $(r_1\times 1)$-vector $\Psi_1=(T^{-1},\ldots,T^{-1})$ and by the $(r_0\times 1)$-vector $\Psi_0=\left(T^{-1/2},\ldots,T^{-1/2}\right)$, so that $\Psi F'F\Psi=O_p(1)$. Using these elements, we have

$$N^{-1}\sum_{i=1}^{N}T^{-2}x_i^{*\prime}x_i^{*}\xrightarrow{a.s.}\left(\frac{1}{2}-\int_0^1\int_0^1(r\wedge s)\,h(r,s)\,dsdr\right)\Omega_{U_xU_x}.$$

The same applies for $N^{-1}\sum_{i=1}^{N}T^{-2}x_i^{*\prime}y_i^{*}\xrightarrow{a.s.}\left(\frac{1}{2}-\int_0^1\int_0^1(r\wedge s)\,h(r,s)\,dsdr\right)\Omega_{U_xU_y}$, so that $\hat{\beta}\xrightarrow{p}\Omega_{U_xU_x}^{-1}\Omega_{U_xU_y}=\beta$ as above. Consequently, having a combination of I(0) and I(1) common factors does not alter the result about the consistency of the pooled estimator.

A.1.2 Constant term

In this section we consider the deterministic specification given by Model 1 through the definition of $D_{i,t}=\mu_i=\left(\mu_{i,0},\mu_{i,1},\ldots,\mu_{i,k}\right)'$. Using the projection matrix $M_D=I-D\left(D'D\right)^{-1}D'$, where

$D=\iota$ denotes a vector of ones, we define $\tilde{Z}_{i,t}=\left(\tilde{U}_{i,t}',\tilde{F}_t'\right)'$, where $\tilde{U}_i=M_DU_i$ and $\tilde{F}=M_DF$ are the OLS detrended variables. By the continuous mapping theorem we have that as $T\to\infty$ for a fixed N

$$T^{-2}\sum_{t=1}^{T}\tilde{Z}_{i,t}\tilde{Z}_{i,t}'\Rightarrow C_i\left(1\right)\int_0^1 W_i^*\left(r\right)W_i^{*\prime}\left(r\right)dr\ C_i\left(1\right)'=\int_0^1 M_i^*\left(r\right)M_i^{*\prime}\left(r\right)dr,$$

where $W_i^*\left(r\right)=W_i\left(r\right)-\int_0^1 W_i\left(s\right)ds$ and $M_i^*\left(r\right)=M_i\left(r\right)-\int_0^1 M_i\left(s\right)ds$ are demeaned Brownian motion vectors. As above,

$$\begin{aligned}E\left(\int_0^1 M_i^*\left(r\right)M_i^{*\prime}\left(r\right)dr\right) &= E\left(C_i\left(1\right)E\left(\int_0^1 W_i^*\left(r\right)W_i^{*\prime}\left(r\right)dr\right)C_i\left(1\right)'\right)\\ &= E\left(C_i\left(1\right)\frac{1}{6}I_{(1+k+r)}C_i\left(1\right)'\right)\\ &= \frac{1}{6}\Omega.\end{aligned}$$

The developments carried out in the previous section follow here replacing $W_i\left(r\right)$ by $W_i^*\left(r\right)$. Note that now

$$\begin{aligned}T^{-2}x_i^{*\prime}x_i^* &= T^{-2}\tilde{x}_i'M_{\tilde{F}}\tilde{x}_i=T^{-2}\tilde{U}_{x_i}'M_{\tilde{F}}\tilde{U}_{x_i}\\ &= T^{-2}\tilde{U}_{x_i}'\tilde{U}_{x_i}-T^{-2}\tilde{U}_{x_i}'\tilde{F}\left(T^{-2}\tilde{F}'\tilde{F}\right)^{-1}T^{-2}\tilde{F}'\tilde{U}_{x_i},\end{aligned}$$

so that, in the limit,

$$\begin{aligned}T^{-2}x_i^{*\prime}x_i^* \Rightarrow\ & C_{U_{x_i}}\left(1\right)\left(\int_0^1 W_{U_{x_i}}^*\left(r\right)W_{U_{x_i}}^{*\prime}\left(r\right)dr\middle| K\right)C_{U_{x_i}}\left(1\right)'\\ & -\left[\left(C_{U_{x_i}}\left(1\right)\left(\int_0^1 W_{U_{x_i}}^*\left(r\right)W_F^{*\prime}\left(r\right)dr\middle| K\right)C_F\left(1\right)'\right)\right.\\ & \left(C_F\left(1\right)\left(\int_0^1 W_F^*\left(r\right)W_F^{*\prime}\left(r\right)dr\middle| K\right)C_F\left(1\right)'\right)^{-1}\\ & \left.\left(C_F\left(1\right)\left(\int_0^1 W_F^*\left(r\right)W_{U_{x_i}}^{*\prime}\left(r\right)dr\middle| K\right)C_{U_{x_i}}\left(1\right)'\right)\right].\end{aligned}$$

From Lemma 2, $E\left(W_{U_i}^*\left(r\right)W_{U_i}^{*\prime}\left(s\right)\right)=((r\wedge s)-\left(2r-r^2\right)/2-\left(2s-s^2\right)/2+1/3)I_{1+k}$, so that, conditional on K, we obtain $N^{-1}\sum_{i=1}^N T^{-2}x_i^{*\prime}x_i^*\overset{a.s.}{\to}(1/6-\int_0^1\int_0^1((r\wedge s)-\left(2r-r^2\right)/2$ $-\left(2s-s^2\right)/2+1/3)h\left(r,s\right)dsdr)\Omega_{U_xU_x}$ and $N^{-1}\sum_{i=1}^N T^{-2}x_i^{*\prime}y_i^*\overset{a.s.}{\to}(1/6-\int_0^1\int_0^1((r\wedge s)-\left(2r-r^2\right)/2-\left(2s-s^2\right)/2+1/3)h\left(r,s\right)dsdr)\Omega_{U_xU_y}$, where now $h\left(r,s\right)=W_F^{*\prime}\left(r\right)C_F\left(1\right)'\left(C_F\left(1\right)\right.$ $\left(\int_0^1 W_F^*\left(r\right)W_F^{*\prime}\left(r\right)dr\right)C_F\left(1\right)'\right)^{-1}C_F\left(1\right)W_F^*\left(s\right)$ with $W_F^*\left(r\right)=W_F\left(r\right)-\int_0^1 W_F\left(s\right)ds$. Therefore, $\hat{\beta}\overset{p}{\to}\Omega_{U_xU_x}^{-1}\Omega_{U_xU_y}=\beta$, as above. Following the steps given in the previous subsection, it can be shown that the same result is obtained if we use joint limits, where the only difference is that we use demeaned Brownian motions instead of standard Brownian motions – to be specific, we need to consider that in this case $N^{-1}\sum_{i=1}^N Q_{i,t}\overset{p}{\to}\frac{1}{6}\Omega$ and the rest of the proof applies. As above, note that the conditioning variables that appear in the numerator and denominator of the estimator cancel out so that the conditional limit of the estimator is also the unconditional limit.

A.1.3 Time trend

In this section we consider the deterministic specification given by Model 2, i.e., $D_{i,t} = (1, t)\,[\delta_{i,0}, \delta_{i,1}, \ldots, \delta_{i,k}]$, with $\delta_{i,j} = \left(\mu_{i,j}, \eta_{i,j}\right)'$, $j = 0, 1, \ldots, k$. Using the projection matrix $M_D = I - D\left(D'D\right)^{-1} D'$, where $D = [\iota\ \tau]$ with ι a vector of ones and $\tau = (1, 2, \ldots, T)'$. We define $\tilde{Z}_{i,t} = \left(\tilde{U}'_{i,t}, \tilde{F}'_t\right)'$, where $\tilde{U}_i = M_D U_i$ and $\tilde{F} = M_D F$ are the OLS detrended variables. By the continuous mapping theorem we have that as $T \to \infty$ for a fixed N

$$T^{-2} \sum_{t=1}^{T} \tilde{Z}_{i,t} \tilde{Z}'_{i,t} \Rightarrow C_i\left(1\right) \int_0^1 W_i^*\left(r\right) W_i^{*\prime}\left(r\right) dr\ C_i\left(1\right)' = \int_0^1 M_i^*\left(r\right) M_i^{*\prime}\left(r\right) dr,$$

where $W_i^*\left(r\right) = W_i\left(r\right) - \left(4 - 6r\right) \int_0^1 W_i\left(s\right) ds - \left(-6 + 12r\right) \int_0^1 sW_i\left(s\right) ds$ and $M_i^*\left(r\right) = M_i\left(r\right) - \left(4 - 6r\right) \int_0^1 M_i\left(s\right) ds - \left(-6 + 12r\right) \int_0^1 sM_i\left(s\right) ds$ are detrended Brownian motion vectors. In this case,

$$\begin{aligned} E\left(\int_0^1 M_i^*\left(r\right) M_i^{*\prime}\left(r\right) dr\right) &= E\left(C_i\left(1\right) E\left(\int_0^1 W_i^*\left(r\right) W_i^{*\prime}\left(r\right) dr\right) C_i\left(1\right)'\right) \\ &= E\left(C_i\left(1\right) \frac{1}{15} I_{(1+k+r)} C_i\left(1\right)'\right) \\ &= \frac{1}{15}\Omega. \end{aligned}$$

From Lemma 2, $E\left(W_{U_i}^*\left(r\right) W_{U_i}^{*\prime}\left(s\right)\right) = ((r \wedge s) + 2r^3 s - r^3 - 3r^2 s + 2r^2 + 2rs^3 - 3rs^2 + \frac{6}{5}rs - \frac{11}{10}r - s^3 + 2s^2 - \frac{11}{10}s + \frac{2}{15})I_{1+k}$, so that, conditional on K, $N^{-1} \sum_{i=1}^{N} T^{-2} x_i^{*\prime} x_i^* \overset{a.s.}{\to} (\frac{1}{15} - \int_0^1 \int_0^1 ((r \wedge s) + 2r^3 s - r^3 - 3r^2 s + 2r^2 + 2rs^3 - 3rs^2 + \frac{6}{5}rs - \frac{11}{10}r - s^3 + 2s^2 - \frac{11}{10}s + \frac{2}{15}) h\left(r, s\right) dsdr)\, \Omega_{U_x U_x}$ and $N^{-1} \sum_{i=1}^{N} T^{-2} x_i^{*\prime} y_i^* \overset{a.s.}{\to} (\frac{1}{15} - \int_0^1 \int_0^1 ((r \wedge s) + 2r^3 s - r^3 - 3r^2 s + 2r^2 + 2rs^3 - 3rs^2 + \frac{6}{5}rs - \frac{11}{10}r - s^3 + 2s^2 - \frac{11}{10}s + \frac{2}{15}) h\left(r, s\right) dsdr) \Omega_{U_x U_y}$, with $h\left(r, s\right) = W_F^{*\prime}\left(r\right) C_F\left(1\right)' (C_F\left(1\right) \left(\int_0^1 W_F^*\left(r\right) W_F^{*\prime}\left(r\right) dr\right) C_F\left(1\right)')^{-1} C_F\left(1\right) W_F^*\left(s\right)$ with $W_F^*\left(r\right) = W_F\left(r\right) - \left(4 - 6r\right) \int_0^1 W_F\left(s\right) ds - \left(-6 + 12r\right) \int_0^1 sW_F\left(s\right) ds$. Consequently, $\hat{\beta} \overset{p}{\to} \Omega_{U_x U_x}^{-1} \Omega_{U_x U_y} = \beta$. Following the steps given above, it can be shown that the same result is obtained if we use joint limits, where the only difference is that we use detrended Brownian motions instead of standard Brownian motions – to be specific, in this case $N^{-1} \sum_{i=1}^{N} Q_{i,t} \overset{p}{\to} \frac{1}{15}\Omega$ and the rest of the proof applies. As above, note that the conditioning variables that appear in the numerator and denominator of the estimator cancel out so that the conditional limit of the estimator is also the unconditional limit.

A.2 Proof of Theorem 2

In order to prove Theorem 2, we begin by defining the projection matrix $\bar{M} = I - \bar{H}\left(\bar{H}'\bar{H}\right)^{-1} \bar{H}'$, where $\bar{H} = \bar{z}$ for Model 0, $\bar{H} = [\iota\ \bar{z}]$ for Model 1 and $\bar{H} = [\iota\ \tau\ \bar{z}]$ for Model 2, with $\bar{z} = [\bar{x}\ \bar{y}]$ being the $(T \times (k+1))$ matrix of cross-section averages. Further, let us define $M_g = I - G\left(G'G\right)^{-1} G'$ and $M_q = I - Q\left(Q'Q\right)^{+} Q'$.

In the case of Model 0 $G = F$ denotes the $(T \times r)$ matrix of unobserved factors, $Q = G\bar{P}$

with $\bar{P} = \bar{\pi}$, and A^{+} is the Moore-Penrose inverse of matrix A. For Model 1, $G = [\iota\ F]$, and

$$\bar{P} = \begin{bmatrix} 1 & \bar{\delta} \\ 0 & \bar{\pi} \end{bmatrix}.$$

Finally, for Model 2 we have $G = [\iota\ \tau\ F]$, and

$$\bar{P} = \begin{bmatrix} I_2 & \bar{\delta} \\ 0 & \bar{\pi} \end{bmatrix}.$$

The pooled estimator is computed as:

$$\begin{aligned}
\hat{\beta} &= \left[\frac{1}{N}\sum_{i=1}^{N} T^{-2}\left(x_i^{*\prime}x_i^{*}\right)\right]^{-1}\frac{1}{N}\sum_{i=1}^{N} T^{-2}\left(x_i^{*\prime}y_i^{*}\right) \\
&= \left[\frac{1}{N}\sum_{i=1}^{N} T^{-2}\left(x_i'\bar{M}x_i\right)\right]^{-1}\frac{1}{N}\sum_{i=1}^{N} T^{-2}\left(x_i'\bar{M}y_i\right).
\end{aligned}$$

In order to show consistency of the pooled estimator, we need to establish that the quadratic form involving the projection matrix using the cross-section averages is asymptotically equivalent to that defined by using the true factors. That is, we need to analyze

$$T^{-2}\left\|Y_i'\bar{M}Y_i - Y_i'M_qY_i\right\|,$$

where

$$Y_i = F\pi_i' + U_i$$

denotes the model defined in (1) in matrix notation, assuming no deterministic terms to simplify the notation.

Note that

$$\begin{aligned}
T^{-2}\left\|Y_i'\bar{M}Y_i - Y_i'M_qY_i\right\| &\leq T^{-2}\left\|\left(Y_i'\bar{H} - Y_i'Q\right)\left(\bar{H}'\bar{H}\right)^{-1}\bar{H}'Y_i\right\| \\
&\quad + T^{-2}\left\|Y_i'Q\left(\left(\bar{H}'\bar{H}\right)^{-1} - \left(Q'Q\right)^{-1}\right)\bar{H}'Y_i\right\| \\
&\quad + T^{-2}\left\|Y_i'Q\left(Q'Q\right)^{-1}\left(\bar{H}'Y_i - Q'Y_i\right)\right\| \\
&= I + II + III.
\end{aligned}$$

Consider part I and recall that $\bar{H} = Q + \bar{U}$.

We then have

$$T^{-2}\left\|\left(Y_i'\bar{H} - Y_i'Q\right)\left(\bar{H}'\bar{H}\right)^{-1}\bar{H}'Y_i\right\| \leq \left\|\frac{Y_i'\bar{U}}{T^2}\right\|\left\|\left(\frac{\bar{H}'\bar{H}}{T^2}\right)^{-1}\frac{\bar{H}'Y_i}{T^2}\right\|,$$

where

$$\left\|\frac{Y_i'\bar{U}}{T^2}\right\| = \left\|\frac{1}{N}\sum_{j=1}^{N}\frac{Y_i'U_j}{T^2}\right\| = O_p\left(\frac{1}{\sqrt{N}}\right),$$

given that $T^{-2}Y_i'U_j = O_p(1)$ and $N^{-1/2}\sum_{j=1}^{N} T^{-2}Y_i'U_j = O_p(1)$. Assuming that the rank condition $rank(\bar{\pi}) = r \leq (1+k)$ for all N as $T, N \to \infty$ holds, and provided that $T^{-2}\bar{H}'\bar{H} = O_p(1)$ and $T^{-2}\bar{H}'Y_i = O_p(1)$, we have that $I = O_p\left(N^{-1/2}\right)$.

For part II we have

$$II \leq \left\| \frac{\bar{U}'\bar{U}}{T^2} + \frac{Q'\bar{U}}{T^2} + \frac{\bar{U}'Q}{T^2} \right\| \left\| \frac{Y_i'Q}{T^2} \left(\frac{\bar{H}'\bar{H}}{T^2}\right)^{-1} \right\| \left\| \left(\frac{Q'Q}{T^2}\right)^{-1} \frac{\bar{H}'Y_i}{T^2} \right\|,$$

where $T^{-2}Y_i'Q = O_p(1)$ and $T^{-2}Q'Q = O_p(1)$. Note that

$$\left\| \frac{\bar{U}'\bar{U}}{T^2} \right\| = \left\| \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{U_i'U_j}{T^2} \right\| = O_p\left(\frac{1}{\sqrt{N}}\right),$$

given that $T^{-2}U_i'U_j = O_p(1)$ and $N^{-1/2}\sum_{j=1}^{N} T^{-2}U_i'U_j = O_p(1)$. Similarly, $\left\|T^{-2}Q'\bar{U}\right\| = O_p\left(N^{-1/2}\right)$, so that $II = O_p\left(N^{-1/2}\right)$.

Part III is given by

$$III \leq \left\| \frac{Y_i'Q}{T^2} \left(\frac{Q'Q}{T^2}\right)^{-1} \right\| \left\| \frac{Y_i'\bar{U}}{T^2} \right\|.$$

Using the elements above, it may be easily shown that $III = O_p\left(N^{-1/2}\right)$.

Consequently, when the rank condition $rank(\bar{\pi}) = r \leq (1+k)$ holds for all N as $T, N \to \infty$, $M_g = M_q$, so that, conditional on K, we have

$$T^{-2}\left\|Y_i'\bar{M}Y_i - Y_i'M_gY_i\right\| \leq O_p\left(\frac{1}{\sqrt{N}}\right),$$

uniformly over i. Therefore, conditional on K, we have

$$\hat{\beta} = \left[\frac{1}{N}\sum_{i=1}^{N} T^{-2}\left(x_i'M_gx_i\right) + O_p\left(\frac{1}{\sqrt{N}}\right)\right]^{-1} \left[\frac{1}{N}\sum_{i=1}^{N} T^{-2}\left(x_i'M_gy_i\right) + O_p\left(\frac{1}{\sqrt{N}}\right)\right],$$

so that as $T, N \to \infty$

$$\hat{\beta} \xrightarrow{p} \beta = \Omega_{U_xU_x}^{-1}\Omega_{U_xU_y},$$

a result that was already established in Theorem 1 and confirmed in the Monte Carlo experiment that has been conducted above.

Table A.1: Mean, median and root mean square error of the $\hat{\beta}_{PCCE}$

			$N=10$			$N=20$			$N=50$			$N=100$		
ϕ_i	ρ	T	Mean	Median	RMSE	Mean	Median	RMSE	Mean	Median	RMSE	Mean	Median	RMSE
1	1	50	0.996	0.996	0.144	0.999	0.999	0.092	1.000	1.000	0.059	1.000	1.000	0.041
1	1	100	1.000	0.998	0.140	1.000	1.000	0.092	1.002	1.001	0.059	1.000	0.999	0.040
1	1	250	1.001	1.001	0.146	0.999	1.001	0.092	0.999	0.999	0.059	1.001	1.001	0.042
1	0.99	50	0.996	0.996	0.144	0.999	1.000	0.092	1.000	1.000	0.059	1.000	1.000	0.041
1	0.99	100	1.000	0.997	0.140	1.000	1.001	0.092	1.002	1.001	0.059	1.000	0.999	0.040
1	0.99	250	1.000	1.000	0.147	0.999	1.000	0.093	0.999	1.000	0.059	1.001	1.001	0.042
1	0.95	50	0.996	0.995	0.145	0.999	0.999	0.092	1.000	1.000	0.060	1.000	1.000	0.041
1	0.95	100	1.000	0.998	0.143	1.000	1.001	0.094	1.002	1.003	0.059	1.000	0.999	0.041
1	0.95	250	1.000	1.003	0.148	0.999	1.001	0.095	0.999	1.000	0.060	1.001	1.001	0.043
1	0.9	50	0.997	0.996	0.145	0.999	0.999	0.094	0.999	0.999	0.060	1.000	1.000	0.041
1	0.9	100	1.000	0.999	0.144	1.000	1.001	0.095	1.002	1.003	0.060	1.000	0.999	0.042
1	0.9	250	1.001	1.003	0.147	0.999	1.000	0.096	0.999	1.000	0.060	1.001	1.001	0.043
0.9	1	50	0.996	0.996	0.117	0.998	0.998	0.076	1.000	1.000	0.048	1.000	1.000	0.033
0.9	1	100	1.001	1.001	0.087	1.000	1.000	0.057	1.001	1.001	0.036	1.000	1.000	0.025
0.9	1	250	1.000	0.999	0.049	1.000	1.000	0.032	1.000	1.000	0.020	1.000	1.000	0.014
0.9	0.99	50	0.996	0.996	0.117	0.998	0.999	0.076	1.000	1.000	0.048	1.000	1.000	0.033
0.9	0.99	100	1.001	1.000	0.087	1.000	1.000	0.057	1.001	1.001	0.036	1.000	1.000	0.025
0.9	0.99	250	1.000	1.000	0.049	1.000	1.000	0.032	1.000	1.000	0.020	1.000	1.000	0.014
0.9	0.95	50	0.996	0.996	0.118	0.998	0.999	0.076	1.000	1.000	0.048	1.000	1.000	0.033
0.9	0.95	100	1.000	1.000	0.087	1.000	1.000	0.057	1.001	1.001	0.036	1.000	1.000	0.025
0.9	0.95	250	1.000	1.000	0.048	1.000	1.000	0.032	1.000	1.000	0.019	1.000	1.000	0.014
0.9	0.9	50	0.996	0.996	0.117	0.998	0.999	0.076	1.000	0.999	0.048	1.000	1.000	0.034
0.9	0.9	100	1.000	1.000	0.087	1.000	1.000	0.057	1.001	1.001	0.035	1.000	1.000	0.025
0.9	0.9	250	1.000	1.000	0.048	1.000	1.000	0.031	1.000	1.000	0.019	1.000	1.000	0.014

Table A.2: Empirical size and power of the panel cointegration tests with normalized spatial dependence, $N = 10$

		SAR				SMA			
		$\vartheta = 0.4$		$\vartheta = 0.8$		$\vartheta = 0.4$		$\vartheta = 0.8$	
ϕ_i	T	Z_τ	$CADF_P$	Z_τ	$CADF_P$	Z_τ	$CADF_P$	Z_τ	$CADF_P$
1	50	0.131	0.064	0.135	0.126	0.134	0.062	0.153	0.096
1	100	0.142	0.069	0.140	0.123	0.140	0.061	0.161	0.090
1	250	0.128	0.060	0.131	0.126	0.130	0.061	0.170	0.095
0.99	50	0.136	0.066	0.139	0.126	0.132	0.062	0.155	0.098
0.99	100	0.152	0.074	0.155	0.137	0.158	0.063	0.180	0.097
0.99	250	0.217	0.096	0.211	0.173	0.225	0.099	0.254	0.136
0.95	50	0.178	0.101	0.173	0.168	0.171	0.094	0.197	0.134
0.95	100	0.384	0.220	0.360	0.286	0.372	0.203	0.376	0.246
0.95	250	0.956	0.923	0.931	0.845	0.953	0.913	0.894	0.880
0.9	50	0.296	0.203	0.272	0.281	0.291	0.207	0.298	0.237
0.9	100	0.788	0.728	0.725	0.673	0.791	0.718	0.720	0.685
0.9	250	1.000	1.000	0.999	1.000	1.000	1.000	0.997	1.000

		SEC1				SEC2			
		$\vartheta = 0.4$		$\vartheta = 0.8$		$\vartheta = 0.4$		$\vartheta = 0.8$	
ϕ_i	T	Z_τ	$CADF_P$	Z_τ	$CADF_P$	Z_τ	$CADF_P$	Z_τ	$CADF_P$
1	50	0.091	0.050	0.095	0.050	0.097	0.049	0.115	0.052
1	100	0.061	0.050	0.060	0.048	0.061	0.047	0.066	0.047
1	250	0.054	0.047	0.054	0.046	0.052	0.044	0.056	0.049
0.99	50	0.092	0.050	0.096	0.050	0.102	0.052	0.118	0.052
0.99	100	0.073	0.053	0.076	0.053	0.075	0.051	0.080	0.052
0.99	250	0.159	0.077	0.155	0.077	0.152	0.081	0.154	0.078
0.95	50	0.152	0.080	0.153	0.079	0.158	0.078	0.163	0.078
0.95	100	0.450	0.177	0.455	0.178	0.458	0.176	0.452	0.177
0.95	250	0.997	0.938	0.997	0.941	0.998	0.942	0.998	0.940
0.9	50	0.337	0.176	0.331	0.177	0.335	0.182	0.308	0.181
0.9	100	0.972	0.734	0.972	0.736	0.972	0.738	0.959	0.742
0.9	250	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

		SEC3			
		$\vartheta = 0.4$		$\vartheta = 0.8$	
ϕ_i	T	Z_τ	$CADF_P$	Z_τ	$CADF_P$
1	50	0.138	0.056	0.128	0.070
1	100	0.123	0.048	0.135	0.059
1	250	0.095	0.052	0.141	0.064
0.99	50	0.143	0.057	0.134	0.070
0.99	100	0.130	0.052	0.157	0.063
0.99	250	0.188	0.082	0.236	0.097
0.95	50	0.174	0.083	0.180	0.096
0.95	100	0.380	0.180	0.391	0.205
0.95	250	0.973	0.934	0.961	0.920
0.9	50	0.282	0.192	0.308	0.199
0.9	100	0.815	0.739	0.820	0.721
0.9	250	1.000	1.000	1.000	1.000

Table A.3: Empirical size and power of the panel cointegration tests with normalized spatial dependence, $N = 50$

		SAR				SMA			
		$\vartheta = 0.4$		$\vartheta = 0.8$		$\vartheta = 0.4$		$\vartheta = 0.8$	
ϕ_i	T	Z_τ	$CADF_P$	Z_τ	$CADF_P$	Z_τ	$CADF_P$	Z_τ	$CADF_P$
1	50	0.073	0.064	0.131	0.126	0.064	0.061	0.083	0.075
1	100	0.083	0.069	0.156	0.139	0.076	0.064	0.105	0.084
1	250	0.060	0.062	0.138	0.134	0.065	0.065	0.091	0.085
0.99	50	0.083	0.070	0.138	0.128	0.078	0.062	0.093	0.080
0.99	100	0.137	0.078	0.195	0.157	0.120	0.077	0.162	0.100
0.99	250	0.404	0.153	0.391	0.235	0.423	0.141	0.428	0.169
0.95	50	0.346	0.133	0.266	0.196	0.344	0.120	0.359	0.145
0.95	100	0.952	0.496	0.795	0.501	0.961	0.488	0.938	0.488
0.95	250	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.9	50	0.936	0.432	0.630	0.451	0.942	0.417	0.918	0.427
0.9	100	1.000	1.000	0.999	0.982	1.000	1.000	1.000	0.998
0.9	250	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

		SEC1				SEC2			
		$\vartheta = 0.4$		$\vartheta = 0.8$		$\vartheta = 0.4$		$\vartheta = 0.8$	
ϕ_i	T	Z_τ	$CADF_P$	Z_τ	$CADF_P$	Z_τ	$CADF_P$	Z_τ	$CADF_P$
1	50	0.054	0.048	0.053	0.046	0.051	0.044	0.054	0.048
1	100	0.061	0.054	0.059	0.053	0.061	0.056	0.063	0.053
1	250	0.049	0.057	0.048	0.056	0.047	0.057	0.048	0.052
0.99	50	0.060	0.049	0.057	0.049	0.057	0.047	0.062	0.052
0.99	100	0.104	0.064	0.104	0.065	0.101	0.066	0.111	0.064
0.99	250	0.398	0.137	0.399	0.135	0.397	0.132	0.395	0.130
0.95	50	0.334	0.107	0.333	0.108	0.338	0.105	0.335	0.105
0.95	100	0.974	0.493	0.973	0.493	0.974	0.493	0.969	0.490
0.95	250	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.9	50	0.960	0.433	0.960	0.429	0.959	0.429	0.955	0.433
0.9	100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.9	250	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

		SEC3			
		$\vartheta = 0.4$		$\vartheta = 0.8$	
ϕ_i	T	Z_τ	$CADF_P$	Z_τ	$CADF_P$
1	50	0.059	0.054	0.071	0.061
1	100	0.062	0.048	0.079	0.052
1	250	0.053	0.055	0.059	0.066
0.99	50	0.071	0.055	0.081	0.063
0.99	100	0.111	0.061	0.122	0.064
0.99	250	0.397	0.133	0.399	0.144
0.95	50	0.352	0.114	0.349	0.127
0.95	100	0.968	0.488	0.959	0.482
0.95	250	1.000	1.000	1.000	1.000
0.9	50	0.957	0.432	0.938	0.428
0.9	100	1.000	0.999	1.000	1.000
0.9	250	1.000	1.000	1.000	1.000