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The evolution to localized and front solutions in a non-Lipschitz reaction-diffusion Cauchy problem with trivial initial data

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Abstract

In this paper, we establish the existence of spatially inhomogeneous classical self-similar solutions to a non-Lipschitz semi-linear parabolic Cauchy problem with trivial initial data. Specifically we consider bounded solutions to an associated two-dimensional non-Lipschitz non-autonomous dynamical system, for which, we establish the existence of a two-parameter family of homoclinic connections on the origin, and a heteroclinic connection between two equilibrium points. Additionally, we obtain bounds and estimates on the rate of convergence of the homoclinic connections to the origin.

Keywords: semi-linear parabolic PDE, self-similar, non-Lipschitz, homoclinic connection, heteroclinic connection 2000 MSC: 35K58, 34C37

1. Introduction

In this paper, we study classical bounded solutions $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ to the non-Lipschitz semi-linear parabolic Cauchy problem

$$u_t - u_{xx} = u |u|^{p-1}$$
 on $\mathbb{R} \times (0, T]$, (1)

$$u = 0 \quad \text{on } \mathbb{R} \times \{0\},\tag{2}$$

with 0 and <math>T > 0 (which we henceforth refer to as [CP]). The primary achievement of the paper is the establishment of the existence of a two-parameter family of localized spatially inhomogeneous solutions to [CP] for which $u(x,t) \rightarrow 0$ as $|x| \rightarrow \infty$ uniformly for $t \in [0,T]$; the secondary achievement of the paper is the establishment of front solutions to [CP], which approach $\pm (1-p)^{1/(1-p)}t$ as $|x| \rightarrow \pm \infty$ uniformly for $t \in [0,T]$. We note here that for $p \ge 1$ in (1), then the unique bounded classical solution with initial data (2) is the trivial solution, see for example [1, Theorem 4.5].

Qualitative properties of non-negative (non-positive) solutions to (1) when 0 , with non-negative (non-positive) initial data, and for which <math>u(x,t) is bounded as $|x| \to \infty$ uniformly for $t \in [0, T]$, have been determined in [2], [3], [4], [5] and [6]. However, we note that any non-negative (non-positive) classical bounded solution to [CP] must be spatially homogeneous for $t \in [0, T]$, see for example [2, Corollary 2.6]. Thus, the solutions constructed in this paper are two signed on $\mathbb{R} \times [0, T]$. The authors are currently unaware of any studies of two signed solutions to (1)-(2) with $0 . Generic local results for spatial homogeneity of solutions to semi-linear parabolic Cauchy problems with homogeneous initial data depend upon uniqueness results, see for example, [6]. For results concerning the related problem of asymptotic homogeneity (in general, asymptotic symmetry) as <math>t \to \infty$ of non-negative (non-positive) global solutions to semi-linear parabolic Cauchy problems, we refer the reader to the survey article [7].

Non-negative (non-positive), spatially inhomogeneous solutions to (1) for p > 1 have been considered in [8], [9] [10], [11], [12], [13], [14], [15], [16] and [17] with the focus primarily on critical exponents for finite time blow-up of solutions, and conditions for the existence of global solutions (see the review articles [18] and [19]). Moreover,

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for p > 1, solutions to (1) with two signed initial data have been considered in [20] and [21], whilst boundary value problems have been studied in [22] and [23].

The paper is structured as follows; in Section 2 we introduce the self-similar solution structure for [CP], and hence, determine an ordinary differential equation related to (1); the remainder of the paper concerns the study of particular solutions to this ordinary differential equation, which is re-written as an equivalent two-dimensional non-autonomous dynamical system. Specifically, in Section 3 we establish the existence of a two-parameter family of homoclinic connections on the equilibrium (0,0). Additionally, we determine bounds and estimates on the asymptotic approach of these solutions to (0,0). In Section 4, we establish the existence of a heteroclinic connection between the equilibrium points $(\pm(1-p)^{1/(1-p)}, 0)$.

2. Self-Similar Structure

With 0 and <math>T > 0, we refer to $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ as a solution to [CP] when u satisfies (1)-(2) with regularity,

$$u \in L^{\infty}(\mathbb{R} \times [0,T]) \cap C(\mathbb{R} \times [0,T]) \cap C^{2,1}(\mathbb{R} \times (0,T]).$$

$$(3)$$

Observe that $u^{\pm} : \mathbb{R} \times [0, T] \to \mathbb{R}$ given by

$$u^{\pm}(x,t) = \pm ((1-p)t)^{1/(1-p)} \quad \forall (x,t) \in \mathbb{R} \times [0,T]$$

are the maximal and minimal solutions to [CP] (see [1, Chapter 8]), and hence any solution $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ to [CP] must satisfy,

$$u^{-}(x,t) \le u(x,t) \le u^{+}(x,t) \quad \forall (x,t) \in \mathbb{R} \times [0,T].$$

$$\tag{4}$$

To construct spatially inhomogeneous solutions to [CP], we consider, for any fixed $x_0 \in \mathbb{R}$, self-similar solutions $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ of the form,

$$u(x,t) = \begin{cases} w\left(\frac{x-x_0}{t^{1/2}}\right) t^{1/(1-p)} &, (x,t) \in \mathbb{R} \times \{0,T\}, \\ 0 &, (x,t) \in \mathbb{R} \times \{0\}, \end{cases}$$
(5)

with $w : \mathbb{R} \to \mathbb{R}$ to be determined. Now, $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ given by (5) is a solution to [CP] if and only if there exist constants $\alpha, \beta \in \mathbb{R}$ such that $w : \mathbb{R} \to \mathbb{R}$ satisfies the following zero-value problem, namely,

$$w'' + \frac{1}{2}\eta w' + w|w|^{p-1} - \frac{1}{(1-p)}w = 0 \quad \forall \eta \in \mathbb{R},$$
(6)

$$w(0) = \alpha, \quad w'(0) = \beta, \tag{7}$$

$$w \in C^2(\mathbb{R}) \cap L^{\infty}(\mathbb{R}).$$
(8)

Here $\eta = (x - x_0)/t^{1/2}$, and we observe that the ordinary differential equation (6) is both non-autonomous and non-Lipschitz. It is convenient to introduce

$$x = w, \quad y = w',$$

after which the problem (6)-(8) is equivalent to the zero-value problem for the two-dimensional, non-Lipschitz, non-autonomous, dynamical system,

$$x' = y \tag{9}$$

$$y' = \frac{1}{(1-p)} x - x|x|^{p-1} - \frac{1}{2}\eta y \quad \forall \eta \in \mathbb{R},$$
(10)

$$(x(0), y(0)) = (\alpha, \beta),$$
 (11)

$$(x,y) \in C^{1}(\mathbb{R}) \cap L^{\infty}(\mathbb{R}).$$
(12)

We refer to the equivalent zero-value problems in (6)-(8) and (9)-(12) as (S). Our objective is now to investigate those $(\alpha, \beta) \in \mathbb{R}^2$ for which (S) has a non-trivial solution. It is instructive to note, at this stage, via (4), that we may conclude that any solution to (S) must satisfy the inequality,

$$-(1-p)^{\frac{1}{(1-p)}} \le w(\eta) \le (1-p)^{\frac{1}{(1-p)}} \quad \forall \eta \in \mathbb{R},$$
(13)

whilst, following [2, Corollary 2.6], any non-constant solution to (S) must be two-signed in w.

3. Homoclinic Connections

In this section we establish the existence of a two parameter family of homoclinic connections for (S) on the equilibrium point (0,0) of the dynamical system (9)-(10), and establish decay rates to the equilibrium point (0,0) as $|\eta| \to \infty$ on these homoclinic connections.

3.1. Existence

In this subsection, we establish the existence of homoclinic connections attached to the equilibrium point (x, y) = (0, 0) of the dynamical system (9)-(10). To begin, observe that $\mathbf{Q} : \mathbb{R}^3 \to \mathbb{R}^2$, where

$$\mathbf{Q}(x, y, \eta) = (Q_1, Q_2)(x, y, \eta) = \left(y, \frac{1}{(1-p)}x - x|x|^{p-1} - \frac{1}{2}\eta y\right) \quad \forall (x, y, \eta) \in \mathbb{R}^3$$
(14)

is such that $\mathbf{Q} \in C(\mathbb{R}^3)$, but also that \mathbf{Q} is not locally Lipschitz continuous on \mathbb{R}^3 (note that \mathbf{Q} is locally Lipschitz continuous on $\mathbb{R}^3 \setminus N$, with *N* any neighbourhood of the plane x = 0). We now have,

Theorem 1. The problem (S) with zero-value $(\alpha, \beta) \in \mathbb{R}^2$ has a solution for $\eta \in [-\delta, \delta]$ (not necessarily unique), where $\delta = 1/(1 + M)$ and

$$M = \max_{(x,y,\eta)\in R} |\mathbf{Q}(x,y,\eta)|$$

with

$$R = \{(x, y, \eta) \in \mathbb{R}^3 : |x - \alpha| \le 1, |y - \beta| \le 1, |\eta| \le 1\}.$$

Proof. This follows immediately from the Cauchy-Peano Local Existence Theorem (see [24, Chapter 1, Theorem 1.2]) since $\mathbf{Q} : \mathbb{R}^3 \to \mathbb{R}^2$ is such that $\mathbf{Q} \in C(\mathbb{R}^3)$.

Remark 1. When $\alpha \neq 0$, then the solution to (S) with zero-value $(\alpha,\beta) \in \mathbb{R}^2$ is unique for $\eta \in [-\delta',\delta']$ for some $0 < \delta' \leq \delta$. In addition, the problem (S) with zero-value $(\pm (1-p)^{1/(1-p)}, 0)$ has the unique global solution

$$(x(\eta), y(\eta)) = (\pm (1-p)^{1/(1-p)}, 0) \quad \forall \eta \in \mathbb{R}.$$
(15)

This follows since **Q** is locally Lipschitz in a neighbourhood of $(\pm (1 - p)^{1/(1-p)}, 0)$ respectively. Also, the problem (S) with zero-value (0,0) has the unique global solution,

$$(x(\eta), y(\eta)) = (0, 0) \quad \forall \eta \in \mathbb{R}.$$

In this case uniqueness does not follows immediately, since \mathbf{Q} is not locally Lipschitz continuous in any neighborhood of (0,0), but instead follows after further qualitative results have been established for solutions to (S) (see Remark 2).

We now introduce the function $V : \mathbb{R}^2 \to \mathbb{R}$ defined by,

$$V(x,y) = \frac{1}{2}y^2 - \frac{1}{2(1-p)}x^2 + \frac{1}{(1+p)}|x|^{1+p} \quad \forall (x,y) \in \mathbb{R}^2.$$
(16)

We observe immediately that

$$V \in C^{1,1}(\mathbb{R}^2),\tag{17}$$



Figure 1: A qualitative sketch of the level curves of V

with

$$\nabla V(x,y) = \left(\frac{-1}{(1-p)}x + x|x|^{p-1}, y\right) \quad \forall (x,y) \in \mathbb{R}^2.$$

$$\tag{18}$$

We now examine the structure of the level curves of V in \mathbb{R}^2 , namely, the family of curves in \mathbb{R}^2 defined by

$$V(x,y) = c, (19)$$

for $-\infty < c < \infty$. It is straightforward to establish that the family of level curves of *V* are qualitatively as sketched in Figure 1, with \mathcal{H} representing the two level curves connecting $(-(1-p)^{1/1-p}, 0)$ to $((1-p)^{1/1-p}, 0)$ and enclosing the origin. In Figure 3.1, on the red curve $V = (1-p)^{2/(1-p)}/(2(1+p))$, whilst on the blue curves V = 0. At $(\pm(1-p)^{1/(1-p)}, 0)$ then $V = (1-p)^{2/(1-p)}/(2(1+p))$, whilst at (0,0) then V = 0. Inside \mathcal{H} , the level curves are simple closed curves concentric with the origin (0,0), and *V* is increasing from V = 0 at the origin (0,0), as each level curve is crossed, when moving out from the origin (0,0) to the boundary curve \mathcal{H} , on which $V = (1-p)^{2/(1-p)}/(2(1+p))$. Thus, inside \mathcal{H} , *V* has a minimum at the origin (0,0) and is increasing on moving radially away from the origin (0,0) to the boundary \mathcal{H} . On the level curves exterior and above or below \mathcal{H} , then $V > (1-p)^{2/(1-p)}/(2(1+p))$, whilst on the level curves to the left and right side of \mathcal{H} , then $V < (1-p)^{2/(1-p)}/(2(1+p))$, with V = 0 on the blue level curves. We now focus on the level curves of *V* on and inside \mathcal{H} , which have

$$0 \le c \le c^*(p), \tag{20}$$

where

$$c^{*}(p) = \frac{(1-p)^{2/(1-p)}}{2(1+p)}.$$
(21)

These are concentric closed curves surrounding the origin (0,0). We will label the interior of the level curve V = c by D_c , with the level curve V = c labelled as ∂D_c , for $0 \le c \le c^*(p)$. In addition, we label the set

$$\bar{D}'_{c^*(p)} = \bar{D}_{c^*(p)} \setminus \{ (\pm (1-p)^{1/(1-p)}, 0), (0,0) \}.$$

Now let $(x^*(\eta), y^*(\eta))$ be any solution to (S) for $\eta \in [-E, E]$ (any E > 0) with zero-value $(\alpha, \beta) \in \mathbb{R}^2$, and define $F : [-E, E] \to \mathbb{R}$ as,

$$F(\eta) = V(x^*(\eta), y^*(\eta)) \quad \forall \eta \in [-E, E].$$

$$(22)$$

Then $F \in C^1([-E, E])$, and via (9), (10) and (14),

$$F'(\eta) = \nabla V(x^*(\eta), y^*(\eta)) . (x^{*'}(\eta), y^{*'}(\eta)) = \nabla V(x^*(\eta), y^*(\eta)) . \mathbf{Q}(x^*(\eta), y^*(\eta), \eta) \quad \forall \eta \in [-E, E].$$

It then follows, via (18) and (14) that,

$$F'(\eta) = -\frac{1}{2}\eta(y^*(\eta))^2 \quad \forall \eta \in [-E, E].$$
(23)

It follows from (23) that

$$F(\eta)$$
 is non-increasing for $\eta \in [0, E]$, (24)

$$F(\eta)$$
 is non-decreasing for $\eta \in [-E, 0]$. (25)

We can now establish the following,

Lemma 2. Let $(x^*(\eta), y^*(\eta))$ be any solution to (S) on [-E, E] (any E > 0) with zero-value $(\alpha, \beta) \in \overline{D}'_{c^*(p)}$. Then

 $(x^*(\eta), y^*(\eta)) \in D_c \quad \forall \eta \in [-E, E] \setminus \{0\},$

where $c = V(\alpha, \beta)$.

Proof. Let the zero-value $(\alpha,\beta) \in \partial D_c \setminus \{\pm ((1-p)^{\frac{1}{1-p}}, 0)\}$ with $0 < c = V(\alpha,\beta) \le c^*(p)$. We first consider the case when $\beta \ne 0$. It follows from (23)-(25) that,

$$F(\eta) < F(0) \quad \forall \eta \in [-E, E] \setminus \{0\}.$$

$$(26)$$

Therefore, via (26),

$$V(x^*(\eta), y^*(\eta)) < c \quad \forall \eta \in [-E, E] \setminus \{0\},$$

and so

$$(x^*(\eta), y^*(\eta)) \in D_c \quad \forall \eta \in [-E, E] \setminus \{0\},$$

as required. Now consider the case when $\beta = 0$. Then $0 < |\alpha| < (1 - p)^{1/(1-p)}$ and therefore, via (10) $y^{*'}(0) \neq 0$ after which a similar argument completes the proof.

We now have:

Theorem 3. For each $(\alpha,\beta) \in \overline{D}'_{c^*(p)}$, then (S) with zero-value (α,β) has a solution $(x^*(\eta), y^*(\eta))$ on [-E, E] (any E > 0). Moreover, every such solution satisfies $(x^*(\eta), y^*(\eta)) \in D_c$ for all $\eta \in [-E, E] \setminus \{0\}$, where $c = V(\alpha, \beta)$.

Proof. For any $(\alpha, \beta) \in \overline{D}'_{c^*(p)}$, Lemma 2 establishes that (S) with zero-value (α, β) is a priori bounded. The result then follows by a finite number of applications of the Cauchy-Peano Local Existence Theorem (see [24, Chapter 1, Theorem 1.2]), with $\delta = 1/(1 + M)$ and

$$M = \max_{(x,y,\eta)\in R'} |\mathbf{Q}(x,y,\eta)|$$

whilst

$$R' = \{ (x, y, \eta) \in \mathbb{R}^3 : |x| \le 2(1-p)^{1/(1-p)}, |y| \le 2\sqrt{2c^*(p)}, |\eta| \le 2E \}$$

The final statement follows immediately from Lemma 2.

We can now establish a global existence result for (S), namely

Corollary 4. For $(\alpha,\beta) \in \overline{D}'_{c^*(p)}$ then (S) with zero-value (α,β) has a solution $(x^*(\eta), y^*(\eta))$ on \mathbb{R} . Moreover, every such solution satisfies $(x^*(\eta), y^*(\eta)) \in D_c$ for all $\eta \in \mathbb{R} \setminus \{0\}$, where $c = V(\alpha, \beta)$.

Proof. Since Theorem 3 holds for any E > 0, the result follows immediately.

Remark 2. Let $(x^*(\eta), y^*(\eta))$ be any solution to (S) on [-E, E] with zero-value (0, 0). It follows from (16), (22) and (23) that

$$V(x^{*}(\eta), y^{*}(\eta)) = F(\eta) \le F(0) = V(0, 0) = 0 \quad \forall \eta \in [-E, E].$$
(27)

Thus $(x^*(\eta), y^*(\eta)) \in S$ for all $\eta \in [-E, E]$, with S being a connected subset of

$$\{(x, y) \in \mathbb{R}^2 : V(x, y) \le 0\}$$

for which $(0,0) \in S$. It follows that $S = \{(0,0)\}$ and so $(x^*(\eta), y^*(\eta)) = (0,0)$ for all $\eta \in [-E, E]$. We conclude that the unique solution to (S) with zero-value (0,0) is given by,

$$(x^*(\eta), y^*(\eta)) = (0, 0) \quad \forall \eta \in \mathbb{R}$$

We next introduce the function $H : \mathbb{R} \to \mathbb{R}$ such that

$$H(x) = \frac{1}{(1-p)} x - x|x|^{p-1} \quad \forall x \in \mathbb{R},$$
(28)

and observe that

$$H \in C(\mathbb{R}).$$
⁽²⁹⁾

We have,

Lemma 5. Let $(\alpha,\beta) \in \overline{D}'_{c^*(p)}$, and let $(x^*(\eta), y^*(\eta))$ for $\eta \in \mathbb{R}$ be a global solution to (S) with zero-value (α,β) . Then

$$y^*(\eta) \to 0 \text{ as } |\eta| \to \infty.$$

Proof. We establish the result for $\eta \to \infty$; the result for $\eta \to -\infty$ follows similarly. Now, from (10),

$$y^{*'}(\eta) = H(x^{*}(\eta)) - \frac{1}{2}\eta y^{*}(\eta) \quad \forall \eta \in [0, \infty).$$
 (30)

It then follows from (30) that,

$$y^{*}(\eta) = \beta e^{-\frac{1}{4}\eta^{2}} + e^{-\frac{1}{4}\eta^{2}} \int_{0}^{\eta} H(x^{*}(s)) e^{\frac{1}{4}s^{2}} ds \quad \forall \eta \in [0, \infty).$$
(31)

Thus,

$$|y^{*}(\eta)| \leq |\beta|e^{-\frac{1}{4}\eta^{2}} + e^{-\frac{1}{4}\eta^{2}} \int_{0}^{\eta} |H(x^{*}(s))|e^{\frac{1}{4}s^{2}} ds \quad \forall \eta \in [0,\infty).$$
(32)

However, via Corollary 4, $(x^*(\eta), y^*(\eta)) \in \overline{D}_{c^*(p)}$ for $\eta \in [0, \infty)$, and so, via (29), there exists a constant $M_H \ge 0$ such that

$$|H(x^*(s))| \le M_H \quad \forall s \in [0, \infty).$$
(33)

It then follows from (32) and (33) that

$$|y^{*}(\eta)| \leq |\beta|e^{-\frac{1}{4}\eta^{2}} + M_{H}e^{-\frac{1}{4}\eta^{2}} \int_{0}^{\eta} e^{\frac{1}{4}s^{2}} ds \quad \forall \eta \in [0,\infty).$$
(34)

Now a simple application of Watson's Lemma (see [25, Proposition 2.1]), gives,

$$\int_0^{\eta} e^{\frac{1}{4}s^2} ds \sim \frac{2}{\eta} e^{\frac{1}{4}\eta^2} \text{ as } \eta \to \infty.$$
(35)

We then have, via (34) and (35), that

$$|y^*(\eta)| \le |\beta|e^{-\frac{1}{4}\eta^2} + \frac{4M_H}{\eta} \text{ as } \eta \to \infty.$$
(36)

It follows from (36) that

$$y^*(\eta) \to 0 \text{ as } \eta \to \infty,$$

as required

We next have,

Lemma 6. Let $(x^*(\eta), y^*(\eta))$ for $\eta \in \mathbb{R}$ be a global solution to (S) with zero-value $(\alpha, \beta) \in \overline{D}'_{c^*(p)}$, and $F : \mathbb{R} \to \mathbb{R}$ as in (22). Then $F(\eta)$ is non-increasing for $\eta \in (0, \infty)$ and non-decreasing for $\eta \in (-\infty, 0)$, with

$$F(\eta) \to \begin{cases} F_{\infty} & \text{as } \eta \to \infty \\ F_{-\infty} & \text{as } \eta \to -\infty \end{cases}$$

where $F_{\infty}, F_{-\infty} \in [0, F(0))$.

Proof. We observe from Corollary 4 that

$$(x^*(\eta), y^*(\eta)) \in D_c \quad \forall \eta \in \mathbb{R} \setminus \{0\},$$
(37)

with $c = V(\alpha, \beta) = F(0)$, and so,

$$0 \le F(\eta) < F(0) \quad \forall \eta \in \mathbb{R} \setminus \{0\}.$$
(38)

In addition, it follows from (38), (24) and (25), since $F \in C^1(\mathbb{R})$, that there exist $F_{\infty}, F_{-\infty} \in \mathbb{R}$, such that

$$F(\eta) \to \begin{cases} F_{\infty} & \text{as } \eta \to \infty \\ F_{-\infty} & \text{as } \eta \to -\infty \end{cases}$$

where $F_{\infty}, F_{-\infty} \in [0, F(0))$, as required.

We now have,

Theorem 7. Let $(x^*(\eta), y^*(\eta))$ for $\eta \in \mathbb{R}$ be a global solution to (S) with zero-value $(\alpha, \beta) \in \overline{D}'_{c^*(p)}$. Then,

 $(x^*(\eta), y^*(\eta)) \to (0,0) \text{ as } |\eta| \to \infty.$

Proof. We establish the result for $\eta \to \infty$. The result for $\eta \to -\infty$ follows similarly. We first recall from Corollary 4 that,

$$(x^*(\eta), y^*(\eta)) \in D_{c^*(p)} \quad \forall \eta \in \mathbb{R} \setminus \{0\},$$
(39)

and from Lemma 5 that,

$$y^*(\eta) \to 0 \text{ as } \eta \to \infty.$$
 (40)

In addition, we have from Lemma 6 that,

$$V(x^*(\eta), y^*(\eta)) \to F_{\infty} \text{ as } \eta \to \infty$$
(41)

for some $F_{\infty} \in [0, c^*(p))$. It follows from (39)-(41) that

$$x^*(\eta) \to x_{\infty} \text{ or } x^*(\eta) \to -x_{\infty} \text{ as } \eta \to \infty$$
 (42)

where x_{∞} is the single non-negative root of

$$V(x,0) = F_{\infty}$$
 with $x \in [0, (1-p)^{1/(1-p)})$.

Without loss of generality we will suppose that

$$(x^*(\eta), y^*(\eta)) \to (x_{\infty}, 0) \text{ as } \eta \to \infty.$$
(43)

However, it follows from (10) that,

$$y^{*}(\eta) = \beta e^{-\frac{1}{4}\eta^{2}} + e^{-\frac{1}{4}\eta^{2}} \int_{0}^{\eta} H(x^{*}(s)) e^{\frac{1}{4}s^{2}} ds \quad \eta \in [0, \infty)$$
(44)

with $H : \mathbb{R} \to \mathbb{R}$ given by (28), and

$$H(x_{\infty}) \le 0. \tag{45}$$

Using (42), it is straightforward to establish that, when,

$$H(x_{\infty}) < 0, \tag{46}$$

then from (44),

$$y^*(\eta) \sim \frac{2H(x_\infty)}{\eta} \text{ as } \eta \to \infty.$$
 (47)

In addition, from (9), we have,

$$x^*(\eta) = \alpha + \int_0^{\eta} y^*(s) ds \quad \forall \eta \in [0, \infty),$$
(48)

which gives, via (47), that

$$\alpha^*(\eta) \sim 2H(x_\infty) \log \eta$$
, as $\eta \to \infty$,

which contradicts (42). We conclude that (46) cannot hold, and so, via (45), we must have

$$H(x_{\infty}) = 0, \tag{49}$$

which, since $x_{\infty} \in [0, (1-p)^{1/(1-p)})$, requires $x_{\infty} = 0$. It then follows from (43) that,

$$(x^*(\eta), y^*(\eta)) \to (0,0)$$
 as $\eta \to \infty$,

as required.

We conclude from Corollary 4 and Theorem 7 that the problem (S) has a two parameter family of nontrivial, distinct homoclinic connections on the equilibrium point (0,0), parametrized by $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$ which we will denote by $w_{\alpha\beta} : \mathbb{R} \to \mathbb{R}$ for each $(\alpha,\beta) \in \bar{D}'_{c^*(p)}$. Here $w = w_{\alpha\beta}(\eta), \eta \in \mathbb{R}$, has zero-values $w(0) = \alpha, w'(0) = \beta$. Moreover,

$$(w_{\alpha,\beta}(\eta), w'_{\alpha,\beta}(\eta)) \in D_{V(\alpha,\beta)} \quad \forall \eta \in \mathbb{R} \setminus \{0\}.$$

Additionally, note that $w_{0,\beta}(\eta)$ is an odd function of η whilst $w_{\alpha,0}(\eta)$ is an even function of η . Furthermore, it also follows from the comments below (13) that $w_{\alpha,\beta}(\eta)$ must be two signed for $\eta \in \mathbb{R}$.

3.2. Decay Bounds and Estimates

In this section, we establish results concerning the rate of decay to zero of $w_{\alpha\beta}(\eta)$ as $\eta \to \pm \infty$. Specifically, we establish algebraic bounds on the rate of decay of $w_{\alpha\beta}(\eta)$ as $\eta \to \pm \infty$, and hence, determine that $w_{\alpha\beta} \in L_q(\mathbb{R})$ for each q > (1 - p)/2. From these bounds we may infer that the corresponding solution to [CP], say $u_{\alpha\beta} : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, satisfies $u(\cdot, t) \in L_q(\mathbb{R})$ for each $t \in [0, \infty)$ and q > (1 - p)/2. To complement the algebraic bounds, we also provide a rational asymptotic approximation to the decay rate of $w_{\alpha\beta}(\eta)$ as $\eta \to \pm \infty$, which, in fact suggests exponential decay as $\eta \to \pm \infty$.

To begin, observe that $w = w_{\alpha,\beta}(\eta)$ for $\eta \in \mathbb{R}$, via (6), satisfies

$$(e^{\frac{1}{4}\eta^2}w')' = H(w)e^{\frac{1}{4}\eta^2} \quad \forall \eta \in \mathbb{R}$$

It follows from two successive integrations, that

$$w'(\eta) = \beta e^{-\frac{1}{4}\eta^2} + e^{-\frac{1}{4}\eta^2} \int_0^{\eta} H(w(s)) e^{\frac{1}{4}s^2} ds \quad \forall \eta \in \mathbb{R}$$
(50)

whilst,

$$w(\eta) = \alpha + \int_0^{\eta} \beta e^{-\frac{1}{4}t^2} dt + \int_0^{\eta} e^{-\frac{1}{4}t^2} \int_0^t H(w(s)) e^{\frac{1}{4}s^2} ds dt \quad \forall \eta \in \mathbb{R}.$$
 (51)

We now have,

Proposition 8. Let $w : \mathbb{R} \to \mathbb{R}$ be a solution to (S) with zero-value $(\alpha, \beta) \in \overline{D}'_{c^*(p)}$. Suppose that

$$|w(\eta)| \leq \frac{c_1}{(1+|\eta|)^{\sigma}} \quad \forall \eta \in \mathbb{R}.$$

with $\sigma \ge 0$ and $c_1 > 0$ (independent of α and β). Then, there exists $c_2 > 0$, which depends on c_1 , σ and p, (independent of α and β) such that,

$$|w'(\eta)| \leq \frac{c_2}{(1+|\eta|)^{\sigma p+1}} \quad \forall \eta \in \mathbb{R}$$

Proof. We give a proof for $\eta \ge 0$; the result for $\eta < 0$ follows similarly. Observe that

$$|H(w(\eta))| = \left|\frac{1}{(1-p)}w(\eta) - |w(\eta)|^{p-1}w(\eta)\right| \le \frac{c_1^p}{(1+\eta)^{\sigma p}} \quad \forall \eta \in [0,\infty),$$
(52)

since, via Corollary 4, $|w(\eta)| < (1-p)^{\frac{1}{(1-p)}}$ for $\eta \in [0, \infty)$. Thus, via (50) and (52), we have,

$$|w'(\eta)| \le |\beta|e^{-\frac{1}{4}\eta^2} + c_1^p e^{-\frac{1}{4}\eta^2} \int_0^\eta \frac{1}{(1+s)^{\sigma p}} e^{\frac{1}{4}s^2} ds \quad \forall \eta \in [0,\infty).$$
(53)

Now, the second term on the right hand side of (53) is a non-negative continuous function for $\eta \in [0, \infty)$, with asymptotic form,

$$c_1^p e^{-\frac{1}{4}\eta^2} \int_0^\eta \frac{1}{(1+s)^{\sigma p}} e^{\frac{1}{4}s^2} ds \sim \frac{2c_1^p}{\eta^{\sigma p+1}} \text{ as } \eta \to \infty.$$

It follows that,

$$c_1^p e^{-\frac{1}{4}\eta^2} \int_0^\eta \frac{1}{(1+s)^{\sigma p}} e^{\frac{1}{4}s^2} ds \le \frac{4c_1^p}{\eta^{\sigma p+1}} \text{ as } \eta \to \infty.$$

We conclude that there exists a positive constant c_2 , depending upon c_1 , p, and σ , such that

$$|w'(\eta)| \leq \frac{c_2}{(1+\eta)^{\sigma p+1}} \quad \forall \eta \in [0,\infty),$$

as required.

We next have,

Proposition 9. Let $w : \mathbb{R} \to \mathbb{R}$ be a solution to (S) with zero-value $(\alpha, \beta) \in \overline{D}'_{c^*(p)}$. Then,

$$(w(\eta), w'(\eta)) \to (0,0) \text{ as } \eta \to \pm \infty,$$

and moreover,

$$|w'(\eta)| \leq \frac{c_2}{(1+|\eta|)} \quad \forall \eta \in \mathbb{R},$$

with $c_2 > 0$ dependent upon p (independent of α and β).

Proof. The first conclusion follows directly from Theorem 7. Additionally, it follows from Corollary 4 that

$$(w(\eta), w'(\eta)) \in \mathcal{H} \quad \forall \eta \in \mathbb{R}$$

and hence, it follows from Proposition 8 (with $\sigma = 0$, $c_1 = (1 - p)^{1/(1-p)}$) that

$$|w'(\eta)| \leq rac{c_2}{(1+|\eta|)} \quad \forall \eta \in \mathbb{R}$$

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as required.

We now demonstrate that every solution $w : \mathbb{R} \to \mathbb{R}$ to (S) with zero-value $(\alpha, \beta) \in \overline{D}'_{c^*(p)}$ decays to zero as $\eta \to \pm \infty$, with decay rate which is at least algebraic in η as $\eta \to \pm \infty$. In particular, we demonstrate that $w : \mathbb{R} \to \mathbb{R}$ is contained in $L^q(\mathbb{R})$ for any q > (1-p)/2. The proof is based on the decay bounds obtained in [8].

Theorem 10. Let $w : \mathbb{R} \to \mathbb{R}$ be a solution to (S) with zero-value $(\alpha, \beta) \in \overline{D}'_{c^*(p)}$. Then, for any $\epsilon > 0$, there exists $c_{1\epsilon}, c_{2\epsilon} > 0$ (dependent generally on α, β, p and ϵ) such that

$$|w(\eta)| < \frac{c_{1\epsilon}}{(1+|\eta|)^{\frac{2}{(1-\rho)}-\epsilon}} \quad \forall \eta \in \mathbb{R},$$
$$|w'(\eta)| < \frac{c_{2\epsilon}}{(1+|\eta|)^{\frac{(1+\rho)}{(1-\rho)}-\epsilon}} \quad \forall \eta \in \mathbb{R}.$$

Proof. We give a proof for $\eta \ge 0$; the argument for $\eta < 0$ follows similarly. Observe on multiplying (6) by $\eta^{-1}w(\eta)$, we have,

$$\frac{1}{\eta} \left[|w(\eta)|^{1+p} - \frac{(w(\eta))^2}{(1-p)} \right] = -\left[\frac{(w(\eta))^2}{4} + \frac{w(\eta)w'(\eta)}{\eta} \right]' + \frac{(w'(\eta))^2}{\eta} - \frac{w(\eta)w'(\eta)}{\eta^2}$$
(54)

for $\eta \in (0, \infty)$. Additionally, via Proposition 9, it follows that there exists $\eta_* \in (0, \infty)$ such that,

$$|w(\eta)| \le \left(\frac{2p(1-p)}{(1+p)}\right)^{\frac{1}{(1-p)}} \quad \forall \eta \in [\eta_*, \infty),$$
(55)

and for $F: [0, \infty) \to \mathbb{R}$ given by

$$F(\eta) = V(w(\eta), w'(\eta)) \quad \forall \eta \in [0, \infty),$$

that

$$0 \le F(\eta) \le \left(\frac{4(c(p))^{\frac{2}{(1+p)}}}{C(p)}\right)^{(1+p)/(1-p)} \quad \eta \in [\eta_*, \infty),$$
(56)

where

$$c(p) = \frac{1}{(1+p)} - \frac{1}{2}$$
, and $C(p) = \frac{2(1+p)}{(1-p)} + 1.$ (57)

Thus, it follows from (54) that

$$\frac{F(\eta)}{\eta} = \frac{(w'(\eta))^2}{2\eta} + \frac{1}{\eta} \left[-\frac{(w(\eta))^2}{2(1-p)} + \frac{|w(\eta)|^{1+p}}{(1+p)} \right] \\
\leq \frac{(w'(\eta))^2}{2\eta} + \frac{1}{\eta} \left[-\frac{(w(\eta))^2}{(1-p)} + |w(\eta)|^{1+p} \right] \\
= \frac{3(w'(\eta))^2}{2\eta} - \left[\frac{(w(\eta))^2}{4} + \frac{w(\eta)w'(\eta)}{\eta} \right]' - \frac{w(\eta)w'(\eta)}{\eta^2},$$
(58)

for $\eta \in [\eta_*, \infty)$. Since $F(\eta) \ge 0$ for all $\eta \in [\eta^*, \infty)$, together with the decay estimates in Proposition 9, it follows that we may integrate inequality (58) from $\eta (\ge \eta^*)$ to l, and then allow $l \to \infty$, to obtain,

$$\int_{\eta}^{\infty} \frac{F(t)}{t} dt \le \frac{(w(\eta))^2}{4} + \frac{2}{\eta} \sup_{t \ge \eta} |w(t)w'(t)| + \frac{3}{2} \int_{\eta}^{\infty} \frac{(w'(t))^2}{t} dt$$
(59)

for $\eta \in [\eta_*, \infty)$. We also note, that since, via Corollary 4, $|w(\eta)| < (1-p)^{1/(1-p)}$, we have,

$$F(\eta) \ge |w(\eta)|^{1+p} c(p) \ge 0,$$
 (60)

for $\eta \in [\eta_*, \infty)$. It therefore follows from (59) and (60) that

$$0 \le \int_{\eta}^{\infty} \frac{F(t)}{t} dt \le \frac{1}{4} \left(\frac{F(\eta)}{c(p)} \right)^{\frac{2}{(1+p)}} + \frac{2}{\eta} \sup_{t \ge \eta} |w(t)w'(t)| + \frac{3}{2} \int_{\eta}^{\infty} \frac{(w'(t))^2}{t} dt$$
(61)

for $\eta \in [\eta_*, \infty)$. We observe that the right hand side of (61) is uniformly bounded for $\eta \in [\eta_*, \infty)$ via Proposition 9. Now suppose that there exists k > 0 such that

$$F(\eta) \le \frac{k}{\eta^{\sigma}} \quad \forall \eta \in [\eta_*, \infty)$$
(62)

for some $\sigma \ge 0$ (note that (62) holds when $\sigma = 0$ via Proposition 9). Then, via (60), it follows that there exists $c_1 > 0$ such that

$$|w(\eta)| \le \frac{c_1}{\eta^{\frac{\sigma}{(1+p)}}} \quad \forall \eta \in [\eta_*, \infty)$$
(63)

and so, via Proposition 8, there exists $c_2 > 0$ such that

$$|w'(\eta)| \le \frac{c_2}{\eta^{\frac{\sigma_p}{(1+p)}+1}} \quad \forall \eta \in [\eta_*, \infty).$$
(64)

Thus, it follows from (61)-(64) and (56), that there exists $c_3, c_4, c_5 > 0$ such that

$$\int_{\eta}^{\infty} \frac{F(t)}{t} dt \leq \frac{1}{4} \left(\frac{F(\eta)}{c(p)} \right)^{\frac{2}{(1+p)}} + \frac{c_3}{\eta^{\sigma+2}} + \frac{c_4}{\eta^{\frac{2\sigma p}{(1+p)}+2}} \leq \frac{F(\eta)}{C(p)} + \frac{c_5}{\eta^{\frac{2\sigma p}{(1+p)}+2}}$$
(65)

for $\eta \in [\eta_*, \infty)$. Upon setting $G : [\eta_*, \infty) \to \mathbb{R}$ to be

$$G(\eta) = \int_{\eta}^{\infty} \frac{F(t)}{t} dt \quad \forall \eta \in [\eta_*, \infty),$$

it follows from (65) that G satisfies,

$$\left(t^{C(p)}G(t)\right)' \leq \frac{c_6}{t^{\frac{2\sigma p}{(1+p)}+3-C(p)}} \quad \forall t \in [\eta_*, \infty),$$
(66)

with $c_6 > 0$ constant. An integration of (66) gives

$$G(\eta) \leq \frac{c_7}{\eta^{\frac{2\sigma_p}{(1+p)}+2}} + \frac{c_8}{\eta^{C(p)}} \quad \forall \eta \in [\eta_*, \infty),$$
(67)

with $c_7, c_8 > 0$ constants. Also, recalling, via Lemma 6, that $F(\eta)$ is non-increasing on $[\eta^*, \infty)$, we have,

$$G(\eta) \ge \int_{\eta}^{2\eta} \frac{F(t)}{t} dt \ge \frac{1}{2} F(2\eta), \quad \forall \eta \in [\eta_*, \infty).$$
(68)

Thus, it follows from (67) and (68) that there exist constants c_9 , $c_{10} > 0$ such that

$$F(\eta) \leq \frac{c_9}{\eta^{\frac{2\sigma p}{(1+p)}+2}} + \frac{c_{10}}{\eta^{C(p)}} \quad \forall \eta \in [\eta_*, \infty).$$
(69)

Since (62) holds for $\sigma = 0$, it follows that there exists sequences $\{\sigma_n\}_{n \in \mathbb{N}}$ and $\{k_n\}_{n \in \mathbb{N}}$ given by

$$\sigma_{1} = 0, \quad \sigma_{n+1} = \min\left\{\frac{2\sigma_{n}p}{(1+p)} + 2, \ C(p)\right\}$$
(70)
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such that

$$F(\eta) \le \frac{k_n}{\eta^{\sigma_n}} \quad \forall \eta \in [\eta_*, \infty).$$
(71)

We obtain from (70) and (57) that,

$$\sigma_n = \frac{2(1+p)}{(1-p)} - \frac{4p}{(1-p)} \left(\frac{2p}{(1+p)}\right)^{n-2} \quad \forall n \in \mathbb{N}$$

and hence σ_n is increasing with

$$\sigma_n \to \frac{2(1+p)}{(1-p)} \quad \text{as } n \to \infty.$$
 (72)

Therefore it follows, via (60) and (70)-(72), that for each $\epsilon > 0$ there exists $c_{1\epsilon} > 0$ such that

$$|w(\eta)| \le \frac{c_{1\epsilon}}{(1+\eta)^{\frac{2}{(1-p)}-\epsilon}} \quad \forall \eta \in [0,\infty),$$
(73)

recalling that $w(\eta)$ is bounded on $[0, \eta^*]$. The bound on $|w'(\eta)|$ follows immediately from (73) and Proposition 8. \Box

The algebraic bounds in Theorem 10 are the tightest decay rates we have been able to establish rigorously. However, the following asymptotic argument indicates that, in fact, $w = w_{\alpha,\beta}(\eta)$ decays exponentially in η as $|\eta| \to \infty$, accompanied by rapid oscillatory behaviour. To this end, we now consider the asymptotic structure of $w = w_{\alpha,\beta}(\eta)$ as $\eta \to \infty$, with the same structure following as $\eta \to -\infty$. Now, for $\eta >> 1$, then $w = w_{\alpha,\beta}(\eta)$ satisfies,

$$w'' + \frac{1}{2}\eta w' + w|w|^{p-1} - \frac{1}{(1-p)}w = 0 \quad \eta >> 1$$
(74)

$$w(\eta), w'(\eta) \to 0 \quad \text{as } \eta \to \infty,$$
 (75)

via (6) and Proposition 9. On using (75), the dominant form of (74) when $\eta >> 1$ is

$$w'' + w|w|^{p-1} = 0. (76)$$

Every solution to (76) is periodic and may be written (up to translation in η) as,

$$w(\eta, a) = aW\left(a^{-\frac{1}{2}(1-p)}\eta\right), \quad \forall \eta \in \mathbb{R},$$
(77)

where $a \in \mathbb{R}^+$ is a parameter and $W : \mathbb{R} \to \mathbb{R}$ is that unique periodic function which satisfies the problem,

$$W'' + W|W|^{p-1} = 0, \quad \zeta \in \mathbb{R}$$
 (78)

$$W(0) = 1, \quad W'(0) = 0.$$
 (79)

The period of $W(\zeta)$ is given by

$$T(p) = 2^{3/2} (1+p)^{1/2} \int_0^1 \frac{d\lambda}{(1-\lambda^{(1+p)})^{1/2}}$$
(80)

whilst,

$$W(\zeta) = -W\left(\frac{1}{2}T(p) - \zeta\right) = W(-\zeta) \quad \forall \zeta \in \mathbb{R}.$$
(81)

Via an integration, the solution to (78)-(79) satisfies

$$\frac{(W'(\eta))^2}{2} + \frac{|W(\eta)|^{1+p}}{(1+p)} = \frac{1}{(1+p)} \quad \forall \eta \in \mathbb{R},$$

which represents a periodic orbit in the (W, W') phase plane, as illustrated in Figure 3.2. It follows from (77) that $w(\eta, a)$ has amplitude a > 0 and period

$$T_a(p) = a^{\frac{1}{2}(1-p)}T(p).$$
(82)



Figure 2: Phase paths for solutions to (78)-(79) for $p_k = (0.1)k$ for k = 1...9. Here the phase path for p_k encloses the phase path for p_{k+1} for k = 1...8.

For any fixed $a \in \mathbb{R}^+$, (77) cannot represent the asymptotic structure to (74) and (75) since W is periodic. The remaining terms in (74) must induce decay as $\eta \to \infty$. However, we observe from (82) that the oscillations in $w(\eta, a)$ becomes increasingly rapid as the amplitude $a \to 0^+$. This suggests that we seek the asymptotic structure of (74)-(75) as $\eta \to \infty$ in the form,

$$w(\eta) \sim a(\eta) W(a(\eta)^{-\frac{1}{2}(1-p)}\eta) \text{ as } \eta \to \infty,$$
(83)

with $a(\eta) > 0$ and,

$$a(\eta), a'(\eta) \to 0 \text{ as } \eta \to \infty.$$
 (84)

Now, the rate of change of amplitude of oscillation in (83), $a'(\eta)$, approaches zero as $\eta \to \infty$, whilst the frequency of oscillation becomes unbounded as $\eta \to \infty$. We can thus use an averaging approach to determine an evolution equation for the amplitude $a(\eta)$ as $\eta \to \infty$. We substitute (83) into (6) and make use of (78). We then integrate the resulting ordinary differential equation over *one* period of $W(\cdot)$, over which, we may hold *a* fixed. We obtain the leading order amplitude equation as,

$$a'' + \frac{1}{2}\eta a' - \frac{1}{(1-p)}a = 0, \quad \eta >> 1,$$
(85)

$$a(\eta), a'(\eta) \to 0 \text{ as } \eta \to \infty.$$
 (86)

The linear ordinary differential equation (85) has two basis functions $a_+ : \mathbb{R} \to \mathbb{R}$ and $a_- : \mathbb{R} \to \mathbb{R}$ which have

$$a_+(\eta) \sim \eta^{-\left(1+\frac{2}{(1-p)}\right)} e^{-\frac{1}{4}\eta^2}, \quad a_-(\eta) \sim \eta^{\frac{2}{(1-p)}} \quad \text{as } \eta \to \infty.$$

It follows that

$$a(\eta) \sim A_{\infty} \eta^{-\left(1 + \frac{2}{(1-p)}\right)} e^{-\frac{1}{4}\eta^2} \quad \text{as } \eta \to \infty,$$
(87)

with A_{∞} being a positive globally determined constant dependent, in general, on α , β and p. Thus, from (83), we have

$$w_{\alpha,\beta}(\eta) \sim a(\eta) W(a(\eta)^{-\frac{1}{2}(1-p)}\eta) \text{ as } \eta \to \infty,$$
(88)

with, $\alpha(\eta)$ having the asymptotic form (87) as $\eta \to \infty$. The same argument leads to the same (up to the constant A_{∞}) asymptotic structure as $\eta \to -\infty$. As a consequence of (87) and (88), we anticipate that $w_{\alpha\beta}(\eta)$ decays to zero at a Gaussian rate as $|\eta| \to \infty$, whilst oscillating about zero with a local frequency which increases at a Gaussian rate as $|\eta| \to \infty$. This indicates that, in fact, $w_{\alpha\beta} \in L^q(\mathbb{R})$ for any q > 0.

3.3. Localized Solutions to [CP]

Following Corollary 4 and Theorem 7, for each $(\alpha,\beta) \in \overline{D}'_{c^*(p)}$, we have constructed a non-trivial, localized, global solution $u_{\alpha\beta} : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ to [CP], namely,

$$u_{\alpha,\beta}(x,t) = \begin{cases} t^{\frac{1}{(1-p)}} w_{\alpha,\beta}\left(\frac{x}{t^{1/2}}\right) &, (x,t) \in \mathbb{R} \times (0,\infty) \\ 0 &, (x,t) \in \mathbb{R} \times \{0\}. \end{cases}$$
(89)

With this two parameter family of solutions to [CP], each solution is distinct, and is not a spatial translate of any other solution in the family. However, we observe that $u_{\alpha,\beta}(x - x_0, t)$ is also a global solution to [CP] for any fixed $x_0 \in \mathbb{R}$. A trivial calculation from (89) establishes that

$$(u_{\alpha,\beta})_{x}(x,t) = t^{\frac{1}{(1-p)} - \frac{1}{2}} w'_{\alpha,\beta}\left(\frac{x}{t^{1/2}}\right),\tag{90}$$

$$(u_{\alpha,\beta})_t(x,t) = \frac{1}{(1-p)} t^{\frac{1}{(1-p)}-1} \left(w_{\alpha,\beta}\left(\frac{x}{t^{1/2}}\right) - \frac{1}{2}(1-p)\left(\frac{x}{t^{1/2}}\right) w'_{\alpha,\beta}\left(\frac{x}{t^{1/2}}\right) \right),\tag{91}$$

for $(x, t) \in \mathbb{R} \times (0, \infty)$, whilst from (1),

$$(u_{\alpha\beta})_{xx}(x,t) = (u_{\alpha\beta})_t(x,t) - (u_{\alpha\beta}|u_{\alpha\beta}|^{p-1})(x,t),$$
(92)

for $(x, t) \in \mathbb{R} \times (0, \infty)$. It then follows immediately from Theorem 7 that,

$$(u_{\alpha,\beta})_x, (u_{\alpha,\beta})_t, (u_{\alpha,\beta})_{xx} \to 0 \text{ as } t \to 0^+ \text{ uniformly for } x \in \mathbb{R},$$

and so, in fact,

$$u_{\alpha,\beta} \in L^{\infty}(\mathbb{R} \times [0,T]) \cap C(\mathbb{R} \times [0,T]) \cap C^{2,1}(\mathbb{R} \times [0,T]).$$

$$(93)$$

It follows from (93) that for each $(\alpha,\beta) \in \overline{D}'_{c^*(p)}$, and any $\tau > 0$, then $u^{\tau}_{\alpha,\beta} : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ such that

$$u_{\alpha,\beta}^{\tau}(x,t) = \begin{cases} (t-\tau)^{\frac{1}{(1-p)}} w_{\alpha,\beta}\left(\frac{x}{(t-\tau)^{1/2}}\right) &, (x,t) \in \mathbb{R} \times (\tau,\infty) \\ 0 &, (x,t) \in \mathbb{R} \times [0,\tau] \end{cases}$$

is also a non-trivial, localized, global solution to [CP]. Finally, we observe, via Theorem 10 that for each q > (1-p)/2, then $u_{\alpha,\beta}(\cdot,t) \in L^q(\mathbb{R} \text{ for each } t \ge 0$. Moreover, (87) and (88) suggest that the localization is Gaussian in x for each t > 0.

4. Heteroclinic Connections

In this section we establish the existence of at least one heteroclinic connection for (S) from the equilibrium point $(-(1-p)^{1/(1-p)}, 0)$ to the equilibrium point $((1-p)^{1/(1-p)}, 0)$.

4.1. Existence

We first consider solutions to the problem (S) for $\eta \in [0, \infty)$ and which remain in the region $\Omega \subset \mathbb{R}^2$, given as

$$\Omega = \left\{ (x, y) : 0 < x < (1 - p)^{1/(1 - p)}, y > 0 \right\}$$
(94)

with boundary $\partial \Omega = \overline{\Omega} \setminus \Omega$. We also define the following subset of $\partial \Omega$, namely,

$$\partial \Omega_1 = \{(x, y) : x = 0, y > 0\}.$$
(95)

Specifically, we consider (S) for $\eta \in [0, \infty)$ and demonstrate that there exists a solution $(x, y) : [0, \infty) \to \overline{\Omega}$ with zero-value $(0,\beta) \in \partial\Omega_1$ and which satisfies

$$(x(\eta), y(\eta)) \in \Omega \quad \forall \eta \in (0, \infty),$$
(96)

$$(x(\eta), y(\eta)) \to ((1-p)^{1/(1-p)}, 0) \text{ as } \eta \to \infty.$$
 (97)

To begin with, it is readily established that for each zero-value $(0,\beta) \in \partial\Omega_1$, then (S) has a local solution $(x, y) : [0, \delta] \to \mathbb{R}^2$ (for some $\delta > 0$). Moreover, $(x(\eta), y(\eta)) \in \Omega$ for $\eta \in (0, \delta]$, and $x(\eta)$ is monotone increasing whilst $y(\eta)$ is monotone decreasing, with $\eta \in (0, \delta]$. It is then straightforward to establish that $(x(\eta), y(\eta))$ can be *uniquely* continued beyond $\eta = \delta$ and must satisfy one of the following three possibilities:

- (i) There exists $\eta_{\beta} > 0$ such that $(x(\eta), y(\eta)) \in \Omega$ for all $\eta \in (0, \eta_{\beta})$ and $(x(\eta_{\beta}), y(\eta_{\beta})) = ((1-p)^{\frac{1}{1-p}}, y_{\beta})$ with $0 < y_{\beta} < \beta$, whilst $x'(\eta_{\beta}) = y_{\beta} > 0$, and so there exists $\epsilon_{\beta} > 0$ such that $(x(\eta), y(\eta)) \notin \overline{\Omega} \cup (\{0\} \times \mathbb{R})$ for $\eta \in (\eta_{\beta}, \eta_{\beta} + \epsilon_{\beta}]$.
- (ii) There exists η_β > 0 such that (x(η), y(η)) ∈ Ω for all η ∈ (0, η_β) and (x(η_β), y(η_β)) = (x_β, 0) with 0 < x_β < (1-p)^{1/2}/_{1-p}, whilst y'(η_β) < 0 and so there exists ε_β > 0 such that (x(η), y(η)) ∉ Ω̄∪({0}×ℝ) for η ∈ (η_β, η_β + ε_β].
 (iii) (x(η), y(η)) ∈ Ω for all η ∈ (0,∞) and (x(η), y(η)) → ((1-p)^{1/2}/_{1-p}, 0) as η → ∞.

Our aim now is to obtain a uniqueness result for (S) with zero-value in
$$\partial \Omega_1$$
, and from this a continuous dependence
result. This is non-trivial, since **Q** in (14) is not locally Lipschitz continuous in any neighborhood of $(0,\beta) \in \partial \Omega_1$,
and so standard uniqueness and continuous dependence theory fail to apply. To begin with, we provide a local a priori
bound for any solution of (S) with zero-value $(0,\beta) \in \partial \Omega_1$.

Proposition 11. Let $(x, y) : [0, \eta_{\beta}] \to \mathbb{R}^2$ be any solution to (S) with zero-value $(0, \beta) \in \partial \Omega_1$ and which satisfies either *case (i) or (ii). Then,*

$$\eta_{\beta} > \min\left\{ \left(\frac{2}{\beta}\right) \left(m_{H} + \sqrt{m_{H}^{2} + \frac{(\beta)^{2}}{2}} \right), \ \frac{(1-p)^{1/(1-p)}}{\beta} \right\} = \eta^{*}$$
(98)

with

$$m_{H} = \inf_{\lambda \in [0, (1-p)^{1/(1-p)}]} H(\lambda),$$
(99)

Proof. Let $(x, y) : [0, \eta_{\beta}] \to \mathbb{R}^2$ be any solution to (S) with zero-value $(0, \beta) \in \partial \Omega_1$, and which satisfies either case (i) or case (ii). Suppose that $\eta_{\beta} \le \eta^*$. Since $(x(\eta), y(\eta)) \in \Omega$ for all $\eta \in (0, \eta_{\beta})$, it follows from (10) that

$$\beta + m_H \eta - \frac{\beta}{4} \eta^2 < y(\eta) < \beta \quad \forall \eta \in (0, \eta_\beta].$$
(100)

However, $\eta_{\beta} \leq \eta^*$ and so, via (100),

$$\frac{\beta}{2} < y(\eta) < \beta \quad \forall \eta \in (0, \eta_{\beta}].$$
(101)

An integration of (9) using (101), then gives,

$$\frac{\beta\eta}{2} < x(\eta) < \beta\eta \quad \forall \eta \in (0, \eta_{\beta}].$$
(102)

It finally follows from (101) and (102), since $\eta_{\beta} \leq \eta^*$, that,

$$x(\eta_{\beta}) < \beta \eta_{\beta} \leq \beta \eta^* \leq (1-p)^{\frac{1}{1-p}}, \quad y(\eta_{\beta}) > \frac{\beta}{2},$$

and so $(x(\eta_{\beta}), y(\eta_{\beta})) \in \Omega$, which is a contradiction. We conclude that $\eta_{\beta} > \eta^*$, as required.

Therefore, we have,

Corollary 12. Let $(x, y) : [0, \eta^*] \to \mathbb{R}^2$ be a solution to (S) with zero-value $(0, \beta) \in \partial \Omega_1$ with η^* given by (98). Then,

$$\frac{\beta\eta}{2} < x(\eta) < (1-p)^{1/(1-p)}, \quad \frac{\beta}{2} < y(\eta) < \beta \quad \forall \eta \in [0,\eta^*],$$

Proof. For cases (i) and (ii), the result follows from Proposition 11, with case (iii) following immediately. \Box

The a priori bounds in Corollary 12, allow us to establish the following local uniqueness result for (S) with zero-value $(0,\beta) \in \partial\Omega_1$. The proof is based on the uniqueness argument in [2].

Proposition 13. The problem (S) with zero-value $(0,\beta) \in \partial \Omega_1$ has at most one solution on $[0,\eta^*]$, with $\eta^* > 0$ given by (98).

Proof. To begin, fix $(0,\beta) \in \partial\Omega_1$. Suppose that $(x,y), (x^*, y^*) : [0,\eta^*] \to \mathbb{R}^2$ are solutions to (S) with zero-value $(0,\beta)$. It follows from Corollary 12 that

$$(x(\eta), y(\eta)), (x^*(\eta), y^*(\eta)) \in \overline{\Omega} \quad \forall \eta \in [0, \eta^*],$$
(103)

whilst from Corollary 12,

$$|x(\eta) - x^{*}(\eta)| < (1-p)^{1/(1-p)}, \quad |y(\eta) - y^{*}(\eta)| < \beta \quad \forall \eta \in [0, \eta^{*}].$$
(104)

Additionally, we observe that for $(X, Y) \in [0, (1-p)^{1/(1-p)}] \times [0,\beta]$, then

$$X + X^{p} + Y < \left(2 + \beta^{1-p}\right) (X + Y)^{p},$$
(105)

since 0 . Now, via (9) and (10) respectively, we have,

$$|x(\eta) - x^{*}(\eta)| \leq \int_{0}^{\eta} |y(s) - y^{*}(s)| ds$$
(106)

$$|y(\eta) - y^{*}(\eta)| \le \int_{0}^{\eta} \left(\frac{1}{(1-p)} |x(s) - x^{*}(s)| + |x(s) - x^{*}(s)|^{p} + \frac{s}{2} |y(s) - y^{*}(s)| \right) ds \tag{107}$$

for all $\eta \in [0, \eta^*]$. We next introduce $v : [0, \eta^*] \to \mathbb{R}$ as,

$$v(\eta) = |x(\eta) - x^{*}(\eta)| + |y(\eta) - y^{*}(\eta)| \quad \forall \eta \in [0, \eta^{*}].$$
(108)

Therefore, via (103)-(108), it follows that

$$\begin{aligned} v(\eta) &\leq \int_{0}^{\eta} \left(\frac{1}{(1-p)} |x(s) - x^{*}(s)| + |x(s) - x^{*}(s)|^{p} + \left(\frac{s}{2} + 1\right) |y(s) - y^{*}(s)| \right) ds \\ &\leq \int_{0}^{\eta} \frac{1}{(1-p)} \left(\frac{\eta^{*}}{2} + 1 \right) \left(|x(s) - x^{*}(s)| + |x(s) - x^{*}(s)|^{p} + |y(s) - y^{*}(s)| \right) ds \\ &\leq \int_{0}^{\eta} \frac{1}{(1-p)} \left(\frac{\eta^{*}}{2} + 1 \right) \left(2 + \beta^{1-p} \right) \left(v(s) \right)^{p} ds \end{aligned} \tag{109}$$

for all $\eta \in [0, \eta^*]$, where the final inequality is due to (104) and (105). Also, via Corollary 12 and (98), η^* is dependent on *p* and β only, and hence, it follows from (109) that

$$v(\eta) \le \int_0^{\eta} K(p,\beta) \left(v(s)\right)^p ds \tag{110}$$

for all $\eta \in [0, \eta^*]$, where the constant $K(p, \beta)$ is given by,

$$K(p,\beta) = \frac{1}{(1-p)} \left(\frac{\eta^*}{2} + 1\right) (2+\beta^{1-p}).$$

We now introduce the function $\overline{H} : [0, \eta^*] \to \overline{\mathbb{R}}_+$ given by,

$$\bar{H}(\eta) = \int_0^{\eta} K(p,\beta) \left(v(s) \right)^p ds \quad \forall \eta \in [0,\eta^*].$$
(111)

It follows from (111) that \overline{H} is non-negative, non-decreasing and differentiable on $[0, \eta^*]$, and via (110), satisfies

$$(\bar{H}(s))' \le K(p,\beta)(\bar{H}(s))^p \quad \forall s \in [0,\eta^*].$$

$$(112)$$

Upon integrating (112) from 0 to η , we obtain

$$\bar{H}(\eta) \le \left((1-p)K(p,\beta)\eta \right)^{1/(1-p)} \quad \forall \eta \in [0,\eta^*]$$
(113)

and it follows from (113), (111) and (110) that

$$v(\eta) \le \delta \quad \forall \eta \in [0, \eta_{\delta}], \tag{114}$$

where $\delta > 0$ is chosen sufficiently small so that

$$\eta_{\delta} = \frac{\delta^{1-p}}{(1-p)K(p,\beta)} < \eta^*.$$

Now, from Corollary 12, we have

$$\min\{x^*(\eta), x(\eta)\} \ge \frac{\beta\eta}{2} \quad \forall \eta \in [0, \eta^*].$$
(115)

Moreover, it follows from (14), (115) and the mean value theorem, that there exists $\theta(s) \ge \min\{x^*(s), x(s)\}$, for which,

$$\begin{aligned} |Q_{2}(x(s), y(s), s) - Q_{2}(x^{*}(s), y^{*}(s), s)| \\ &\leq \frac{1}{(1-p)} |x(s) - x^{*}(s)| + |x(s)^{p} - x^{*}(s)^{p}| + \frac{s}{2} |y(s) - y^{*}(s)| \\ &\leq \frac{1}{(1-p)} |x(s) - x^{*}(s)| + p(\theta(s))^{p-1} |x(s) - x^{*}(s)| + \frac{\eta^{*}}{2} |y(s) - y^{*}(s)| \\ &\leq \left(\frac{1}{(1-p)} + p\left(\frac{\beta s}{2}\right)^{p-1}\right) |x(s) - x^{*}(s)| + \frac{\eta^{*}}{2} |y(s) - y^{*}(s)| \\ &\leq \left(\frac{1}{(1-p)} + p\left(\frac{\beta s}{2}\right)^{p-1} + \frac{\eta^{*}}{2}\right) v(s) \end{aligned}$$
(116)

for all $s \in (0, \eta^*]$. Now, via (9), (10), (14), (105), (116) and (114), we have,

$$\begin{aligned} v(\eta) &\leq \int_{0}^{\eta} \left(|Q_{1}(x(s), y(s), s) - Q_{1}(x^{*}(s), y^{*}(s), s)| + |Q_{2}(x(s), y(s), s) - Q_{2}(x^{*}(s), y^{*}(s), s)| \right) ds \\ &\leq \int_{0}^{\eta_{\delta}} K(p, \beta) (v(s))^{p} ds + \int_{\eta_{\delta}}^{\eta} \left(1 + \frac{1}{(1-p)} + p \left(\frac{\beta s}{2}\right)^{p-1} + \frac{\eta^{*}}{2} \right) v(s) ds \\ &\leq \frac{\delta}{(1-p)} + \int_{\eta_{\delta}}^{\eta} \left(1 + \frac{1}{(1-p)} + p \left(\frac{\beta s}{2}\right)^{p-1} + \frac{\eta^{*}}{2} \right) v(s) ds \end{aligned}$$
(117)

for all $\eta \in [\eta_{\delta}, \eta^*]$. An application of Gronwall's Lemma [26, Corollary 6.2] to (117), gives

$$v(\eta) \le \frac{\delta}{(1-p)} e^{\left(\eta^* \left(1 + \frac{1}{(1-p)} + \left(\frac{\beta\eta^*}{2}\right)^{p-1} + \frac{\eta^*}{2}\right)\right)}$$
(118)

for all $\eta \in [\eta_{\delta}, \eta^*]$. Since v is non-negative and η^* is independent of δ , it follows from (118) and (114), upon letting $\delta \to 0$, that

$$v(\eta) = 0 \quad \forall \eta \in [0, \eta^*]. \tag{119}$$

Finally, it follows from (119) and (108) that

$$(x(\eta), y(\eta)) = (x^*(\eta), y^*(\eta)) \quad \forall \eta \in [0, \eta^*],$$

as required.

We can now state the following uniqueness result.

Lemma 14. For each $(0,\beta) \in \partial\Omega_1$ then (S) with zero-value $(0,\beta)$ has exactly one solution $(x,y) : I \to \mathbb{R}^2$. This solution satisfies exactly one of the cases: (i) (with $I = [0, \eta_\beta + \epsilon_\beta]$), (ii) (with $I = [0, \eta_\beta + \epsilon_\beta]$) or (iii) (with $I = [0, \infty)$).

Proof. We have established earlier that for each $(0,\beta) \in \partial\Omega_1$, then (S) with zero-value $(0,\beta)$ has at least one solution $(x, y) : I \to \mathbb{R}^2$, and that the solution satisfies one of the cases (i)-(iii). It follows from Proposition 13 that this solution is unique for $\eta \in [0, \eta^*]$, (with η^* depending only upon β and p) and, moreover, in whichever case of (i)-(iii) it falls, that $(x(\eta), y(\eta)) \notin \{(0, \lambda) : \lambda \in \mathbb{R}\}$ for any $\eta \in I \setminus [0, \eta^*]$. Repeated application of the classical uniqueness theorem [24, Chapter 1, Theorem 2.2] then completes the uniqueness result for $\eta \in I \setminus [0, \eta^*]$.

We immediately obtain a continuous dependence result for solutions of (S) with zero-value in $\partial \Omega_1$, namely,

Corollary 15. Let $(0,\beta^*) \in \partial \Omega_1$ and suppose that the unique solution to (S) with zero-value $(0,\beta^*)$, say $(x^*, y^*) : I \to \mathbb{R}$, satisfies case (i) or (ii), with $I = [0, \eta_{\beta^*} + \epsilon_{\beta^*}]$. Then, given $\epsilon' > 0$, there exists $\delta' > 0$ such that for all $\beta > 0$ satisfying $|\beta - \beta^*| < \delta'$, the corresponding unique solution to (S) with zero-value $(0,\beta)$, say $(x,y) : I' \to \mathbb{R}$, has I' = I and satisfies the corresponding case (i) or (ii), with,

$$|x(\eta) - x^*(\eta)| + |y(\eta) - y^*(\eta)| < \epsilon' \quad \forall \eta \in I.$$

Proof. We first recall that (for a suitable choice of β^*) then

$$|x^*(\eta)| \leq (1-p)^{\frac{1}{1-p}} + 1, \quad |y^*(\eta)| \leq \beta^* + 1 \quad \forall \eta \in [0, \eta_{\beta^*} + \epsilon_{\beta^*}],$$

and, via (14), that $\mathbf{Q}(x, y, \eta)$ is continuous (and therefore bounded) on the rectangle

$$R = \left\{ (x, y, \eta) : |x| \le (1 - p)^{\frac{1}{1 - p}} + 1, \quad |y| \le \beta^* + 1, \quad 0 \le \eta \le \eta_{\beta^*} + \epsilon_{\beta^*} \right\}.$$

The *uniqueness result* in Lemma 14 then allows for an application of the result [24, Theorem 4.3, pp. 59] which completes the proof. \Box

It is now convenient to introduce the three sets E_1 , E_2 and E_3 , where

$$E_1 = \{(0,\beta) \in \partial \Omega_1: \text{ the unique solution to (S) with zero-value } (0,\beta) \text{ satisfies case (i)} \},\$$

with E_2 and E_3 defined similarly for cases (ii) and (iii) respectively. It follows from Lemma 14 that

$$E_i \cap E_j = \emptyset \text{ for } i, j = 1, 2, 3 \text{ with } i \neq j,$$
(120)

whilst

$$E_1 \cup E_2 \cup E_3 = \partial \Omega_1. \tag{121}$$

We now establish that E_1 and E_2 are both nonempty.

Proposition 16. The set E_1 is non-empty and is such that $(0,\beta) \in E_1$ for each

$$\beta > \sqrt{2\left(\left((1-p)^{1/(1-p)} - m_H\right)^2 - m_H^2\right)},\tag{122}$$

with m_H given by (99).

Proof. Let $(x, y) : I \to \mathbb{R}^2$ be the unique solution to (S) with zero-value $(0,\beta) \in \partial\Omega_1$ and β satisfying (122). Since $(x(\eta), y(\eta)) \in \overline{\Omega}$ for all $\eta \in I'$ (where $I' = [0, \eta_\beta]$ for cases (i) and (ii), and $I' = [0, \infty)$ for case (iii)) then, via (9) and (10), we have,

$$\frac{\beta}{2} \le y(\eta) \le \beta, \quad x(\eta) \ge \frac{\beta\eta}{2} \quad \forall \eta \in [0, \bar{\eta}_{\beta}],$$
(123)

with,

$$\bar{\eta}_{\beta} = \begin{cases} \min\{\eta_{\beta}, \eta_{\beta}'\} &: \text{ cases (i) and (ii)} \\ \eta_{\beta}' &: \text{ case (iii)} \end{cases}$$
(124)

and

$$\eta_{\beta}' = \frac{2}{\beta} \left(m_H + \sqrt{m_H^2 + \frac{\beta^2}{2}} \right).$$

Now suppose case (iii) occurs, then $(x(\eta'_{\beta}), y(\eta'_{\beta})) \in \Omega$. However,

$$x(\eta'_{\beta}) \ge \frac{\beta \eta'_{\beta}}{2} = m_H + \sqrt{m_H^2 + \frac{\beta^2}{2}} > (1-p)^{\frac{1}{1-p}}$$

via (123) and (122), and we arrive at a contradiction. We can therefore eliminate case (iii). Next suppose case (ii) occurs. It follows from (123)₂ and (124) that $\eta_{\beta} \le \eta'_{\beta}$, and so $\bar{\eta}_{\beta} = \eta_{\beta}$. Thus, via (123)₁,

$$y(\eta_{\beta}) \geq \frac{\beta}{2} > 0.$$

However, in case (ii), $y(\eta_{\beta}) = 0$, and we arrive at a contradiction. We conclude finally that case (i) must occur, as required.

We can also establish a similar result for E_2 .

Proposition 17. The set E_2 is non-empty and is such that $(0,\beta) \in E_2$ for each

$$0 < \beta < \sqrt{\frac{(1-p)^{2/(1-p)}}{(1+p)}}.$$
(125)

Proof. It follows from (16)-(21) that for β satisfying the inequality (125), then $(0,\beta) \in D_{c^*(p)}$. It then follows from Corollary 4 that (S) with zero-value $(0,\beta)$ has a global solution which lies in D_{c_β} for all $\eta \in (0,\infty)$ with $c_\beta = V(0,\beta) < c^*(p)$, and so the solution to (S) in $\eta \ge 0$ must satisfy case (ii). Therefore, $(0,\beta) \in E_2$, as required.

We next establish that both E_1 and E_2 are open subsets of $\partial \Omega_1$.

Proposition 18. The sets E_1 and E_2 are open subsets of $\partial \Omega_1$.

Proof. We will prove the result for E_1 . The proof for E_2 is similar. Let $(0,\beta^*) \in E_1$. Then, via Lemma 14, (S) with zero-value $(0,\beta^*)$ has a unique solution $(x^*, y^*) : [0, \eta_{\beta^*} + \epsilon_{\beta^*}] \to \mathbb{R}^2$, with

$$(x^*(\eta), y^*(\eta)) \in \Omega \quad \forall \eta \in (0, \eta_{\beta^*})$$
(126)

and

$$(x^{*}(\eta_{\beta^{*}}), y^{*}(\eta_{\beta^{*}})) = ((1-p)^{1/(1-p)}, y_{\beta^{*}})$$
(127)

for some $0 < y_{\beta^*} < \beta^*$, whilst

$$(x^*(\eta), y^*(\eta)) \notin \overline{\Omega} \quad \forall \eta \in (\eta_{\beta^*}, \eta_{\beta^*} + \epsilon_{\beta^*}].$$
(128)

Now consider the family of open balls

 $B(x^*(\eta), y^*(\eta); \epsilon')$ with $\eta \in [0, \eta_{\beta^*} + \epsilon_{\beta^*}]$

and via (126)-(128), choose ϵ' sufficiently small so that

$$B(x^*(\eta_{\beta^*} + \epsilon_{\beta^*}), y^*(\eta_{\beta^*} + \epsilon_{\beta^*}); \epsilon') \cap \bar{\Omega} = \emptyset$$
(129)

and

$$\bigcup_{\lambda \in [0,\eta_{\theta^*} + \epsilon_{\theta^*}]} B(x^*(\lambda), y^*(\lambda); \epsilon') \cap (\partial \Omega \setminus \partial \Omega_1) \subset \{((1-p)^{1/(1-p)}, \lambda) : \lambda > 0\}.$$
(130)

It then follows from Corollary 15 that there exists $\delta' > 0$ such that the corresponding unique solution to (S) with zero-value $(0,\beta) \in \partial\Omega_1$, satisfying $|\beta - \beta^*| < \delta'$, say $(x, y) : [0, \eta_{\beta^*} + \epsilon_{\beta^*}] \to \mathbb{R}^2$ has

$$(x(\eta), y(\eta)) \in \bigcup_{\lambda \in [0, \eta_{\beta^*} + \epsilon_{\beta^*}]} B(x^*(\lambda), y^*(\lambda); \epsilon') \quad \forall \eta \in [0, \eta_{\beta^*} + \epsilon_{\beta^*}]$$
(131)

Therefore, via (129)-(131), $\{(0,\beta) : |\beta - \beta^*| < \delta'\} \subseteq E_1$, and so E_1 is an open subset of $\partial \Omega_1$, as required.

Finally, we have

Corollary 19. *The set* E_3 *is a non-empty closed subset of* $\partial \Omega_1$ *.*

Proof. Via Propositions 16 and 17, E_1 and E_2 are both nonempty subsets of $\partial\Omega_1$. Moreover, via (120) E_1 and E_2 are disjoint. Suppose that E_3 is empty, then via (121) and Proposition 18, E_1 and E_2 form an open partition of $\partial\Omega_1$. However, $\partial\Omega_1$ is a connected subset of \mathbb{R}^2 , and we arrive at a contradiction. Hence E_3 must be nonempty. Finally, $E_3 = \partial\Omega_1 \setminus (E_1 \cup E_2)$ and is therefore a closed subset of $\partial\Omega_1$.

Remark 3. In Corollary 19, the existence of at least one point in E_3 has been established. However, it has not been established that this is the only point in E_3 .

To conclude this section, we arrive at our main result, namely,

Theorem 20. There exists a solution $(x, y) : \mathbb{R} \to \mathbb{R}^2$ to (S) with zero-value $(0,\beta) \in \partial\Omega_1$, for some

$$\sqrt{\frac{(1-p)^{2/(1-p)}}{(1+p)}} \le \beta \le \sqrt{2\left(((1-p)^{1/(1-p)} - m_H)^2 - m_H^2\right)},$$

which satisfies

and

$$(x(\eta), y(\eta)) \to (\pm (1-p)^{1/(1-p)}, 0) \text{ as } \eta \to \pm \infty$$
 (132)

$$|x(\eta)| < (1-p)^{1/(1-p)}, \quad 0 < y(\eta) \le \beta \quad \forall \eta \in \mathbb{R}.$$
 (133)

Proof. It follows directly from Corollary 19 and (iii) that there exists $(x^*, y^*) : [0, \infty) \to \mathbb{R}^2$ which is a solution to (S) with zero-value $(0, \beta^*)$, such that

$$(x^*(\eta), y^*(\eta)) \to ((1-p)^{1/(1-p)}, 0) \text{ as } \eta \to \infty,$$
 (134)

$$(x^*(\eta), y^*(\eta)) \in \Omega \quad \forall \eta \in (0, \infty).$$
(135)

It follows from (125) and (122), that

$$\sqrt{\frac{(1-p)^{1/(1-p)}}{(1+p)}} \le \beta^* \le \sqrt{2\left(((1-p)^{1/(1-p)} - m_H)^2 - m_H^2\right)}.$$

Now, define the function $(x, y) : \mathbb{R} \to \mathbb{R}^2$ to be

$$(x(\eta), y(\eta)) = \begin{cases} (x^*(\eta), y^*(\eta)) & ; \eta \in [0, \infty) \\ (-x^*(-\eta), y^*(-\eta)) & ; \eta \in (-\infty, 0). \end{cases}$$
(136)

It follows from (136) that $(x, y) : \mathbb{R} \to \mathbb{R}^2$ is a solution to (S) with zero-value $(0, \beta^*)$, and via (94) and (iii), (since $y(\eta)$ is monotone decreasing for $\eta \in (0, \infty)$) that this solution satisfies (132) and (133).

We conclude from Theorem 20 that the problem (S) has at least one heteroclinic connection from the equilibrium point $(-(1-p)^{1/(1-p)}, 0)$ $(\eta = -\infty)$ to the equilibrium point $((1-p)^{1/(1-p)}, 0)$ $(\eta = \infty)$, which we denote by $w_{\beta^*} : \mathbb{R} \to \mathbb{R}$. Here $w = w_{\beta^*}(\eta), \eta \in \mathbb{R}$, has zero-value $w(0) = 0, w'(0) = \beta^*$ for some

$$\sqrt{\frac{(1-p)^{2/(1-p)}}{(1+p)}} \leq \beta^* \leq \sqrt{2\left(((1-p)^{1/(1-p)}-m_H)^2-m_H^2\right)},$$

and

$$|w_{eta^*}(\eta)| < (1-p)^{1/(1-p)}, \quad 0 < w_{eta^*}'(\eta) \le eta^* \quad \forall \eta \in \mathbb{R},$$

recalling also, that $w_{\beta^*}(\eta)$ is an odd function of $\eta \in \mathbb{R}$. Finally, a straightforward linearization as $|\eta| \to \infty$ establishes that,

$$w_{\beta^*}(\eta) \sim \pm (1-p)^{1/(1-p)} - \frac{A_{\infty}}{\eta^3} e^{-\frac{1}{4}\eta^2} \text{ as } \eta \to \pm \infty,$$

with A_{∞} being a globally determined constant.

4.2. Front Solutions to [CP]

Following Theorem 20, with $\beta = \beta^*$ we have constructed the front-like global solution $u_{\beta^*} : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ to [CP], namely,

$$u_{\beta^*}(x,t) = \begin{cases} t^{\frac{1}{(1-p)}} w_{\beta^*}\left(\frac{x}{t^{1/2}}\right) &, (x,t) \in \mathbb{R} \times (0,\infty) \\ 0 &, (x,t) \in \mathbb{R} \times \{0\}. \end{cases}$$
(137)

We again observe that $u_{\beta^*}(x - x_0, t)$ is also a global solution to [CP] for any fixed $x_0 \in \mathbb{R}$. In addition, following Section 3.3, we conclude that, for any $\tau > 0$, $u_{\beta^*}^{\tau} : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ such that

$$u_{\beta^*}^{\tau}(x,t) = \begin{cases} \left(t-\tau\right)^{\frac{1}{(1-p)}} w_{\beta^*}\left(\frac{x}{(t-\tau)^{1/2}}\right) &, (x,t) \in \mathbb{R} \times (\tau,\infty) \\ 0 &, (x,t) \in \mathbb{R} \times [0,\tau] \end{cases}$$

is also a front-like global solution to [CP].

5. Discussion

There are two questions that arise naturally from this study. The first being how one can rigorously establish the decay rate of the homoclinic solutions $w : \mathbb{R} \to \mathbb{R}$ to (S) as $\eta \to \pm \infty$, that is suggested by (87) and (88); the second being whether or not for the problem (S), there is a unique heteroclinic connection from the equilibrium point $(-(1-p)^{1/(1-p)}, 0)$ to the equilibrium point $((1-p)^{1/(1-p)}, 0)$ which has zero value in $\partial \Omega_1$ (Theorem 20 guarantees that there exists at least one connection).

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