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STABILITY OF THE BRASCAMP–LIEB CONSTANT AND APPLICATIONS

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Abstract. We prove that the best constant in the general Brascamp–Lieb inequality is a locally bounded function of the underlying linear transformations. As applications we deduce certain very general Fourier restriction, Kakeya-type, and nonlinear variants of the Brascamp–Lieb inequality which have arisen recently in harmonic analysis.

1. Introduction

The celebrated Brascamp–Lieb inequality, which simultaneously generalises many important multilinear inequalities in analysis, including the Hölder, Loomis–Whitney and Young convolution inequalities, takes the form

$$\int_H \prod_{j=1}^m (f_j \circ L_j)^{p_j} \leq C \prod_{j=1}^m \left(\int_{H_j} f_j \right)^{p_j}. \tag{1}$$

Here m denotes a positive integer, H and H_j denote euclidean spaces of finite dimensions n and $n_j \leq n$ respectively, equipped with Lebesgue measure for each $1 \leq j \leq m$. The maps $L_j : H \to H_j$ are surjective linear transformations, and the exponents $0 \leq p_j \leq 1$ are real numbers. This inequality is often referred to as multilinear, since it is equivalent to

$$\int_H \prod_{j=1}^m f_j \circ L_j \leq C \prod_{j=1}^m \|f_j\|_{L^{q_j}(H_j)} \tag{2}$$

where $q_j = p_j^{-1}$ for each j.

Following the notation introduced in [10] we denote by $\mathrm{BL}(\mathbf{L}, \mathbf{p})$ the smallest constant C for which (1) holds for all nonnegative input functions $f_j \in L^1(\mathbb{R}^{n_j})$, $1 \leq j \leq m$. Here $\mathbf{L}$ and $\mathbf{p}$ denote the m-tuples $(L_j)_{j=1}^m$ and $(p_j)_{j=1}^m$ respectively. We refer to $(\mathbf{L}, \mathbf{p})$ as the *Brascamp–Lieb datum*, and $\mathrm{BL}(\mathbf{L}, \mathbf{p})$ as the *Brascamp–Lieb constant.* To avoid completely degenerate cases, where $\mathrm{BL}(\mathbf{L}, \mathbf{p})$ is easily seen to be infinite, it is natural to restrict attention to data $(\mathbf{L}, \mathbf{p})$ for which

$$\bigcap_{j=1}^m \ker L_j = \{0\}.$$

In [30] Lieb proved that $\mathrm{BL}(\mathbf{L}, \mathbf{p})$ is exhausted by centred gaussian inputs

$$f_j(x) = \exp(-\pi \langle A_j x, x \rangle),$$

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for arbitrary positive-definite transformations $A_j : H_j \to H_j$, and thus

$$\mathrm{BL}(\mathbf{L}, \mathbf{p}) = \sup \frac{\prod_{j=1}^m (\det A_j)^{p_j/2}}{\det \left(\sum_{j=1}^m p_j L_j^* A_j L_j \right)^{1/2}},$$

where the supremum is taken over all such A_j, $1 \leq j \leq m$. While this considerably reduces the complexity of computing the Brascamp–Lieb constant for a given datum, it does not provide a transparent characterisation of the data for which it is finite. This problem was addressed in [10] and [11] (see also the forerunner [26] in the rank one setting), where it was shown that $\mathrm{BL}(\mathbf{L}, \mathbf{p})$ is finite if and only if the scaling condition

$$\sum_{j=1}^m p_j n_j = n \tag{3}$$

and the dimension condition

$$\dim(V) \leq \sum_{j=1}^m p_j \dim(L_j V) \tag{4}$$

hold for all subspaces $V \subseteq H$.

In this note we turn our attention to the *stability* of the constant $\mathrm{BL}(\mathbf{L}, \mathbf{p})$ as a function of the linear maps $\mathbf{L}$, establishing the following basic result:

Theorem 1.1. *Suppose that $(\mathbf{L}^0, \mathbf{p})$ is a Brascamp–Lieb datum for which $\mathrm{BL}(\mathbf{L}^0, \mathbf{p}) < \infty$. Then there exists $\delta > 0$ and a constant $C < \infty$ such that*

$$\mathrm{BL}(\mathbf{L}, \mathbf{p}) \leq C$$

whenever $\|\mathbf{L} - \mathbf{L}^0\| < \delta$.

Of course Theorem 1.1 tells us that for fixed $\mathbf{p}$, the finiteness set

$$F(\mathbf{p}) := \{\mathbf{L} : \mathrm{BL}(\mathbf{L}, \mathbf{p}) < \infty\}$$

is open, and that the function $\mathbf{L} \mapsto \mathrm{BL}(\mathbf{L}, \mathbf{p})$ is locally bounded. We refer to the concurrent work of Bourgain and Demeter [21] for some interesting applications of this result in the setting of Weyl sums and Diophantine equations.

Under certain additional constraints on the kernels of the linear maps L_j^0, the conclusion of Theorem 1.1 may be seen quite directly. For instance, in the rank one case ($n_j = 1$ for all j) this follows quickly via Barthe's characterisation of the extreme points of the Brascamp–Lieb polytope

$$\Pi(\mathbf{L}) := \{\mathbf{p} : \mathrm{BL}(\mathbf{L}, \mathbf{p}) < \infty\},$$

combined with the tautological statement that $\mathbf{p} \in \Pi(\mathbf{L})$ if and only if $\mathbf{L} \in F(\mathbf{p})$; see [2]. A similar understanding may be reached in the co-rank one case ($n_j = n - 1$ for all j) using the characterisation of extreme points in Valdimarsson [32]. It is also pertinent to note that when the kernels of the maps satisfy the basis condition

$$\bigoplus_{j=1}^m \ker L_j^0 = H, \tag{5}$$

a condition which is stable under perturbations of the L_j, there is an explicit expression for the Brascamp–Lieb constant $\mathrm{BL}(\mathbf{L}, \mathbf{p})$, from which the conclusion of Theorem 1.1 (and indeed the *smoothness* of $\mathbf{L} \mapsto \mathrm{BL}(\mathbf{L}, \mathbf{p})$) is manifest; see [8].

If we restrict attention to the so-called *simple* Brascamp–Lieb data, that is, data $(\mathbf{L},\mathbf{p})$ for which (4) holds with strict inequality for all nonzero proper subspaces V, much more can be said. In particular, it was shown by Valdimarsson in [33] that the set

$$F_S(\mathbf{p}) := \{\mathbf{L} \in F(\mathbf{p}) : (\mathbf{L},\mathbf{p}) \text{ is simple}\}$$

is open, and that the Brascamp–Lieb constant $\mathbf{L} \mapsto \mathrm{BL}(\mathbf{L},\mathbf{p})$ is in fact differentiable there. Since Valdimarsson's argument is based on an application of the implicit function theorem, this regularity conclusion may be pushed even as far as analyticity. However, if $(\mathbf{L},\mathbf{p})$ is not simple, that is, there exists a nonzero proper subspace V of H for which (4) holds with equality (such subspaces are referred to as *critical subspaces*), the situation appears to be much more delicate. In particular, since $F_S(\mathbf{p})$ is open, the mere existence of a critical subspace is unstable under perturbations of $\mathbf{L}$. This makes a more standard inductive approach to Theorem 1.1, via factoring the Brascamp–Lieb constant through critical subspaces, appear quite problematic.

In this paper we also prove local boundedness for certain *localised* versions of the Brascamp–Lieb constant, including $\mathrm{BL}_{\mathrm{loc}}(\mathbf{L},\mathbf{p})$, the best constant C in the inequality

$$\int_{\|x\|_H \le 1} \prod_{j=1}^m (f_j \circ L_j)^{p_j} \le C \prod_{j=1}^m \Big(\int_{H_j} f_j \Big)^{p_j}. \tag{6}$$

These inequalities have also been the subject of considerable attention; see [30] and [10] for a gaussian-localised variant, and the more recent [11] for a characterisation of finiteness of the best constant.

When $\mathbf{p}$ satisfies the scaling condition (3), $\mathrm{BL}_{\mathrm{loc}}(\mathbf{L},\mathbf{p}) = \mathrm{BL}(\mathbf{L},\mathbf{p})$. Thus the stability result Theorem 1.1 will follow from the corresponding result for $\mathrm{BL}_{\mathrm{loc}}(\mathbf{L},\mathbf{p})$.

Theorem 1.1 and its local variants (see the forthcoming Theorems 2.1 and 2.3) are motivated by certain seemingly quite difficult "perturbed" versions of the Brascamp–Lieb inequality that have arisen in harmonic analysis over the past decade. For such applications we take H and H_j to be $\mathbb{R}^n$ and $\mathbb{R}^{n_j}$, respectively.

The first conjectural generalisation is combinatorial in nature, and takes the form

$$\int_{\mathbb{R}^n} \prod_{j=1}^m \Big(\sum_{\alpha_j \in \mathcal{A}_j} f_{j,\alpha_j} \circ L_{j,\alpha_j} \Big)^{p_j} \le C \prod_{j=1}^m \Big(\sum_{\alpha_j \in \mathcal{A}_j} \int_{\mathbb{R}^{n_j}} f_{j,\alpha_j} \Big)^{p_j}, \tag{7}$$

where, for each $1 \le j \le m$, the linear mappings $(L_{j,\alpha_j})_{\alpha_j \in \mathcal{A}_j}$ are required to be close to a *fixed* surjection $L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$. Here, for each $1 \le j \le m$, $\mathcal{A}_j$ indexes the linear maps and arbitrary integrable functions $f_{j,\alpha_j} : \mathbb{R}^{n_j} \to \mathbb{R}_+$, and the fixed maps $\mathbf{L} = (L_j)_{1 \le j \le m}$ are such that $\mathrm{BL}(\mathbf{L},\mathbf{p}) < \infty$. Such a generalisation is known to hold in some very special cases, the most notable being when the fixed maps $\mathbf{L}$ and exponents $\mathbf{p}$ correspond to the Loomis–Whitney datum. This is easily seen to be equivalent to the endpoint multilinear Kakeya inequality of Guth [27]; see also the non-endpoint versions in [12], [7], [28] and applications beginning with [22]. Considering indexing sets $\mathcal{A}_j$, with each consisting of just one element, reveals the statement of Theorem 1.1 as a necessary feature for such a combinatorial generalisation to hold.

Inequality (7) is best understood via an equivalent formulation obtained by testing it on finite sums of characteristic functions of δ-balls, upon which it may be expressed as

$$\int_{\mathbb{R}^n} \prod_{j=1}^m \Big(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j} \Big)^{p_j} \le C \delta^n \prod_{j=1}^m (\#\mathbb{T}_j)^{p_j}, \tag{8}$$

uniformly in δ, where for each j, $\mathbb{T}_j$ denotes an arbitrary finite collection of δ-neighbourhoods of n_j'-dimensional affine subspaces of $\mathbb{R}^n$ which, modulo translations, are close to the fixed subspace $V_j := \ker L_j$. Here $n_j' := n - n_j$, and we use the standard metric on the Grassmann manifold of n_j'-dimensional subspaces of $\mathbb{R}^n$. Notice that the characterisation of finiteness of $\mathrm{BL}(\mathbf{L}^0, \mathbf{p})$, given by (3) and (4), depends only on the *kernels* of the linear maps L_j. In particular, for $V_j := \ker L_j$, the condition (4) may be rewritten as

$$\dim(V) \leq \sum_{j=1}^{m} p_j \dim(V \cap V_j^{\perp}).$$

A second generalisation of the Brascamp–Lieb inequality is oscillatory in nature, and belongs to the restriction theory of the Fourier transform. To describe this suppose that, for each $1 \leq j \leq m$, $\Sigma_j : U_j \to \mathbb{R}^n$ is a smooth parametrisation of a n_j-dimensional submanifold S_j of $\mathbb{R}^n$ by a neighbourhood U_j of the origin in $\mathbb{R}^{n_j}$. We associate to each Σ_j the *extension operator*

$$E_j g_j(\xi) := \int_{U_j} e^{2\pi i \xi \cdot \Sigma_j(x)} g_j(x) dx,$$

where $\xi \in \mathbb{R}^n$. In this setting it is natural to conjecture that if $\mathrm{BL}(\mathbf{L}, \mathbf{p}) < \infty$, where $L_j := (\mathrm{d}\Sigma_j(0))^*$ for each j, then provided the neighbourhoods U_j of 0 are chosen small enough, the inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} |E_j g_j|^{2p_j} \leq C \prod_{j=1}^{m} \|g_j\|_{L^2(U_j)}^{2p_j} \tag{9}$$

holds for all $g_j \in L^2(U_j)$, $1 \leq j \leq m$. The weaker inequality

$$\int_{B(0,R)} \prod_{j=1}^{m} |E_j g_j|^{2p_j} \leq C_\varepsilon R^\varepsilon \prod_{j=1}^{m} \|g_j\|_{L^2(U_j)}^{2p_j},$$

involving an arbitrary $\varepsilon > 0$ loss was established in the particular case when $(\mathbf{L}, \mathbf{p})$ is the Loomis–Whitney datum in [12], and has had extensive applications and developments beginning with [22]; see also [15], [16], [23], [19], [17], [18], [20], [24], [21], [29]. The endpoint (9) is only known in very elementary situations, and is easily seen to be best possible in the sense that $\mathrm{BL}(\mathbf{L}, \mathbf{p}) < \infty$ provides a necessary condition on the p_j by taking *linear* Σ_j; see [7] for further discussion.

A third, seemingly more modest generalisation of the Brascamp–Lieb inequality, originating in [13], involves dropping the linearity requirement on the maps L_j, and instead considering B_j smooth submersions in a neighbourhood of a point $x_0 \in \mathbb{R}^n$. In this context it seems natural to conjecture (see [8]) that provided

$$\mathrm{BL}_{\mathrm{loc}}(\mathrm{d}\mathbf{B}(x_0), \mathbf{p}) < \infty,$$

there exists a neighbourhood U of x_0 and a finite constant C such that

$$\int_U \prod_{j=1}^{m} (f_j \circ B_j)^{p_j} \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{p_j}, \tag{10}$$

or equivalently

$$\int_U \prod_{j=1}^{m} f_j \circ B_j \leq C \prod_{j=1}^{m} \|f_j\|_{L^{q_j}(\mathbb{R}^{n_j})}, \tag{11}$$

where, as in (2), $q_j = p_j^{-1}$. Here $\mathrm{d}\mathbf{B}(x_0) = (\mathrm{d}B_j(x_0))$, where $\mathrm{d}B_j(x_0) : \mathbb{R}^n \to \mathbb{R}^{n_j}$ denotes the derivative of B_j at the point x_0. Such a generalisation has been shown to hold under the basis condition (5) on the derivative maps $\mathrm{d}B_j(x_0)$; we refer to [8], [13], [6], [4] for this and applications to problems in euclidean harmonic analysis and dispersive PDE. An elementary scaling and limiting argument shows that if (10) holds then there exists a neighbourhood U' of x_0 such that

$$\sup_{x\in U'} \mathrm{BL}_{\mathrm{loc}}(\mathrm{d}\mathbf{B}(x),\mathbf{p})<\infty,$$

a statement which is closely related to the local boundedness of the (linear) localised Brascamp–Lieb constant; see [9] for further details. The local variant of Theorem 1.1 (see Theorem 2.1) may thus be viewed as a modest first step towards the general form of this nonlinear Brascamp–Lieb conjecture.

Our applications of Theorem 1.1 consist of proving certain weak forms of the generalised Brascamp–Lieb inequalities (8), (9) and (10), where one accepts some arbitrarily small loss in regularity of the input functions. All of these combine our stability results with well-known variants of the induction-on-scales method.

Our application to the variant (7) is best expressed in terms of the equivalent geometric formulation (8), and is the following.

Theorem 1.2. *Suppose* $(\mathbf{L},\mathbf{p})$ *is a Brascamp–Lieb datum for which* $\mathrm{BL}(\mathbf{L},\mathbf{p})<\infty$*. Then there exists* $\nu>0$ *such that for every* $\varepsilon>0$*,*

$$\int_{[-1,1]^n}\prod_{j=1}^m\Big(\sum_{T_j\in\mathbb{T}_j}\chi_{T_j}\Big)^{p_j}\le C_\varepsilon\delta^{n-\varepsilon}\prod_{j=1}^m(\#\mathbb{T}_j)^{p_j} \tag{12}$$

holds for all finite collections $\mathbb{T}_j$ *of* δ*-neighbourhoods of* n_j'*-dimensional affine subspaces of* $\mathbb{R}^n$ *which, modulo translations, are within a distance* ν *of the fixed subspace* $V_j:=\ker L_j$*.*

In the particular case of *gaussian-extremisable* Brascamp–Lieb data $(\mathbf{L},\mathbf{p})$, the above theorem may be seen as a consequence of Corollary 4.2 in [12]; see [10] for a characterisation of such data. As in the case where $(\mathbf{L},\mathbf{p})$ is the Loomis–Whitney datum (see [12]), the above theorem implies the following very general restriction theorem. We refer to [21] for recent number-theoretic applications of these "multilinear" restriction and Kakeya-type inequalities.

Theorem 1.3. *Suppose that* $\mathrm{BL}(\mathbf{L},\mathbf{p})<\infty$*, where* $L_j:=(\mathrm{d}\Sigma_j(0))^*$ *for each* j*. Then there exist neighbourhoods* U_j *of* $0\in\mathbb{R}^{n_j}$*,* $1\le j\le m$*, such that for every* $\varepsilon>0$*,*

$$\int_{B(0,R)}\prod_{j=1}^m|E_jg_j|^{2p_j}\le C_\varepsilon R^\varepsilon\prod_{j=1}^m\|g_j\|_{L^2(U_j)}^{2p_j} \tag{13}$$

holds for all $g_j\in L^2(U_j)$*,* $1\le j\le m$*, and all* $R\ge 1$*.*

Our application to the variant (10) is best expressed in terms of the equivalent formulation (11), and involves a regularity loss that may be captured in the scale of the classical Sobolev spaces. (Absorbing this loss in the scale of L^p spaces appears to be less clear.)

Theorem 1.4. *Suppose* $(\mathbf{L},\mathbf{p})$ *is a Brascamp–Lieb datum for which* $\mathrm{BL}(\mathbf{L},\mathbf{p})<\infty$*, and that for each* $1\le j\le m$*,* $B_j:\mathbb{R}^n\to\mathbb{R}^{n_j}$ *is a smooth submersion in a neighbourhood of the origin satisfying* $\mathrm{d}B_j(0)=L_j$*. Then there exists a neighbourhood* U *of the origin in* $\mathbb{R}^n$ *such that for every* $\varepsilon>0$*,*

$$\int_U\prod_{j=1}^m f_j\circ B_j\le C_\varepsilon\prod_{j=1}^m\|f_j\|_{L^{q_j}_\varepsilon(\mathbb{R}^{n_j})}, \tag{14}$$

where $q_j=p_j^{-1}$*.*

Here, we use the notation $\|f\|_{L^q_\varepsilon(\mathbb{R}^n)} = \|(1-\Delta)^{\varepsilon/2} f\|_{L^q(\mathbb{R}^n)}$ for $q \geq 1$ and $n \in \mathbb{N}$. It is worth noting that the proof allows the smoothness condition on the B_j to be relaxed to $C^{1,\beta}$ for any $\beta > 0$.

Structure of the paper. In Section 2 we prove Theorem 1.1 via the corresponding statement for the local Brascamp–Lieb constant $\mathrm{BL}_{\mathrm{loc}}(\mathbf{L}, \mathbf{p})$. We conclude Section 2 by unifying these results in the setting of partially-localised Brascamp–Lieb constants. We prove Theorems 1.2–1.4 in Section 3.

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2. Stability of the Brascamp–Lieb constant

2.1. **Openness.** Although the proof of the local boundedness of $\mathbf{L} \mapsto \mathrm{BL}(\mathbf{L}, \mathbf{p})$ simultaneously establishes the openness of $F(\mathbf{p})$, the latter permits a much more elementary approach. We describe this first.

It suffices to prove that if the dimension condition (4) holds for $\mathbf{L} = \mathbf{L}^0$ then there exists $\delta > 0$ such that (4) holds whenever $\|\mathbf{L} - \mathbf{L}^0\| < \delta$. For each $1 \leq k \leq n$ let $\mathcal{E}_k$ denote the compact set of all orthonormal sets $\mathbf{e} := \{e_1, \ldots, e_k\}$ in H. This notation allows us to rewrite (4) as

$$k \leq \sum_{j=1}^m p_j \dim(\langle L_j e_1, \ldots, L_j e_k \rangle) \tag{15}$$

for all $\mathbf{e} \in \mathcal{E}_k$ and $1 \leq k \leq n$.

Fix k and let $\mathbf{e} \in \mathcal{E}_k$. Since (15) holds with $\mathbf{L} = \mathbf{L}^0$, for each $1 \leq j \leq m$ we may choose a subset $I_j \subseteq \{1, \ldots, k\}$ satisfying $|I_j| = \dim(\langle L_j^0 e_1, \ldots, L_j^0 e_k \rangle)$,

$$k \leq \sum_{j=1}^m p_j |I_j| \tag{16}$$

and

$$\bigwedge_{i \in I_j} L_j^0 e_i \neq 0.$$

Since

$$(\mathbf{L}, \mathbf{e}') \mapsto \bigwedge_{i \in I_j} L_j e_i' \in \Lambda^{|I_j|}(H_j)$$

is continuous for each j, there exist $\varepsilon(\mathbf{e}), \delta(\mathbf{e}) > 0$ such that

$$\bigwedge_{i \in I_j} L_j e_i' \neq 0$$

for each j, whenever $\|\mathbf{e}' - \mathbf{e}\| < \varepsilon(\mathbf{e})$ and $\|\mathbf{L} - \mathbf{L}^0\| < \delta(\mathbf{e})$. In particular, $\dim(\langle L_j e_1', \ldots, L_j e_k' \rangle) \geq |I_j|$ for each j, and so by (16),

$$k \leq \sum_{j=1}^m p_j \dim(\langle L_j e_1', \ldots, L_j e_k' \rangle)$$

whenever $\|\mathbf{e}' - \mathbf{e}\| < \varepsilon(\mathbf{e})$ and $\|\mathbf{L} - \mathbf{L}^0\| < \delta(\mathbf{e})$. Since $\mathcal{E}_k$ is compact there exists a finite collection $\mathbf{e}^1, \ldots, \mathbf{e}^N \in \mathcal{E}_k$ such that the sets

$$\{\mathbf{e}' \in \mathcal{E}_k : \|\mathbf{e}' - \mathbf{e}^\ell\| < \varepsilon(\mathbf{e}^\ell)\},$$

with $\ell = 1, \ldots, N$, cover $\mathcal{E}_k$. Finally, choosing $\delta = \min\{\delta(\mathbf{e}^1), \ldots, \delta(\mathbf{e}^N)\}$ we conclude that (15) holds whenever $\|\mathbf{L} - \mathbf{L}^0\| < \delta$ and $\mathbf{e} \in \mathcal{E}_k$. Since there are boundedly many such k, the claimed openness follows.

2.2. **Local boundedness for localised data.** In this section we prove the following local version of Theorem 1.1:

Theorem 2.1. *Suppose that $(\mathbf{L}^0, \mathbf{p})$ is a Brascamp–Lieb datum such that $\mathrm{BL}_{\mathrm{loc}}(\mathbf{L}^0, \mathbf{p}) < \infty$. Then there exists $\delta > 0$ and a constant $C < \infty$ such that*

$$\mathrm{BL}_{\mathrm{loc}}(\mathbf{L}, \mathbf{p}) \leq C$$

whenever $\|\mathbf{L} - \mathbf{L}^0\| < \delta$.

In [10] it is shown that $\mathrm{BL}_{\mathrm{loc}}(\mathbf{L}^0, \mathbf{p})$ is finite if and only if

$$\operatorname{codim}_H(V) \geq \sum_{j=1}^m p_j \operatorname{codim}_{H_j}(L_j^0 V) \quad \text{for all subspaces } V \subset H. \tag{17}$$

Note that Theorem 1.1 is a direct corollary of Theorem 2.1 in the case where the condition (3) is satisfied by a scaling argument.

Our proof of Theorem 2.1 amounts to an appropriately uniform version of the proof of the finiteness characterisation theorem for the Gaussian localised version in [10]. The advantage of this approach over the alternative in [11] is that it avoids reference to critical subspaces, objects whose existence is unstable under perturbations of $\mathbf{L}$. Our argument fails to yield a more quantitative statement, such as something closer to upper semi-continuity for the Brascamp–Lieb constant, due to the crucial role played by the compactness of appropriately nondegenerate bases for H.

As in the proof of the openness of $F(\mathbf{p})$ given in Section 2.1, we shall exploit the finiteness condition (17) through the consideration of an appropriate set of bases of H. The key tool is a uniform version of Lemma 5.1 from [10].

Lemma 2.2. *Suppose $(\mathbf{L}^0, \mathbf{p})$ is a Brascamp–Lieb datum such that $\mathrm{BL}_{\mathrm{loc}}(\mathbf{L}^0, \mathbf{p}) < \infty$. Then there exist real numbers $c, \delta > 0$ such that for every $\mathbf{e} \in \mathcal{E}_n$ and every $\mathbf{L}$ satisfying $\|\mathbf{L} - \mathbf{L}^0\| \leq \delta$, there exists a set $I_j \subseteq \{1, \ldots, n\}$ with $|I_j| = \dim(H_j)$ for each $1 \leq j \leq m$, such that*

$$\sum_{j=1}^m p_j |I_j \cap \{1, \ldots, k\}| \leq k \text{ for all } 1 \leq k \leq n, \tag{18}$$

and

$$\Big| \bigwedge_{i \in I_j} L_j e_i \Big| \geq c \text{ for all } 1 \leq j \leq m. \tag{19}$$

In the above lemma, and throughout, we identify $\bigwedge_{i \in I_j} L_j e_i$ with a real number via Hodge duality.

Proof. Let $\mathcal{I}$ denote the set of all m-tuples $(I_1, \ldots, I_m)$ of subsets of $\{1, \ldots, n\}$ satisfying $|I_j| = \dim(H_j)$ and (18). Define

$$h(\mathbf{L}, \mathbf{e}) = \max_{(I_1, \ldots, I_m) \in \mathcal{I}} \min_{1 \leq j \leq m} \Big| \bigwedge_{i \in I_j} L_j e_i \Big|.$$

We begin by proving that $h(\mathbf{L}^0,\mathbf{e}) \geq c'$ for all $\mathbf{e} \in \mathcal{E}_n$ and some $c' > 0$. By the continuity of h and the compactness of $\mathcal{E}_n$, it is enough to verify that $h(\mathbf{L}^0,\mathbf{e}) \neq 0$ for all $\mathbf{e} \in \mathcal{E}_n$. From the definition of h, it suffices to show that there exists $(I_1,\ldots,I_m) \in \mathcal{I}$ for which

$$\bigwedge_{i\in I_j} L_j^0 e_i \neq 0 \text{ for all } 1 \leq j \leq m. \tag{20}$$

Proceeding as in [10], we fix j and select I_j by a backwards greedy algorithm, firstly by putting i_0 in I_j, where

$$i_0 = \max\{i \in \{1,\ldots,n\} : L_j^0 e_i \neq 0\}$$

and then choosing indices $i \in \{1,\ldots,i_0-1\}$ for which $L_j^0 e_i$ is not in the linear span of $\{L_j^0 e_{i'} : i < i' \leq n\}$. By construction (20) holds, and since L_j^0 is surjective, $|I_j| = \dim(H_j)$.

To prove (18), we apply the codimension condition (17) with V equal to the span of $\{e_{k+1},\ldots,e_n\}$, to obtain

$$\sum_j p_j \dim(H_j/L_j^0 V) \leq k.$$

By construction of I_j, $\dim(L_j^0 V) = |I_j \cap \{k+1,\ldots,n\}|$ and hence $\dim(H_j/L_j^0 V) = |I_j \cap \{1,\ldots,k\}|$.

Now let $\mathcal{K}$ be a compact set of linear maps with $\mathbf{L}^0$ belonging to its interior. Since the function h is uniformly continuous on the compact set $\mathcal{K}\times\mathcal{E}_n$, there exists $\delta > 0$ such that

$$|h(\mathbf{L},\mathbf{e}) - h(\mathbf{L}^0,\mathbf{e})| \leq \frac{c'}{2}$$

and $\mathbf{L} \in \mathcal{K}$ whenever $\mathbf{e} \in \mathcal{E}_n$ and $\|\mathbf{L}-\mathbf{L}^0\| \leq \delta$. Therefore, $h(\mathbf{L},\mathbf{e}) \geq \frac{c'}{2}$ whenever $\mathbf{e} \in \mathcal{E}_n$ and $\|\mathbf{L}-\mathbf{L}^0\| \leq \delta$. The lemma now follows from the definition of h. □

Proof of Theorem 2.1. We assume, as we may, that $p_j > 0$ and $H_j \neq \{0\}$ for each j. Let c and δ be those given by Lemma 2.2. We emphasise that these quantities depend only on the fixed datum $(\mathbf{L}^0,\mathbf{p})$. To further emphasise uniformity we include the explicit constant factors arising in the remainder of the argument.

The constant $\mathrm{BL}_{\mathrm{loc}}(\mathbf{L},\mathbf{p})$ is bounded above by a fixed multiple of the best constant in the Gaussian localised case,

$$\int_H e^{-\pi|\cdot|^2} \prod_{j=1}^m (f_j \circ L_j)^{p_j} \leq C \prod_{j=1}^m \Big(\int_{H_j} f_j\Big)^{p_j}.$$

By an application of Lieb's Theorem (Theorem 6.2 in [30]), we have

$$\mathrm{BL}_{\mathrm{loc}}(\mathbf{L},\mathbf{p}) \leq C \sup \frac{\prod_{j=1}^m (\det A_j)^{p_j/2}}{\det(M+Id)^{1/2}},$$

where $M = \sum_{j=1}^m p_j L_j^* A_j L_j$, Id is the identity matrix, and the supremum is taken over all positive definite $A_j : H_j \to H_j$, $1 \leq j \leq m$. It will thus suffice to prove that there exists a constant $C > 0$ such that

$$\prod_{j=1}^m (\det A_j)^{p_j} \leq C \det(M+Id) \tag{21}$$

for all data $(\mathbf{L},\mathbf{p})$ such that $\|\mathbf{L}-\mathbf{L}^0\| \leq \delta$ and all such positive definite A_j.

Since $p_j > 0$ and $\bigcap_{j=1}^m \ker L_j = \{0\}$, we have that M and $M+Id$ are positive definite. Let $e_1,\ldots,e_n$ be an orthonormal basis of eigenvectors for $M+Id$, ordered so that their corresponding eigenvalues satisfy $\mu_1 \geq \ldots \geq \mu_n > 1$.

For each $1 \le i \le n$ we have that

$$\langle e_i, Me_i\rangle_H = \mu_i - \langle e_i, e_i\rangle_H \le \mu_i,$$

and so for each $1 \le i \le n$ and $1 \le j \le m$,

$$\langle A_j L_j e_i, L_j e_i\rangle_{H_j} = \langle e_i, L_j^* A_j L_j e_i\rangle_H \le \frac{1}{p_j}\langle e_i, Me_i\rangle_H \le \mu_i/p_j.$$

Applying Lemma 2.2, for each $1 \le j \le m$, there exists $I_j \subseteq \{1, \dots, n\}$ of cardinality $|I_j| = \dim(H_j)$ such that (18) and (19) hold. For fixed $1 \le j \le m$, if we consider $L_j^* A_j L_j$ acting on the subspace spanned by $\{e_i : i \in I_j\}$, then, since the determinant of a positive semi-definite transformation is at most the product of its diagonal entries,

$$\det(A_j) \le \Big| \bigwedge_{i\in I_j} L_j e_i \Big|^{-2} \prod_{i \in I_j} \langle L_j^* A_j L_j e_i, e_i\rangle_H.$$

Thus

$$\det(A_j) \le \big(c^2 p_j^{n_j}\big)^{-1} \prod_{i\in I_j} \mu_i,$$

where $c > 0$ is the constant given by (19), and this implies

$$\prod_{j=1}^m (\det A_j)^{p_j} \le \left(c^{2\sum_{j=1}^m p_j} \prod_{j=1}^m p_j^{p_j n_j} \right)^{-1} \prod_{i=1}^n \mu_i^{a_i},$$

where $a_i := \sum_{j=1}^m p_j |I_j \cap \{i\}|$. By telescoping we may write

$$\prod_{i=1}^n \mu_i^{a_i} = \det(M + Id) \prod_{k=1}^n \left(\frac{\mu_{k+1}}{\mu_k}\right)^{k - \sum_{i=1}^k a_i}$$

since $\det(M + Id) = \prod_{i=1}^n \mu_i$, and where we have defined $\mu_{n+1} := 1$. Applying (18), $k - \sum_{i=1}^k a_i \ge 0$ and, by construction, $\frac{\mu_{k+1}}{\mu_k} \le 1$ for all $1 \le k \le n$. Hence (21) holds with constant $C = (c^{2\sum_{j=1}^m p_j} \prod_{j=1}^m p_j^{p_j n_j})^{-1}$. □

2.3. **Local boundedness for partially localised data.** In this section we prove a generalisation of Theorem 1.1 for partially localised Brascamp–Lieb constants (see [30] and more recently [11]).

Let $(\mathbf{L}, \mathbf{p})$ be a Brascamp–Lieb datum, let $H_0 \subseteq H$ be a subspace of H, and let G be a positive semi-definite linear map whose kernel is H_0. The associated partially localised Brascamp–Lieb inequality is

$$\int_{\{x\in H : |\langle Gx, x\rangle| < 1\}} \prod_{j=1}^m (f_j \circ L_j)^{p_j} \le C \prod_{j=1}^m \left(\int_{H_j} f_j\right)^{p_j}. \tag{22}$$

Denote the best constant in the above inequality by $\mathrm{BL}_G(\mathbf{L}, \mathbf{p})$ (not to be confused with $\mathrm{BL}_{\mathbf{g}}$ from [10]).

In [11] it is shown that $\mathrm{BL}_G(\mathbf{L}, \mathbf{p})$ is finite if and only if

$$\dim(V) \le \sum_{j=1}^m p_j \dim(L_j V) \quad \text{for all subspaces } V \subseteq H_0 \tag{23}$$

and

$$\mathrm{codim}_H(V) \ge \sum_{j=1}^m p_j\, \mathrm{codim}_{H_j}(L_j V) \quad \text{for all subspaces } V \subset H. \tag{24}$$

It is tempting to believe that the partially localised case should follow easily from the localised case, Theorem 2.1, by a scaling argument as Theorem 1.1 does. However, the scaling argument in the partially localised setting requires an anisotropic dilation, which changes the initial Brascamp–Lieb datum nontrivially. Nevertheless, a version of Theorem 1.1 holds in this case as well. Again our proof is an appropriately uniform version of the finiteness characterisation in [10] combining the methods from the fully-local and fully-global cases. This gives a proof of the characterisation of finiteness for partially localised data which does not require factoring through critical subspaces as in [11].

Theorem 2.3. *Suppose that* $(\mathbf{L}^0, \mathbf{p})$ *is a Brascamp–Lieb datum and* G *is a positive semi-definite linear map such that* $\mathrm{BL}_G(\mathbf{L}^0, \mathbf{p}) < \infty$. *Then there exists* $\delta > 0$ *and a constant* $C < \infty$ *such that*

$$\mathrm{BL}_G(\mathbf{L}, \mathbf{p}) \leq C$$

whenever $\|\mathbf{L} - \mathbf{L}^0\| < \delta$.

As in the proof of Theorem 2.1, we shall exploit the conditions (23) and (24) through the consideration of an appropriate set of bases of H. However, rather than using orthonormal bases, it will be important to use classes which have some alignment with the distinguished subspace $\ker G = H_0$, and to permit bases which are not quite orthonormal.

For $0 < \alpha \leq 1$ let $\mathcal{V}_\alpha$ denote the set of all $\mathbf{v} = (v_1, \ldots, v_n) \in H^n$ such that $\|v_i\| \leq 1$ for all $1 \leq i \leq n$, and

$$\left| \bigwedge_{i=1}^n v_i \right| \geq \alpha.$$

Further, for each $\ell \in \mathbb{N}$ with $n - \dim H_0 \leq \ell < n$, let

$$\mathcal{V}_{\alpha,\ell} := \{\mathbf{v} \in \mathcal{V}_\alpha : v_{\ell+1}, \cdots, v_n \in H_0\},$$

and $\mathcal{V}_{\alpha,n} := \mathcal{V}_\alpha$. We thus interpret an element $\mathbf{v}$ of $\mathcal{V}_{\alpha,\ell}$ as a certain (ordered) basis for H with a lower bound on its degeneracy.

Clearly $\mathcal{V}_{\alpha,\ell} \subseteq \mathcal{V}_{\alpha,\ell+1}$ for each ℓ. Note also that $\mathcal{V}_{\alpha,\ell}$ is compact for each α and ℓ.

Lemma 2.4. *Suppose that* $n - \dim H_0 \leq \ell \leq n$ *and* $\alpha \in (0, 1]$, *and that* $(\mathbf{L}^0, \mathbf{p})$ *is a Brascamp–Lieb datum for which* $\mathrm{BL}_G(\mathbf{L}^0, \mathbf{p}) < \infty$. *Then there exist real numbers* $c_\ell, \delta_\ell > 0$ *such that for every* $\mathbf{v} \in \mathcal{V}_{\alpha,\ell}$ *and every* $\mathbf{L}$ *satisfying* $\|\mathbf{L} - \mathbf{L}^0\| \leq \delta_\ell$, *there exists a set* $I_j \subseteq \{1, \ldots, n\}$ *with* $|I_j| = \dim(H_j)$ *for each* $1 \leq j \leq m$, *such that*

$$\sum_{j=1}^m p_j |I_j \cap \{1, \ldots, k\}| \leq k \text{ for all } 0 \leq k \leq n, \tag{25}$$

$$\sum_{j=1}^m p_j |I_j \cap \{k+1, \ldots, n\}| \geq n - k \text{ for all } \ell \leq k \leq n, \tag{26}$$

and

$$\left| \bigwedge_{i \in I_j} L_j v_i \right| \geq c_\ell \text{ for all } 1 \leq j \leq m. \tag{27}$$

Proof. Let $\mathcal{I}_\ell$ denote the set of all m-tuples $(I_1, \ldots, I_m)$ of subsets of $\{1, \ldots, n\}$ satisfying $|I_j| = \dim(H_j)$, (25), and (26). Define

$$h_\ell(\mathbf{L}, \mathbf{v}) = \max_{(I_1,\ldots,I_m) \in \mathcal{I}_\ell} \min_{1 \leq j \leq m} \left| \bigwedge_{i \in I_j} L_j v_i \right|.$$

We begin by proving that $h_\ell(\mathbf{L}^0,\mathbf{v}) \geq c'_\ell$ for all $\mathbf{v} \in \mathcal{V}_{\alpha,\ell}$ and some $c'_\ell > 0$. By the continuity of h_ℓ and the compactness of $\mathcal{V}_{\alpha,\ell}$, it is enough to verify that $h_\ell(\mathbf{L}^0,\mathbf{v}) \neq 0$ for all $\mathbf{v} \in \mathcal{V}_{\alpha,\ell}$. From the definition of h_ℓ it suffices to show that there exists $(I_1,\dots,I_m) \in \mathcal{I}_\ell$ for which

$$\bigwedge_{i\in I_j} L_j^0 v_i \neq 0 \text{ for all } 1 \leq j \leq m. \tag{28}$$

Again, we select each I_j by a backwards greedy algorithm, and (25) follows as before. To prove (26), we let $\ell \leq k \leq n$ and apply (23), which is a consequence of our hypothesis that $\mathrm{BL}_G(\mathbf{L}^0,\mathbf{p}) < \infty$, with V equal to the span of $\{v_{k+1},\dots,v_n\} \subset H_0$ to obtain

$$\sum_{j=1}^m p_j \dim(L_j^0 V) \geq n-k.$$

By construction of I_j, we have $\dim(L_j^0V) = |I_j \cap \{k+1,\dots,n\}|$. Thus $(I_1,...,I_m) \in \mathcal{I}_\ell$ satisfies (28), as required.

Now let $\mathcal{K}$ be a compact set of linear maps which contains $\mathbf{L}^0$ in its interior. Since the function h_ℓ is uniformly continuous on the compact set $\mathcal{K}\times\mathcal{V}_{\ell,\alpha}$, there exists $\delta_\ell > 0$ such that

$$|h_\ell(\mathbf{L},\mathbf{v}) - h_\ell(\mathbf{L}^0,\mathbf{v})| \leq \frac{c'_\ell}{2}$$

and $\mathbf{L} \in \mathcal{K}$ whenever $\mathbf{v} \in \mathcal{V}_{\alpha,\ell}$ and $\|\mathbf{L}-\mathbf{L}^0\| \leq \delta_\ell$. Therefore, $h_\ell(\mathbf{L},\mathbf{v}) \geq \frac{c'_\ell}{2}$ whenever $\mathbf{v} \in \mathcal{V}_{\alpha,\ell}$ and $\|\mathbf{L}-\mathbf{L}^0\| \leq \delta_\ell$. The lemma now follows from the definition of h_ℓ. □

Proof of Theorem 2.3. We assume, as we may, that $p_j > 0$ and $H_j \neq \{0\}$ for each j. By applying a linear transformation we may also assume that G is the orthogonal projection of H onto $H_0^\perp$. We may also reduce to the case where $n \geq 2$ as when $n = 1$, G is either the identity or 0.

Fix $\alpha \in (0,1)$ and let

$$c := \min_\ell c_\ell, \quad \delta := \min_\ell \delta_\ell, \tag{29}$$

where c_ℓ, δ_ℓ are those given by Lemma 2.4. We emphasise that these quantities depend only on H_0, α, and the fixed datum $(\mathbf{L}^0,\mathbf{p})$.

It will suffice to prove that there exists a constant $C > 0$ such that

$$\int_H e^{-\pi\langle Gx,x\rangle} \prod_{j=1}^m (f_j\circ L_j)^{p_j} \leq C \prod_{j=1}^m \left(\int_{H_j} f_j\right)^{p_j}$$

whenever $\|\mathbf{L}-\mathbf{L}^0\| < \delta$. By Lieb's Theorem (Theorem 6.2 in [30]), this is equivalent to proving that

$$\prod_{j=1}^m (\det A_j)^{p_j} \leq C \det(M+G) \tag{30}$$

holds uniformly for such $\mathbf{L}$ and all positive definite $A_j : H_j \to H_j$, $1 \leq j \leq m$, where $M = \sum_{j=1}^m p_j L_j^* A_j L_j$. To this end we fix an auxiliary quantity

$$\gamma = \min\left\{ \left(\frac{1-\alpha}{n}\right)^2, \left(\frac{c}{2\max_j n_j(\|L_j^0\|+\delta)^{n_j}}\right)^2 \right\},$$

which, of course, only depends on the fixed datum $(\mathbf{L}^0,\mathbf{p})$, δ and our choice of α.

Since $p_j > 0$ and $\bigcap_{j=1}^m \ker L_j = \{0\}$, we have that M, and thus $M+G$, is positive definite. Let $e_1,\dots,e_n$ be an orthonormal basis of eigenvectors for $M+G$, ordered so that their corresponding eigenvalues satisfy $\mu_1 \geq \cdots \geq \mu_n > 0$. As in the proof of Theorem 2.1, we have that $\langle e_i, Me_i\rangle_H \leq \mu_i$ and $\langle A_j L_j e_i, L_j e_i\rangle_{H_j} \leq \frac{\mu_i}{p_j}$ for each $1 \leq i \leq n$ and $1 \leq j \leq m$.

Suppose first that $\mu_n \geq \gamma$. This case is much the same as the fully localised case, except that the lower bound on the eigenvalues is γ rather than 1. Thus, by rescaling and using the argument in the localised case, it follows that

$$\prod_{j=1}^m (\det A_j)^{p_j} \leq \gamma^{(\sum_{j=1}^m p_j n_j)-n} \left(c^{2\sum_{j=1}^m p_j} \prod_{j=1}^m p_j^{p_j n_j} \right)^{-1} \det(M+G).$$

Otherwise $\mu_n < \gamma$. In this case, the argument combines the approaches from the localised and the non-localised cases in [10]. We define

$$\ell := \min\{i \in \{1,\dots,n\} : \mu_i < \gamma\}.$$

As $n \geq 2$, by construction we have that $\gamma \leq 1/4$, and so since G is an orthogonal projection,

$$|Ge_i|^2 = \langle Ge_i, e_i\rangle \leq \langle (M+G)e_i, e_i\rangle \leq \mu_i \leq \gamma \leq 1/4 \tag{31}$$

whenever $\ell \leq i \leq n$. Thus as $e_\ell, \dots, e_n$ are orthonormal in H, the vectors

$$e_\ell - Ge_\ell, \dots, e_n - Ge_n$$

are linearly independent in H_0. Thus in particular we have $\ell \geq n - \dim(H_0) + 1$.

Next, define $(v_1, \dots, v_n)$ by $v_i := e_i$ for $1 \leq i \leq \ell - 1$ and by $v_i := e_i - Ge_i \in H_0$ for $\ell \leq i \leq n$. By definition, $\|v_i\| \leq 1$ for all $1 \leq i \leq n$. Moreover, using the multilinearity of the wedge product and expanding in a suitable telescoping sum, we have

$$\left| \bigwedge_{i=1}^n v_i - \bigwedge_{i=1}^n e_i \right| \leq (n - \ell + 1) \max_{1 \leq i \leq n} \|e_i - v_i\|.$$

For $\ell \leq i \leq n$ we have $e_i - v_i = Ge_i$, and so arguing as in (31) it follows that $\|e_i - v_i\|^2 = |Ge_i|^2 \leq \gamma$, and therefore

$$\max_{1 \leq i \leq n} \|e_i - v_i\| \leq \gamma^{1/2}. \tag{32}$$

Hence by our choice of γ,

$$\left| \bigwedge_{i=1}^n v_i \right| \geq 1 - n\gamma^{1/2} \geq \alpha.$$

Whence, $(v_1, \dots, v_n) \in \mathcal{V}_{\alpha,\ell}$.

By Lemma 2.4 applied to $\mathbf{v} = (v_1, \dots, v_n)$, there exists $I_j \subseteq \{1, \dots, n\}$ for each $1 \leq j \leq m$ of cardinality $|I_j| = \dim(H_j)$ such that all three of (25), (26), and (27) hold for $\mathbf{v}$.

Now observe that

$$\left| \bigwedge_{i \in I_j} L_j v_i - \bigwedge_{i \in I_j} L_j e_i \right| \leq n_j \|L_j\|^{n_j} \max_{i \in I_j} \|e_i - v_i\| \leq n_j \|L_j\|^{n_j} \gamma^{1/2}$$

where the second inequality follows from (32), and hence our choice of γ guarantees that for $\|L_j - L_j^0\| \leq \delta$ we have

$$\left| \bigwedge_{i \in I_j} L_j e_i \right| \geq c - \max_j n_j \|L_j\|^{n_j} \gamma^{1/2} \geq c/2.$$

Here we are assuming that we have chosen norms so that $\|L_j\|$ is bounded above by $\|\mathbf{L}\|$. This is of course quite natural, although since all choices of norms are equivalent, there is no loss of generality in doing this. As before, we set $a_i := \sum_{j=1}^m p_j |I_j \cap \{i\}|$ and obtain

$$\prod_{j=1}^m (\det A_j)^{p_j} \leq \left((c/2)^{2\sum_{j=1}^m p_j} \prod_{j=1}^m p_j^{p_j n_j} \right)^{-1} \prod_{i=1}^n \mu_i^{a_i}$$

and a telescoping argument yields

$$\prod_{i=1}^{\ell-1} \mu_i^{a_i} \leq \gamma^{(\sum_{i=1}^{\ell-1} a_i)-(\ell-1)} \prod_{i=1}^{\ell-1} \mu_i.$$

For the terms with $i \geq \ell$, similarly to the global case in [10], we write

$$\prod_{i=\ell}^n \mu_i^{a_i} = \mu_\ell^{a_{\geq \ell}} \prod_{i=\ell}^{n-1} \left(\frac{\mu_{i+1}}{\mu_i} \right)^{a_{\geq i+1}}.$$

where $a_{\geq \ell} := \sum_{j=1}^m p_j |I_j \cap \{\ell, \dots, n\}|$.

By (26), for $\ell \leq i \leq n-1$ we have $a_{\geq i+1} \geq n-i$ and as $\frac{\mu_{i+1}}{\mu_i} \leq 1$, this yields

$$\prod_{i=\ell}^n \mu_i^{a_i} \leq \mu_\ell^{a_{\geq \ell}} \prod_{i=\ell}^{n-1} \left(\frac{\mu_{i+1}}{\mu_i} \right)^{n-i},$$

which on reversing the telescoping gives,

$$\prod_{i=\ell}^n \mu_i^{a_i} \leq \mu_\ell^{a_{\geq \ell}-(n-\ell+1)} \prod_{i=\ell}^n \mu_i.$$

Recall that $\ell - 1 \geq n - \dim(H_0)$, so that we may apply (26) to conclude that $a_{\geq \ell} \geq n - \ell + 1$, and therefore $\prod_{i=\ell}^n \mu_i^{a_i} \leq \prod_{i=\ell}^n \mu_i$. Finally, we obtain

$$\prod_{j=1}^m (\det A_j)^{p_j} \leq \gamma^{(\sum_{i=1}^{\ell-1} a_i)-(\ell-1)} \left((c/2)^{2\sum_{j=1}^m p_j} \prod_{j=1}^m p_j^{p_j n_j} \right)^{-1} \det(M+G) \tag{33}$$

which concludes the proof. □

3. The proofs of Theorems 1.2–1.4

As we shall see in this section, the local boundedness of the Brascamp–Lieb constant established in Theorem 1.1 is a natural requirement for the induction-on-scales method to yield Theorems 1.2–1.4. Within harmonic analysis at least, the induction-on-scales arguments that we use go back to Bourgain [14], and have been used extensively since; see in particular [8], [6], [12], [7], [28] for very similar arguments in the context of the Loomis–Whitney and multilinear Kakeya inequalities. In this Brascamp–Lieb setting, these inductive arguments are manifestations of a fundamental multi-scale inequality of Ball [1], and are closely related to heat-flow monotonicity and semigroup interpolation; see [26], [10], [12], [8] for further discussion of this perspective.

We warn that the function $\mathcal{C}$ will have a different definition in each of the sections 3.1–3.3 below.

3.1. **Generalised multilinear Kakeya inequalities and Theorem 1.2.** Here we prove Theorem 1.2 using the induction-on-scales argument in Guth [28]. The role of Theorem 1.1 in this argument is to effectively change the order of the quantifiers in the hypothesis of Theorem 1.2. By suitably partitioning the families $\mathbb{T}_j$, $1 \leq j \leq m$, and applying Theorem 1.1, it suffices to prove the following weaker variant of Theorem 1.2. The deduction of Theorem 1.2 in this way incurs a cost in the size of the constant C, but in a way which only depends on ε.

Theorem 3.1. *Suppose* $(\mathbf{L}, \mathbf{p})$ *is a Brascamp–Lieb datum for which* $\mathrm{BL}(\mathbf{L}, \mathbf{p}) < \infty$*, and* $\varepsilon > 0$*. Then there exists* $\nu = \nu(\varepsilon) > 0$ *and* $C = C(\varepsilon) < \infty$ *(both independent of* δ*) such that*

$$\int_{[-1,1]^n} \prod_{j=1}^m \Big(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j} \Big)^{p_j} \leq C\delta^{n-\varepsilon} \prod_{j=1}^m (\#\mathbb{T}_j)^{p_j}$$

holds for all finite collections $\mathbb{T}_j$ *of* δ*-neighbourhoods of* n_j'*-dimensional affine subspaces of* $\mathbb{R}^n$ *which, modulo translations, are within a distance* ν *of the fixed subspace* $V_j := \ker L_j$*.*

For each $0 < \delta, \nu \leq 1$, let $\mathcal{C}(\delta, \nu)$ denote the smallest constant C in the inequality

$$\int_{[-1,1]^n} \prod_{j=1}^m \Big(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j} \Big)^{p_j} \leq C\delta^{n} \prod_{j=1}^m (\#\mathbb{T}_j)^{p_j} \tag{34}$$

over all such families $\mathbb{T}_j$, $1 \leq j \leq m$, as in the statement of Theorem 3.1. We are required to show that given any $\varepsilon > 0$, there exists $\nu = \nu(\varepsilon) > 0$ (independent of δ) such that $\mathcal{C}(\delta, \nu) \lesssim_\varepsilon \delta^{-\varepsilon}$.

Proposition 3.2. *There is a constant* $\kappa < \infty$*, independent of* δ *and* ν*, such that*

$$\mathcal{C}(\delta, \nu) \leq \kappa \mathcal{C}(\delta/\nu, \nu).$$

Iterating Proposition 3.2 we obtain $\mathcal{C}(\delta, \nu) \leq \kappa^\ell \mathcal{C}(\delta/\nu^\ell, \nu)$ for each $\ell \in \mathbb{N}$. We choose ℓ such that $\delta/\nu^\ell \sim 1$, and ν such that $\varepsilon \log(1/\nu) = \log \kappa$, so that $\kappa^\ell \lesssim_\varepsilon \delta^{-\varepsilon}$ and hence $\mathcal{C}(\delta, \nu) \lesssim_\varepsilon \delta^{-\varepsilon}$, as required.

Proof of Proposition 3.2. We begin by decomposing $[-1,1]^n$ into a grid of axis-parallel cubes Q of sidelength δ/ν and write

$$\int_{[-1,1]^n} \prod_{j=1}^m \Big(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j} \Big)^{p_j} = \sum_Q \int_Q \prod_{j=1}^m \Big(\sum_{T_j \in \mathbb{T}_j(Q)} \chi_{T_j \cap Q} \Big)^{p_j},$$

where $\mathbb{T}_j(Q) := \{T_j \in \mathbb{T}_j : T_j \cap Q \neq \emptyset\}$. Observe that since T_j is a δ-neighbourhood of an affine n_j'-dimensional subspace of $\mathbb{R}^n$ which, modulo translations, is within a distance ν of $V_j := \ker L_j$, there exists an $O(\delta)$-neighbourhood T_j' with $T_j \cap Q \subseteq T_j' \cap Q$ and T_j' parallel to V_j. Since $\mathrm{BL}(\mathbf{L}, \mathbf{p}) < \infty$,

$$\begin{aligned}
\int_Q \prod_{j=1}^m \Big(\sum_{T_j \in \mathbb{T}_j(Q)} \chi_{T_j \cap Q} \Big)^{p_j} &\leq \int_Q \prod_{j=1}^m \Big(\sum_{T_j \in \mathbb{T}_j(Q)} \chi_{T_j'} \Big)^{p_j} \\
&\lesssim \delta^n \prod_{j=1}^m (\#\mathbb{T}_j(Q))^{p_j} \\
&\leq \delta^n \prod_{j=1}^m \Big(\sum_{T_j \in \mathbb{T}_j} \chi_{\widetilde{T}_j}(x_Q) \Big)^{p_j}
\end{aligned}$$

uniformly in $x_Q \in Q$. Here $\widetilde{T}_j = T_j + B(0, c\delta/\nu)$, with factor c chosen large enough so that $T_j \cap Q \neq \emptyset \Rightarrow Q \subseteq \widetilde{T}_j$. Averaging the above over $x_Q \in Q$ gives

$$\int_Q \prod_{j=1}^m \Big(\sum_{T_j \in \mathbb{T}_j(Q)} \chi_{T_j \cap Q} \Big)^{p_j} \lesssim \nu^n \int_Q \prod_{j=1}^m \Big(\sum_{T_j \in \mathbb{T}_j} \chi_{\widetilde{T}_j} \Big)^{p_j},$$

which on summing in Q results in

$$\begin{aligned} \int_{[-1,1]^n} \prod_{j=1}^m \Big(\sum_{T_j \in \mathbb{T}_j(Q)} \chi_{T_j \cap Q} \Big)^{p_j} &\lesssim \nu^n \int_{[0,1]^n} \prod_{j=1}^m \Big(\sum_{T_j \in \mathbb{T}_j} \chi_{\widetilde{T}_j} \Big)^{p_j} \\ &\lesssim \delta^n \mathcal{C}(\delta/\nu, \nu) \prod_{j=1}^m (\#\mathbb{T}_j)^{p_j}, \end{aligned}$$

from which the proposition follows. □

3.2. **Generalised multilinear restriction inequalities and Theorem 1.3.** The deduction of Theorem 1.3 from Theorem 1.2 is a routine generalisation of the argument in [12] (see also [7]) in the setting of the Loomis–Whitney datum. We provide a sketch of the argument here for the sake of completeness.

Proof of Theorem 1.3 from Theorem 1.2. We begin with an observation. For each $\varepsilon > 0$ and $R \geq 1$, applying Theorem 1.2 with $\delta = R^{-1/2}$, and using rescaling and limiting arguments we obtain

$$\int_{B(0,R)} \prod_{j=1}^m \left(\sum_{T_{j,R} \in \mathbb{T}_{j,R}} \frac{\chi_{T_j}}{|T_j|} * g_{T_j} \right)^{p_j} \lesssim_\varepsilon R^{\frac{\varepsilon}{2} - \sum_{j=1}^m p_j n_j'} \prod_{j=1}^m \left(\sum_{T_j \in \mathbb{T}_{j,R}} \|g_{T_j}\|_1 \right)^{p_j} \tag{35}$$

for all nonnegative $g_{T_j} \in L^1(\mathbb{R}^n)$, $T_j \in \mathbb{T}_{j,R}$, $1 \leq j \leq m$. Here $\mathbb{T}_{j,R}$ is any finite collection of rectangles in $\mathbb{R}^n$ with n_j sides of length $O(R^{1/2})$ and n_j' sides of length $O(R)$, with the property that each $T_j \in \mathbb{T}_{j,R}$ is contained in an $O(R^{1/2})$-neighbourhood of an n_j'-dimensional subspaces of $\mathbb{R}^n$ which is (modulo translations) within a distance $\nu > 0$ (given by Theorem 1.2) of $\ker L_j$.

In order to prove Theorem 1.3 it will suffice to show that

$$\int_{B(0,R)} \prod_{j=1}^m |G_j|^{2p_j} \lesssim_\varepsilon R^{\varepsilon - \sum_{j=1}^m p_j n_j'} \prod_{j=1}^m \|G_j\|_2^{2p_j} \tag{36}$$

for all $G_j \in L^2(\mathbb{R}^n)$ such that $\operatorname{supp} \widehat{G_j} \subseteq S_j + O(R^{-1})$, $1 \leq j \leq m$, and all $R \geq 1$. To see that (36) implies (13) we first observe that $E_j g_j = \widehat{h_j d\sigma_j}$, where the S_j-carried measure σ_j is defined by

$$\int_{\mathbb{R}^n} \psi d\sigma_j := \int_{U_j} \psi(\Sigma_j(x)) dx$$

and h_j by $g_j = h_j \circ \Sigma_j$. Let ϕ be a smooth bump function supported in $B(0,1)$ with Fourier transform bounded below on $B(0,1)$, and let $\phi_R(x) = R^n \phi(Rx)$. Setting $G_j = (E_j g_j)\widehat{\phi}_R$ reveals that $\operatorname{supp} \widehat{G_j} \subseteq S_j + O(R^{-1})$ and $\|G_j\|_2 \sim R^{n_j'/2} \|h_j\|_2 \sim R^{n_j'/2} \|g_j\|_2$ uniformly in R for each $1 \leq j \leq m$. Applying (36) to these functions G_j establishes (13); see [31] for further details of this reduction in a bilinear setting.

Next we let $\mathcal{C}(R)$ denote the smallest constant C in the inequality

$$\int_{B(0,R)} \prod_{j=1}^m |G_j|^{2p_j} \leq C R^{-\sum_{j=1}^m p_j n_j'} \prod_{j=1}^m \|G_j\|_2^{2p_j} \tag{37}$$

over all $G_j \in L^2(\mathbb{R}^n)$ such that $\operatorname{supp} \widehat{G_j} \subseteq S_j + O(R^{-1})$, $1 \leq j \leq m$. In these terms (36) becomes $\mathcal{C}(R) \lesssim_\varepsilon R^\varepsilon$.

Upon iterating and using the elementary fact that $\mathcal{C}(100) < \infty$, it will be enough to prove that for each $\varepsilon > 0$, there exists a constant c_ε, independent of R, such that

$$\mathcal{C}(R) \leq c_\varepsilon R^\varepsilon \mathcal{C}(R^{1/2}) \tag{38}$$

for all $R \geq 1$.

Let $x \in B(0,R)$ and $\phi^x_{R^{1/2}} : \mathbb{R}^n \to \mathbb{R}$ be given by $\phi^x_{R^{1/2}}(y) = e^{-2\pi i x\cdot y}\phi_{R^{1/2}}(y)$, where $\phi_{R^{1/2}}$ is defined above; observe that the Fourier transform of $\phi^x_{R^{1/2}}$ is bounded below on $B(x, R^{1/2})$ uniformly in x and R. Applying (37) on $B(x, R^{1/2})$, using the modulation-invariance of the inequality, we obtain

$$\int_{B(x,R^{1/2})} \prod_{j=1}^m |G_j|^{2p_j} \lesssim \mathcal{C}(R^{1/2}) R^{-\frac{1}{2}\sum_{j=1}^m p_j n_j'} \prod_{j=1}^m \|\widehat{G}_j * \phi^x_{R^{1/2}}\|_2^{2p_j}$$

uniformly in x and R. Averaging this over all $|x| \leq R$ yields

$$\int_{B(0,R)} \prod_{j=1}^m |G_j|^{2p_j} \lesssim \mathcal{C}(R^{1/2}) R^{-\frac{n}{2}-\frac{1}{2}\sum_{j=1}^m p_j n_j'} \int_{B(0,R)} \prod_{j=1}^m \|\widehat{G}_j * \phi^x_{R^{1/2}}\|_2^{2p_j}\,dx.$$

Defining $\widehat{G_j^{\rho_j}} = \widehat{G_j}\chi_{\rho_j}$ for caps ρ_j with diameter $R^{-1/2}$ which together provide a cover of $S_j + O(R^{-1})$ with bounded overlap, we may write

$$\int_{B(0,R)} \prod_{j=1}^m |G_j|^{2p_j} \lesssim \mathcal{C}(R^{1/2}) R^{-\frac{n}{2}-\frac{1}{2}\sum_{j=1}^m p_j n_j'} \int_{B(0,R)} \prod_{j=1}^m \Big(\sum_{\rho_j} \|\widehat{G_j^{\rho_j}} * \phi^x_{R^{1/2}}\|_2^2\Big)^{p_j} dx.$$

Using the rapid decay of the function $\phi^x_{R^{1/2}}$ it now suffices to show that

$$\int_{B(0,R)} \prod_{j=1}^m \Big(\sum_{\rho_j} \|G_j^{\rho_j}\|^2_{L^2(B(x,R^{1/2}))}\Big)^{p_j} dx \lesssim R^{\varepsilon+\frac{n}{2}-\frac{1}{2}\sum_{j=1}^m p_j n_j'} \prod_{j=1}^m \|G_j\|_2^{2p_j}. \tag{39}$$

Now let $\widetilde{G}_j^{\rho_j}$ be given by $G_j^{\rho_j} = \widetilde{G}_j^{\rho_j} * \widehat{\psi}_{\rho_j}$, where ψ_{ρ_j} is a Schwartz function which satisfies $\psi_{\rho_j} \sim 1$ on ρ_j and

$$|\widehat{\psi}_{\rho_j}(x+y)| \lesssim \frac{\chi_{\rho_j^*}(x)}{|\rho_j^*|}$$

uniformly in $x \in \mathbb{R}^n, y \in B(0, R^{1/2})$. Here ρ_j^* is a rectangle in $\mathbb{R}^n$ with n_j sides of length $O(R^{1/2})$ and n_j' sides of length $O(R)$, lying in an $O(R^{1/2})$-neighbourhood of an n_j'-dimensional affine subspace of $\mathbb{R}^n$, which, if the neighbourhoods U_j are chosen sufficiently small, is within distance ν of $\ker L_j$ (modulo translations). Applying the Cauchy–Schwarz inequality and integrating in $y \in B(0, R^{1/2})$, we have

$$\|G_j^{\rho_j}\|^2_{L^2(B(x,R^{1/2}))} \lesssim R^{n/2} \frac{\chi_{\rho_j^*}}{|\rho_j^*|} * |\widetilde{G}_j^{\rho_j}|^2(x)$$

uniformly in ρ_j, x and R. An application of (35), the scaling condition (3), followed by the bounded overlap property of the caps ρ_j, completes the proof of (39), and hence (38). □

It is interesting to note that in the particular case of the Loomis–Whitney datum, one can recover Theorem 1.2 from Theorem 1.3 by a simple Rademacher function argument (see [12]). However, when the codimensions $n_j' \neq 1$ this argument fails due to certain orientation restrictions on the dual objects ρ_j^* arising in the above wave-packet analysis.

3.3. **Nonlinear Brascamp–Lieb inequalities and Theorem 1.4.** The induction-on-scales argument we use here is very closely related to the one used in Section 3.1. We begin by introducing, for each $0 < \delta \leq 1$ and $n \in \mathbb{N}$, the class of functions

$$L^1(\mathbb{R}^n;\delta) := \{f \in L^1(\mathbb{R}^n) : f \geq 0 \text{ and } \tfrac{1}{2}f(y) \leq f(x) \leq 2f(y) \text{ whenever } |x-y| \leq \delta\}. \tag{40}$$

As remarked in [8], $u(c\delta,\cdot) := P_{c\delta} * \mu \in L^1(\mathbb{R}^n;\delta)$ for every finite Borel measure μ on $\mathbb{R}^n$, where P_t denotes the Poisson kernel and c a suitable dimensional constant. The defining property of a function $f \in L^1(\mathbb{R}^n;\delta)$ states that f is "essentially constant at scale δ", and in the context of the harmonic function, u, is a manifestation of the Harnack principle.

Proposition 3.3. *Under the hypotheses of Theorem 1.4, there exists a neighbourhood U of the origin in $\mathbb{R}^n$ and a constant $\kappa < \infty$ such that*

$$\int_U \prod_{j=1}^m (f_j \circ B_j)^{p_j} \lesssim \left(\log\left(\frac{1}{\delta}\right)\right)^\kappa \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j\right)^{p_j} \tag{41}$$

for all functions $f_j \in L^1(\mathbb{R}^{n_j};\delta)$, $1 \leq j \leq m$, and all $0 < \delta \leq 1$.

Before proving Proposition 3.3 we indicate how it implies Theorem 1.4. We begin with a simple observation. For each $1 \leq j \leq m$ let ψ_j be a Schwartz function on $\mathbb{R}^{n_j}$, and for each $\delta_1,\ldots,\delta_m \geq \delta > 0$, let $\psi_{j,\delta_j}(x) := \delta_j^{-n_j}\psi_j(\delta_j^{-1}x)$. Bounding $|\psi_j|$ by a suitably normalised Poisson kernel, as we may, it follows that for each nonnegative $g_j \in L^1(\mathbb{R}^{n_j})$ there is a $\widetilde{g}_j \in L^1(\mathbb{R}^{n_j};\delta)$ such that $|\psi_{j,\delta_j}| * g_j \lesssim \widetilde{g}_j$ and $\int \widetilde{g}_j \lesssim \int g_j$, with implicit constants uniform in $\delta_1,\ldots,\delta_m$ and δ. Thus by Proposition 3.3,

$$\int_U \prod_{j=1}^m \left((|\psi_{j,\delta_j}| * g_j) \circ B_j\right)^{p_j} \lesssim \left(\log\left(\frac{1}{\delta}\right)\right)^\kappa \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} g_j\right)^{p_j} \tag{42}$$

for all nonnegative $g_j \in L^1(\mathbb{R}^{n_j})$, $1 \leq j \leq m$, and $\delta_1,\ldots,\delta_m \geq \delta > 0$.

Let $\varepsilon > 0$. For each $1 \leq j \leq m$ let $\{P_{j,k}\}_{k=0}^\infty$ be the standard annular Littlewood–Paley projection operators on $\mathbb{R}^{n_j}$ with associated convolution kernels $\{\phi_{j,k}\}_{k=0}^\infty$. We choose these kernels such that for $k > 0$, $\widehat{\phi}_{j,k}(\xi) = \widehat{\phi}_j(2^{-k}\xi)$ for some fixed Schwartz function ϕ_j on $\mathbb{R}^{n_j}$ with Fourier support in the annulus $\{\xi \in \mathbb{R}^{n_j} : 1/4 \leq |\xi| \leq 2\}$, and such that $\phi_{j,0}$ is a Schwartz function with Fourier support in the unit ball of $\mathbb{R}^{n_j}$. Furthermore the functions $\{\widehat{\phi}_{j,k}\}_{k=0}^\infty$ are taken to form a partition of unity on $\mathbb{R}^{n_j}\backslash\{0\}$, so that $\sum_{k\geq 0} P_{j,k}$ is the identity for each j. For each $1 \leq j \leq m$ let $\widetilde{\phi}_j$ be a Schwartz function whose Fourier transform is equal to 1 on the Fourier support of ϕ_j, and define $\widetilde{\phi}_{j,k}$ in a similar way to $\phi_{j,k}$. Observe that $\widetilde{\phi}_{j,k} * \phi_{j,k} = \phi_{j,k}$ for all j,k.

We may thus write

$$\int_U \prod_{j=1}^m f_j \circ B_j = \sum_{k_1,\ldots,k_m \geq 0} \int_U \prod_{j=1}^m (P_{j,k_j} f_j) \circ B_j.$$

By symmetry we need only consider the above sum for $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$. Writing $P_{j,k_j} f_j = \widetilde{\phi}_{j,k_j} * (P_{j,k_j} f_j)$ and applying Hölder's inequality we have

$$\begin{aligned}\int_U \prod_{j=1}^m |P_{j,k_j} f_j| \circ B_j &\leq \int_U \left(|\widetilde{\phi}_{j,k_j}| * |P_{j,k_j} f_j|\right) \circ B_j \\ &\lesssim \int_U \prod_{j=1}^m \left(|\widetilde{\phi}_{j,k_j}| * |P_{j,k_j} f_j|^{q_j}\right)^{p_j} \circ B_j,\end{aligned}$$

which by (42) is, up to a bounded factor, bounded above by

$$2^{\varepsilon k_1/2}\prod_{j=1}^m \|P_{j,k_j}f_j\|_{q_j}.$$

Thus

$$\begin{aligned}\sum_{k_1\geq\cdots\geq k_m\geq 0}\int_U\prod_{j=1}^m|P_{j,k_j}|\circ B_j &\lesssim \sum_{k_1\geq\cdots\geq k_m\geq 0} 2^{\varepsilon k_1/2}\prod_{j=1}^m \|P_{j,k_j}f_j\|_{q_j}\\ &\lesssim \sum_{k_1\geq\cdots\geq k_m\geq 0} 2^{-\varepsilon k_1/2}\|f_1\|_{L^{q_1}_\varepsilon}\prod_{j=2}^m\|f_j\|_{q_j}\\ &\lesssim \sum_{k_1\geq\cdots\geq k_m\geq 0} 2^{-\varepsilon k_1/(2m)}\cdots 2^{-\varepsilon k_m/(2m)}\|f_1\|_{L^{q_1}_\varepsilon(\mathbb{R}^{d_1})}\prod_{j=2}^m\|f_j\|_{q_j}\\ &\lesssim \prod_{j=1}^m\|f_j\|_{L^{q_j}_\varepsilon(\mathbb{R}^{n_j})},\end{aligned}$$

as required.

Proof of Proposition 3.3. Let η be a positive real number to be determined. For $\delta>0$ let $\mathcal{C}(\delta)$ denote the best constant C in the inequality

$$\int_{[-\eta,\eta]^n}\prod_{j=1}^m(f_j\circ B_j)^{p_j}\leq C\prod_{j=1}^m\left(\int_{\mathbb{R}^{n_j}}f_j\right)^{p_j} \tag{43}$$

over all functions $f_j\in L^1(\mathbb{R}^{n_j};\delta)$, $1\leq j\leq m$. Of course Proposition 3.3 states that for some choice of η, depending only on the nonlinear maps $B_1,\ldots,B_m$ and exponents $p_1,\ldots,p_m$, there is a $\kappa<\infty$ for which $\mathcal{C}(\delta)\lesssim(\log(1/\delta))^\kappa$. This will follow upon iterating $O(\log\log(1/\delta))$ times the recursive inequality

$$\mathcal{C}(\delta)\lesssim\mathcal{C}(\sqrt{\delta}). \tag{44}$$

There will be more than one constraint placed on η, although the most significant will be a consequence of the local boundedness of the classical Brascamp–Lieb constant, established in Theorem 1.1. Since the B_j are smooth in a neighbourhood of the origin, and $\mathrm{d}B_j(0)=L_j$, we have that $\|\mathrm{d}B_j(x)-L_j\|\lesssim|x|$ in this neighbourhood. Thus, by Theorem 1.1, there exists $\eta_0>0$ such that $\mathrm{BL}((\mathrm{d}B_j(x))_{j=1}^m,\mathbf{p})<\infty$ uniformly in $|x|\leq\eta_0$.

In order to prove (44), we first partition $[-\eta,\eta]^n$ into a disjoint union of axis-parallel cubes Q of sidelength $\sqrt{\delta}$, and write

$$\int_{[-\eta,\eta]^n}\prod_{j=1}^m(f_j\circ B_j)^{p_j}=\sum_Q\int_Q\prod_{j=1}^m((f_j\chi_{B_j(Q)})\circ B_j)^{p_j}.$$

Taylor expanding B_j about $x_Q\in Q$ we obtain

$$B_j(x)=B_j(x_Q)+\mathrm{d}B_j(x_Q)(x-x_Q)+O(|x-x_Q|^2),$$

and so if $x\in Q$,

$$B_j(x)-(B_j(x_Q)+\mathrm{d}B_j(x_Q)(x-x_Q))=O(\delta).$$

Since $f_j\in L^1(\mathbb{R}^{n_j};\delta)$ we have that

$$f_j(B_j(x))\lesssim f_j(B_j(x_Q)+\mathrm{d}B_j(x_Q)(x-x_Q)),$$

uniformly in $x \in Q$ and $Q \subseteq [-\eta, \eta]^n$. By translation-invariance we have

$$\int_Q \prod_{j=1}^m ((f_j \chi_{B_j(Q)}) \circ B_j)^{p_j} \lesssim \mathrm{BL}((\mathrm{d}B_j(x_Q))_{j=1}^m, \mathbf{p}) \prod_{j=1}^m \left(\int_{B_j(Q)} f_j \right)^{p_j}$$

for all Q. Choosing $\eta \leq \eta_0$ we obtain

$$\int_{[-\eta,\eta]^n} \prod_{j=1}^m (f_j \circ B_j)^{p_j} \lesssim \sum_Q \prod_{j=1}^m \left(\int_{B_j(Q)} f_j \right)^{p_j}. \tag{45}$$

Since $\mathrm{d}B_j(0) = L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ is an isometry, and B_j is smooth in a neighbourhood of the origin, we have that (making $\eta > 0$ smaller if necessary), $|B_j(Q)| \sim \delta^{n_j/2}$ and

$$\frac{1}{|B_j(Q)|} \int_{B_j(Q)} f_j \lesssim P_{c\sqrt{\delta}} * f_j(B_j(x_Q))$$

uniformly in $x_Q \in Q$ and $Q \subseteq [-\eta, \eta]^n$. Here, as before, P_t denotes the Poisson kernel on $\mathbb{R}^{n_j}$ (the dimension dictated by context), and c a dimensional constant to be chosen momentarily. Thus

$$\int_{[-\eta,\eta]^n} \prod_{j=1}^m (f_j(B_j(x)))^{p_j} dx \lesssim \delta^{\frac{1}{2}(p_1 n_1 + \cdots + p_m n_m)} \sum_Q \prod_{j=1}^m (P_{c\sqrt{\delta}} * f_j(B_j(x_Q)))^{p_j}.$$

Averaging in the choices of $x_Q \in Q$, and using the scaling condition $\sum_{j=1}^m p_j n_j = n$, yields

$$\begin{aligned} \int_{[-\eta,\eta]^n} \prod_{j=1}^m (f_j(B_j(x)))^{p_j} dx &\lesssim \sum_Q \int_Q \prod_{j=1}^m (P_{c\sqrt{\delta}} * f_j(B_j(x)))^{p_j} dx \\ &\lesssim \int_{[-\eta,\eta]^n} \prod_{j=1}^m (P_{c\sqrt{\delta}} * f_j(B_j(x)))^{p_j} dx. \end{aligned}$$

Choosing $c > 0$ appropriately ensures that $P_{c\sqrt{\delta}} * f_j \in L^1(\mathbb{R}^{n_j}; \sqrt{\delta})$ for each j. The claimed inequality (44) follows. □

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