# Locally finite groups in which every non-cyclic subgroup is self-centralizing

Delizia , Costantino; Parker, Christopher; Moravec, Primoz ; Nicotera, Chiara ; Jezernik, Urban

*DOI:*
10.1016/j.jpaa.2016.06.015

# Locally finite groups in which every non-cyclic subgroup is self-centralizing

Costantino Delizia[a,*], Urban Jezernik[b], Primož Moravec[c], Chiara Nicotera[a], Chris Parker[d]

[a] *University of Salerno, Italy*
[b] *Institute of Mathematics, Physics, and Mechanics, Ljubljana, Slovenia*
[c] *University of Ljubljana, Slovenia*
[d] *University of Birmingham, United Kingdom*

**Abstract**

Locally finite groups having the property that every non-cyclic subgroup contains its centralizer are classified.

*Keywords:* Self-centralizing subgroup, Frobenius group, locally finite group

*2010 MSC:* 20F50, 20E34, 20D25

## 1. Introduction

A subgroup $H$ of a group $G$ is *self-centralizing* if the centralizer $C_G(H)$ is contained in $H$. In [1] it has been remarked that a locally graded group in which all non-trivial subgroups are self-centralizing has to be finite; therefore it has to be either cyclic of prime order or non-abelian of order being the product of two different primes.

In this article, we consider the more extensive class $\mathfrak{X}$ of all groups in which every non-cyclic subgroup is self-centralizing. In what follows we use the term $\mathfrak{X}$-groups in order to denote groups in the class $\mathfrak{X}$. The study of properties of $\mathfrak{X}$-groups was initiated in [1]. In particular, the first four authors determined the structure of finite $\mathfrak{X}$-groups which are either nilpotent, supersoluble or simple.

*Corresponding author

*Email addresses:* cdelizia@unisa.it (Costantino Delizia), urban.jezernik@imfm.si (Urban Jezernik), primoz.moravec@fmf.uni-lj.si (Primož Moravec), cnicoter@unisa.it (Chiara Nicotera), c.w.parker@bham.ac.uk (Chris Parker)

In this paper, Theorem 2.1 gives a complete classification of finite $\mathfrak{X}$-groups. We remark that this result does not depend on classification of the finite simple groups rather only on the classification of groups with dihedral or semidihedral Sylow 2-subgroups. We also determine the infinite soluble $\mathfrak{X}$-groups, and the locally finite $\mathfrak{X}$-groups the results being presented in Theorems 3.6 and 3.7. It turns out that these latter groups are suitable finite extensions either of the infinite cyclic group or of a Prüfer $p$-group, $\mathbb{Z}_{p^\infty}$, for some prime $p$.

We follow [2] for basic group theoretical notation. In particular, we note that $F^*(G)$ denotes the generalized Fitting subgroup of $G$, that is the subgroup of $G$ generated by all subnormal nilpotent or quasisimple subgroups of $G$. The latter subgroups are the components of $G$. We see from [2, Section 31] that distinct components commute. The fundamental property of the generalized Fitting subgroup that we shall use is that it contains its centralizer in $G$ [2, (31.13)]. We denote the alternating group and symmetric group of degree $n$ by $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$ respectively. We use standard notation for the classical groups. The notation $\mathrm{Dih}(n)$ denotes the dihedral group of order $n$ and $\mathrm{Q}_8$ is the quaternion group of order 8. The term quaternion group will cover groups which are often called generalized quaternion groups. The cyclic group of order $n$ is represented simply by $n$, so for example $\mathrm{Dih}(12) \cong 2 \times \mathrm{Dih}(6) \cong 2 \times \mathrm{Sym}(3)$. Finally $\mathrm{Mat}(10)$ denotes the Mathieu group of degree 10. The Atlas [3] conventions are used for group extensions. Thus, for example, $p^2{:}\mathrm{SL}_2(p)$ denotes the split extension of an elementary abelian group of order $p^2$ by $\mathrm{SL}_2(p)$.

**Acknowledgment**

We would like to thank Professor Hermann Heineken for pointing out an oversight in the statement of (2.1.3) of Theorem 2.1. We also thank our referee for valuable suggestions.

## 2. Finite $\mathfrak{X}$-groups

In this section we determine all the finite groups belonging to the class $\mathfrak{X}$. The main result is the following.

**Theorem 2.1.** *Let $G$ be a finite $\mathfrak{X}$-group. Then one of the following holds:*

(1) *If $G$ is nilpotent, then either*

(1.1) *$G$ is cyclic;*

(1.2) *$G$ is elementary abelian of order $p^2$ for some prime $p$;*

(1.3) *$G$ is an extraspecial $p$-group of order $p^3$ for some odd prime $p$; or*

(1.4) *$G$ is a dihedral, semidihedral or quaternion 2-group.*

(2) *If $G$ is supersoluble but not nilpotent, then, letting $p$ denote the largest prime divisor of $|G|$ and $P \in \mathrm{Syl}_p(G)$, we have that $P$ is a normal subgroup of $G$ and one of the following holds:*

(2.1) *$P$ is cyclic and either*

(2.1.1) *$G \cong D \ltimes C$, where $C$ is cyclic, $D$ is cyclic and every non-trivial element of $D$ acts fixed point freely on $C$ (so $G$ is a Frobenius group);*

(2.1.2) *$G = D \ltimes C$, where $C$ is a cyclic group of odd order, $D$ is a quaternion group, and $C_G(C) = C \times D_0$ where $D_0$ is a cyclic subgroup of index 2 in $D$ with $G/D_0$ a dihedral group; or*

(2.1.3) *$G = D \ltimes C$, where $D$ is a cyclic $q$-group, $C$ is a cyclic $q'$-group (here $q$ denotes the smallest prime dividing the order of $G$), $1 < Z(G) < D$ and $G/Z(G)$ is a Frobenius group;*

(2.2) *$P$ is extraspecial and $G$ is a Frobenius group with cyclic Frobenius complement of odd order dividing $p-1$.*

(3) *If $G$ is not supersoluble and $F^*(G)$ is nilpotent, then either (3.1) or (3.2) below holds.*

(3.1) *$F^*(G)$ is elementary abelian of order $p^2$, $F^*(G)$ is a minimal normal subgroup of $G$ and one of the following holds:*

(3.1.1) *$p = 2$ and $G \cong \mathrm{Sym}(4)$ or $G \cong \mathrm{Alt}(4)$; or*

(3.1.2) *$p$ is odd and $G = G_0 \ltimes N$ is a Frobenius group with Frobenius kernel $N$ and Frobenius complement $G_0$ which is itself an $\mathfrak{X}$-group. Furthermore, either*

(3.1.2.1) *$G_0$ is cyclic of order dividing $p^2 - 1$ but not dividing $p - 1$;*

(3.1.2.2) *$G_0$ is quaternion;*

(3.1.2.3) *$G_0$ is supersoluble as in (2.1.2) with $|C|$ dividing $p - \epsilon$ where $p \equiv \epsilon \pmod 4$;*

(3.1.2.4) *$G_0$ is supersoluble as in (2.1.3) with $D$ a 2-group, $C_D(C)$ a non-trivial maximal subgroup of $D$ and $|C|$ odd dividing $p - 1$ or $p + 1$;*

(3.1.2.5) *$G_0 \cong \mathrm{SL}_2(3)$;*

(3.1.2.6) *$G_0 \cong \mathrm{SL}_2(3)\cdot 2$ and $p \equiv \pm 1 \pmod 8$; or*

(3.1.2.7) *$G_0 \cong \mathrm{SL}_2(5)$ and 60 divides $p^2 - 1$.*

(3.2) *$F^*(G)$ is extraspecial of order $p^3$ and one of the following holds:*

(3.2.1) *$G \cong \mathrm{SL}_2(3)$ or $G \cong \mathrm{SL}_2(3)\cdot 2$ (with quaternion Sylow 2-subgroups of order 16); or*

(3.2.2) *$G = K \ltimes N$ where $N$ is extraspecial of order $p^3$ and exponent $p$ with $p$ an odd prime, $K$ centralizes $Z(N)$ and is cyclic of odd order dividing $p + 1$. Furthermore, $G/Z(N)$ is a Frobenius group.*

(4) *If $F^*(G)$ is not nilpotent, then either*

(4.1) *$F^*(G) \cong \mathrm{SL}_2(p)$ where $p$ is a Fermat prime, $|G/F^*(G)| \leq 2$ and $G$ has quaternion Sylow 2-subgroups; or*

(4.2) *$G \cong \mathrm{PSL}_2(9)$, $\mathrm{Mat}(10)$ or $\mathrm{PSL}_2(p)$ where $p$ is a Fermat or Mersenne prime.*

*Furthermore, all the groups listed above are $\mathfrak{X}$-groups.*

We make a brief remark about the group $\mathrm{SL}_2(3)\cdot 2$ and the groups appearing in part (4.1) of Theorem 2.1 in the case $G > F^*(G)$. To obtain such groups, take $F = \mathrm{SL}_2(p^2)$, then the groups in question are isomorphic to the normalizer in $F$ of the subgroup isomorphic to $\mathrm{SL}_2(p)$. We denote these groups by $\mathrm{SL}_2(p)\cdot 2$

to indicate that the extension is not split (there are no elements of order 2 in the outer half of the group).

We shall repeatedly use the fact that if $L$ is a subgroup of an $\mathfrak{X}$-group $X$, then $L$ is an $\mathfrak{X}$-group. Indeed, if $H \leq L$ is non-cyclic, then $C_L(H) \leq C_X(H) \leq H$.

The following elementary facts will facilitate our proof that the examples listed are indeed $\mathfrak{X}$-groups.

**Lemma 2.2.** *The finite group $X$ is an $\mathfrak{X}$-group if and only if $C_X(x)$ is an $\mathfrak{X}$-group for all $x \in X$ of prime order.*

*Proof.* If $X$ is an $\mathfrak{X}$-group, then, as $\mathfrak{X}$ is subgroup closed, $C_X(x)$ is an $\mathfrak{X}$-group for all $x \in X$ of prime order. Conversely, assume that $C_X(x)$ is an $\mathfrak{X}$-group for all $x \in X$ of prime order (and hence of any order). Let $H \leq X$ be non-cyclic. We shall show $C_X(H) \leq H$. If $C_X(H) = 1$, then $C_X(H) \leq H$ and we are done. So assume $x \in C_X(H)$ and $x \neq 1$. Then $H \leq C_X(x)$ which is an $\mathfrak{X}$-group. Hence $x \in C_{C_X(x)}(H) \leq H$. Therefore $C_X(H) \leq H$, and $X$ is an $\mathfrak{X}$-group. □

**Lemma 2.3.** *Suppose that $X$ is a Frobenius group with kernel $K$ and complement $L$. If $K$ and $L$ are $\mathfrak{X}$-groups, then $X$ is an $\mathfrak{X}$-group.*

*Proof.* Let $x \in X$ have prime order. Then, as $K$ and $L$ have coprime orders, $x \in K$ or $x$ is conjugate to an element of $L$. But then, since $X$ is a Frobenius group, either $C_X(x) \leq K$ or $C_X(x)$ is conjugate to a subgroup of $L$. Since $K$ and $L$ are $\mathfrak{X}$-groups, $C_X(x)$ is an $\mathfrak{X}$-group. Hence $X$ is an $\mathfrak{X}$-group by Lemma 2.2. □

The rest of this section is dedicated to the proof of Theorem 2.1; therefore $G$ always denotes a finite $\mathfrak{X}$-group. Parts (1) and (2) of Theorem 2.1 are already proved in [1, Theorems 2.2, 2.4, 3.2 and 3.4]. However, our statement in (2.1.3) adds further detail which we now explain. So, for a moment, assume that $G$ is supersoluble, $q$ is the smallest prime dividing $|G|$, $D$ is a cyclic $q$-group and $C$ is a cyclic $q'$-group. In addition, $1 \neq Z(G) = C_D(C)$. Assume that $d \in D \setminus Z(G)$. Then, as $d \notin Z(G)$, $C$ is not centralized by $d$. By coprime action, $C = [C, d] \times C_C(d)$ and so $Y = [C, d]\langle d \rangle$ is centralized by $C_C(d)$. As $Y$

is non-abelian and $C_C(d) \cap Y = 1$, we deduce that $C_C(d) = 1$. Hence $G/Z(G)$ is a Frobenius group. This means that we can assume that (1) and (2) hold and, in particular, we assume that $G$ is not supersoluble.

The following lemma provides the basic case subdivision of our proof.

**Lemma 2.4.** *One of the following holds:*

(i) $F^*(G)$ *is elementary abelian of order* $p^2$ *for some prime* $p$.

(ii) $F^*(G)$ *is extraspecial of order* $p^3$ *for some prime* $p$.

(iii) $F^*(G)$ *is quasisimple.*

*Proof.* Suppose first that $F^*(G)$ is nilpotent. Then its structure is given in part (1) of Theorem 2.1. Suppose that $F^*(G)$ is cyclic. Since $C_G(F^*(G)) = F^*(G)$, we have $G/F^*(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(F^*(G))$. Because the automorphism group of a cyclic group is abelian, we have that $G$ is supersoluble. Therefore, by our assumption concerning $G$, $F^*(G)$ is not cyclic. Hence $F^*(G)$ is either elementary abelian of order $p^2$ for some prime $p$, is extraspecial of order $p^3$ for some odd prime $p$ or $F^*(G)$ is a dihedral, semidihedral or quaternion 2-group. Since the automorphism groups of dihedral, semidihedral and quaternion groups of order at least 16 are 2-groups, we deduce that when $p = 2$ and $F^*(G)$ is non-abelian, $F^*(G)$ is extraspecial. This proves the lemma when $F^*(G)$ is nilpotent.

If $F^*(G)$ is not nilpotent, then there exists a component $K \leq F^*(G)$. As $F^*(G) = C_{F^*(G)}(K)K$ and $K$ is non-abelian, we have $F^*(G) = K$ and this is case (iii). □

**Lemma 2.5.** *Suppose that* $p$ *is a prime and* $F^*(G)$ *is extraspecial of order* $p^3$. *Then one of the following holds:*

(i) $G \cong \mathrm{SL}_2(3)$, $G \cong \mathrm{SL}_2(3){\cdot}2$ *(with quaternion Sylow* 2*-subgroups of order* 16*); or*

(ii) $G = NK$ *where* $N$ *is extraspecial of order* $p^3$ *of exponent* $p$ *with* $p$ *an odd prime,* $K$ *centralizes* $Z(N)$ *and is cyclic of odd order dividing* $p + 1$. *Furthermore,* $G/Z(N)$ *is a Frobenius group.*

*Proof.* Let $N = F^*(G)$. We have that $N$ is extraspecial of order $p^3$ by assumption. Suppose first that $p = 2$, then we have $N \cong \mathrm{Q}_8$ as the dihedral group of order 8 has no odd order automorphisms and $G$ is not a 2-group. Since $\mathrm{Aut}(\mathrm{Q}_8) \cong \mathrm{Sym}(4)$, $G/Z(N)$ is isomorphic to a subgroup of $\mathrm{Sym}(4)$ containing $\mathrm{Alt}(4)$. If $G/Z(N) \cong \mathrm{Alt}(4)$, then $G = NT \cong \mathrm{SL}_2(3)$ where $T$ is a cyclic subgroup of order 3. When $G/Z(N) \cong \mathrm{Sym}(4)$, taking $T \in \mathrm{Syl}_3(G)$, we have $NT \cong \mathrm{SL}_2(3)$, $N_G(T)$ has order 12 and $N_G(T)/Z(N) \cong \mathrm{Sym}(3)$. Since $N_G(T)$ is an $\mathfrak{X}$-group and $N_G(T)$ is supersoluble, we see that $N_G(T)$ is a product $DT$ where $D$ is cyclic of order 4 by (2.1.3). Because the Sylow 2-subgroups of $G$ are either dihedral, semidihedral or quaternion and $D \not\leq N$, we see that $ND$ is quaternion. Thus $G \cong \mathrm{SL}_2(3)\cdot 2$ as claimed in (i).

Assume that $p$ is odd. We know that the outer automorphism group of $N$ is isomorphic to a subgroup of $\mathrm{GL}_2(p)$ and $C_{\mathrm{Aut}(N)}(Z(N))/\mathrm{Inn}(N)$ is isomorphic to a subgroup of $\mathrm{SL}_2(p)$. Since $p$ is odd and the Sylow $p$-subgroups of $G$ are $\mathfrak{X}$-groups, we have $N \in \mathrm{Syl}_p(G)$ and $G/N$ is a $p'$-group by part (1) of Theorem 2.1. Set $Z = Z(N)$. Since $G/N$ and $N$ have coprime orders, the Schur Zassenhaus Theorem says that $G$ contains a complement $K$ to $N$. Set $K_1 = C_K(Z)$. Then $K_1$ commutes with $Z$ and so $K_1$ is cyclic. If $K_1 = 1$, then $|K|$ divides $p-1$ and we find that $G$ is supersoluble, which is a contradiction. Hence $K_1 \neq 1$. Let $x \in K_1$. Then $[N, x]$ and $C_N(x)$ commute by the Three Subgroups Lemma. Hence $C_N(x)$ centralizes $[N, x]\langle x\rangle$ which is non-abelian. It follows that $[N, x] = N$ and $C_N(x) = Z$. If $\langle x\rangle$ does not act irreducibly on $N/Z$, then there exists $Z < N_1 < N$ which is $\langle x\rangle$-invariant. If $N_1$ is cyclic, then, as $\langle x\rangle$ centralizes $\Omega_1(N_1) = Z$, $\langle x\rangle$ centralizes $N_1 > Z$, a contradiction. If $N_1$ is elementary abelian, then, as $\langle x\rangle$ centralizes $Z$, $[N_1, \langle x\rangle]$ has order at most $p$ by Maschke's Theorem. If $[N_1, \langle x\rangle] \neq 1$, then $[N_1, \langle x\rangle]\langle x\rangle$ is non-abelian and $Z$ centralizes $[N_1, \langle x\rangle]\langle x\rangle$, a contradiction. Hence $\langle x\rangle$ centralizes $N_1$ contrary to $C_N(\langle x\rangle) = Z$. We conclude that every element of $K_1$ acts irreducibly on $N/Z(N)$. In particular, since $K_1$ is isomorphic to a subgroup of $\mathrm{SL}_2(p)$, we have that $K_1$ is cyclic of odd order dividing $p+1$. Furthermore, as $K_1$ acts irreducibly on $N/Z(N)$, $N$ has exponent $p$.

By the definition of $K_1$, $|K/K_1|$ divides $|\mathrm{Aut}(Z)| = p-1$. Assume that $K \neq K_1$ and let $y \in K \setminus K_1$ have prime order $r$. Then $r$ does not divide $|K_1|$ and $Z\langle y\rangle$ is non-abelian. Since $K_1$ centralizes $Z$, we have $C_{K_1}(y) = 1$. Let $w \in K_1$ have prime order $q$. Then $\langle y\rangle\langle w\rangle$ is non-abelian and acts faithfully on $V = N/Z$. Therefore [2, 27.18] implies that $C_N(y) \neq 1$. As $C_N(y) \cap Z = 1$ and $C_N(y)$ centralizes $Z\langle y\rangle$, we have a contradiction. Hence $K = K_1$. Finally, we note that $NK/Z(N)$ is a Frobenius group.

It remains to show that the groups listed are $\mathfrak{X}$-groups. We consider the groups listed in (ii) and leave the groups in (i) to the reader. Assume that $H \leq G$ is non-cyclic. We shall show that $C_G(H) \leq H$. If $H \geq N$, then $C_G(H) \leq C_G(N) \leq N \leq H$ and we are done. Suppose that $H \leq N$. Then, as $N$ is extraspecial of exponent $p$, $H$ is elementary abelian of order $p^2$ and $C_N(H) = H$. Since $G/N$ is cyclic of odd order dividing $p+1$, we see that $N_G(H) = N$ and so $C_G(H) = C_N(H) = H$ and we are done in this case. Suppose that $H \not\leq N$ and $N \not\leq H$. Let $h \in H \setminus N$. Then, as $|G/N|$ divides $p+1$ and is odd, we either have $H \cap N = N$ or $H \cap N = Z$. So we must have $H \cap N = Z = Z(G)$. Now $H/Z \cong G/N$ is cyclic of order dividing $p+1$ and so we get that $H$ is cyclic, a contradiction. Thus $G$ is an $\mathfrak{X}$-group. $\square$

**Lemma 2.6.** *Suppose that $N = F^*(G)$ is elementary abelian of order $p^2$. Then one of the following holds:*

(i) $p = 2$, $G \cong \mathrm{Sym}(4)$ *or* $\mathrm{Alt}(4)$*; or*

(ii) *$p$ is odd and $G = NG_0$ is a Frobenius group with Frobenius kernel $N$ and Frobenius complement $G_0$ which is itself an $\mathfrak{X}$-group. Furthermore, either*

  (a) *$G_0$ is cyclic of order dividing $p^2-1$ but not dividing $p-1$;*

  (b) *$G_0$ is quaternion;*

  (c) *$G_0$ is supersoluble as in part* (2.1.2) *of Theorem* 2.1 *with $|C|$ dividing $p-\epsilon$ where $p \equiv \epsilon \pmod 4$;*

  (d) *$G_0$ is supersoluble as in part* (2.1.3) *of Theorem* 2.1 *with $D$ a 2-group, $C_D(C)$ a non-trivial maximal subgroup of $D$ and $|C|$ odd dividing $p-1$ or $p+1$;*

(e) $G_0 \cong \mathrm{SL}_2(3)$*;*

(f) $\mathrm{SL}_2(3)^{\cdot}2$ *and* $p \equiv \pm 1 \pmod 8$*; or*

(g) $G_0 \cong \mathrm{SL}_2(5)$ *and* $60$ *divides* $p^2 - 1$.

*Furthermore, all the groups listed are* $\mathfrak{X}$*-groups.*

*Proof.* We have $N$ has order $p^2$, is elementary abelian and $G/N$ is isomorphic to a subgroup of $\mathrm{GL}_2(p)$. If $p = 2$, then we quickly obtain part (i). So assume that $p$ is odd.

Suppose that $p$ divides the order of $G/N$. Let $P \in \mathrm{Syl}_p(G)$. Then $P$ is extraspecial of order $p^3$ and $P$ is not normal in $G$. Hence by [4, Theorem 2.8.4] there exists $g \in G$ such that $G \geq K = \langle P, P^g \rangle \cong p^2{:}\mathrm{SL}_2(p)$. Let $Z = Z(P)$, $t$ be an involution in $K$, $K_0 = C_K(t)$ and $P_0 = P \cap K_0$. Then, as $t$ inverts $N$, $K_0 \cong \mathrm{SL}_2(p)$, $P_0$ has order $p$ and centralizes $Z\langle t \rangle$, which is a contradiction as $Z\langle t \rangle \cong \mathrm{Dih}(2p)$. Hence $G/N$ is a $p'$-group.

Suppose that $x \in G \setminus N$. If $C_N(x) \neq 1$, then $C_N(x)$ centralizes $[N, x]\langle x \rangle$ which is non-abelian, a contradiction. Thus $C_N(x) = 1$ for all $x \in G \setminus N$. It follows that $G$ is a Frobenius group with Frobenius kernel $N$. Let $G_0$ be a Frobenius complement to $N$. As $G_0 \leq G$, $G_0$ is an $\mathfrak{X}$-group. Recall that the Sylow 2-subgroups of $G_0$ are either cyclic or quaternion and that the odd order Sylow subgroups of $G_0$ are all cyclic [5, V.8.7].

Assume that $N$ is not a minimal normal subgroup of $G$. Then $G/N$ is conjugate in $\mathrm{GL}_2(p)$ to a subgroup of the diagonal subgroup. Therefore $G$ is supersoluble, which is a contradiction. Hence $N$ is a minimal normal subgroup of $G$ and $G_0$ is isomorphic to an irreducible subgroup of $\mathrm{GL}_2(p)$. This completes the general description of the structure of $G$. It remains to determine the structure of $G_0$.

If $G_0$ is nilpotent, then Theorem 2.1 (1) applies to give $G_0$ is either quaternion or cyclic. In the latter case, as $G_0$ acts irreducibly on $N$ it is isomorphic to a subgroup of the multiplicative group of $\mathrm{GF}(p^2)$ and is not of order dividing $p-1$. This gives the structures in (ii) (a) and (b).

If $G_0$ is supersoluble, then the structure of $G_0$ is described in part (2.1) of

Theorem 2.1, as $\mathrm{GL}_2(p)$ contains no extraspecial subgroups of odd order. We adopt the notation from (2.1). By [5, V.8.18 c)], $Z(G_0) \neq 1$. Hence (2.1.1) cannot occur. Case (2.1.2) can occur and, as $C$ commutes with a cyclic subgroup of order at least 4 and $G_0$ is isomorphic to a subgroup of $\mathrm{GL}_2(p)$, $|C|$ divides $p-1$ if $p \equiv 1 \pmod 4$ and $|C|$ divides $p+1$ if $p \equiv 3 \pmod 4$. In the situation described in part (2.1.3) of Theorem 2.1, the groups have no 2-dimensional faithful representations unless $q = 2$ and $C_D(C)$ has index 2. In this case $|C|$ is an odd divisor of $p-1$ or $p+1$.

Suppose that $G_0$ is not supersoluble. Refereing to Lemma 2.4 and using the fact that the Sylow subgroups of $G_0$ are either cyclic or quaternion, we have that $F^*(G_0)$ is either quaternion of order 8 or $F^*(G_0)$ is quasisimple. In the first case we obtain the structures described in parts (b), (e) and (f) from Lemma 2.5 where for part (f) we note that we require $\mathrm{SL}_2(p)$ to have order divisible by 16.

If $F^*(G_0)$ is quasisimple, then Zassenhaus's Theorem [6, Theorem 18.6, p. 204] gives $G_0 = WM$ where $W \cong \mathrm{SL}_2(5)$ and $M$ is metacyclic. Since $G_0$ is an $\mathfrak{X}$-group, this means that $M \leq W$ and $G_0 \cong \mathrm{SL}_2(5)$. Since $\mathrm{SL}_2(5)$ is isomorphic to a subgroup of $\mathrm{GL}_2(p)$ only when $p = 5$ or 60 divides $p^2 - 1$ and $p \neq 5$ part (g) holds.

That $\mathrm{Sym}(4)$ and $\mathrm{Alt}(4)$ are $\mathfrak{X}$-groups is easy to check. The groups listed in (ii) are $\mathfrak{X}$-groups by Lemma 2.3. $\square$

The finite simple $\mathfrak{X}$-groups are determined in [1]. We have to extend the arguments to the cases where $F^*(G)$ is simple or quasisimple. This is relatively elementary.

**Lemma 2.7.** *Suppose that $F^*(G)$ is simple. Then $G \cong \mathrm{SL}_2(4)$, $\mathrm{PSL}_2(9)$, $\mathrm{Mat}(10)$ or $\mathrm{PSL}_2(p)$ where $p$ is a Fermat or Mersenne prime.*

*Proof.* Set $H = F^*(G)$. As $\mathfrak{X}$ is subgroup closed, $H$ is an $\mathfrak{X}$-group and so $H$ is one of the groups listed in the statement by Theorem 3.7 of [1]. Hence we obtain $H \cong \mathrm{SL}_2(4)$, $\mathrm{PSL}_2(9)$ or $\mathrm{PSL}_2(p)$ for $p$ a Fermat or Mersenne prime.

Suppose that $G > H$. If $H \cong \mathrm{SL}_2(4)$, then $G \cong \mathrm{Sym}(5)$ and the subgroup $2 \times \mathrm{Sym}(3)$ witnesses the fact that $\mathrm{Sym}(5)$ is not an $\mathfrak{X}$-group. Suppose $H \cong$

$\mathrm{PSL}_2(9) \cong \mathrm{Alt}(6)$. If $G \geq K \cong \mathrm{Sym}(6)$, then $G$ contains $\mathrm{Sym}(5)$ which is impossible. Therefore $G \cong \mathrm{PGL}_2(9)$ or $G \cong \mathrm{Mat}(10)$. In the first case, $G$ contains a subgroup $\mathrm{Dih}(20) \cong 2 \times \mathrm{Dih}(10)$ which is impossible. Thus $G \cong \mathrm{Mat}(10)$ and this group is easily shown to satisfy the hypothesis as all the centralizer of elements of prime order are $\mathfrak{X}$-groups.

If $H \cong \mathrm{PSL}_2(p)$, $p$ a Fermat or Mersenne prime, then $G \cong \mathrm{PGL}_2(p)$ and contains a dihedral group of order $2(p+1)$ and one of order $2(p-1)$. One of these is not a 2-group and this contradicts $G$ being an $\mathfrak{X}$-group. $\square$

**Lemma 2.8.** *Suppose that $F^*(G)$ is quasisimple but not simple. Then $F^*(G) \cong \mathrm{SL}_2(p)$ where $p$ is a Fermat prime, $|G/H| \leq 2$ and $G$ has quaternion Sylow 2-subgroups.*

*Proof.* Let $H = F^*(G)$ and $Z = Z(H)$. Since $H$ centralizes $Z$, we have $Z$ is cyclic. Let $S \in \mathrm{Syl}_2(H)$. If $Z \not\leq S$, then $S$ must be cyclic. Since groups with a cyclic Sylow 2-subgroup have a normal 2-complement [2, 39.2], this is impossible. Hence $Z \leq S$. In particular, $Z(G) \neq 1$ as the central involution of $H$ is central in $G$. It follows also that all the odd order Sylow subgroups of $G$ are cyclic. By part (1) of Theorem 2.1, $S$ is either abelian, dihedral, semidihedral or quaternion. If $S$ is abelian, then $S/Z$ is cyclic and again we have a contradiction. So $S$ is non-abelian. Thus $S/Z$ is dihedral (including elementary abelian of order 4). Hence $H/Z \cong \mathrm{Alt}(7)$ or $\mathrm{PSL}_2(q)$ for some odd prime power $q$ [4, Theorem 16.3]. Since the odd order Sylow subgroups of $G$ are cyclic, we deduce that $H \cong \mathrm{SL}_2(p)$ for some odd prime $p$. If $p-1$ is not a power of 2, then $H$ has a non-abelian subgroup of order $pr$ where $r$ is an odd prime divisor of $p-1$ which is centralized by $Z$. Hence $p$ is a Fermat prime.

Suppose that $G > H$ with $H \cong \mathrm{SL}_2(p)$, $p$ a Fermat prime. Note $G/H$ has order 2. Let $S \in \mathrm{Syl}_2(G)$. Then $S \cap H$ is a quaternion group. Suppose that $S$ is not quaternion Then there is an involution $t \in S \setminus H$. By the Baer-Suzuki Theorem, there exists a dihedral group $D$ of order $2r$ for some odd prime $r$ which contains $t$. Since $D$ and $Z$ commute, this is impossible. Hence $S$ is quaternion. This gives the structure described in the lemma.

It remains to demonstrate that the groups $\mathrm{SL}_2(p)$ and $\mathrm{SL}_2(p)\cdot 2$ with $p$ a Fermat prime are indeed $\mathfrak{X}$-groups. Let $G$ denote one of these group, $H = F^*(G) \cong \mathrm{SL}_2(p)$. Recall from the comments just after the statement of Theorem 2.1 that $G$ is isomorphic to a subgroup of $X = \mathrm{SL}_2(p^2)$. Let $V$ be the natural $\mathrm{GF}(p^2)$ representation of $X$ and thereby a representation of $G$. Assume that $L \leq G$ is non-cyclic. Since $H$ has no abelian subgroups which are not cyclic, $L$ is non-abelian and $L$ acts irreducibly on $V$. Schur's Lemma implies that $C_X(L)$ consists of scalar matrices and so has order at most 2. If $L$ has even order, then as $G$ has quaternion Sylow 2-subgroups, $L \geq C_G(L)$. So suppose that $L$ has odd order. Then using Dickson's Theorem [7, 260, page 285], as $p$ is a Fermat prime, we find that $L$ is cyclic, a contradiction. Thus $G$ is an $\mathfrak{X}$-group. □

*Proof of Theorem* 2.1. This follows from the combination of the lemmas in this section. □

## 3. Locally finite $\mathfrak{X}$-groups

It has been proved in [1, Theorem 2.2] that an infinite abelian group is in the class $\mathfrak{X}$ if and only if it is either cyclic or isomorphic to $\mathbb{Z}_{p^\infty}$ (the Prüfer $p$-group) for some prime $p$. Moreover, Theorem 2.3 and Theorem 2.5 of [1] imply that every infinite nilpotent $\mathfrak{X}$-group is abelian. We start this section by showing that some extensions of infinite abelian $\mathfrak{X}$-groups provide further examples of infinite $\mathfrak{X}$-groups.

**Lemma 3.1.** *The infinite dihedral group belongs to the class* $\mathfrak{X}$.

*Proof.* Write $G = \langle a, y \mid y^2 = 1, a^y = a^{-1}\rangle$. Then for every non-cyclic subgroup $H$ of $G$ there exist non-zero integers $n$ and $m$ such that $a^n, a^m y \in H$. It easily follows that $C_G(H) = 1$. □

**Lemma 3.2.** *Let* $G = A\langle y\rangle$ *where* $A \cong \mathbb{Z}_{2^\infty}$ *and* $\langle y\rangle$ *has order* 2 *or* 4*, with* $y^2 \in A$ *and* $a^y = a^{-1}$*, for all* $a \in A$*. Then* $G$ *belongs to the class* $\mathfrak{X}$.

*Proof.* It is clear that $G/A$ has order 2, and $A$ is the Fitting subgroup of $G$. Also $C_G(A) = A$ and $Z(G)$ is the subgroup of order 2 of $A$. Let $H$ be a non-cyclic subgroup of $G$ with $H \neq A$. Then $H \not\leq A$ as every proper subgroup of $A$ is cyclic. Pick any element $h \in H \setminus A$. Then $G = A\langle h\rangle$ since $|G : A| = 2$. Therefore by the Dedekind modular law we get $H = C\langle h\rangle$, where $C = A \cap H > 1$ is finite.

Since $h = bv$ with $b \in A$ and $v \in \langle y\rangle \setminus A$, we get $a^h = a^{-1}$ for all $a \in A$. In particular, $C_A(h)$ has order 2 and $C_G(h)$ has order 4. Since $C$ has a unique involution and $h \in C_G(H)$, we conclude that $C_G(H) \leq H$ and so $G$ is an $\mathfrak{X}$-group. □

When $\langle y\rangle$ has order 2, the group $G = A \rtimes \langle y\rangle$ of Lemma 3.2 is a generalized dihedral group.

Let $p$ denote any odd prime. Then, by Hensel's Theorem (see for instance [8, Theorem 127.5]), the group $\mathbb{Z}_{p^\infty}$ has an automorphism of order $p-1$, say $\phi$.

**Lemma 3.3.** *The groups $G = \mathbb{Z}_{p^\infty} \rtimes \langle \phi^j\rangle$ for $1 \leq j \leq p-1$ are $\mathfrak{X}$-groups.*

*Proof.* As $\mathfrak{X}$ is subgroup closed, it suffices to show that $G = \mathbb{Z}_{p^\infty} \rtimes \langle \phi\rangle$ is an $\mathfrak{X}$-group. Write the elements of $G$ in the form $ay$ with $a \in A \cong \mathbb{Z}_{p^\infty}$ and $y \in \langle \phi\rangle$. Suppose there exist non-trivial elements $a \in A$ and $y \in \langle \phi\rangle$ such that $a^y = a$. For a suitable non-negative integer $n$, the element $a^{p^n}$ has order $p$ and it is fixed by $y$. Then $y$ centralizes all elements of order $p$ in $A$, and therefore $y = 1$ by a result due to Baer (see, for instance, [9, Lemma 3.28]). This contradiction shows that $\langle \phi\rangle$ acts fixed point freely on $A$.

Let $H$ be any non-cyclic subgroup of $G$. Then, as $G/A$ is cyclic, $A \cap H \neq 1$. If $H = A$ then of course $C_G(H) = H$. Thus we can assume that there exist non-trivial elements $a, b \in A$ and $y \in \langle \phi\rangle$ such that $a, by \in H$. Let $g \in C_G(H)$. If $g \in A$ then $1 = [g, by] = [g, y]$, so $g = 1$. Now let $g = cz$ with $c \in A$ and $1 \neq z \in \langle \phi\rangle$. Thus $1 = [cz, a] = [z, a]$, and $a = 1$, a contradiction. Therefore $C_G(H) \leq H$ for all non-cyclic subgroups $H$ of $G$, so $G$ is an $\mathfrak{X}$-group. □

**Lemma 3.4.** *An infinite polycyclic group belongs to the class $\mathfrak{X}$ if and only if it is either cyclic or dihedral.*

*Proof.* Arguing as in the proof of Theorem 3.1 of [1], one can easily prove that every infinite polycyclic $\mathfrak{X}$-group is either cyclic or dihedral. On the other hand, the infinite dihedral group belongs to the class $\mathfrak{X}$ by Lemma 3.1. $\square$

**Proposition 3.5.** *A torsion-free soluble group belongs to the class $\mathfrak{X}$ if and only if it is cyclic.*

*Proof.* Let $G$ be a torsion-free soluble $\mathfrak{X}$-group. Then every abelian subgroup of $G$ is cyclic, so $G$ satisfies the maximal condition on subgroups by a result due to Mal'cev (see, for instance, [10, 15.2.1]). Thus $G$ is polycyclic by [10, 5.4.12]. Therefore $G$ has to be cyclic. $\square$

In next theorem we determine all infinite soluble $\mathfrak{X}$-groups.

**Theorem 3.6.** *Let $G$ be an infinite soluble group. Then $G$ is an $\mathfrak{X}$-group if and only if one of the following holds:*

(i) *$G$ is cyclic;*

(ii) *$G \cong \mathbb{Z}_{p^\infty}$ for some prime $p$;*

(iii) *$G$ is dihedral;*

(iv) *$G = A\langle y\rangle$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y\rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$;*

(v) *$G \cong A \rtimes D$, where $A \cong \mathbb{Z}_{p^\infty}$ and $1 \neq D \leq C_{p-1}$ for some odd prime $p$.*

*Proof.* First let $G$ be an $\mathfrak{X}$-group. If $G$ is abelian then (i) or (ii) holds by [1, Theorem 2.2]. Assume $G$ is non-abelian, and let $A$ be the Fitting subgroup of $G$. Then $A \neq 1$ and $C_G(A) \leq A$ as $G$ is soluble. Let $N$ be a nilpotent normal subgroup of $G$. Then $N$ is finite, as, otherwise, using $N$ is self-centralizing and $G/Z(N)$ is a subgroup of $\mathrm{Aut}(N)$, we obtain $G$ is finite, which is a contradiction. Thus [1, Theorems 2.3 and 2.5] imply that $N$ is abelian. In particular, as the product of any two normal nilpotent subgroups of $G$ is again a normal nilpotent subgroup by Fitting's Theorem, we see that the generators of $A = F(G)$ commute. Hence $A$ is abelian. As $A$ is infinite and abelian, $A = C_G(A)$ is either infinite cyclic or isomorphic to $\mathbb{Z}_{p^\infty}$ for some prime $p$. In the former case clearly $G' \leq A$.

In the latter case, let $C$ be any proper subgroup of $A$. Thus $C$ is finite cyclic. Moreover $C$ is characteristic in $A$, so it is normal in $G$, and $G/C_G(C)$ is abelian since it is isomorphic to a subgroup of $\mathrm{Aut}(C)$. It follows that $G' \leq C_G(C)$, and again $G' \leq C_G(A) = A$. Therefore $G/A$ is abelian.

If $A$ is infinite cyclic, then the argument in the proof of Theorem 3.1 of [1] shows that $G$ is dihedral. Thus (iii) holds.

Let $A \cong \mathbb{Z}_{p^\infty}$ for some prime $p$, and suppose there exists an element $x \in G$ of infinite order. Then $x \in G \setminus A$, and so there exists an element $y \in A$ such that $[x, y] \neq 1$. Then $\langle y \rangle$ is a finite normal subgroup of $G$, so conjugation by $x$ induces a non-trivial automorphism of $\langle y \rangle$. Since $\mathrm{Aut}(\langle y \rangle)$ is finite, it follows that there is an integer $n$ such that $[x^n, y] = 1$. Now $y$ is a torsion element and $x^n$ has infinite order and so $\langle x^n, y \rangle$ is neither periodic nor torsion free and this contradicts [1, Theorems 2.2]. Therefore $G$ is periodic, and $G/A$ is isomorphic to a periodic subgroup of automorphisms of $\mathbb{Z}_{p^\infty}$.

It is well-known that $\mathrm{Aut}(\mathbb{Z}_{p^\infty})$ is isomorphic to the multiplicative group of all $p$-adic units. It follows that periodic automorphisms of $\mathbb{Z}_{p^\infty}$ form a cyclic group having order 2 if $p = 2$, and order $p - 1$ if $p$ is odd (see, for instance, [11] for details). In the latter case (v) holds. Finally, let $p = 2$. Then $G/A = \langle yA \rangle$ has order 2, and $G = A\langle y \rangle$ with $y \notin A$ and $y^2 \in A$. Moreover $a^y = a^{-1}$, for all $a \in A$. If $y$ has order 2 then $G = A \rtimes \langle y \rangle$. Otherwise from $y^2 \in A$ it follows $y^2 = (y^2)^y = y^{-2}$, so $y$ has order 4. Therefore $G$ has the structure described in (iv).

On the other hand, Lemmas 3.1 – 3.3 show that the groups listed in (i) – (v) are $\mathfrak{X}$-groups. $\square$

Finally, we determine all infinite locally finite $\mathfrak{X}$-groups.

**Theorem 3.7.** *Let $G$ be an infinite locally finite group. Then $G$ is an $\mathfrak{X}$-group if and only if one of the following holds:*

(i) *$G \cong \mathbb{Z}_{p^\infty}$ for some prime $p$;*

(ii) *$G = A\langle y \rangle$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y \rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$;*

(iii) *$G \cong A \rtimes D$, where $A \cong \mathbb{Z}_{p^\infty}$ and $1 \neq D \leq C_{p-1}$ for some odd prime $p$.*

*Proof.* Any abelian subgroup of $G$ is either finite or isomorphic to $\mathbb{Z}_{p^\infty}$ for some prime $p$, so it satisfies the minimal condition on subgroups. Thus $G$ is a Černikov group by a result due to Šunkov (see, for instance [10, page 436, I]). Hence there exists an abelian normal subgroup $A$ of $G$ such that $A \cong \mathbb{Z}_{p^\infty}$ for some prime $p$, and $G/A$ is finite. It follows that $G$ is metabelian, arguing as in the proof of Theorem 3.6. Therefore the result follows from Theorem 3.6. □

**Corollary 3.8.** *Let $G$ be an infinite locally nilpotent group. Then $G$ is an $\mathfrak{X}$-group if and only if one of the following holds:*

(i) *$G$ is cyclic;*

(ii) *$G \cong \mathbb{Z}_{p^\infty}$ for some prime $p$;*

(iii) *$G = A\langle y\rangle$ where $A \cong \mathbb{Z}_{2^\infty}$ and $\langle y\rangle$ has order 2 or 4, with $y^2 \in A$ and $a^y = a^{-1}$, for all $a \in A$.*

*Proof.* Suppose $G$ is not abelian. Every finitely generated subgroup of $G$ is nilpotent, so it is either abelian or finite. It easily follows that all torsion-free elements of $G$ are central. Thus $G$ is periodic (see [12, Proposition 1]). Therefore $G$ is direct product of $p$-groups (see, for instance, [10, Proposition 12.1.1]). Actually only one prime can occur since $G$ is an $\mathfrak{X}$-group, so $G$ is a locally finite $p$-group. Thus the result follows by Theorem 3.7. □

## References


[1] C. Delizia, U. Jezernik, P. Moravec, C. Nicotera, Groups in which every non-cyclic subgroup contains its centralizer, J. Algebra Appl. 13 (5) (2014) 1350154 (11 pages). doi:10.1142/S0219498813501545.

[2] M. Aschbacher, Finite group theory, Cambridge University Press, 1993.

[3] J. Conway, R. Curtis, S. Norton, R. Parker, R. Wilson, Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J.G. Thackray, Oxford University Press, 1985.

[4] D. Gorenstein, Finite groups, Harper and Row, 1968.

[5] B. Huppert, Endliche Gruppen I, Springer Verlag, 1967.

[6] D. Passman, Permutation groups, Benjamin, 1968.

[7] L. Dickson, Linear groups: With an exposition of the Galois field theory, Dover Publications, 1958.

[8] L. Fuchs, Infinite abelian groups, Volume II, Academic Press, 1973.

[9] D. Robinson, Finiteness conditions and generalized soluble groups, Part 1, Springer-Verlag, 1972.

[10] D. Robinson, A course in the theory of groups, Springer-Verlag, 1996.

[11] H. Hasse, Zahlentheorie, Akademie, 1949.

[12] C. Delizia, Some remarks on aperiodic elements in locally nilpotent groups, Int. J. Algebra 1 (7) (2007) 311–315.