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# Locally finite groups in which every non-cyclic subgroup is self-centralizing 

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#### Abstract

Locally finite groups having the property that every non-cyclic subgroup contains its centralizer are classified.


Keywords: Self-centralizing subgroup, Frobenius group, locally finite group 2010 MSC: 20F50, 20E34, 20D25

## 1. Introduction

A subgroup $H$ of a group $G$ is self-centralizing if the centralizer $C_{G}(H)$ is contained in $H$. In [1] it has been remarked that a locally graded group in which all non-trivial subgroups are self-centralizing has to be finite; therefore it has to

5 be either cyclic of prime order or non-abelian of order being the product of two different primes.

In this article, we consider the more extensive class $\mathfrak{X}$ of all groups in which every non-cyclic subgroup is self-centralizing. In what follows we use the term $\mathfrak{X}$-groups in order to denote groups in the class $\mathfrak{X}$. The study of properties of $\mathfrak{X}$-groups was initiated in [1]. In particular, the first four authors determined the structure of finite $\mathfrak{X}$-groups which are either nilpotent, supersoluble or simple.

[^0]In this paper, Theorem 2.1 gives a complete classification of finite $\mathfrak{X}$-groups. We remark that this result does not depend on classification of the finite simple groups rather only on the classification of groups with dihedral or semidihedral 15 Sylow 2-subgroups. We also determine the infinite soluble $\mathfrak{X}$-groups, and the locally finite $\mathfrak{X}$-groups the results being presented in Theorems 3.6 and 3.7. It turns out that these latter groups are suitable finite extensions either of the infinite cyclic group or of a Prüfer $p$-group, $\mathbb{Z}_{p^{\infty}}$, for some prime $p$.

We follow [2] for basic group theoretical notation. In particular, we note that
${ }_{20} \quad F^{*}(G)$ denotes the generalized Fitting subgroup of $G$, that is the subgroup of $G$ generated by all subnormal nilpotent or quasisimple subgroups of $G$. The latter subgroups are the components of $G$. We see from [2, Section 31] that distinct components commute. The fundamental property of the generalized Fitting subgroup that we shall use is that it contains its centralizer in $G$ [2, (31.13)].
25 We denote the alternating group and symmetric group of degree $n$ by $\operatorname{Alt}(n)$ and $\operatorname{Sym}(n)$ respectively. We use standard notation for the classical groups. The notation $\operatorname{Dih}(n)$ denotes the dihedral group of order $n$ and $\mathrm{Q}_{8}$ is the quaternion group of order 8 . The term quaternion group will cover groups which are often called generalized quaternion groups. The cyclic group of order $n$ is represented simply by $n$, so for example $\operatorname{Dih}(12) \cong 2 \times \operatorname{Dih}(6) \cong 2 \times \operatorname{Sym}(3)$. Finally $\operatorname{Mat}(10)$ denotes the Mathieu group of degree 10. The Atlas [3] conventions are used for group extensions. Thus, for example, $p^{2}: \mathrm{SL}_{2}(p)$ denotes the split extension of an elementary abelian group of order $p^{2}$ by $\mathrm{SL}_{2}(p)$.

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## 2. Finite $\mathfrak{X}$-groups

In this section we determine all the finite groups belonging to the class $\mathfrak{X}$.
${ }_{40}$ The main result is the following.

Theorem 2.1. Let $G$ be a finite $\mathfrak{X}$-group. Then one of the following holds:
(1) If $G$ is nilpotent, then either
(1.1) $G$ is cyclic;
(1.2) $G$ is elementary abelian of order $p^{2}$ for some prime $p$;
(1.3) $G$ is an extraspecial p-group of order $p^{3}$ for some odd prime $p$; or (1.4) $G$ is a dihedral, semidihedral or quaternion 2-group.
(2) If $G$ is supersoluble but not nilpotent, then, letting $p$ denote the largest prime divisor of $|G|$ and $P \in \operatorname{Syl}_{p}(G)$, we have that $P$ is a normal subgroup of $G$ and one of the following holds:
(2.1) $P$ is cyclic and either
(2.1.1) $G \cong D \ltimes C$, where $C$ is cyclic, $D$ is cyclic and every non-trivial element of $D$ acts fixed point freely on $C$ (so $G$ is a Frobenius group);
(2.1.2) $G=D \ltimes C$, where $C$ is a cyclic group of odd order, $D$ is a quaternion group, and $C_{G}(C)=C \times D_{0}$ where $D_{0}$ is a cyclic subgroup of index 2 in $D$ with $G / D_{0}$ a dihedral group; or
(2.1.3) $G=D \ltimes C$, where $D$ is a cyclic $q$-group, $C$ is a cyclic $q^{\prime}$ group (here $q$ denotes the smallest prime dividing the order of $G$ ), $1<Z(G)<D$ and $G / Z(G)$ is a Frobenius group;
(2.2) $P$ is extraspecial and $G$ is a Frobenius group with cyclic Frobenius complement of odd order dividing $p-1$.
(3) If $G$ is not supersoluble and $F^{*}(G)$ is nilpotent, then either (3.1) or (3.2) below holds.
(3.1) $F^{*}(G)$ is elementary abelian of order $p^{2}, F^{*}(G)$ is a minimal normal subgroup of $G$ and one of the following holds:
(3.1.1) $p=2$ and $G \cong \operatorname{Sym}(4)$ or $G \cong \operatorname{Alt}(4)$; or
(3.1.2) $p$ is odd and $G=G_{0} \ltimes N$ is a Frobenius group with Frobenius kernel $N$ and Frobenius complement $G_{0}$ which is itself an $\mathfrak{X}$-group. Furthermore, either
(3.1.2.1) $G_{0}$ is cyclic of order dividing $p^{2}-1$ but not dividing $p-1$;
(3.1.2.2) $G_{0}$ is quaternion;
(3.1.2.3) $G_{0}$ is supersoluble as in (2.1.2) with $|C|$ dividing $p-\epsilon$ where $p \equiv \epsilon(\bmod 4) ;$
(3.1.2.4) $G_{0}$ is supersoluble as in (2.1.3) with $D$ a 2-group, $C_{D}(C) a$ non-trivial maximal subgroup of $D$ and $|C|$ odd dividing $p-1$ or $p+1$;
(3.1.2.5) $G_{0} \cong \mathrm{SL}_{2}(3)$;
(3.1.2.6) $G_{0} \cong \mathrm{SL}_{2}(3) \cdot 2$ and $p \equiv \pm 1(\bmod 8)$; or
(3.1.2.7) $G_{0} \cong \mathrm{SL}_{2}(5)$ and 60 divides $p^{2}-1$.
(3.2) $F^{*}(G)$ is extraspecial of order $p^{3}$ and one of the following holds:
(3.2.1) $G \cong \mathrm{SL}_{2}(3)$ or $G \cong \mathrm{SL}_{2}(3) \cdot 2$ (with quaternion Sylow 2-subgroups of order 16); or
(3.2.2) $G=K \ltimes N$ where $N$ is extraspecial of order $p^{3}$ and exponent $p$ with $p$ an odd prime, $K$ centralizes $Z(N)$ and is cyclic of odd order dividing $p+1$. Furthermore, $G / Z(N)$ is a Frobenius group.
(4) If $F^{*}(G)$ is not nilpotent, then either
(4.1) $F^{*}(G) \cong \mathrm{SL}_{2}(p)$ where $p$ is a Fermat prime, $\left|G / F^{*}(G)\right| \leq 2$ and $G$ has quaternion Sylow 2-subgroups; or
(4.2) $G \cong \mathrm{PSL}_{2}(9)$, $\operatorname{Mat}(10)$ or $\mathrm{PSL}_{2}(p)$ where $p$ is a Fermat or Mersenne prime.

Furthermore, all the groups listed above are $\mathfrak{X}$-groups.

We make a brief remark about the group $\mathrm{SL}_{2}(3) \cdot 2$ and the groups appearing in part (4.1) of Theorem 2.1 in the case $G>F^{*}(G)$. To obtain such groups, take $F=\mathrm{SL}_{2}\left(p^{2}\right)$, then the groups in question are isomorphic to the normalizer in $F$ of the subgroup isomorphic to $\mathrm{SL}_{2}(p)$. We denote these groups by $\mathrm{SL}_{2}(p) \cdot 2$
to indicate that the extension is not split (there are no elements of order 2 in the outer half of the group).

We shall repeatedly use the fact that if $L$ is a subgroup of an $\mathfrak{X}$-group $X$, then $L$ is an $\mathfrak{X}$-group. Indeed, if $H \leq L$ is non- cyclic, then $C_{L}(H) \leq C_{X}(H) \leq H$.

The following elementary facts will facilitate our proof that the examples listed are indeed $\mathfrak{X}$-groups.

Lemma 2.2. The finite group $X$ is an $\mathfrak{X}$-group if and only if $C_{X}(x)$ is an $\mathfrak{X}$-group for all $x \in X$ of prime order.

Proof. If $X$ is an $\mathfrak{X}$-group, then, as $\mathfrak{X}$ is subgroup closed, $C_{X}(x)$ is an $\mathfrak{X}$-group for all $x \in X$ of prime order. Conversely, assume that $C_{X}(x)$ is an $\mathfrak{X}$-group for all $x \in X$ of prime order (and hence of any order). Let $H \leq X$ be non- cyclic. We shall show $C_{X}(H) \leq H$. If $C_{X}(H)=1$, then $C_{X}(H) \leq H$ and we are done. So assume $x \in C_{X}(H)$ and $x \neq 1$. Then $H \leq C_{X}(x)$ which is an $\mathfrak{X}$-group. Hence $x \in C_{C_{X}(x)}(H) \leq H$. Therefore $C_{X}(H) \leq H$, and $X$ is an $\mathfrak{X}$-group.

Lemma 2.3. Suppose that $X$ is a Frobenius group with kernel $K$ and complement L. If $K$ and $L$ are $\mathfrak{X}$-groups, then $X$ is an $\mathfrak{X}$-group.

Proof. Let $x \in X$ have prime order. Then, as $K$ and $L$ have coprime orders, $x \in K$ or $x$ is conjugate to an element of $L$. But then, since $X$ is a Frobenius group, either $C_{X}(x) \leq K$ or $C_{X}(x)$ is conjugate to a subgroup of $L$. Since $K$ and $L$ are $\mathfrak{X}$-groups, $C_{X}(x)$ is an $\mathfrak{X}$-group. Hence $X$ is an $\mathfrak{X}$-group by Lemma 2.2.

The rest of this section is dedicated to the proof of Theorem 2.1 therefore $G$ always denotes a finite $\mathfrak{X}$-group. Parts (1) and (2) of Theorem 2.1 are already proved in [1, Theorems 2.2, 2.4, 3.2 and 3.4]. However, our statement in (2.1.3) adds further detail which we now explain. So, for a moment, assume that $G$ is supersoluble, $q$ is the smallest prime dividing $|G|, D$ is a cyclic $q$-group and $C$ is a cyclic $q^{\prime}$-group. In addition, $1 \neq Z(G)=C_{D}(C)$. Assume that $d \in D \backslash Z(G)$. Then, as $d \notin Z(G), C$ is not centralized by $d$. By coprime action, $C=[C, d] \times C_{C}(d)$ and so $Y=[C, d]\langle d\rangle$ is centralized by $C_{C}(d)$. As $Y$
is non-abelian and $C_{C}(d) \cap Y=1$, we deduce that $C_{C}(d)=1$. Hence $G / Z(G)$ is a Frobenius group. This means that we can assume that (1) and (2) hold and, in particular, we assume that $G$ is not supersoluble.

The following lemma provides the basic case subdivision of our proof.

Lemma 2.4. One of the following holds:
(i) $F^{*}(G)$ is elementary abelian of order $p^{2}$ for some prime $p$.
(ii) $F^{*}(G)$ is extraspecial of order $p^{3}$ for some prime $p$.
(iii) $F^{*}(G)$ is quasisimple.

Proof. Suppose first that $F^{*}(G)$ is nilpotent. Then its structure is given in part (1) of Theorem 2.1. Suppose that $F^{*}(G)$ is cyclic. Since $C_{G}\left(F^{*}(G)\right)=F^{*}(G)$, we have $G / F^{*}(G)$ is isomorphic to a subgroup of $\operatorname{Aut}\left(F^{*}(G)\right)$. Because the automorphism group of a cyclic group is abelian, we have that $G$ is supersoluble. Therefore, by our assumption concerning $G, F^{*}(G)$ is not cyclic. Hence $F^{*}(G)$ is either elementary abelian of order $p^{2}$ for some prime $p$, is extraspecial of order $p^{3}$ for some odd prime $p$ or $F^{*}(G)$ is a dihedral, semidihedral or quaternion 2group. Since the automorphism groups of dihedral, semidihedral and quaternion groups of order at least 16 are 2 -groups, we deduce that when $p=2$ and $F^{*}(G)$ is non-abelian, $F^{*}(G)$ is extraspecial. This proves the lemma when $F^{*}(G)$ is nilpotent.

If $F^{*}(G)$ is not nilpotent, then there exists a component $K \leq F^{*}(G)$. As $F^{*}(G)=C_{F^{*}(G)}(K) K$ and $K$ is non-abelian, we have $F^{*}(G)=K$ and this is case (iii).

Lemma 2.5. Suppose that $p$ is a prime and $F^{*}(G)$ is extraspecial of order $p^{3}$. Then one of the following holds:
(i) $G \cong \mathrm{SL}_{2}(3), G \cong \mathrm{SL}_{2}(3) \cdot 2$ (with quaternion Sylow 2-subgroups of order 16); or
(ii) $G=N K$ where $N$ is extraspecial of order $p^{3}$ of exponent $p$ with $p$ an odd prime, $K$ centralizes $Z(N)$ and is cyclic of odd order dividing $p+1$. Furthermore, $G / Z(N)$ is a Frobenius group.

Proof. Let $N=F^{*}(G)$. We have that $N$ is extraspecial of order $p^{3}$ by assumption. Suppose first that $p=2$, then we have $N \cong \mathrm{Q}_{8}$ as the dihedral group of order 8 has no odd order automorphisms and $G$ is not a 2-group. Since $\operatorname{Aut}\left(\mathrm{Q}_{8}\right) \cong \operatorname{Sym}(4), G / Z(N)$ is isomorphic to a subgroup of $\operatorname{Sym}(4)$ containing Alt(4). If $G / Z(N) \cong \operatorname{Alt}(4)$, then $G=N T \cong \mathrm{SL}_{2}(3)$ where $T$ is a cyclic subgroup of order 3 . When $G / Z(N) \cong \operatorname{Sym}(4)$, taking $T \in \operatorname{Syl}_{3}(G)$, we have $N T \cong \operatorname{SL}_{2}(3), N_{G}(T)$ has order 12 and $N_{G}(T) / Z(N) \cong \operatorname{Sym}(3)$. Since $N_{G}(T)$ is an $\mathfrak{X}$-group and $N_{G}(T)$ is supersoluble, we see that $N_{G}(T)$ is a product $D T$ where $D$ is cyclic of order 4 by (2.1.3). Because the Sylow 2-subgroups of $G$ are either dihedral, semidihedral or quaternion and $D \not \leq N$, we see that $N D$ is quaternion. Thus $G \cong \mathrm{SL}_{2}(3) \cdot 2$ as claimed in (i).

Assume that $p$ is odd. We know that the outer automorphism group of $N$ is isomorphic to a subgroup of $\mathrm{GL}_{2}(p)$ and $C_{\operatorname{Aut}(N)}(Z(N)) / \operatorname{Inn}(N)$ is isomorphic to a subgroup of $\mathrm{SL}_{2}(p)$. Since $p$ is odd and the Sylow $p$-subgroups of $G$ are $\mathfrak{X}$-groups, we have $N \in \operatorname{Syl}_{p}(G)$ and $G / N$ is a $p^{\prime}$-group by part (1) of Theorem 2.1. Set $Z=Z(N)$. Since $G / N$ and $N$ have coprime orders, the Schur Zassenhaus Theorem says that $G$ contains a complement $K$ to $N$. Set $K_{1}=C_{K}(Z)$. Then $K_{1}$ commutes with $Z$ and so $K_{1}$ is cyclic. If $K_{1}=1$, then $|K|$ divides $p-1$ and we find that $G$ is supersoluble, which is a contradiction. Hence $K_{1} \neq 1$. Let $x \in K_{1}$. Then $[N, x]$ and $C_{N}(x)$ commute by the Three Subgroups Lemma. Hence $C_{N}(x)$ centralizes $[N, x]\langle x\rangle$ which is non-abelian. It follows that $[N, x]=N$ and $C_{N}(x)=Z$. If $\langle x\rangle$ does not act irreducibly on $N / Z$, then there exists $Z<N_{1}<N$ which is $\langle x\rangle$-invariant. If $N_{1}$ is cyclic, then, as $\langle x\rangle$ centralizes $\Omega_{1}\left(N_{1}\right)=Z,\langle x\rangle$ centralizes $N_{1}>Z$, a contradiction. If $N_{1}$ is elementary abelian, then, as $\langle x\rangle$ centralizes $Z,\left[N_{1},\langle x\rangle\right]$ has order at most $p$ by Maschke's Theorem. If $\left[N_{1},\langle x\rangle\right] \neq 1$, then $\left[N_{1},\langle x\rangle\right]\langle x\rangle$ is non-abelian and $Z$ centralizes $\left[N_{1},\langle x\rangle\right]\langle x\rangle$, a contradiction. Hence $\langle x\rangle$ centralizes $N_{1}$ contrary to $C_{N}(\langle x\rangle)=Z$. We conclude that every element of $K_{1}$ acts irreducibly on $N / Z(N)$. In particular, since $K_{1}$ is isomorphic to a subgroup of $\mathrm{SL}_{2}(p)$, we have that $K_{1}$ is cyclic of odd order dividing $p+1$. Furthermore, as $K_{1}$ acts irreducibly on $N / Z(N), N$ has exponent $p$.

By the definition of $K_{1},\left|K / K_{1}\right|$ divides $|\operatorname{Aut}(Z)|=p-1$. Assume that $K \neq K_{1}$ and let $y \in K \backslash K_{1}$ have prime order $r$. Then $r$ does not divide $\left|K_{1}\right|$ and $Z\langle y\rangle$ is non-abelian. Since $K_{1}$ centralizes $Z$, we have $C_{K_{1}}(y)=1$. Let $w \in K_{1}$ have prime order $q$. Then $\langle y\rangle\langle w\rangle$ is non-abelian and acts faithfully on $V=N / Z$. Therefore [2, 27.18] implies that $C_{N}(y) \neq 1$. As $C_{N}(y) \cap Z=1$ and $C_{N}(y)$ centralizes $Z\langle y\rangle$, we have a contradiction. Hence $K=K_{1}$. Finally, we note that $N K / Z(N)$ is a Frobenius group.

It remains to show that the groups listed are $\mathfrak{X}$-groups. We consider the groups listed in (ii) and leave the groups in (i) to the reader. Assume that $H \leq G$ is non-cyclic. We shall show that $C_{G}(H) \leq H$. If $H \geq N$, then $C_{G}(H) \leq C_{G}(N) \leq N \leq H$ and we are done. Suppose that $H \leq N$. Then, as $N$ is extraspecial of exponent $p, H$ is elementary abelian of order $p^{2}$ and $C_{N}(H)=H$. Since $G / N$ is cyclic of odd order dividing $p+1$, we see that $N_{G}(H)=N$ and so $C_{G}(H)=C_{N}(H)=H$ and we are done in this case. Suppose that $H \not \leq N$ and $N \not \leq H$. Let $h \in H \backslash N$. Then, as $|G / N|$ divides $p+1$ and is odd, we either have $H \cap N=N$ or $H \cap N=Z$. So we must have $H \cap N=Z=Z(G)$. Now $H / Z \cong G / N$ is cyclic of order dividing $p+1$ and so we get that $H$ is cyclic, a contradiction. Thus $G$ is an $\mathfrak{X}$-group.

Lemma 2.6. Suppose that $N=F^{*}(G)$ is elementary abelian of order $p^{2}$. Then one of the following holds:
(i) $p=2, \quad G \cong \operatorname{Sym}(4)$ or $\operatorname{Alt}(4)$; or
(ii) $p$ is odd and $G=N G_{0}$ is a Frobenius group with Frobenius kernel $N$ and Frobenius complement $G_{0}$ which is itself an $\mathfrak{X}$-group. Furthermore, either
(a) $G_{0}$ is cyclic of order dividing $p^{2}-1$ but not dividing $p-1$;
(b) $G_{0}$ is quaternion;
(c) $G_{0}$ is supersoluble as in part (2.1.2) of Theorem 2.1 with $|C|$ dividing $p-\epsilon$ where $p \equiv \epsilon(\bmod 4) ;$
(d) $G_{0}$ is supersoluble as in part (2.1.3) of Theorem 2.1 with $D$ a 2-group, $C_{D}(C)$ a non-trivial maximal subgroup of $D$ and $|C|$ odd dividing $p-1$ or $p+1$;
(e) $G_{0} \cong \mathrm{SL}_{2}(3)$;
(f) $\mathrm{SL}_{2}(3) \cdot 2$ and $p \equiv \pm 1(\bmod 8)$; or
(g) $G_{0} \cong \mathrm{SL}_{2}(5)$ and 60 divides $p^{2}-1$.

Furthermore, all the groups listed are $\mathfrak{X}$-groups.

Proof. We have $N$ has order $p^{2}$, is elementary abelian and $G / N$ is isomorphic to a subgroup of $\mathrm{GL}_{2}(p)$. If $p=2$, then we quickly obtain part (i). So assume that $p$ is odd.

Suppose that $p$ divides the order of $G / N$. Let $P \in \operatorname{Syl}_{p}(G)$. Then $P$ is extraspecial of order $p^{3}$ and $P$ is not normal in $G$. Hence by [4, Theorem 2.8.4] there exists $g \in G$ such that $G \geq K=\left\langle P, P^{g}\right\rangle \cong p^{2}: \mathrm{SL}_{2}(p)$. Let $Z=Z(P), t$ be an involution in $K, K_{0}=C_{K}(t)$ and $P_{0}=P \cap K_{0}$. Then, as $t$ inverts $N$, $K_{0} \cong \mathrm{SL}_{2}(p), P_{0}$ has order $p$ and centralizes $Z\langle t\rangle$, which is a contradiction as $Z\langle t\rangle \cong \operatorname{Dih}(2 p)$. Hence $G / N$ is a $p^{\prime}$-group.

Suppose that $x \in G \backslash N$. If $C_{N}(x) \neq 1$, then $C_{N}(x)$ centralizes $[N, x]\langle x\rangle$ which is non-abelian, a contradiction. Thus $C_{N}(x)=1$ for all $x \in G \backslash N$. It follows that $G$ is a Frobenius group with Frobenius kernel $N$. Let $G_{0}$ be a Frobenius complement to $N$. As $G_{0} \leq G, G_{0}$ is an $\mathfrak{X}$-group. Recall that the Sylow 2-subgroups of $G_{0}$ are either cyclic or quaternion and that the odd order Sylow subgroups of $G_{0}$ are all cyclic [5, V.8.7].

Assume that $N$ is not a minimal normal subgroup of $G$. Then $G / N$ is conjugate in $\mathrm{GL}_{2}(p)$ to a subgroup of the diagonal subgroup. Therefore $G$ is supersoluble, which is a contradiction. Hence $N$ is a minimal normal subgroup of $G$ and $G_{0}$ is isomorphic to an irreducible subgroup of $\mathrm{GL}_{2}(p)$. This completes the general description of the structure of $G$. It remains to determine the structure of $G_{0}$.

If $G_{0}$ is nilpotent, then Theorem 2.1 (1) applies to give $G_{0}$ is either quaternion or cyclic. In the latter case, as $G_{0}$ acts irreducibly on $N$ it is isomorphic to a subgroup of the multiplicative group of $\operatorname{GF}\left(p^{2}\right)$ and is not of order dividing $p-1$. This gives the structures in (ii) (a) and (b).

If $G_{0}$ is supersoluble, then the structure of $G_{0}$ is described in part (2.1) of

Theorem 2.1. as $\mathrm{GL}_{2}(p)$ contains no extraspecial subgroups of odd order. We adopt the notation from (2.1). By [5, V.8.18 c)], $Z\left(G_{0}\right) \neq 1$. Hence (2.1.1) cannot occur. Case (2.1.2) can occur and, as $C$ commutes with a cyclic subgroup of order at least 4 and $G_{0}$ is isomorphic to a subgroup of $\mathrm{GL}_{2}(p),|C|$ divides $p-1$ if $p \equiv 1(\bmod 4)$ and $|C|$ divides $p+1$ if $p \equiv 3(\bmod 4)$. In the situation described in part (2.1.3) of Theorem 2.1, the groups have no 2-dimensional faithful representations unless $q=2$ and $C_{D}(C)$ has index 2 . In this case $|C|$ is an odd divisor of $p-1$ or $p+1$.

Suppose that $G_{0}$ is not supersoluble. Refereing to Lemma 2.4 and using the fact that the Sylow subgroups of $G_{0}$ are either cyclic or quaternion, we have that $F^{*}\left(G_{0}\right)$ is either quaternion of order 8 or $F^{*}\left(G_{0}\right)$ is quasisimple. In the first case we obtain the structures described in parts (b), (e) and (f) from Lemma 2.5 where for part (f) we note that we require $\mathrm{SL}_{2}(p)$ to have order divisible by 16 .

If $F^{*}\left(G_{0}\right)$ is quasisimple, then Zassenhaus's Theorem [6, Theorem 18.6, p. 204] gives $G_{0}=W M$ where $W \cong \mathrm{SL}_{2}(5)$ and $M$ is metacyclic. Since $G_{0}$ is an $\mathfrak{X}$-group, this means that $M \leq W$ and $G_{0} \cong \mathrm{SL}_{2}(5)$. Since $\mathrm{SL}_{2}(5)$ is isomorphic to a subgroup of $\mathrm{GL}_{2}(p)$ only when $p=5$ or 60 divides $p^{2}-1$ and $p \neq 5$ part (g) holds.

That $\operatorname{Sym}(4)$ and $\operatorname{Alt}(4)$ are $\mathfrak{X}$-groups is easy to check. The groups listed in (ii) are $\mathfrak{X}$-groups by Lemma 2.3 .

The finite simple $\mathfrak{X}$-groups are determined in [1]. We have to extend the arguments to the cases where $F^{*}(G)$ is simple or quasisimple. This is relatively elementary.

Lemma 2.7. Suppose that $F^{*}(G)$ is simple. Then $G \cong \operatorname{SL}_{2}(4), \operatorname{PSL}_{2}(9)$, Mat(10) or $\mathrm{PSL}_{2}(p)$ where $p$ is a Fermat or Mersenne prime.

Proof. Set $H=F^{*}(G)$. As $\mathfrak{X}$ is subgroup closed, $H$ is an $\mathfrak{X}$-group and so $H$ is one of the groups listed in the statement by Theorem 3.7 of [1]. Hence we obtain $H \cong \mathrm{SL}_{2}(4), \mathrm{PSL}_{2}(9)$ or $\mathrm{PSL}_{2}(p)$ for $p$ a Fermat or Mersenne prime.

Suppose that $G>H$. If $H \cong \operatorname{SL}_{2}(4)$, then $G \cong \operatorname{Sym}(5)$ and the subgroup $2 \times \operatorname{Sym}(3)$ witnesses the fact that $\operatorname{Sym}(5)$ is not an $\mathfrak{X}$-group. Suppose $H \cong$
$\operatorname{PSL}_{2}(9) \cong \operatorname{Alt}(6)$. If $G \geq K \cong \operatorname{Sym}(6)$, then $G$ contains $\operatorname{Sym}(5)$ which is impossible. Therefore $G \cong \mathrm{PGL}_{2}(9)$ or $G \cong \operatorname{Mat}(10)$. In the first case, $G$ contains a subgroup $\operatorname{Dih}(20) \cong 2 \times \operatorname{Dih}(10)$ which is impossible. Thus $G \cong \operatorname{Mat}(10)$ and this group is easily shown to satisfy the hypothesis as all the centralizer of elements of prime order are $\mathfrak{X}$-groups.

If $H \cong \operatorname{PSL}_{2}(p), p$ a Fermat or Mersenne prime, then $G \cong \operatorname{PGL}_{2}(p)$ and contains a dihedral group of order $2(p+1)$ and one of order $2(p-1)$. One of these is not a 2 -group and this contradicts $G$ being an $\mathfrak{X}$-group.

Lemma 2.8. Suppose that $F^{*}(G)$ is quasisimple but not simple. Then $F^{*}(G) \cong$ $\mathrm{SL}_{2}(p)$ where $p$ is a Fermat prime, $|G / H| \leq 2$ and $G$ has quaternion Sylow 2 -subgroups.

Proof. Let $H=F^{*}(G)$ and $Z=Z(H)$. Since $H$ centralizes $Z$, we have $Z$ is cyclic. Let $S \in \operatorname{Syl}_{2}(H)$. If $Z \not 又 S$, then $S$ must be cyclic. Since groups with a cyclic Sylow 2-subgroup have a normal 2-complement [2, 39.2], this is impossible. Hence $Z \leq S$. In particular, $Z(G) \neq 1$ as the central involution of $H$ is central in $G$. It follows also that all the odd order Sylow subgroups of $G$ are cyclic. By part (1) of Theorem 2.1. $S$ is either abelian, dihedral, semidihedral or quaternion. If $S$ is abelian, then $S / Z$ is cyclic and again we have a contradiction. So $S$ is non-abelian. Thus $S / Z$ is dihedral (including elementary abelian of order 4). Hence $H / Z \cong \operatorname{Alt}(7)$ or $\mathrm{PSL}_{2}(q)$ for some odd prime power $q$ [4. Theorem 16.3]. Since the odd order Sylow subgroups of $G$ are cyclic, we deduce that $H \cong \mathrm{SL}_{2}(p)$ for some odd prime $p$. If $p-1$ is not a power of 2 , then $H$ has a non-abelian subgroup of order $p r$ where $r$ is an odd prime divisor of $p-1$ which is centralized by $Z$. Hence $p$ is a Fermat prime.

Suppose that $G>H$ with $H \cong \mathrm{SL}_{2}(p), p$ a Fermat prime. Note $G / H$ has order 2. Let $S \in \operatorname{Syl}_{2}(G)$. Then $S \cap H$ is a quaternion group. Suppose that $S$ is not quaternion Then there is an involution $t \in S \backslash H$. By the Baer-Suzuki Theorem, there exists a dihedral group $D$ of order $2 r$ for some odd prime $r$ which contains $t$. Since $D$ and $Z$ commute, this is impossible. Hence $S$ is quaternion. This gives the structure described in the lemma.

It remains to demonstrate that the groups $\mathrm{SL}_{2}(p)$ and $\mathrm{SL}_{2}(p) \cdot 2$ with $p$ a Fermat prime are indeed $\mathfrak{X}$-groups. Let $G$ denote one of these group, $H=F^{*}(G) \cong$ $\mathrm{SL}_{2}(p)$. Recall from the comments just after the statement of Theorem 2.1 that $G$ is isomorphic to a subgroup of $X=\mathrm{SL}_{2}\left(p^{2}\right)$. Let $V$ be the natural $\mathrm{GF}\left(p^{2}\right)$ representation of $X$ and thereby a representation of $G$. Assume that $L \leq G$ is non-cyclic. Since $H$ has no abelian subgroups which are not cyclic, $L$ is non-abelian and $L$ acts irreducibly on $V$. Schur's Lemma implies that $C_{X}(L)$ consists of scalar matrices and so has order at most 2 . If $L$ has even order, then as $G$ has quaternion Sylow 2-subgroups, $L \geq C_{G}(L)$. So suppose that $L$ has odd order. Then using Dickson's Theorem [7, 260, page 285], as $p$ is a Fermat prime, we find that $L$ is cyclic, a contradiction. Thus $G$ is an $\mathfrak{X}$-group.

Proof of Theorem 2.1. This follows from the combination of the lemmas in this section.

## 3. Locally finite $\mathfrak{X}$-groups

It has been proved in [1, Theorem 2.2] that an infinite abelian group is in the class $\mathfrak{X}$ if and only if it is either cyclic or isomorphic to $\mathbb{Z}_{p^{\infty}}$ (the Prüfer $p$-group) for some prime $p$. Moreover, Theorem 2.3 and Theorem 2.5 of [1] imply that every infinite nilpotent $\mathfrak{X}$-group is abelian. We start this section by showing that some extensions of infinite abelian $\mathfrak{X}$-groups provide further examples of infinite $\mathfrak{X}$-groups.

Lemma 3.1. The infinite dihedral group belongs to the class $\mathfrak{X}$.

Proof. Write $G=\left\langle a, y \mid y^{2}=1, a^{y}=a^{-1}\right\rangle$. Then for every non-cyclic subgroup $H$ of $G$ there exist non-zero integers $n$ and $m$ such that $a^{n}, a^{m} y \in H$. It easily follows that $C_{G}(H)=1$.

Lemma 3.2. Let $G=A\langle y\rangle$ where $A \cong \mathbb{Z}_{2 \infty}$ and $\langle y\rangle$ has order 2 or 4 , with $y^{2} \in A$ and $a^{y}=a^{-1}$, for all $a \in A$. Then $G$ belongs to the class $\mathfrak{X}$.

Proof. It is clear that $G / A$ has order 2 , and $A$ is the Fitting subgroup of $G$. Also $C_{G}(A)=A$ and $Z(G)$ is the subgroup of order 2 of $A$. Let $H$ be a non-cyclic subgroup of $G$ with $H \neq A$. Then $H \not 又 A$ as every proper subgroup of $A$ is cyclic. Pick any element $h \in H \backslash A$. Then $G=A\langle h\rangle$ since $|G: A|=2$. Therefore by the Dedekind modular law we get $H=C\langle h\rangle$, where $C=A \cap H>1$ is finite.

Since $h=b v$ with $b \in A$ and $v \in\langle y\rangle \backslash A$, we get $a^{h}=a^{-1}$ for all $a \in A$. In particular, $C_{A}(h)$ has order 2 and $C_{G}(h)$ has order 4 . Since $C$ has a unique involution and $h \in C_{G}(H)$, we conclude that $C_{G}(H) \leq H$ and so $G$ is an $\mathfrak{X}$-group.

When $\langle y\rangle$ has order 2, the group $G=A \rtimes\langle y\rangle$ of Lemma 3.2 is a generalized dihedral group.

Let $p$ denote any odd prime. Then, by Hensel's Theorem (see for instance 8, Theorem 127.5]), the group $\mathbb{Z}_{p^{\infty}}$ has an automorphism of order $p-1$, say $\phi$.

Lemma 3.3. The groups $G=\mathbb{Z}_{p \infty} \rtimes\left\langle\phi^{j}\right\rangle$ for $1 \leq j \leq p-1$ are $\mathfrak{X}$-groups.

Proof. As $\mathfrak{X}$ is subgroup closed, it suffices to show that $G=\mathbb{Z}_{p^{\infty}} \rtimes\langle\phi\rangle$ is an $\mathfrak{X}$-group. Write the elements of $G$ in the form $a y$ with $a \in A \cong \mathbb{Z}_{p^{\infty}}$ and $y \in\langle\phi\rangle$. Suppose there exist non-trivial elements $a \in A$ and $y \in\langle\phi\rangle$ such that $a^{y}=a$. For a suitable non-negative integer $n$, the element $a^{p^{n}}$ has order $p$ and it is fixed by $y$. Then $y$ centralizes all elements of order $p$ in $A$, and therefore $y=1$ by a result due to Baer (see, for instance, [9, Lemma 3.28]). This contradiction shows that $\langle\phi\rangle$ acts fixed point freely on $A$.

Let $H$ be any non-cyclic subgroup of $G$. Then, as $G / A$ is cyclic, $A \cap H \neq 1$. If $H=A$ then of course $C_{G}(H)=H$. Thus we can assume that there exist non-trivial elements $a, b \in A$ and $y \in\langle\phi\rangle$ such that $a, b y \in H$. Let $g \in C_{G}(H)$. 355 If $g \in A$ then $1=[g, b y]=[g, y]$, so $g=1$. Now let $g=c z$ with $c \in A$ and $1 \neq z \in\langle\phi\rangle$. Thus $1=[c z, a]=[z, a]$, and $a=1$, a contradiction. Therefore $C_{G}(H) \leq H$ for all non-cyclic subgroups $H$ of $G$, so $G$ is an $\mathfrak{X}$-group.

Lemma 3.4. An infinite polycyclic group belongs to the class $\mathfrak{X}$ if and only if it is either cyclic or dihedral. every infinite polycyclic $\mathfrak{X}$-group is either cyclic or dihedral. On the other hand, the infinite dihedral group belongs to the class $\mathfrak{X}$ by Lemma 3.1 .

Proposition 3.5. A torsion-free soluble group belongs to the class $\mathfrak{X}$ if and only if it is cyclic.

Proof. Let $G$ be a torsion-free soluble $\mathfrak{X}$-group. Then every abelian subgroup of $G$ is cyclic, so $G$ satisfies the maximal condition on subgroups by a result due to Mal'cev (see, for instance, [10, 15.2.1]). Thus $G$ is polycyclic by [10, 5.4.12]. Therefore $G$ has to be cyclic.

In next theorem we determine all infinite soluble $\mathfrak{X}$-groups.

Theorem 3.6. Let $G$ be an infinite soluble group. Then $G$ is an $\mathfrak{X}$-group if and only if one of the following holds:
(i) $G$ is cyclic;
(ii) $G \cong \mathbb{Z}_{p^{\infty}}$ for some prime $p$;
(iii) $G$ is dihedral;
(iv) $G=A\langle y\rangle$ where $A \cong \mathbb{Z}_{2 \infty}$ and $\langle y\rangle$ has order 2 or 4 , with $y^{2} \in A$ and $a^{y}=a^{-1}$, for all $a \in A ;$
(v) $G \cong A \rtimes D$, where $A \cong \mathbb{Z}_{p^{\infty}}$ and $1 \neq D \leq C_{p-1}$ for some odd prime $p$.

Proof. First let $G$ be an $\mathfrak{X}$-group. If $G$ is abelian then (i) or (ii) holds by [1, Theorem 2.2]. Assume $G$ is non-abelian, and let $A$ be the Fitting subgroup of s. Then $A \neq 1$ and $C_{G}(A) \leq A$ as $G$ is soluble. Let $N$ be a nilpotent normal subgroup of $G$. Then $N$ is finite, as, otherwise, using $N$ is self-centralizing and $G / Z(N)$ is a subgroup of $\operatorname{Aut}(N)$, we obtain $G$ is finite, which is a contradiction. Thus [1, Theorems 2.3 and 2.5] imply that $N$ is abelian. In particular, as the product of any two normal nilpotent subgroups of $G$ is again a normal nilpotent subgroup by Fitting's Theorem, we see that the generators of $A=F(G)$ commute. Hence $A$ is abelian. As $A$ is infinite and abelian, $A=C_{G}(A)$ is either infinite cyclic or isomorphic to $\mathbb{Z}_{p \infty}$ for some prime $p$. In the former case clearly $G^{\prime} \leq A$.

In the latter case, let $C$ be any proper subgroup of $A$. Thus $C$ is finite cyclic. Moreover $C$ is characteristic in $A$, so it is normal in $G$, and $G / C_{G}(C)$ is abelian since it is isomorphic to a subgroup of $\operatorname{Aut}(C)$. It follows that $G^{\prime} \leq C_{G}(C)$, and again $G^{\prime} \leq C_{G}(A)=A$. Therefore $G / A$ is abelian.

If $A$ is infinite cyclic, then the argument in the proof of Theorem 3.1 of [1] shows that $G$ is dihedral. Thus (iii) holds.

Let $A \cong \mathbb{Z}_{p^{\infty}}$ for some prime $p$, and suppose there exists an element $x \in G$ of infinite order. Then $x \in G \backslash A$, and so there exists an element $y \in A$ such that $[x, y] \neq 1$. Then $\langle y\rangle$ is a finite normal subgroup of $G$, so conjugation by $x$ induces a non-trivial automorphism of $\langle y\rangle$. Since $\operatorname{Aut}(\langle y\rangle)$ is finite, it follows that there is an integer $n$ such that $\left[x^{n}, y\right]=1$. Now $y$ is a torsion element and $x^{n}$ has infinite order and so $\left\langle x^{n}, y\right\rangle$ is neither periodic nor torsion free and this contradicts [1, Theorems 2.2]. Therefore $G$ is periodic, and $G / A$ is isomorphic to a periodic subgroup of automorphisms of $\mathbb{Z}_{p^{\infty}}$.

It is well-known that $\operatorname{Aut}\left(\mathbb{Z}_{p^{\infty}}\right)$ is isomorphic to the multiplicative group of all $p$-adic units. It follows that periodic automorphisms of $\mathbb{Z}_{p^{\infty}}$ form a cyclic group having order 2 if $p=2$, and order $p-1$ if $p$ is odd (see, for instance, 11 ] for details). In the latter case (v) holds. Finally, let $p=2$. Then $G / A=\langle y A\rangle$ has order 2 , and $G=A\langle y\rangle$ with $y \notin A$ and $y^{2} \in A$. Moreover $a^{y}=a^{-1}$, for all $a \in A$. If $y$ has order 2 then $G=A \rtimes\langle y\rangle$. Otherwise from $y^{2} \in A$ it follows $y^{2}=\left(y^{2}\right)^{y}=y^{-2}$, so $y$ has order 4 . Therefore $G$ has the structure described in (iv).

On the other hand, Lemmas 3.1-3.3 show that the groups listed in (i) - (v) are $\mathfrak{X}$-groups.

Finally, we determine all infinite locally finite $\mathfrak{X}$-groups.
Theorem 3.7. Let $G$ be an infinite locally finite group. Then $G$ is an $\mathfrak{X}$-group if and only if one of the following holds:
(i) $G \cong \mathbb{Z}_{p^{\infty}}$ for some prime $p$;
(ii) $G=A\langle y\rangle$ where $A \cong \mathbb{Z}_{2 \infty}$ and $\langle y\rangle$ has order 2 or 4 , with $y^{2} \in A$ and $a^{y}=a^{-1}$, for all $a \in A ;$
(iii) $G \cong A \rtimes D$, where $A \cong \mathbb{Z}_{p^{\infty}}$ and $1 \neq D \leq C_{p-1}$ for some odd prime $p$.

Proof. Any abelian subgroup of $G$ is either finite or isomorphic to $\mathbb{Z}_{p^{\infty}}$ for some prime $p$, so it satisfies the minimal condition on subgroups. Thus $G$ is a Cernikov group by a result due to Šunkov (see, for instance [10, page 436, I]). Hence there exists an abelian normal subgroup $A$ of $G$ such that $A \cong \mathbb{Z}_{p \infty}$ for some prime $p$, and $G / A$ is finite. It follows that $G$ is metabelian, arguing as in the proof of Theorem 3.6 Therefore the result follows from Theorem 3.6.

Corollary 3.8. Let $G$ be an infinite locally nilpotent group. Then $G$ is an $\mathfrak{X}$-group if and only if one of the following holds:
(i) $G$ is cyclic;
(ii) $G \cong \mathbb{Z}_{p \infty}$ for some prime $p$;
(iii) $G=A\langle y\rangle$ where $A \cong \mathbb{Z}_{2 \infty}$ and $\langle y\rangle$ has order 2 or 4 , with $y^{2} \in A$ and $a^{y}=a^{-1}$, for all $a \in A$.

Proof. Suppose $G$ is not abelian. Every finitely generated subgroup of $G$ is nilpotent, so it is either abelian or finite. It easily follows that all torsion-free elements of $G$ are central. Thus $G$ is periodic (see [12, Proposition 1]). Therefore $G$ is direct product of $p$-groups (see, for instance, [10, Proposition 12.1.1]). Actually only one prime can occur since $G$ is an $\mathfrak{X}$-group, so $G$ is a locally finite $p$-group. Thus the result follows by Theorem 3.7

## References

[1] C. Delizia, U. Jezernik, P. Moravec, C. Nicotera, Groups in which every non-cyclic subgroup contains its centralizer, J. Algebra Appl. 13 (5) (2014) 1350154 (11 pages). doi:10.1142/S0219498813501545
[2] M. Aschbacher, Finite group theory, Cambridge University Press, 1993.
[3] J. Conway, R. Curtis, S. Norton, R. Parker, R. Wilson, Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J.G. Thackray, Oxford University Press, 1985.
[4] D. Gorenstein, Finite groups, Harper and Row, 1968.
[5] B. Huppert, Endliche Gruppen I, Springer Verlag, 1967.
[6] D. Passman, Permutation groups, Benjamin, 1968.
[7] L. Dickson, Linear groups: With an exposition of the Galois field theory,
[8] L. Fuchs, Infinite abelian groups, Volume II, Academic Press, 1973.
[9] D. Robinson, Finiteness conditions and generalized soluble groups, Part 1, Springer-Verlag, 1972.
[10] D. Robinson, A course in the theory of groups, Springer-Verlag, 1996.
[11] H. Hasse, Zahlentheorie, Akademie, 1949.
[12] C. Delizia, Some remarks on aperiodic elements in locally nilpotent groups, Int. J. Algebra 1 (7) (2007) 311-315.


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