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Good, Christopher; Fernández, Leobardo

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# SHADOWING FOR INDUCED MAPS OF HYPERSPACES

LEOBARDO FERNÁNDEZ AND CHRIS GOOD

ABSTRACT. Given a nonempty compact metric space  $X$  and a continuous function  $f: X \rightarrow X$ , we study shadowing and  $h$ -shadowing for the induced maps on hyperspaces, particularly in symmetric products,  $F_n(X)$ , and the hyperspace of compact subsets of  $X$ ,  $2^X$ . We prove that  $f$  has shadowing ( $h$ -shadowing) if and only if  $2^f$  has shadowing ( $h$ -shadowing).

## 1. INTRODUCTION

A continuous function  $f: X \rightarrow X$  on a compact metric space induces a number of maps on related spaces. There is a close relationship, for example, between the dynamical behaviour of  $f$ , the topological structure of the inverse limit space  $\varprojlim (X, f)$  and induced shift map on  $(X, f)$ . This situation has been extensively studied (see for example [4, 15, 29] and the references contained therein). Over the past few years there has been increasing interest in the study of induced map on the hyperspace of closed subsets and various of its subsets equipped with the Vietoris topology (or Hausdorff metric). This study was initiated by Bauer and Sigmund [5] and it has been argued [11] that, from a computational and domain theoretic point of view, this is the natural approach to dynamical systems.

Given a compact metric space  $X$ ,  $2^X$  is the hyperspace of nonempty closed subsets of  $X$  with the Vietoris topology. A continuous map  $f: X \rightarrow X$  induces a continuous map  $2^f: 2^X \rightarrow 2^X$  defined by  $2^f(A) = f(A)$ . A number of well-studied subspaces (such as the collections  $C_n(X)$  of closed sets with at most  $n$  components,  $F(X)$  of finite subsets, or  $F_n(X)$  of subsets sets with at most  $n$  points) are invariant under this map and therefore form dynamical systems in their own right. It turns out that a number of dynamical properties lift between these systems. For example,  $2^f$  is transitive if and only if it is weakly mixing if and only if  $f$  is weakly mixing [3, 27]. In [12] the authors study chain transitivity, chain recurrence and periodicity of induced maps on  $F_n(X)$  and  $2^X$ . Relationships between the entropy of

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the map  $f$  and the entropy of the induced maps on  $2^X$ ,  $C_n(X)$ ,  $F_n(X)$  and  $F(X)$  are studied in [13] and [18]. In [14] the authors study periodicity, recurrence, quasi periodicity, wandering points, shadowing, exactness and non-wandering for the induced map in the hyperspace  $F_n(X)$ . Induced maps on the symmetric products  $F_n(X)$  are also studied in [19] and [16].

Of particular relevance in the computation of a dynamical system is the notion of shadowing, which is the focus of this paper. Given a map  $f$ , a  $\delta$ -pseudo orbit is a (finite or infinite) sequence of points such that the distance between  $f(x_i)$  and  $x_{i+1}$  is less than  $\delta$ . A typical example of a pseudo orbit would be the points produced computationally in calculating the orbit of a point where there is a round off error. A pseudo orbit is said to be  $\varepsilon$ -shadowed if there is a real orbit whose points track the pseudo orbit within a distance of  $\varepsilon$ . The map  $f$  has the shadowing property if, for a given  $\varepsilon$ , there is a  $\delta$  such that  $\delta$ -pseudo orbits are  $\varepsilon$  shadowed. Shadowing has been studied in the context of numerical analysis [8, 7, 25], at times being cited as a prerequisite to achieving accurate mathematical models, and extensively as a property in its own right [9, 30, 20, 24, 26, 28]. Bowen was one of the first to consider this property in [6], where he used it in the study of  $\omega$ -limit sets of Axiom A diffeomorphisms.

Some work on the shadowing of induced hyperspace maps has been done. In [14] it is proved that, for any  $n \geq 1$ , if the restriction  $f_n$  of  $2^f$  to  $F_n(X)$  has shadowing, then  $f$  has shadowing. The authors also prove that if  $f$  has shadowing, then  $f_2$  has shadowing but give an example ( $z \mapsto z^2$  on  $S^1$ ) for which  $f$  has shadowing but  $f_n$  does not have shadowing for any  $n \geq 3$ . Interestingly, we prove below that the pseudo orbits in this example that cannot be shadowed in  $F_n(X)$  can be shadowed in  $F_m(X)$  for some  $m > n$ . Sakai [30] proves that a positively expansive map on a compact metric space has shadowing if and only if it is open. In [31] it is shown that the induced map  $2^f$  of a positively expansive open map  $f$  is open but need not be positively expansive. However, the authors show that such induced maps do have shadowing.

In this paper we show that, in fact  $2^f$  has shadowing if and only if  $f$  has shadowing.

If a map  $f: X \rightarrow X$  has shadowing, then the restriction  $f^{<\omega}$  of  $2^f$  to  $F(X)$  has shadowing for finite pseudo orbits. Since  $F(X)$  is not compact, this is not enough to show that  $F(X)$  has shadowing. However  $F(X)$  is dense in  $2^X$  and invariant under  $2^f$ , and this is enough, via a general result on shadowing in dense subspaces to prove that  $f$  has shadowing if and only

if  $2^f$  has shadowing. Using slightly different arguments we prove a similar result which says that  $f$  has the much stronger property of  $h$ -shadowing if and only if  $2^f$  does.

## 2. PRELIMINARIES

We start with some definitions.

**Definition 2.1.** Let  $X$  be a compact metric space. Consider the following hyperspaces of  $X$ :

- $2^X = \{A \subseteq X : A \text{ is nonempty and closed}\}$  is the *hyperspace of closed nonempty subsets of  $X$* .
- $C(X) = \{A \in 2^X : A \text{ is connected}\}$  is the *hyperspace of subcontinua of  $X$* .
- $F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\}$  is the  *$n$ -fold symmetric product of  $X$* .
- $F(X) = \bigcup_{n=1}^{\infty} F_n(X)$  is the collection of all finite subsets of  $X$ .

**Definition 2.2.** Given a map  $f: X \rightarrow Y$  between compact metric spaces, the induced maps are given in the following way:

- The *induced map*  $2^f: 2^X \rightarrow 2^Y$  is given by  $2^f(A) = f(A)$ .
- The *induced map*  $C(f): C(X) \rightarrow C(Y)$  is given by  $C(f) = 2^f|_{C(X)}$ .
- The *induced map*  $f_n: F_n(X) \rightarrow F_n(Y)$  is given by  $f_n = 2^f|_{F_n(X)}$ .
- The *induced map*  $f^{<\omega}: F(X) \rightarrow F(X)$  is given by  $f^{<\omega} = 2^f|_{F(X)}$ .

Given a metric space  $X$  with metric  $d$ . For any  $r > 0$  and any  $A \in 2^X$ , the *open ball about  $A$  of radius  $r$*  is given by:

$$N_X(A, r) = \{x \in X : d(x, A) < r\}.$$

For the special case when  $A = \{x\}$  we write  $N_X(x, r)$ . If  $X$  is a compact metric space with metric  $d$ , then (see for example [21])  $2^X$  is a compact metric space when equipped with the Hausdorff metric

$$H(A, B) = \inf\{\varepsilon > 0 : A \subseteq N_X(B, \varepsilon) \text{ and } B \subseteq N_X(A, \varepsilon)\}.$$

The topology generated by  $H$  coincides with the Vietoris topology.

## 3. SHADOWING

It is shown in [31] that if  $f$  is a positively expansive open map then  $2^f$  has shadowing. Here we prove that if one of the induced maps  $f_n$ ,  $C(f)$ ,  $2^f$  or  $f^{<\omega}$  has shadowing, then  $f$  has shadowing. Also we prove that if  $f$  has

shadowing, then  $f^{<\omega}$  has finite shadowing which, in turn, implies that  $2^f$  has shadowing.

We start with basic definitions. Let  $X$  be a compact metric space and let  $f: X \rightarrow X$  be a continuous function. For  $\delta > 0$ , the (finite or infinite) sequence  $\Gamma = \langle x_0, x_1, \dots \rangle$  of points in  $X$  is a  $\delta$ -pseudo orbit if  $d(f(x_i), x_{i+1}) < \delta$  for every  $i \geq 0$ . If  $\varepsilon > 0$ , we say that the sequence  $\langle y_0, y_1, \dots \rangle$   $\varepsilon$ -shadows  $\Gamma$  provided  $d(y_i, x_i) < \varepsilon$  for every  $i$ . If  $y_i = f^i(y)$  for some point  $y \in X$ , we say that  $y$  shadows the sequence  $\Gamma$ . We say that  $f$  has shadowing if for every  $\varepsilon > 0$  there is  $\delta > 0$  such that every  $\delta$ -pseudo orbit is  $\varepsilon$ -shadowed by some point in  $X$ . In the case that only finite pseudo orbits are shadowed, we say that  $f$  has finite shadowing. If  $X$  is compact,  $f$  has shadowing if and only if  $f$  has finite shadowing (see, for example, [2, Remark 1]).

We first prove a general result about shadowing that we assume to be well known.

**Lemma 3.1.** *Let  $X$  be a compact metric space, let  $f: X \rightarrow X$  be a continuous function and let  $Y$  be a dense invariant subset of  $X$ . Then  $f$  has finite shadowing if and only if  $f|_Y$  has finite shadowing.*

*Proof.* Assume first that  $f$  has shadowing. Let  $\varepsilon > 0$  and choose  $\delta$  such that every  $\delta$ -pseudo orbit in  $X$  is  $\varepsilon/2$ -shadowed. Let  $\Gamma = \langle y_0, y_1, \dots, y_r \rangle$  be a  $\delta$ -pseudo orbit in  $Y$ . Then  $\Gamma$  is a  $\delta$ -pseudo orbit in  $X$ . Since  $f$  has shadowing, there is a point  $x \in X$  which  $\varepsilon/2$ -shadows  $\Gamma$ , i.e.,  $d(f^i(x), y_i) < \frac{\varepsilon}{2}$ , for every  $i \in \{0, 1, 2, \dots, r\}$ . Since  $f$  is continuous, there is  $\eta_{r-1} > 0$ , with  $\eta_{r-1} < \frac{\varepsilon}{2}$  and  $f(N_X(f^{r-1}(x), \eta_{r-1})) \subseteq N_X(f^r(x), \frac{\varepsilon}{2})$ . Also, there is  $\eta_{r-2} > 0$ , with  $\eta_{r-2} < \eta_{r-1}$  and  $f(N_X(f^{r-2}(x), \eta_{r-2})) \subseteq N_X(f^{r-1}(x), \eta_{r-1})$ . Continuing this process, there is  $\eta_1 > 0$ , with  $\eta_1 < \eta_2$  and  $f(N_X(f(x), \eta_1)) \subseteq N_X(f^2(x), \eta_2)$ . Finally, there is  $\eta_0 > 0$ , with  $\eta_0 < \eta_1$  and  $f(N_X(x, \eta_0)) \subseteq N_X(f(x), \eta_1)$ . By construction, every  $y \in N_X(x, \eta_0) \cap Y$   $\varepsilon$ -shadows  $\Gamma$ .

Now assume that  $f|_Y$  has finite shadowing, let  $\varepsilon > 0$  and let  $\Gamma = \langle x_0, x_1, \dots, x_r \rangle$  be a  $\delta/3$ -pseudo orbit in  $X$ , where  $\delta$  is given by shadowing in  $f|_Y$  for  $\varepsilon/2$ . Since  $f$  is continuous and  $X$  is compact,  $f$  is uniformly continuous and there exists  $\eta > 0$  with  $\eta < \frac{\delta}{3}$  and  $\eta < \frac{\varepsilon}{2}$  such that if  $d(x, y) < \eta$ , then  $d(f(x), f(y)) < \frac{\delta}{3}$ . For each  $i \in \{0, 1, \dots, r\}$ , let  $y_i \in N_X(x_i, \eta) \cap Y$ . Hence  $d(f(x_i), f(y_i)) < \frac{\delta}{3}$ . Thus,  $\Gamma^* = \langle y_0, y_1, \dots, y_r \rangle$  is a  $\delta$ -pseudo orbit in  $Y$  because:

$$\begin{aligned} d(f(y_i), y_{i+1}) &\leq d(f(y_i), f(x_i)) + d(f(x_i), x_{i+1}) + d(x_{i+1}, y_{i+1}) \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta. \end{aligned}$$

Since  $f|_Y$  has shadowing, there is a point  $y \in Y$  which  $\frac{\varepsilon}{2}$ -shadows  $\Gamma^*$ . But then  $d(f^i(y), x_i) < d(f^i(y), y_i) + d(y_i, x_i) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ . Therefore,  $y$   $\varepsilon$ -shadows  $\Gamma$  and  $f$  has finite shadowing.  $\square$

Turning now to induced maps on hyperspaces, we start with a simple observation.

**Theorem 3.2.** *Let  $X$  be a compact metric space and let  $f: X \rightarrow X$  be a continuous function. Let  $n \geq 1$ . If any of  $f_n$ ,  $C(f)$ ,  $2^f$  or  $f^{<\omega}$  has shadowing, then  $f$  has shadowing.*

*Proof.* The proof is the identical in each case, so we prove it for  $2^f$ . So suppose that  $2^f$  has shadowing. Let  $\varepsilon > 0$  and let  $\delta > 0$  given by shadowing for  $2^f$ . Let  $\Gamma = \langle x_0, x_1, \dots, x_r \rangle$  be a  $\delta$ -pseudo orbit in  $X$ . Then  $\Gamma^* = \langle \{x_0\}, \{x_1\}, \dots, \{x_r\} \rangle$  is a  $\delta$ -pseudo orbit in  $2^X$ . Since  $2^f$  has shadowing, there is a point  $A \in 2^X$  which  $\varepsilon$ -shadows  $\Gamma^*$ . But then every point  $x$  of  $A$   $\varepsilon$ -shadows  $\Gamma$ .  $\square$

As mentioned above, in [14] it is shown that  $f$  has shadowing if and only if  $f_2$  has shadowing but that there is a map  $f$  with shadowing for which certain pseudo orbits in  $F_n(X)$  can only be shadowed in  $F_m(X)$  for some  $m > n$ . The fact that finite sets can always be shadowed by larger finite sets turns out to be a general property of shadowing maps.

**Theorem 3.3.** *Let  $X$  be a compact metric space and let  $f: X \rightarrow X$  be a continuous function. If  $f$  has shadowing, then  $f^{<\omega}$  has finite shadowing.*

*Proof.* Fix  $\varepsilon > 0$  and let  $\delta > 0$  be given by shadowing of  $f$ . Let  $\Gamma = \langle A_0, A_1, \dots, A_r \rangle$  be a finite  $\delta$ -pseudo orbit in  $F(X)$  and assume that  $|A_i| = n_i$  for each  $i \in \{0, 1, 2, \dots, r\}$ . We will construct a family of  $\delta$ -pseudo orbits in  $X$ , denoted  $\{\Gamma_j : j \leq n\}$ , for some  $n$ , such that, writing

$$\Gamma_j = \langle a_0^j, a_1^j, a_2^j, \dots, a_{r-1}^j, a_r^j \rangle,$$

we have  $A_i = \{a_i^j : j \leq n\}$  for each  $i \leq r$ .

To this end, suppose that  $A_r = \{a_r^1, a_r^2, \dots, a_r^{n_r}\}$ . For each  $j$ , with  $1 \leq j \leq n_r$ , we first construct a  $\delta$ -pseudo orbit in  $X$  with  $i$ th element in  $A_i$ , whose final element is  $a_r^j$ . Since  $\Gamma$  is a  $\delta$ -pseudo orbit, we can choose  $a_{r-1}^j \in A_{r-1}$  such that  $d(f(a_{r-1}^j), a_r^j) < \delta$ . Again, there is some  $a_{r-2}^j \in A_{r-2}$  such that  $d(f(a_{r-2}^j), a_{r-1}^j) < \delta$ . Continuing in this way, we have  $\delta$ -pseudo orbits

$$\Gamma_j = \langle a_0^j, a_1^j, a_2^j, \dots, a_{r-1}^j, a_r^j \rangle,$$

for each  $j \leq n_r$ , such that  $A_r = \{a_r^j : j \leq n_r\}$  and  $\{a_i^j : j \leq n_r\} \subseteq A_i$  for each  $i \leq r$ .

Let  $k = \max \{i < r : A_i \neq \{a_i^j : j \leq n_r\}\}$  (if no such  $k$  exists then we are done) and write  $A_k - \{a_k^j : j \leq n_r\} = \{a_k^j : n_r < j \leq n'_k\}$ . Exactly as for  $A_r$ , for each  $n_r < j \leq n'_k$ , we can construct a  $\delta$ -pseudo orbit  $\Gamma'_j = \langle a_0^j, a_1^j, \dots, a_k^j \rangle$  such that  $a_i^j \in A_i$ , for  $i \leq k$ . Clearly  $A_k = \{a_k^j : j \leq n'_k\}$ . Now, since  $f(a_k^j) \in f^{<\omega}(A_k)$  and  $H(f^{<\omega}(A_k), A_{k+1}) < \delta$ , there is  $a_{k+1}^j \in A_{k+1}$  such that  $d(f(a_k^j), a_{k+1}^j) < \delta$ . Similarly for each  $n_r < j \leq n'_k$  and  $k < i < r$ , there are  $a_i^j \in A_i$  such that  $d(f(a_i^j), a_{i+1}^j) < \delta$ , so that we can extend  $\Gamma'_j$  to a  $\delta$ -pseudo orbit  $\Gamma_j$  which starts in  $A_0$  and ends in  $A_r$ .

Repeating this process, it is clear then that we can construct the collection  $\{\Gamma_j : j \leq n\}$  of  $\delta$ -pseudo orbits in  $X$ . Since  $f$  has shadowing, for each  $\Gamma_j$  there is a point  $b_j \in X$  which  $\varepsilon$ -shadows  $\Gamma_j$ . Let  $B = \{b_0, b_1, \dots, b_m\}$ . By construction,  $B$   $\varepsilon$ -shadows  $\Gamma$ .  $\square$

Our main theorem now follows easily.

**Theorem 3.4.** *Let  $X$  be a compact metric space and let  $f: X \rightarrow X$  be a continuous function. Then  $f$  has shadowing if and only if  $2^f$  has shadowing.*

*Proof.* By Theorem 3.2, if  $2^f$  has shadowing, then  $f$  has shadowing. Conversely, if  $f$  has shadowing then  $f^{<\omega}$  has finite shadowing by Theorem 3.3, but  $F(X)$  is an invariant dense subset of  $2^X$ , so by Lemma 3.1  $2^f$  has shadowing.  $\square$

It also follows immediately from Theorems 3.2 and 3.3 that  $f^{<\omega}$  has finite shadowing whenever  $f_n$  has shadowing for some positive integer  $n$ . Example 3.5 is an example in which  $f_n$  has shadowing for every positive integer  $n$  but  $f^{<\omega}$  does not have infinite shadowing (recall that  $F(X)$  is not a compact space). The proof of this fact isolates the fundamental idea in the Example 12 of [14]. The fact that this system has shadowing is well-known folk lore, though we include a proof for completeness.

**Example 3.5.** Let  $X = \{\frac{1}{2^n} : n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$ , and let  $f: X \rightarrow X$  given by:  $f(0) = 0$ ,  $f(1) = 1$  and for every  $n \in \mathbb{N}$ ,  $n \geq 1$ ,  $f(\frac{1}{2^n}) = \frac{1}{2^{n-1}}$ . To see that  $f$  has shadowing let  $\varepsilon > 0$ . Let  $k_0$  be such that  $\frac{1}{2^{k_0+1}} < \varepsilon \leq \frac{1}{2^{k_0}}$  and choose  $\delta < \frac{1}{2^{k_0}} - \frac{1}{2^{k_0+1}}$ . Let  $\Gamma = \langle x_0, x_1, x_2, \dots \rangle$  be a  $\delta$ -pseudo orbit in  $X$ . Notice that if  $x_m = \frac{1}{2^{k_0}}$  for some  $m \geq 0$ , then  $\langle x_m, x_{m+1}, x_{m+2}, \dots \rangle$  must be a real orbit because of the choice of  $\delta$ . There are two cases to consider:  $\Gamma \subseteq [0, \varepsilon)$  or  $\Gamma \cap [\varepsilon, 1] \neq \emptyset$ . In the first case,  $y = 0$   $\varepsilon$ -follows  $\Gamma$ . In the second case, let  $m$  be the least non-negative integer such that  $x_m > \varepsilon$ . Either  $m = 0$  and  $\Gamma$  is a real orbit (which shadows itself), or  $x_m = \frac{1}{2^{k_0}}$  and so  $y = \frac{1}{2^{k_0+m}}$   $\varepsilon$ -follows  $\Gamma$ .

To see that  $f^{<\omega}$  does not have infinite shadowing let  $\varepsilon = \frac{1}{8}$  and let  $\delta > 0$ . Since  $\delta > 0$ , there is a positive number  $N \geq 3$  such that  $\frac{1}{2^N} < \delta$ . Let  $A_0 = \{0, 1\}$ ,  $A_1 = \{0, \frac{1}{2^N}, 1\}$ ,  $A_2 = f^{<\omega}(A_1) = \{0, \frac{1}{2^{N-1}}, 1\}$ ,  $A_3 = (f^{<\omega})^2(A_1) = \{0, \frac{1}{2^{N-2}}, 1\}$ ,  $\dots$ ,  $A_N = (f^{<\omega})^{N-1}(A_1) = \{0, \frac{1}{2^{N-(N-1)}}, 1\} = \{0, \frac{1}{2}, 1\}$ ,  $A_{N+1} = (f^{<\omega})^N(A_1) = \{0, 1\} = A_0$ . By construction:

$$\Gamma = \langle A_0, A_1, A_2, \dots, A_N, A_0, A_1, A_2, \dots \rangle$$

is a  $\delta$ -pseudo orbit in  $F(X)$  (which actually is a  $\delta$ -pseudo orbit in  $F_3(X)$ ). It is not difficult to see that the sets that  $\varepsilon$ -follows  $\Gamma$  are sets of the form  $B = \{0, \frac{1}{2^{kN}}, \frac{1}{2^{(k-1)N}}, \frac{1}{2^{(k-2)N}}, \dots, \frac{1}{2^N}, 1\}$ . The number of iterations that  $B$  is going to  $\varepsilon$ -follow  $\Gamma$  depend on the number  $k$ .

#### 4. $h$ -SHADOWING

The following definition was introduced in [2] and is motivated by the fact that shifts of finite type actually possess a stronger shadowing property,  $h$ -shadowing, or shadowing with exact hit, which happens to coincide with shadowing in shift spaces (but not necessarily in other systems). In fact [1], it turns out that open maps that are expanding (in the sense that, for some  $\mu > 1$  and small enough  $\varepsilon$ ,  $B_{\mu\varepsilon}(f(x)) \subseteq f(B_\varepsilon(x))$ ) have  $h$ -shadowing.

**Definition 4.1.** Let  $X$  be a compact metric space and let  $f: X \rightarrow X$  be a continuous function. We say that  $f$  has  $h$ -shadowing if for every  $\varepsilon > 0$  there is  $\delta > 0$  such that every finite  $\delta$ -pseudo orbit  $\Gamma = \langle x_0, x_1, \dots, x_r \rangle$ , there is a point  $x \in X$  such that  $d(f^i(x), x_i) < \varepsilon$  for every  $i < r$  and  $f^r(x) = x_r$ .

The proofs of the following two theorems are similar to the proofs of Theorem 3.2 and 3.3, respectively.

**Theorem 4.2.** *Let  $X$  be a compact metric space and let  $f: X \rightarrow X$  be a continuous function. If  $f_n$ ,  $2^f$  or  $f^{<\omega}$  has  $h$ -shadowing, then  $f$  has  $h$ -shadowing.*

**Theorem 4.3.** *Let  $X$  be a compact metric space and let  $f: X \rightarrow X$  be a continuous function. If  $f$  has  $h$ -shadowing, then  $f^{<\omega}$  has  $h$ -shadowing.*

Also, it follows immediately from Theorems 4.2 and 4.3 that if  $f_n$  has  $h$ -shadowing for every positive integer  $n$ , then  $f^{<\omega}$  has  $h$ -shadowing.

**Lemma 4.4.** *Let  $X$  be a compact metric space, let  $f: X \rightarrow X$  be a continuous function and let  $Y$  be a dense invariant subset of  $X$ . If  $f|_Y$  has  $h$ -shadowing, then  $f$  has  $h$ -shadowing.*



*Proof.* Suppose that  $f|_Y$  has  $h$ -shadowing. Let  $\varepsilon > 0$  and choose  $\delta > 0$  so that every finite  $\delta$ -pseudo orbit in  $Y$  is  $\varepsilon/2$   $h$ -shadowed some  $y \in Y$ . Let  $\Gamma = \langle x_0, x_1, \dots, x_r \rangle$  be a  $\delta/3$ -pseudo orbit in  $X$ . By the argument of Lemma 3.1, for each  $n > 0$  there is a  $\delta$ -pseudo orbit in  $Y$ ,  $\Gamma_n^* = \langle y_{n,0}, y_{n,1}, \dots, y_{n,r} \rangle$ , such that  $d(x_i, y_{n,i}) \leq \varepsilon/2$  and  $d(x_r, y_{n,r}) < 1/2^n$ . By  $h$ -shadowing in  $Y$ , there is a point  $y_n \in Y$  which  $\varepsilon/2$  shadows  $\Gamma_n^*$ . Then, if  $y$  is the limit in  $X$  of a convergent subsequence from  $\{y_n : n \geq 1\}$ ,  $y$   $\varepsilon$ - $h$ -shadows  $\Gamma$ .  $\square$

The converse of Lemma 4.4 is not true. To see this, let  $f: [0, 1] \rightarrow [0, 1]$  be the full tent map with slope 2. Then, according to [1, Example 5.4],  $f$  has  $h$ -shadowing. Let  $Y = ([0, 1] - \mathbb{Q}) \cup \{0, 1\}$ . Then  $Y$  is a dense invariant (but not strongly invariant) subset of  $[0, 1]$ , but  $f|_Y$  does not have  $h$ -shadowing because for any  $\delta$  there are  $\delta$ -pseudo orbits ending in 1, which obviously cannot be shadowed by an orbit that ends in 1. However, it is true that  $f$  has  $h$ -shadowing if and only if  $2^f$  has shadowing.

**Theorem 4.5.** *Let  $X$  be a compact metric space and let  $f: X \rightarrow X$  be a continuous function. Then  $2^f: 2^X \rightarrow 2^X$  has  $h$ -shadowing if and only if  $f^{<\omega}: F(X) \rightarrow F(X)$  has  $h$ -shadowing.*

*Proof.* Assume first that  $2^f$  has  $h$ -shadowing, let  $\varepsilon > 0$  and let  $\delta > 0$  be given by  $h$ -shadowing for  $2^f$ . Let  $\Gamma = \{A_0, A_1, A_2, \dots, A_r\}$  be a  $\delta$ -pseudo orbit in  $F(X)$ . Then  $\Gamma$  is a  $\delta$ -pseudo orbit in  $2^X$ . Since  $2^f$  has  $h$ -shadowing, there is a point  $C$  in  $2^X$  such that  $H(f^i(C), A_i) < \varepsilon/2$  for  $i \in \{0, 1, \dots, r-1\}$  and  $f^r(C) = A_r$ . Let  $B_r = A_r$  and assume that  $B_r = \{b_r^1, b_r^2, \dots, b_r^{n_r}\}$ . Since  $B_r = f^r(C)$ , then for each point  $b_r^j$  in  $B_r$  there is a point  $b_{r-1}^j$  in  $f^{r-1}(C)$  such that  $f(b_{r-1}^j) = b_r^j$ . Let  $B_{r-1}^* = \{b_{r-1}^1, b_{r-1}^2, \dots, b_{r-1}^{n_r}\}$ . If  $H(B_{r-1}^*, f^{r-1}(C)) < \varepsilon/2$ , let  $B_{r-1} = B_{r-1}^*$ . Otherwise, if  $H(B_{r-1}^*, f^{r-1}(C)) \geq \varepsilon/2$ , then there are finitely many points  $b_{r-1}^{n_r+1}, b_{r-1}^{n_r+2}, \dots, b_{r-1}^{n_r+k}$  in  $f^{r-1}(C) \setminus N_X(B_r, \varepsilon/2)$  such that if  $B_{r-1} = \{b_{r-1}^1, b_{r-1}^2, \dots, b_{r-1}^{n_r}, b_{r-1}^{n_r+1}, b_{r-1}^{n_r+2}, \dots, b_{r-1}^{n_r+k}\}$  then  $B_{r-1} \subseteq f^{r-1}(C)$  and  $H(B_{r-1}, f^{r-1}(C)) < \varepsilon/2$ , which implies  $H(B_{r-1}, A_{r-1}) < \varepsilon$ . Rename the points in  $B_{r-1}$  as follows:  $B_{r-1} = \{b_{r-1}^1, b_{r-1}^2, \dots, b_{r-1}^{n_{r-1}}\}$ . Continuing this process we obtain  $B_0 = \{b_0^1, b_0^2, \dots, b_0^{n_0}\}$ , finite subset of  $C$ , which  $\varepsilon$ -shadows  $\Gamma$  and, by construction,  $f^r(B_0) = A_r$ . Thus,  $f^{<\omega}$  has  $h$ -shadowing.

For the converse just recall that  $2^f|_{F(X)} = f^{<\omega}$ , therefore, if  $f$  has  $h$ -shadowing then so does  $2^f$  by Lemma 4.4.  $\square$

As a consequence of Theorem 4.3 and Theorem 4.5 we have the following theorem.

**Theorem 4.6.** *Let  $X$  be a compact metric space and let  $f: X \rightarrow X$  be a continuous function. Then  $f$  has  $h$ -shadowing if and only if  $2^f$  has  $h$ -shadowing.*

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SCHOOL OF MATHEMATICS; UNIVERSITY OF BIRMINGHAM; BIRMINGHAM, UNITED KINGDOM B15 2TT

*E-mail address:* l.fernandez@bham.ac.uk, leobardof@ciencias.unam.mx

SCHOOL OF MATHEMATICS; UNIVERSITY OF BIRMINGHAM; BIRMINGHAM, UNITED KINGDOM B15 2TT

*E-mail address:* c.good@bham.ac.uk