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A 2-LOCAL IDENTIFICATION OF $P\Omega_8^+(3)$

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ABSTRACT. This paper is devoted to the proof of an identification theorem for $\Omega_8^+(2)$ and $P\Omega_8^+(3)$. The main theorem will be applied in the programme aimed at determining the almost simple groups which have parabolic characteristic 2. The proof involves a novel application of a recent theorem due to Meierfrankenfeld, Weiss and the second author which identifies Lie type groups by their residual structure.

1. INTRODUCTION

In projects aimed at classifying finite simple groups, naming the group is an essential part and it often leads to that most delicate group theoretical investigations. The generic finite simple group is a Lie type group. In this article we present one such identification theorem which is important in the investigation of groups of parabolic characteristic 2 and so for the on-going research to classify such groups independently of the classification of the finite simple groups. The proof of our main theorem involves an application of a uniqueness theorem of Meierfrankenfeld, Weiss and the second author [9] to recognize $P\Omega_8^+(3)$. Traditionally, to recognize this group either a Phan system would need to be constructed or, once the centralizer of an involution has been determined, Aschbacher's Classical Involution Theorem [1, 2] could be invoked. The advantage of recognising the group via [9] is that the proof of the main theorem in [9] is entirely elementary. The proof of our Theorem 1 below, can be seen as an exemplar for the proof of theorems where Lie type groups need to be identified.

We recall that a simple Lie type group G defined in characteristic p, p a prime, has the feature that for $S \in \mathrm{Syl}_p(G)$ and $1 \neq X \leq S$,

$$C_G(O_p(N_G(X))) \;\leq\; O_p(N_G(X)) \tag{1}$$

where $O_p(N_G(X))$ is the largest normal p-subgroup of $N_G(X)$. The groups $N_G(X)$ are called *p-local subgroups* of G. Condition (1) roughly says that $O_p(N_G(X))$ is relatively large.

An arbitrary group which satisfies property (1) for all non-trivial $X \leq S$ is said to have *local characteristic* p and, if we restrict this requirement just to non-trivial subgroups X which are normal in S, then such groups are said to have *parabolic characteristic* p.

Our main theorem is as follows.

Theorem 1. *Suppose that G is a group of parabolic characteristic* 2 *and H is a subgroup of G of odd index. If $F^*(H) \cong \Omega_8^+(2)$ and $H = N_G(F^*(H))$, then $F^*(G) \cong \Omega_8^+(2)$ or $\mathrm{P}\Omega_8^+(3)$.*

The hypothesis of Theorem 1 arises naturally in the approach adopted to classifying groups G of parabolic characteristic p (p a prime) led by Meierfrankenfeld, Stellmacher and Stroth and exemplified by the work in [8] which aims at a deeper understanding of groups of local and parabolic characteristic p. The main theorem of this paper is applicable in the situation when $p = 2$ in [8].

Let us first explain the general idea behind the approach to groups G of parabolic characteristic p for arbitrary primes p taken in [8]. Initially their attention is targeted on the p-local subgroups L of G which do not centralize a p-central element (a p-element with centralizer of p' index). They do not completely determine the structure of L, but they give precise information about the structure of $L/O_p(L)$. Going beyond [8], the next step is to determine large parts of the centralizer of a p-central element of order p. The generic examples of groups of parabolic characteristic p are groups of Lie type in characteristic p. Let X be such a simple group of rank at least 3 and fix a set $\{P_1, \dots, P_n\}$ of maximal parabolic subgroups containing a fixed Borel subgroup of X. The objective of the work in [8] and the upcoming research is to discover a collection of subgroups $L_1, \dots, L_n$ which contain a common Sylow p-subgroup such that the quotient groups $L_i/O_p(L_i)$ match up with the quotients $O^{p'}(P_i/O_p(P_i))$, $1 \leq i \leq n$. It is not important that $L_1, \dots, L_n$ are p-local subgroups and they do not need to contain a full Sylow p-subgroup of G. Once this goal is attained, the intention is to use the main result from [9] to find that $\langle L_1, \dots, L_n\rangle$ is isomorphic to X. Furthermore it follows that $H = N_G(\langle L_1, \dots, L_n\rangle)$ contains a Sylow p-subgroup of G and $\langle L_1, \dots, L_n\rangle = F^*(H)$. The proof of Theorem 1 shows that this method od argument is practical and demonstrates how this also might be achieved in more general situations.

The result in [9] roughly says the following. Let X be a finite group of Lie type in characteristic p and rank at least 3, the target group, with Coxeter diagram Π and G be an unknown finite group. Let $\mathcal{D}$ be a collection of subsets of size at least two of the set of vertices of Π containing all connected ones of size exactly two. Set $H = \langle G_D \mid D \in \mathcal{D}\rangle$, where G_D are certain subgroups of G having the automorphism groups of the corresponding residue of the building belonging to X as a homomorphic image and in a compatible way. Adding a few technical assumptions on the kernels of this homomorphism and $O_p(H) = 1$, then $H \cong X$. Once such a subgroup H is constructed, it remains to show that $G = H$, unless of course it is not. This is the main goal of the work of the authors together with G. Pientka and A. Seidel in [11] which identifies groups G of local or parabolic characteristic p provided they contain a Lie type group $F^*(H)$ defined in characteristic p and such that $H = N_G(F^*(H))$ is of p' index. In this context Theorem 1 handles one of the more difficult cases when it is possible that in fact $G \neq H$.

In the present paper, we start with a group G of parabolic characteristic 2. We assume that the approach described in the previous paragraph provides a subgroup H of G such that $F^*(H) \cong \Omega_8^+(2)$. In this case, we know that H contains a Sylow 2-subgroup of G but it may be that the 2-local subgroups of H are not 2-local subgroups of G (the subgroups $L_1, \dots, L_n$ are chosen so that [9] is applicable). We know that one possible conclusion of our theorem is that $F^*(G) \cong \mathrm{P}\Omega_8^+(3)$. Of course, this group is not a Lie type group in characteristic 2. Thus to apply [9] we need to develop the 3-local structure of G. We first pin down subgroups $F_1, \dots, F_4$ of G, which share a common Sylow 3-subgroup and satisfy, for $i = 1, \dots, 4$, $C_{F_i}(O_3(F_i)) \leq O_3(F_i)$ and $F_i/O_3(F_i)$ resembles a large part of the corresponding maximal parabolic subgroup of $\mathrm{P}\Omega_8^+(3)$. It is very important for the approach that they do not have to contain a Sylow 3-subgroup of G. Indeed, recall that there is an outer automorphism of order 3 of $F^*(H)$, which might be induced by an element of order 3 from G and which is not contained in any of the F_i. Application of [9] then gives us that $\langle F_1, F_2, F_3, F_4\rangle \cong \mathrm{P}\Omega_8^+(3)$.

Before going into details of the proof we repeat that the paper has two purposes, first it solves one of the most difficult cases of [11], which is part of the MSS programme and secondly it shows how the method of identifying groups using [9] is practical.

We now give a sketch of the proof of Theorem 1 which elucidates the strategy of the proof while hiding the technical details. So suppose that G is a group of parabolic characteristic 2 and that $H \leq G$ is such that $H = N_G(F^*(H))$, $F^*(H) \cong \Omega_8^+(2)$ and $|G : H|$ is odd. We let $S \in \mathrm{Syl}_2(F^*(H))$ and $\langle r\rangle = Z(S)$ (which has order 2). Note that the subgroup structure of H is an open book whereas the subgroup structure of G is a mystery. Because of Lemma 2.2, which is a general fact about groups of parabolic characteristic p, we may assume that G is a simple group. An elementary application of the Thompson Transfer Lemma then shows $S \in \mathrm{Syl}_2(G)$ and that $|H : F^*(H)|$ divides 3; a fact that we have to carry through with us till the very end of the proof, but surprisingly does not cause any real difficulty. Holt's Theorem [7] (see also Theorem 2.7) provides a way to show that a subgroup of G is in fact equal to G. We employ this result twice, the first time, in Proposition 3.4, to show that if $C_G(r) = C_H(r)$ then $G = F^*(H)$. From this stage we know that $C_G(r) > C_H(r)$.

Remember we know everything about $C_H(r)$ and $C_G(r)$ is to be discovered. Here the fact that $C_G(r)$ satisfies condition (1) plays a pivotal role and separates $\mathrm{P}\Omega_8^+(3)$ from its counterparts $L = \mathrm{P}\Omega_8^+(q)$ for q an odd prime with $q \equiv 3, 5 \pmod 8$ (these groups also contains $\Omega_8^+(2)$). In all these groups $O^2(C_L(r)) = \Omega_4^+(q) \circ \Omega_4^+(q)$ and only when $q = 3$ do we obtain a group which satisfies (1). Anyway, up to the discrepancy caused by the possibility that $H \neq F^*(H)$, we find the explicit structure of $C_G(r)$ in Lemma 3.5. Thus $C_G(r)$ has a normal subgroup $X = K_1K_2K_3K_4$ where $K_i \cong \mathrm{SL}_2(3)$ and $[K_i, K_j] = 1$ when $i \neq j$. We intend to build the 3-local structure of G. So for $a \in I = \{1, 2, 3, 4\}$, let $\langle \tau_a \rangle \in \mathrm{Syl}_3(K_a)$ and set $C_a = C_G(\tau_a)$. Our global hypothesis only refers to 2-local subgroups, so 3-local subgroups have to be constructed with some delicacy. As r centralizes τ_a, we can quickly show that $r \in Z(C_a/D_a)$ and that r inverts $D_a/\langle \tau_a \rangle$ where $D_a = O(C_a)$ is the largest normal subgroup of odd order in C_a. It is this fact that allows us to see that $D_a \neq \langle \tau_a \rangle$ as we shall see. Now D_a turns out to be normalized by an elementary abelian group $\langle r, j \rangle$ of order 4 with j and rj conjugate in G. As r acts fixed point freely on $D_a/\langle \tau_a \rangle$, we get $D_a = C_{D_a}(j)C_{D_a}(rj)$. Now $C_H(j)$ has shape $2^6{:}\mathrm{Sym}(6)$ and because we know $C_G(r)$ is not contained in H, we can show that $M = C_G(j)$ is not contained in H. A delicate argument involving two applications of the Thompson Transfer Lemma shows that M has a normal subgroup M^* of index 4. We can also show $C_{M^*/\langle j \rangle}(r\langle j \rangle)$ has a structure and embedding required to apply a theorem of Aschbacher (Theorem 2.8) which then reveals that $M^* \cong \Omega_6^-(3) \cong 2{\cdot}\mathrm{U}_4(3)$. Now the centralizer of τ_a in M has a normal subgroup D which is extraspecial of order 3^5 and exponent 3. Furthermore, $D = [D, r]$ and so, as $rD_a \in Z(C_a/D_a)$, we discover that $D \leq D_a$ and this information can be further exploited to show that D_a is extraspecial of order 3^9. We gather all our facts together to show in Lemma 3.11 that $C_G(\tau_a)$ has a subgroup of index dividing 3 of shape $3^{1+8}.2^{1+6}.3^3$. This subgroup has the same shape as the centralizer of a 3-central element of $\mathrm{P}\Omega_8^+(3)$. We next embark on the construction of further 3-local subgroups. This involves groups generated by various subgroups of D_a as a varies through I. For example, one important subgroup is $E_4 = E_{4a}E_{4b}E_{4c}$ where $E_{4a} = \langle \tau_a, \tau_4 \rangle (D_a \cap D_4)$ and $\{a, b, c\} = \{1, 2, 3\}$. Eventually we assemble a collection of subgroup $P_1, \ldots, P_4$ which correspond to the minimal parabolic subgroups of $\mathrm{P}\Omega_8^+(3)$ and we control the groups generated by pairs of these subgroups. Once this is done we are ready to apply [9] to find a subgroup P of G with $P \cong \mathrm{P}\Omega_8^+(3)$. A short final argument using Holt's Theorem again establishes our main goal.

Our notation is standard and is in line with that presented in [6].

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2. Preliminary results

We begin by collecting some results about groups of parabolic characteristic p.

Lemma 2.1. *Let p be a prime, G be a group and S be a Sylow p-subgroup of G. Then G is of parabolic characteristic p if and only if $C_G(O_p(C_G(z))) \le O_p(C_G(z))$ for all $z \in \Omega_1(Z(S))^\#$.*

Proof. Suppose $C_G(O_p(C_G(z))) \le O_p(C_G(z))$ for all $z \in \Omega_1(Z(S))^\#$. Let X be a non-trivial normal subgroup of S and set

$$E = E(N_G(X))O_{p'}(N_G(X)).$$

Let $z \in (X \cap Z(S))^\#$ and put $R = O_p(C_G(z))$. Then EX acts on R. We have $C_R(X)$ normalizes and is normalized by E, so $[C_R(X), E]$ is a normal p-subgroup of E and is consequently contained in $Z(E)$. Hence

$$[C_R(X), E] = [C_R(X), E, E] = 1.$$

But then $E \le C_G(R)$ by the $A \times B$-Lemma. As $C_G(R) \le R$ by assumption, we have $E = 1$ and so $F^*(N_G(X)) = O_p(N_G(X))$. This means that

$$C_G(O_p(N_G(X))) \le O_p(N_G(X))$$

as required. The converse is clear. □

Lemma 2.2. *Suppose that p is a prime, G has parabolic characteristic p and $S \in \mathrm{Syl}_p(G)$. If G_1 is a subnormal subgroup of G and $Z(S \cap G_1) \le Z(S)$ is non-trivial, then G_1 has parabolic characteristic p.*

Proof. Let $S_1 = S \cap G_1$ and let $z \in Z(S_1)^\#$. Then $z \in Z(S)$ and so $C_G(z)$ has characteristic p. Since $C_{G_1}(z)$ is subnormal in $C_G(z)$ and G

has parabolic characteristic p, we have

$$F^*(C_{G_1}(z)) \le F^*(C_G(z)) = O_p(C_G(z)).$$

Hence $F^*(C_{G_1}(z)) = O_p(C_{G_1}(z))$ and so $C_{G_1}(O_p(C_{G_1}(z))) \le O_p(C_{G_1}(z))$. Thus G_1 has parabolic characteristic p by Lemma 2.1. □

Lemma 2.3. *If G has parabolic characteristic 2 and $Z(G/O(G)) = 1$, then $O(G) = 1$.*

Proof. Assume $O(G) \ne 1$. Let S be a Sylow 2-subgroup of G and $T = C_S(O(G))$. If $C_G(O(G)) \not\le O(G)$, then $T \ne 1$ and T is normal in S. But then $[O(G), O_2(N_G(T))] \le O(G) \cap O_2(N_G(T)) = 1$ and as G has parabolic characteristic 2,

$$O(G) \le C_G(O_2(N_G(T))) \le O_2(N_G(T)),$$

which is a contradiction. Thus $C_G(O(G)) \le O(G)$. Now let $z \in \Omega_1(Z(S))^{\#}$. Then $F^*(C_G(z)) = O_2(C_G(z))$ again as G has parabolic characteristic 2. Hence $C_{O(G)}(z) = 1$, so z inverts $O(G)$ and then $C_G(O(G)) = O(G)$. Since z inverts $O(G)$, we have that z induces an automorphism of $O(G)$ which is central in $\mathrm{Aut}(O(G))$. In particular,

$$1 \ne zO(G) \in Z(G/O(G)) = 1,$$

a contradiction. □

The next lemma plays a central role in the proof of Theorem 1 when we build $C_G(r)$ for a 2-central involution $r \in H$.

Lemma 2.4. *Let V be an 8-dimensional $\mathrm{GF}(2)$-space, $X \cong \mathrm{Sp}_2(2) \wr \mathrm{Sym}(3)$, and $B = B_1 \times B_2 \times B_3$ with $B_i \cong \mathrm{Sp}_2(2)$ the base group of X. For $1 \le i \le 3$, let $a_i \in B_i$ have order 2 and let W_i be the natural B_i-module. Then*

(i) *There is a unique faithful irreducible representation of B on V and so V can be identified with $W_1 \otimes W_2 \otimes W_3$. Furthermore, this representation extends to X.*

(ii) *V admits a B-invariant quadratic form q of plus type preserved by X, so $X \le \mathrm{O}(V, q)$.*

(iii) (a) *$[V, a_1]$ is totally singular of dimension 4.*
(b) *$[V, a_1a_2]$ is totally singular of dimension 4.*
(c) *$[V, a_1a_2a_3]$ is totally isotropic of dimension 4 but is not totally singular.*

Proof. The tensor product $W_1 \otimes W_2 \otimes W_3$ is an irreducible B-module. Conversely assume that B acts irreducibly on V. Then, by Clifford's Theorem, V is a semisimple B_i-module for each $1 \le i \le 3$. Furthermore, each irreducible B_i-submodule is the natural module for

B_i, which is absolutely irreducible. Therefore V is a homogeneous B_i-module and indeed the tensor product module for B. As $X = \mathrm{Aut}(B)$ and the representation is unique, it extends to X. This proves (i).

For $i = 1, 2, 3$, let W_i have symplectic basis $\{e_i, f_i\}$ and associated symplectic form $(\ , \ )_i$. Then V supports a symmetric bilinear form

$$(\ , \ ) = \prod_{i=1}^{3} (\ , \ )_i$$

and an associated quadratic form q which is entirely defined by specifying that all the pure tensors are singular. In this way B embeds into $\mathrm{O}_8^+(2)$. This form is preserved by X. This proves (ii).

We may suppose that $a_i \in B_i$ centralizes e_i and sends f_i to $e_i + f_i$. Then

$$[V, a_1] = \langle e_1 \otimes x \otimes y \mid x \in \{e_2, f_2\}, y \in \{e_3, f_3\} \rangle$$

which is totally singular. Similarly

$$[V, a_1 a_2] = \langle e_1 \otimes e_2 \otimes y_1, e_1 \otimes f_2 \otimes y_2 + f_1 \otimes e_2 \otimes y_2 \mid y_1, y_2 \in \{e_3, f_3\} \rangle$$

is also totally singular. Finally

$$\begin{aligned} [V, a_1 a_2 a_3] \quad = \quad & \langle e_1 \otimes e_2 \otimes e_3, f_1 \otimes e_2 \otimes e_3 + e_1 \otimes f_2 \otimes e_3, \\ & e_1 \otimes e_2 \otimes f_3 + e_1 \otimes f_2 \otimes e_3, \\ & e_1 \otimes e_2 \otimes f_3 + f_1 \otimes f_2 \otimes e_3 + e_1 \otimes f_2 \otimes f_3 + f_1 \otimes e_2 \otimes f_3 \rangle. \end{aligned}$$

We observe that this space is isotropic but that

$$\begin{gathered} \mathrm{q}(e_1 \otimes e_2 \otimes f_3 + f_1 \otimes f_2 \otimes e_3 + e_1 \otimes f_2 \otimes f_3 + f_1 \otimes e_2 \otimes f_3) = \\ (e_1 \otimes e_2 \otimes f_3, f_1 \otimes f_2 \otimes e_3) = 1 \end{gathered}$$

which means that this space is not totally singular. This proves (iii) (a), (b) and (c). □

Lemma 2.5. *Suppose that V and B are as in Lemma 2.4 and let $C = \langle a_1, a_2, a_3 \rangle \in \mathrm{Syl}_2(B)$. Assume that D is a subgroup of the isometry group of V and that $D > B$. If $|D : B|$ is odd, then there is a perpendicular decomposition of V into non-degenerate 2-spaces of minus-type*

$$V = V_1 \perp V_2 \perp V_3 \perp V_4$$

which is preserved by D. Furthermore,

(i) *$D = O_3(D) N_D(C)$.*
(ii) *$O_3(D)$ is elementary abelian of order 3^4 or 3^3, contains $O_3(B)$ and is inverted by $a_1 a_2 a_3$.*
(iii) *$\langle a_1 a_2, a_2 a_3 \rangle$ permutes $\{V_1, \dots, V_4\}$ regularly.*

(iv) $D = O_3(D)C$ *or* $|D : O_3(D)C| = 3$ *with* $D/O_3(D) \cong 2 \times \mathrm{Alt}(4)$.
(v) $O_3(B) = [O_3(D), [D, D]]$ *is normal in* D.
(vi) $\dim C_V(C) = 1$.

Proof. Set $Y = O_3(B)$ and $U = O_3(D)$. As $|D : B|$ is odd and $C \in \mathrm{Syl}_2(B)$, Lemma 2.4 (i), (ii) and (iii) yields that $e = a_1a_2a_3$ is the unique element in C such that $[V, e]$ is not totally singular. Thus $e^D \cap C = \{e\}$. Application of the Z^*-theorem [5] yields

$$eO(D) \in Z(D/O(D)).$$

In particular, we have

$$Y = [Y, e] \leq O(D)$$

and so $O(D)C$ acts irreducibly on V. By Clifford Theory, $V|_{O(D)}$ is a direct sum of irreducible representations of $O(D)$ all of equal dimension. Let U be an irreducible summand. Then $U \otimes k$, where k is an algebraically closed field of characteristic 2, is a direct sum of irreducible $kO(D)$-modules $W_1, \ldots, W_m$ of equal dimension. Since $|O(D)|$ is odd, so is $w = \dim W_j$. Hence w divides $\dim V = 8$ and so $w = 1$ and we infer that $O(D)$ is abelian. Since, by [4], elements of order 5 and 7 in $\Omega_8^+(2)$ are not centralized by a group of order $3^3 = |Y|$, $O(D) = U$.

Because Y is elementary abelian, V decomposes as a direct sum of centralizers of maximal subgroups of Y and so, as YS acts irreducibly on V, there are maximal subgroups $Y_1, \ldots, Y_4$ of Y such that

$$V = V_1 \perp \cdots \perp V_4 \qquad (2)$$

where $V_i = C_V(Y_i)$ and V_i admits Y acting irreducibly. These subspace are each normalized by U and so U is elementary abelian of order at most 3^4.

Since e inverts each Y/Y_i and B acts irreducibly on V, we also see that $C/\langle e\rangle$ permutes $\{V_1, V_2, V_3, V_4\}$ regularly. Now $C_D(Y) \geq U$ fixes each V_i and so we deduce that $C_D(Y) = U$ and $|U : Y| \leq 3$, U is elementary abelian and e inverts U. Furthermore D preserves the decomposition (2). It follows that D is isomorphic to subgroup of $\mathrm{O}_2^-(2) \wr \mathrm{Sym}(4)$ and so D is a $\{2, 3\}$-group. Therefore, as C is abelian, $UC = O_{3,2}(D)$ and the Fratinni argument implies

$$D = UN_D(C).$$

Since $C \in \mathrm{Syl}_2(D)$, it only remains to remark that the action of e together with (iii) yields $\dim C_V(C) = 1$ and we are done. □

Lemma 2.6. *Suppose that $X = \mathrm{Sp}_6(2)$ and $T \in \mathrm{Syl}_2(X)$. Then, $N_X(T) = T$, $Z(T)$ is elementary abelian of order 4, $Z(T) \le \Phi(T)$ and every automorphism of T centralizes $Z(T)$.*

Proof. As T is a Borel subgroup of X by Lie Theory, we have $N_X(T) = T$. The character table of X [4, page 46] hows that X possesses three classes of 2-central elements, in particular, as $N_X(T) = T$, $Z(T)$ has order 4. In addition, the character table shows that the transvection r in $Z(T)$ is not a square while the other two involutions in $Z(T)$ are squares. In particular, the transvection in $Z(T)$ is centralized by every automorphism of T and $Z(T) \le \Phi(T)$.

Let V be the symplectic space upon which X acts. Then $\dim[V, r] = 1$ and the stabiliser in X of $[V, r]$ has shape $2^5{:}\mathrm{Sp}_4(2)$ [4, page 46] which we write as UL with U elementary abelian of order 2^5 and $L \cong \mathrm{Sp}_4(2) \cong \mathrm{Sym}(6)$. We have $U/\langle r\rangle$ is the symplectic module for L, which is isomorphic to the irreducible 4-dimensional section of the 6-dimensional permutation module, and $T_1 = T \cap L \cong 2 \times \mathrm{Dih}(8)$. Set $D = [T, T]\langle r\rangle$. Then D is characteristic in T and

$$D = [UT_1, UT_1]\langle r\rangle = [U, T_1][T_1, T_1]\langle r\rangle$$

where $|[T_1, T_1]| = 2$. As in the permutation module $U/\langle r\rangle$, the commutator with T_1 is a hyperplane of $U_1/\langle r\rangle$, we see $[U, T_1]\langle r\rangle$ has order 2^4. Now $\langle u\rangle = [T_1, T_1]$ is an involution in $\mathrm{Alt}(6)$, which inverts elements of order 5, so $[U/\langle r\rangle, u] = C_{U/\langle r\rangle}(u)$ is of order 4. This implies that $[T_1, T_1]$ centralizes a maximal subgroup in $[U, T_1]$. Thus

$$|[[U, T_1], [T_1, T_1]]| = 2.$$

It follows that $|[D, D]| = 2$, is characteristic in T and is not equal to $\langle r\rangle$. Thus any automorphism of T centralizes $\langle r\rangle$ and $[D, D]$ and thus centralizes $Z(T)$. □

As mentioned in the Introduction, the next result due to Holt [7] will be used to show that certain potentially proper subgroups of a group are in fact the entire group.

Theorem 2.7 (Holt)**.** *Suppose that K is a simple group, P is a proper subgroup of K and r is a 2-central element of K. If $r^K \cap P = r^P$ and $C_K(r) \le P$, then $K \cong \mathrm{PSL}_2(2^a)$ $(a \ge 2)$, $\mathrm{PSU}_3(2^a)$ $(a \ge 2)$, ${}^2\mathrm{B}_2(2^a)$ $(a \ge 3$ and odd$)$ or $\mathrm{Alt}(n)$ $(n \ge 5)$ where in the first three cases P is a Borel subgroup of K and in the last case $P \cong \mathrm{Alt}(n-1)$.*

Proof. This formulation of Holt's Theorem can be found as stated here in [10]. □

We also require the following theorem from Aschbacher's book [3].

Theorem 2.8 (Aschbacher)**.** *Suppose that G is a finite group, $z \in G$ is an involution and $H = C_G(z)$. Assume that $O^2(H) = H_1H_2$ with $H_i \cong \mathrm{SL}_2(3)$, H_i normal in $O^2(H)$ and $H_1 \cap H_2 = \langle z\rangle$. Set $Q = O_2(O^2(H))$. If $Q = F^*(H)$, $|H| = 2^7 \cdot 3^2$, $z^G \cap Q \neq \{z\}$ and $|Q \cap Q^g| = 4$ for all $z^g \in Q \setminus \langle z\rangle$, then $G \cong \mathrm{PSU}_4(3)$.*

Proof. The hypotheses of this theorem state that G is of type $\mathrm{PSU}_4(3)$ (a combination of [3, Hypothesis 45.1] and the additional requirements on [3, page 244]). The theorem is then the content of [3, Lemma 45.11]. □

For the remainder of this article we assume that G satisfies the hypothesis of Theorem 1. Thus G has parabolic characteristic 2 and H is a subgroup of G such that

(i) $H = N_G(F^*(H))$;
(ii) $|G : H|$ is odd; and
(iii) $F^*(H) \cong \Omega_8^+(2)$.

We also fix the following

Notation 2.9. (i) $S_0 \in \mathrm{Syl}_2(H)$;
(ii) $S = S_0 \cap F^*(H) \in \mathrm{Syl}_2(F^*(H))$;
(iii) $r \in Z(S)^{\#}$; and
(iv) $Q = O_2(C_H(r)) = F^*(C_H(r))$.

The next proposition introduces further notation which we shall also adhere to for the remainder of the paper. The proposition concerns the 2-local structure of $F^*(H)$. We first of all recall that $H/F^*(H)$ embeds into $\mathrm{Out}(\Omega_8^+(2)) \cong \mathrm{Sym}(3)$. The elements of order 3 in $\mathrm{Out}(\Omega_8^+(2))$ are called triality automorphisms.

Proposition 2.10. (i) *$F^*(H)$ has five conjugacy classes of involutions. One is represented by r, there are three classes j_1, j_2 and j_3 which are fused by the triality automorphism of $F^*(H)$ and one further class j_4. We choose representatives j_i, $1 \leq i \leq 4$, so that $C_S(j_i) \in \mathrm{Syl}_2(C_{F^*(H)}(j_i))$ and observe that $|C_S(j_i)| = 2^{10}$.*

(ii) *The group Q is extraspecial of order 2^{1+8} and plus-type. Furthermore*

$$\begin{aligned} C_{F^*(H)}(r)/Q &\cong \Omega_4^+(2) \times \mathrm{SL}_2(2) \\ &\cong \mathrm{Sym}(3) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3) \end{aligned}$$

and $Q/\langle r\rangle$ is the tensor product module of the natural modules for $\Omega_4^+(2)$ and $\mathrm{SL}_2(2)$ (as described in Lemma 2.4). Furthermore, $C_{F^*(H)}(r)$ splits over Q.

(iii) *There are three elementary abelian normal subgroups* U_1, U_2, U_3 *in* S, *such that* $N_{F^*(H)}(U_i)/U_i \cong \Omega_6^+(2) \cong \mathrm{Alt}(8)$. *Furthermore, as a* $\mathrm{GF}(2)\mathrm{Alt}(8)$*-module,* U_i *is the non-trivial irreducible section of the* 8*-dimensional permutation module. These subgroups are fused by the triality automorphism.*

(iv) *For* $i = 1, 2, 3$, *we have* $C_{F^*(H)}(j_i) = U_i L_i$ *where* $L_i \cong \mathrm{Sym}(6)$ *and* U_i *are as in part (iii).*

(v) *For* $i = 1, 2, 3$, *every involution in* U_i *is* $F^*(H)$*-conjugate to either* r *or* j_i. *In particular,* U_i *contains* 35 $F^*(H)$*-conjugates of* r *and* 28 $F^*(H)$*-conjugates of* j_i.

(vi) $|C_{F^*(H)}(j_4)| = 2^{10}{\cdot}3$, $|C_{U_1}(j_4)| = 2^4$ *and* $C_{N_{F^*(H)}(U_1)}(j_4)/C_{U_1}(j_4)$ *is isomorphic to the centralizer of a* 2*-central involution in* $\mathrm{Alt}(8)$.

(vii) *For* $i = 1, 2, 3, 4$, $Z(C_S(j_i)) = \langle r, j_i\rangle$.

(viii) *The second centre of* S, $Z_2(S)$, *is elementary abelian of order* 4 *and is normalized by a parabolic subgroup of order* $2^{12} \cdot 3$ *in* $F^*(H)$ *(corresponding to the middle node in the* D_4 *Dynkin diagram).*

(ix) *There is an involution* $i \in \mathrm{Aut}(F^*(H)) \setminus \mathrm{F}^*(H)$ *such that* $C_{F^*(H)}(i) \cong \mathrm{Sp}_6(2) \cong \Omega_7(2)$ *and* $Z(S) = Z(S_0)$.

Proof. We can read most parts of this proposition from [4, page 75]. In fact (i) and (iii) can be found directly in [4]. Indeed, for (iii), we may assume that $N_H(U_1)$ is the point stabilizer in the natural orthogonal representation of $F^*(H)$ and so U_1 is the orthogonal module for $\Omega_6^+(2) \cong \mathrm{Alt}(8)$. This module is isomorphic to the irreducible section of the 8-dimensional permutation module. That the same is true for all U_i follows from the fact that the three groups are conjugate under $\mathrm{Aut}(F^*(H))$. For part (ii), we have to argue a little bit. By [4] we have that Q is extraspecial of order 2^9 and plus type, $C_{F^*(H)}(r)/Q \cong \mathrm{Sym}(3)\times\mathrm{Sym}(3)\times\mathrm{Sym}(3)$ and the extension splits. We also use [4] to see that $N_H(U_i)$ is a split extension and so $C_{N_{F^*(H)}(U_1)}(r)$ is also a split extension of U_1 by a group $H_1 \cong 2^4{:}\Omega_4^+(2)$, where H_1 is the point stabilizer in $\mathrm{Alt}(8)$ on U_1. As $C_{N_{F^*(H)}(U_1)}(r)/O_2(C_{N_{F^*(H)}(U_1)}(r))$ acts irreducibly on $(U_1 \cap Q)/\langle r\rangle$ and on $Q/(Q \cap U_1)$, we see that $C_{F^*(H)}(r)$ acts irreducibly on $Q/\langle r\rangle$. Thus we may apply Lemma 2.4. This proves (ii).

Now (iv) and (v) follow immediately from the action described in part (ii).

For part (vi) we first note that by [4] we have $|C_{F^*(H)}(j_4)| = 2^{10} \cdot 3$. We may choose notation such that $N_{F^*(H)}(U_1)$ contains a Sylow 2-subgroup of $C_{F^*(H)}(j_4)$. By (v), $j_4 \notin U_1$. As there are no transvections in Alt(8) on U_1, we see $|C_{U_1}(j_4)| = 2^4$. Hence $j_4 = iu$, where i is a 2-central involution in Alt(8) and $u \in U_1$. As any involution in the coset $U_1 i$ is centralized by an element of order three in Alt(8), we see that j_4 is centralized by such an element and so $C_{N_{F^*(H)}(U_1)}(j_4)/C_{U_1}(j_4)$ is isomorphic to the centralizer of i in Alt(8), which yields (vi).

To prove (vii) assume that $1 \le i \le 3$. Then by (iv), $Z(C_S(j_i))$ is the centralizer of a Sylow 2-subgroup of Sym(6) in U_i, and so is equal to $\langle r, j_i \rangle$.

We now consider $C_S(j_4)$. As the centralizer of a Sylow 2-subgroup of Alt(8) in U_1 is one dimensional and $C_S(j_4)/C_{U_1}(j_4)$ is a Sylow 2-subgroup of Alt(8), we get $Z(C_S(j_4)) = \langle j_4, r \rangle$. This proves (vii).

By (ii), Lemma 2.4 and Lemma 2.5(vi), we see that $|Z_2(S)| = 4$ and $C_{F^*(H)}(Z_2(S)) \le S$. Let P be the minimal parabolic subgroup of $F^*(H)$ containing S and corresponding to the middle node in the D_4 Dynkin diagram. Then $P/O_2(P) \cong \mathrm{Sym}(3)$. As $P \not\le C_{F^*(H)}(r)$, we have that $\Omega_1(Z(O_2(P))) = \langle Z(S)^P \rangle \ne Z(S)$. Hence $|\Omega_1(Z(O_2(P)))| = 4$, which gives $Z_2(S) = \Omega_1(Z(O_2(P)))$ and then $P \le N_{F^*(H)}(Z_2(S))$, which is (viii).

Finally (ix) follows from [4].

□

3. Proof of Theorem 1

In this section we will prove Theorem 1. We continue with the notation introduced at the end of Section 2. By Lemma 2.3 we have that $O(G) = 1$. Because $|G : H|$ is odd and $O_2(H) = 1$, we have $O_2(G) = 1$, so $F^*(G) = E(G)$. By Proposition 2.10 (ix) we have $Z(S) = Z(S_0) = \langle r \rangle$. As $Z(S_0) \cap E(G) \ne 1$ and $Z(S_0) = \langle r \rangle$, we obtain $F^*(H) = \langle r^H \rangle \le E(G)$. Using $\langle r \rangle = Z(S)$ is of order 2, it follows that $F^*(G)$ is a simple group. Furthermore, as $Z(S) = Z(S_0)$, Lemma 2.2 implies $F^*(G)$ has parabolic characteristic 2 and so we may assume that $G = F^*(G)$ is a simple group.

Lemma 3.1. *We have that $O_2(C_G(r)) = Q$ is extraspecial of order 2^{1+8}. In particular, $N_G(Q) = C_G(r)$.*

Proof. We know that Q is extraspecial of order 2^{1+8} and $C_H(r)$ acts irreducibly on $Q/\langle r \rangle$. Therefore, as $O_2(C_G(r)) \le Q$, if $Q \ne O_2(C_G(r))$, then $O_2(C_G(r))$ has order 2. But this contradicts the assumption that G has parabolic characteristic 2. Therefore $O_2(C_G(r)) = Q$. As $\langle r \rangle = Z(Q)$, we then have $N_G(Q) = C_G(r)$. □

Lemma 3.2. *We have that r is not G-conjugate to j_1, j_2 or j_3.*

Proof. Suppose that for some $1 \le i \le 3$, j_i and r are G-conjugate. Let $Q_i = O_2(C_G(j_i))$ and $S_i \in \mathrm{Syl}_2(C_G(j_i))$. Then Q_i is extraspecial of order 2^{1+8} and $2^3 \le |S_i/Q_i| \le 2^4$. As $C_{F^*(H)}(j_i)/U_i \cong \mathrm{Sym}(6)$ by Proposition 2.10 (iv), we have that $Q_i \cap C_S(j_i) \le O_2(C_H(j_i)) = V_i$ with $|V_i : U_i| \le 2$ and $|S_i/Q_i| \ge |C_S(j_i)/V_i|$, so $|S_i : Q_i| = 2^4$, and $U_i \le Q_i$. But Q_i does not contain an elementary abelian subgroup of order 2^6, a contradiction. Thus r and j_i are not G-conjugate. $\square$

Lemma 3.3. *The group S is equal to S_0 and S/Q is elementary abelian of order 8. In particular, either $H = F^*(H)$ or $|H/F^*(H)| = 3$.*

Proof. Assume that the statement is false. Then $|S_0/S| = 2$ and, by Proposition 2.10 (ix) there is an involution $i \in S_0 \setminus S$ such that $C_H(i) \cong \langle i \rangle \times U$, where $U \cong \mathrm{Sp}_6(2)$. Let T be a Sylow 2-subgroup of U. Since G has no subgroup of index 2, the Thompson Transfer Lemma [6, Lemma 15.16] implies that i is conjugate to an involution $x \in S \le F^*(H)$ such that $C_{S_0}(x) \in \mathrm{Syl}_2(C_G(x))$. As $C_G(r)$ is of characteristic 2 and $|S_0 : Q| = 16$, we have that x is not conjugate to r. Therefore x is conjugate in H to one of $\{j_1, j_2, j_3, j_4\}$ and we have $2^{10} \le |C_{S_0}(x)| \le 2^{11}$. Furthermore, as i does not centralize a subgroup of order 2^{11} in H, as $Z(C_S(x)) = \langle x, r\rangle$ by Proposition 2.10 (vii) and $|C_{S_0}(x)| \ge 4$, we also have $Z(C_{S_0}(x)) = \langle x, r\rangle$. Because $Z(T\langle i\rangle) = \langle i \rangle Z(T)$ is elementary abelian of order 8 by Lemma 2.6 and $|T\langle i \rangle| = 2^{10}$, we infer that $C_{S_0}(x)$ has order 2^{11} and $T\langle i\rangle$ has index 2 in a Sylow 2-subgroup of $C_G(i)$. Let $T_1 \le C_G(i)$ with $|T_1 : T \times \langle i \rangle| = 2$. Then T_1 normalize $Z(T\langle i\rangle)$ and $\Phi(T\langle i \rangle) \cap Z(T\langle i\rangle)$. Since T_1 centralizes i, T_1 induce automorphism on $T\langle i\rangle/\langle i\rangle \cong T$. Lemma 2.6 implies that T_1 centralizes $Z(T\langle i\rangle)/\langle i\rangle$ and so T_1 centralizes $\Phi(T\langle i\rangle) \cap Z(T\langle i \rangle) = Z(T)$. This shows T_1 centralizes $Z(T\langle i\rangle)$. However, this contradicts $Z(C_{S_0}(x)) = \langle x, r\rangle$ and therefore completes the proof of the lemma. $\square$

Proposition 3.4. *If $C_G(r) \le H$, then $G = F^*(H)$.*

Proof. If
$$r^G \cap H = r^H,$$
then, as G is simple and $C_G(r) \le H$ by hypothesis, we may apply Theorem 2.7. As H is not soluble and not isomorphic to an alternating group, this yields $G = H$ and then $G = F^*(H)$. Thus we may assume that r is G-conjugate to an element of $\{j_1, j_2, j_3, j_4\}$ and so, by Lemma 3.2, r is G-conjugate to $t = j_4$. We seek a contradiction.

Set $Q_t = O_2(C_G(t))$ and note that $C_G(t)/Q_t$ has elementary abelian Sylow 2-subgroups of order 8 by Lemma 3.3. As $|C_S(t)| = 2^{10}$ and $|S/Q| = 8$, $|C_S(t) \cap Q| \ge 2^7$. Especially this means
$$\langle r \rangle = (C_S(t) \cap Q)' \le Q_t.$$

Now $|C_{Q_t}(r)| = 2^8$ and so we also have

$$\langle t\rangle = C_{Q_t}(r)' \le Q.$$

The containment

$$(Q_t \cap Q)' \le \langle r\rangle \cap \langle t\rangle = 1,$$

yields $|Q_t \cap Q| \le 2^5$ and so $|C_{Q_t}(r)C_Q(t)| \ge 2^8 \cdot 2^8/2^5 = 2^{11}$. Because

$$C_{Q_t}(r)C_Q(t) \le C_G(r) \cap C_G(t) \le H$$

and $|C_H(t)|_2 = 2^{10}$ by Proposition 2.10 (i) and Lemma 3.3, this is impossible. This contradiction completes the proof of the proposition. □

From now on we assume that

$$C_G(r) \not\le H$$

and intend to prove that $F^*(G) \cong \mathrm{P}\Omega_8^+(3)$. To this end, our first aim is to determine the structure of $C_G(r)$.

Lemma 3.5. *We have that*

$$C_G(r) = XC_H(r)$$

where $X = O_{2,3}(C_G(r))$. Furthermore there are normal subgroup $K_1, \dots, K_4$ of X such that $K_i \cong \mathrm{SL}_2(3)$, $X = K_1K_2K_3K_4$ is a commuting product and we have the following structural information about $C_G(r)$

(i) *The subgroups K_i are the only subnormal subgroups isomorphic to $\mathrm{SL}_2(3)$ in $C_G(r)$.*

(ii) *The quotient S/Q is elementary abelian of order 8 and permutes the subgroups K_i transitively.*

(iii) *There is some $e^* \in S/Q$ such that e^* inverts X/Q. We have $S/Q = \langle e^*\rangle \times A$, where A acts regularly on $\{K_1, K_2, K_3, K_4\}$.*

(iv) *Either $H = F^*(H)$ and $SX = C_G(r)$ or $|H/F^*(H)| = 3$ and $C_G(r)/X \cong 2 \times \mathrm{Alt}(4)$.*

Proof. Set $X = O_{2,3}(C_G(r))$ and $Y = O_{2,3}(C_H(r))$. From Lemma 3.1 we have $Q = O_2(C_G(r)) = O_2(C_H(r))$. Set $V = Q/\langle r\rangle$. Then by Proposition 2.10 (ii), we have

$$C_{F^*(H)}(r)/Q = SY/Q \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3)$$

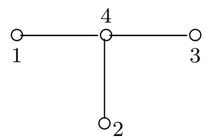
acts irreducibly on V which can be identified with an orthogonal space of plus-type. Since $C_G(r) > C_H(r)$ and since $|C_G(r) : C_H(r)|$ is odd, we may apply Lemma 2.5 to find the structure of $C_G(r)$. In particular, we have

$$C_G(r) = XN_{C_G(r)}(S),$$
$$V = V_1 \perp \cdots \perp V_4$$

is preserved by $C_G(r)$ and Y is normalized by $C_G(r)$. If $C_G(r) = XS$, then we have $C_G(r) = XC_H(r)$. Suppose that $|C_G(r) : XS| = 3$.

Since Y is normalized by $C_G(r)$, $N_{C_G(r)}(S)$ normalizes YS. We need to demonstrate that $N_{C_G(r)}(S) \leq H$.

Let P be the minimal parabolic subgroup of $F^*(H)$ which contains S and corresponds to the middle node of

$\Pi =$ 

We know from Proposition 2.10 (viii) that the second centre of S, $Z_2(S)$, has order 4 and is normalized P. Moreover, P induces $\mathrm{SL}_2(2)$ on $Z_2(S)$ and so all the involutions in $Z_2(S)$ are P-conjugate. Therefore, as $QO_2(P) = S$ and $Q = O_2(C_G(r))$,

$$P = \langle Q^g \mid g \in G, r^g \in Z_2(S) \rangle S$$

is normalized by $N_G(S)$. Since $F^*(H) = \langle P, YS \rangle$ and $N_{C_G(r)}(S)$ normalizes YS we now have

$$N_{C_G(r)}(S) \leq N_G(F^*(H)) = H$$

as we require. Therefore $C_G(r) = XC_H(r)$.

Since $C_G(r) \not\leq H$, we now have $|X/Q| = 3^4$. Hence there exist subgroups $F_1, \ldots F_4 \leq X$ of order 3 such that F_i centralizes V/V_i. It follows that $K_i = [Q, F_i]F_i \cong \mathrm{SL}_2(3)$ and $X = K_1K_2K_3K_4$ is a commuting product as claimed. □

Before we can start to construct the 3-local structure of G, we need to determine the centralizers for the involutions j_i, $i = 1, 2, 3$. We do this through a sequence of three lemmas. Recall from Proposition 2.10 (iii) that j_1 ,j_2, and j_3 are conjugate under the action of $\mathrm{Aut}(F^*(H))$. Thus, by Lemma 3.3, for $i = 1, 2, 3$,

$$C_H(j_i) = C_{F^*(H)}(j_i)$$

and, by Proposition 2.10 (iv), $C_{F^*(H)}(j_i) = U_iL_i$ with U_i elementary abelian of order 2^6 and $L_i \cong \mathrm{Sym}(6)$. Furthermore, $N_H(U_i)/U_i \cong \mathrm{Alt}(8)$. We fix the subgroups K_k, $1 \leq k \leq 4$ and

$$X = O_{2,3}(C_G(r)) = K_1K_2K_3K_4,$$

from Lemma 3.5. Note that X has shape $2_+^{1+8}.3^4$.

Lemma 3.6. *Suppose that j is H-conjugate to one of j_1, j_2 or j_3. Then $C_G(j) \not\leq H$. Furthermore, if j is chosen so that $C_S(j) \in \mathrm{Syl}_2(C_H(j))$, then*

(i) *$j \in Q$ and the elements j and jr are $C_G(r)$-conjugate;*
(ii) *$C_S(j)$ is a Sylow 2-subgroup of $C_G(j)$; and*
(iii) *there are $s, t \in \{1, 2, 3, 4\}$ with $s \neq t$ such that*

$$C_G(r) \cap C_G(j) = C_G(\langle j, r\rangle) = K_s K_t C_S(j).$$

Proof. We may assume that j is H-conjugate to j_1 and is chosen so that $C_S(j) \in \mathrm{Syl}_2(C_H(j))$. Thus $C_S(j)$ has order 2^{10} and $Z(C_S(j)) = \langle r, j\rangle$ by Proposition 2.10 (i) and (vii). If $j \notin Q$, then $|C_Q(j)| \leq 2^5$ by Lemma 2.4 (i), (ii) and (iii) and so $|C_S(j)| \leq 2^8$, a contradiction. Hence $j \in Q$. Therefore, j is Q-conjugate to jr and, as $Z(C_S(j)) = \langle r, j\rangle$, and as $r \not\sim j$ by Lemma 3.2, $N_G(C_S(j))$ is contained in $C_G(r)$. This shows that (i) and (ii) hold.

The involutions in $Q \setminus \langle r\rangle$ are products of an even number of elements taken from $O_2(K_i) \setminus \langle r\rangle$ for different values of $i \in \{1, 2, 3, 4\}$. Therefore, X acts on the involutions of $Q \setminus \langle r\rangle$ with one orbit of length 162 and the remaining orbits have length 18. Recalling that $C_H(j) = U_1 L_1$ and $C_{L_1}(r) \cong \mathrm{Sym}(4) \times 2$, we obtain $L_1 \cap X$ has order divisible by 3 from Lemma 3.5 (iv). Thus $C_X(j)$ has order divisible by 3 and consequently j is in an X-orbit of length 18. In particular 3^2 divides $|C_G(j) \cap C_G(r)|$ and so $C_G(j) \cap C_G(r) \not\leq H$.

Furthermore as the involutions in $Q \setminus \langle r\rangle$, which are in an orbit of length 18 are centralized by two of the K_i, we have that $C_G(j) \cap C_G(r)$ contains a subgroup $K_s K_t$ for suitable s, t. As $C_S(j) \in \mathrm{Syl}_2(C_G(j))$ we have assertion (iii) via Lemma 3.5. Arguing similarly for j conjugate to j_2 or j_3, we obtain Lemma 3.6. □

Let j be H-conjugate to one of j_1, j_2 or j_3 be selected so that $C_S(j) \in \mathrm{Syl}_2(C_H(j))$. Define

$$M = C_G(j).$$

Our intention is to show that $F^*(M) \cong \Omega_6^-(3) \cong 2.\mathrm{PSU}_4(3)$. Before we can do this, we need to show that M has a subnormal subgroup of index 4. We set

$$X_j = K_s K_t$$

so that

$$C_G(j) \cap C_G(r) = X_j C_S(j)$$

is as in Lemma 3.6 (iii).

Lemma 3.7. *The group M has a subnormal subgroup M^* of index 4.*

Proof. Again we may assume that j is H-conjugate to j_1. The results for j_2 and j_3 will follow similarly. From Proposition 2.10 (iv), $C_H(j) = U_1 L_1$ with $L_1 \cong \mathrm{Sym}(6)$, where U_1 is the non-trivial irreducible part of the GF(2)-permutation module for $N_H(U_1)/U_1 \cong \mathrm{Alt}(8)$. Therefore

$L_1' \cong \mathrm{Alt}(6)$, U_1 is isomorphic to the 6-point permutation module and the "transpositions" from $L_1 \cong \mathrm{Sym}(6)$ operate on U_1 with commutators of dimension 2 containing $\langle j \rangle$. Put

$$J = C_H(j)' = C_{F^*(H)}(j)'.$$

Then $J/\langle j \rangle \cong 2^4{:}\mathrm{Alt}(6)$ and, as $C_H(j) = U_1L_1$, $C_H(j)/J$ is elementary abelian of order 4. Set

$$J_1 = O_2(J)L_1.$$

Then $J_1 \cong 2^5{:}\mathrm{Sym}(6)$. Let $\widetilde{L} \leq L_1$, with

$$\widetilde{L} \cong \mathrm{Alt}(5)$$

and $S \cap \widetilde{L} \in \mathrm{Syl}_2(\widetilde{L})$ (so $r \in [U, \widetilde{L} \cap C_S(j)]$). More concretely, if $\mathrm{Alt}(8)$ acts on $\{1, \ldots, 8\}$ and L_1' on $\{1, \ldots, 6\}$, then we choose $\widetilde{L}$ to act on $\{1, \ldots, 5\}$. In particular, $\widetilde{L}$ induces the non-trivial section of the 5-point permutation module on $O_2(J)/\langle j \rangle$. Then, as this $\mathrm{Alt}(5)$-module is projective, it follows that

$$U_1 = [U_1, \widetilde{L}] \times C_{U_1}(\widetilde{L})$$

with $|[U_1, \widetilde{L}]| = 2^4$, $|C_{U_1}(\widetilde{L})| = 2^2$ and $j \in C_{U_1}(\widetilde{L})$. In $N_H(U_1)/U_1 \cong \mathrm{Alt}(8)$, $\widetilde{L}U_1/U_1 \cong \mathrm{Alt}(5)$ is centralized by a 3-cycle $\omega = (6, 7, 8)$ which acts trivially on $[U_1, \widetilde{L}]$ and permutes the involutions in $C_{U_1}(\widetilde{L})$ transitively. In particular, as $r \in [U_1, \widetilde{L}]$, we obtain $\omega \in N_{C_G(r)}(C_{U_1}(\widetilde{L}))$. Note that, as $[U_1, \widetilde{L}] \leq J$ and $U_1 \not\leq J$,

$$C_{U_1}(\widetilde{L}) \not\leq J.$$

Let $u \in C_{U_1}(\widetilde{L}) \setminus J$. Then $u \neq j$ and the elements u, uj and j are $\langle \omega \rangle$-conjugate. If $C_{C_H(j)}(u)/U_1 \cong \mathrm{Sym}(5)$, there is some non-trivial 2-element x, which centralizes $C_{U_1}(\widetilde{L})$ and induces $\widetilde{L}\langle x \rangle \cong \mathrm{Sym}(5)$ on $[U_1, \widetilde{L}]$, which gives that x acts as a transvection on $[U_1, \widetilde{L}]$ and so also on U_1. As there are no transvections in L_1 on U_1, we get

$$C_{C_H(j)}(u) = U_1\tilde{L}.$$

Because ω acts non-trivially on $C_{U_1}(\widetilde{L})$, $w \in C_G(r)$ and $j \in Q$, we obtain $C_{U_1}(\widetilde{L}) \leq Q$ and the elements u, j, uj, ur, jr and ujr are all G-conjugate. Therefore

$$Z(C_{C_S(j)}(u)) = \langle j, r, u \rangle$$

and this subgroup contains a unique G-conjugate of r by Lemma 3.2.

Notice that $C_{U_1\widetilde{L}}(r)$ has shape $2^6{:}\mathrm{Alt}(4)$ and this group embeds into $C_M(r) = X_jC_S(j)$ by Lemma 3.6. Thus as an element τ of order 3 in $C_{U_1\widetilde{L}}(r)$ has commutator of order at least 16 on $O_2(C_{U_1\widetilde{L}}(r))$, we infer that this element is actually a diagonal element in $X_j = K_sK_t$. Therefore $u \in C_Q(\tau) = C_Q(X_j) = O_2(K_aK_b)$ where $\{a, b, s, t\} = \{1, 2, 3, 4\}$. In particular,

$$[X_j, u] = 1.$$

Suppose that $\widetilde{M}$ has index at most 2 in M and contains $U_1L_1' \cong 2^6$:Alt(6). We claim that $\widetilde{M}$ has a subgroup of index 2. We claim that u is not M-conjugate to an element x of $S \cap J_1 \cap \widetilde{M}$ such that $C_{C_S(j)}(x)$ is a Sylow 2-subgroup of $C_M(x)$. As $S \cap J_1 \cap \widetilde{M}$ is a subgroup of index 2 in $S \cap \widetilde{M}$, the Thompson Transfer Lemma [6, Lemma 15.16] will then yield $\widetilde{M}$ has a subgroup of index 2.

Assume that $x \in S \cap J_1 \cap \widetilde{M}$ is M-conjugate to u. Since u and j are G-conjugate, u is not conjugate to r by Lemma 3.2. Notice that $Z(C_{C_S(j)}(x)) = \langle j, r, x\rangle$. Assume that $C_{C_S(j)}(x)$ is a Sylow 2-subgroup of $C_M(x)$. Then there is $g \in M$ with $x^g = u$ and $C_{C_S(j)}(u) \le C_{C_S(j)}(x)^g$. Then r^g centralizes $U_1 \le C_{C_S(j)}(u)$ and so, as G has parabolic characteristic 2 and U_1 is normal and self-centralizing in S, $r^g \in U_1 \le C_{C_S(j)}(u)$. Hence

$$r^g \in Z(C_{C_S(j)}(u)) = \langle j, r, u\rangle.$$

Because r is the only G-conjugate of r in $Z(C_{C_S(j)}(u))$, we have $r^g = r$ and so $g \in C_G(r)$. Since $[X_j, u] = 1$ and X_j is normal in $M \cap C_G(r) = X_jC_S(j)$, we have $[X_j, x] = 1$ which means that

$$x^g \in x^{C_S(j)} \subseteq x^{C_H(j)}.$$

But $x^{C_H(j)}$ is contained in J_1, a contradiction as $u \notin J$. This proves the claim.

Taking $\widetilde{M} = M$, we see M has a subgroup $\widehat{M}$ of index 2. If $\widehat{M}$, contains U_1L_1' then we may take $\widetilde{M} = \widehat{M}$ and obtain a subgroup of index 4, establishing the lemma in this case. Thus we may assume $\widehat{M}$ is a subgroup of M of index 2 and $\widehat{M}$ does not contain U_1L_1'. Hence we may assume that $J_2 = \widehat{M} \cap C_H(j) > J$ has shape 2^5.Sym(6). Notice that this group contains L_1' but might be a non-split extension. Let

$$U_0 = U_1 \cap J_2 = O_2(J).$$

To simplify notation for the ensuing argument, we set

$$\overline{M} = M/\langle j\rangle.$$

With this notation we have $\overline{U_0}$ is elementary abelian of order 2^4, $\overline{J} \cong 2^4$:Alt(6) and $\overline{J}_2$ has shape 2^4.Sym(6).

As all involutions in a coset of $\overline{U_0}$ in $\overline{J}$ are $\overline{J}$-conjugate, $\overline{J}$ contains exactly two conjugacy classes of involutions.

Since $X_j = O^2(X_j) \leq \widehat{M}$, and $O_2(X_j)$ is non-abelian and is normalized by $C_J(r)$ where $C_J(r)/U_0 \cong \mathrm{Sym}(4)$, we obtain

$$O_2(X_j)U_0 \leq J$$

and

$$|O_2(X_j)U_0/U_0| = 2^2.$$

In particular, every involution in $\overline{J}$ is conjugate to one in $\overline{O_2(X_j)} \cong 2_+^{1+4}$. Since X_j is transitive on the involutions in $O_2(X_j) \setminus \{r\}$, we see that all the involutions in $\overline{J}$ are conjugate to $\overline{r}$.

Let $\rho \in K_sK_t \cap C_J(r)$ have order 3, and P be a Sylow 3-subgroup of J containing ρ. Then, as $C_{U_0}(P) = \langle j \rangle$, we get $\overline{N_{J_2}(P)} \cong \mathrm{Sym}(3) \wr 2$. Hence there exists $w \in J_2 \setminus J$ with $w^2 \in \langle j \rangle$ such that $[w, \rho] = 1$. Furthermore,

$$\langle \rho \rangle U_0/U_0 \leq C_{J/U_0}(w) \cong \mathrm{Sym}(4)$$

and as, w induces a transvection on $\overline{U_0}$ but not on U_1, we have w and wj are M-conjugate. As r and rj are not G-conjugate by Lemma 3.6 (ii), we have that $\overline{w}$ and $\overline{r}$ are not $\overline{M}$-conjugate. This shows that $\overline{w}$ is not conjugate to an element of $\overline{J}$ and therefore $\widehat{M}$ has a subgroup M^* of index 2 by the Thompson Transfer Lemma [6, Lemma 15.16]. This completes the proof of Lemma 3.7. □

We let M^* be the subnormal subgroup of G of index 4 and prove the following lemma.

Lemma 3.8. *We have $M^*/\langle j \rangle \cong \mathrm{PSU}_4(3)$.*

Proof. Set $\overline{M} = M/\langle j \rangle$. We intend to show that $M^*/\langle j \rangle$ satisfies the hypothesis of Aschbacher's Theorem 2.8. Since M^* is subnormal in G, we have $X_j = O^2(C_M(r)) \leq M^*$ and $J = C_H(j)' = O^2(C_H(j)) \leq M^*$. Furthermore, $C_S(j)M^* = M$ and so $|C_S(j) \cap M^*| = 2^8$. Then as $|J|_2 = 2^8$, $C_S(j) \cap M^* = S \cap J$. Thus we have

$$|C_{\overline{M^*}}(\overline{r})| = 2^7 \cdot 3^2,$$

and, as $j \notin O_2(X_j)$,

$$O^2(C_{\overline{M^*}}(\overline{r})) = \overline{X_j} = \overline{K_sK_t}$$

with $\overline{K_s} \cong \overline{K_t} \cong \mathrm{SL}_2(3)$ and $\overline{K_s} \cap \overline{K_t} = \langle \overline{r} \rangle$.

Set

$$W = O_2(X_j) = O_2(K_sK_t) \cong 2_+^{1+4}$$

and put

$$U_0 = O_2(J).$$

Observe that for $u \in U_0 \setminus W\langle j\rangle$ we have $u \notin O_2(X_j\langle u\rangle)$. As $\overline{J} \cong 2^4{:}\mathrm{Alt}(6)$ and $j \notin W$, $|WU_0/U_0| = 2^2$ and $(W \cap U_0)\langle j\rangle$ is a maximal subgroup of U_0. Then as $C_S(j) \cap M^* = S \cap J$ we infer that

$$O_2(C_{M^*}(r)) = C_Q(j) \cap M^* = W\langle j\rangle.$$

Hence

$$O_2(C_{\overline{M^*}}(\overline{r})) = \overline{W} \cong 2_+^{1+4}$$

and $|\overline{W} \cap \overline{U_0}| = 2^3$. Since $\overline{J}$ acts transitively on the involutions in $\overline{U_0}$, $\overline{r}^{\overline{M^*}} \cap \overline{W} \neq \{\overline{r}\}$. To complete the verification of the hypothesis of Theorem 2.8, it remains to show that if $\overline{r}^{\overline{g}} \in \overline{W} \setminus \{\overline{r}\}$, then $|\overline{W} \cap \overline{W}^{\overline{g}}| = 2^2$.

Assume that $\overline{r}^{\overline{g}} \in \overline{W}$ for some $g \in M^*$. As all the involutions in $\overline{W} \setminus \langle \overline{r}\rangle$ are conjugate in $C_{\overline{M^*}}(\overline{r})$ and $W \cap U_0$ has order 2^3, we may as well suppose that $\overline{r}^{\overline{g}} \in \overline{W}^{\overline{g}} \cap \overline{U_0}$. Since $\overline{J}$ acts transitively on the involutions in $\overline{U_0}$ and $\overline{W}^{\overline{g}} = O_2(C_{\overline{M^*}}(\overline{r}^{\overline{g}}))$, we may also suppose that $g \in J$. Since $C_{\overline{W}}(\overline{r}^g) \cong \mathrm{Dih}(8) \times 2$, we have

$$\langle \overline{r}\rangle = C_{\overline{W}}(\overline{r}^g)' \leq \overline{W}^g$$

as $C_{\overline{M^*}}(\overline{r}^{\overline{g}})/\overline{W}^{\overline{g}}$ has abelian Sylow 2-subgroups. Hence

$$|\overline{W} \cap \overline{W}^g| \geq 2^2.$$

Assume that $WU_0 = W^gU_0$. Then $C_{U_0}(W) = \langle r, j\rangle = \langle r^g, j\rangle$ which means that $\overline{r} = \overline{r}^g$ and then $\overline{W} = \overline{W}^{\overline{g}}$, a contradiction. Thus $WU_0 \neq W^gU_0$. As in $J/U_0 \cong \mathrm{Alt}(6)$ two conjugate fours groups are either equal or intersect in the identity subgroup, we now have $\overline{W} \cap \overline{W}^g \leq \overline{U_0}$. Assume that $\overline{W} \cap \overline{W}^g$ has order 2^3. Then $\overline{W} \cap U_0 = \overline{W}^g \cap U_0$. It follows that

$$[\overline{U_0}, \overline{W}] = \overline{U_0} \cap \overline{W} = \overline{U_0} \cap \overline{W}^{\overline{g}} = [\overline{U_0}, \overline{W}^{\overline{g}}].$$

But then $\langle \overline{W}, \overline{W}^g\rangle$ centralizes $\overline{U_0}/[\overline{U_0}, \overline{W}]$ and this implies that $\overline{W} = \overline{W}^{\overline{g}}$, a contradiction. Therefore $|\overline{W} \cap \overline{W}^{\overline{g}}| = 4$. We have verified all the assumptions of Theorem 2.8 and so $\overline{M^*} \cong \mathrm{PSU}_4(3)$. □

We now establish more notation:

Notation 3.9. *Set $I = \{1, 2, 3, 4\}$ and let $a \in I$. Then*

(i) *$\langle \tau_a\rangle \in \mathrm{Syl}_3(K_a)$;*
(ii) *$e \in C_G(r)$ has order 2 is chosen so that e inverts τ_b for all $b \in I$ (so e projects to e^* from Lemma 3.5 and can be seen in a complement to Q in $C_H(r)$);*
(iii) *$C_a = C_G(\tau_a)$;*

(iv) $D_a = O(C_a)$ *is the largest normal subgroup of* C_a *of odd order; and*

(v) $W_a = K_bK_cK_d$ *where* $\{a, b, c, d\} = I$.

From the structure of $X = K_1K_2K_3K_4 = K_aW_a$, we know that W_a commutes with τ_a and we have W_a has shape $2_-^{1+6}.3^3$. Also, from Lemmas 3.3 and 3.5, we recall

$$|C_G(r) : XS| = |H : F^*(H)| \text{ divides } 3$$

and $C_{C_G(r)}(\tau_a) = \langle\tau_a\rangle \times W_a$ if $C_G(r) = XS$ and otherwise $\langle\tau_a\rangle W_a$ is of index 3 in $C_{C_G(r)}(\tau_a)$ and permutes the set $\{K_b, K_c, K_d\}$ transitively. We also note that $K_a\langle e\rangle \cong \mathrm{GL}_2(3)$. Our next objective is to determine the structure of the centralizer C_a.

Lemma 3.10. *Suppose that* $a \in I$. *Then* $rD_a \in Z(C_a/D_a)$. *In particular,*

$$C_a = C_{C_G(r)}(\tau_a)D_a$$

with $|C_a : D_aW_a| = |H : F^*(H)|$ *and, if* $B \le C_a$ *is normalized by* r, *then* $[B, r] \le D_a$.

Proof. By definition $C_a \ge W_a$ and $O_2(W_a)$ is extraspecial of order 2^7. Hence, as $O_2(W_a) \in \mathrm{Syl}_2(C_{C_G(r)}(\tau_a))$, we also have $O_2(W_a) \in \mathrm{Syl}_2(C_G(\tau_a))$. Assume that r is C_a-conjugate to $t \in O_2(W_a)$. Then $C_{O_2(W_a)}(t)$ has derived subgroup $\langle r\rangle$ whereas it is also contained in $\langle t\rangle$. Hence $r = t$ and we conclude

$$r^{C_a} \cap C_{C_a}(r) = \{r\}.$$

Now the first assertion follows from the Z^*-Theorem [5]. Moreover, we have

$$|C_a : D_aW_a| = |C_{C_G(r)}(\tau_a) : \langle\tau_a\rangle W_a| = |H : F^*(H)|.$$

Finally, as $r \in Z(C_a/D_a)$, if $B \le C_a$, then $[B, r] \le D_a$. □

Lemma 3.11. *For* $a \in I$, *the subgroup* D_a *is extraspecial of exponent* 3 *and order* 3^9. *Furthermore,*

(i) $D_a/\langle\tau_a\rangle$ *is inverted by* r; *and*

(ii) G *has parabolic characteristic* 3.

Proof. Let $j \in W_a$ be an involution with $j \ne r$ that commutes with K_b for some $b \in I \setminus \{a\}$. Then j and jr are G-conjugate to one of j_1, j_2 or j_3 and $\langle r, j\rangle \le W_a$ acts on D_a. Set $M^* = O^2(C_G(j))$ so that $M^*/\langle j\rangle \cong \mathrm{PSU}_4(3)$ by Lemma 3.8.

As

$$C_{D_a}(r) \le O(C_{C_G(r)}(\tau_a)) = \langle\tau_a\rangle,$$

we see that r inverts $D_a/\langle \tau_a \rangle$. Now coprime action gives

$$D_a = C_{D_a}(j)C_{D_a}(rj) \text{ with } C_{D_a}(j) \cap C_{D_a}(rj) = \langle \tau_a \rangle$$

and so

$$[C_{D_a}(j), C_{D_a}(rj)] = [C_{D_a}(j), [D_a, j]] = 1$$

by the Three Subgroups Lemma.

To determine the structure of D_a we consider $C_{M^*}(\tau_a)$. We have that $\tau_a \in M^*$ and τ_a centralizes $\langle r, j \rangle$ and so, by [4, pages 52 and 54], $O_3(C_{M^*}(\tau_a))$ is an extraspecial group of order 3^5 and exponent 3 extended by $K_b = O^2(K_b)$ for some $b \in I$. In particular, using Lemma 3.10 yields

$$\langle \tau_a \rangle < O_3(C_{M^*}(\tau_a)) = [O_3(C_{M^*}(\tau_a)), r] \le D_a$$

and so $|C_{D_a}(j)| = 3^5$. Similarly $|C_{D_a}(rj)| = 3^5$ and so D_a is extraspecial of order 3^9 and exponent 3.

If $C_{C_a}(D_a) \not\le D_a$, then $C_{C_a}(D_a)$ has even order and is normal in C_a. Thus $r \in C_{C_a}(D_a)$, whereas we know r inverts $D_a/\langle \tau_a \rangle$. Hence $C_{C_a}(D_a) \le D_a$, so, in particular, $\langle \tau_a \rangle = Z(S_a)$ for $S_a \in \mathrm{Syl}_3(C_a)$ and then part (iii) follows from Lemma 2.1. □

We proceed to define some additional notation.

Notation 3.12. *Assume that $I = \{a, b, c, d\}$. We make the following definitions.*

(i) *For $\emptyset \ne J \subseteq I$,*
$$D_J = \bigcap_{k \in J} D_k.$$
(ii) $E_{ab} = D_{\{a,b\}}\langle \tau_a, \tau_b \rangle$.
(iii) $E_a = E_{ab}E_{ac}E_{ad}$.
(iv) $V_a = \langle D_I, \tau_a \rangle$.
(v) $T_a = D_aE_a = D_a\langle \tau_b, \tau_c, \tau_d \rangle$.

Lemma 3.13. *Assume that $I = \{a, b, c, d\}$. Then we have*

(i) *E_{ab} is elementary abelian of order 3^6;*
(ii) *E_{ab} is normalised by $\langle D_a, D_b \rangle K_cK_d\langle e \rangle$ where the element e is as in Notation 3.9.*
(iii) *$E_{ab} \cap E_{ac} = \langle \tau_a \rangle D_{\{a,b,c\}}$, $D_{\{a,b,c\}}$ has order 3^2 and is normalized by K_d.*
(iv) *V_a is centralized by E_a, $[V_a, D_a] = \langle \tau_a \rangle$ and $[V_a, D_{\{b,c,d\}}] = D_I$.*

Proof. To make the argument as clear as possible we assume that $a = 1$ and $b = 2$. Then

$$K_3K_4 \le C_1 \cap C_2.$$

In particular K_3K_4, which has shape $2_+^{1+4}.3^2$, acts on $D_{\{1,2\}}$. Since r inverts $D_1/\langle\tau_1\rangle$ by Lemma 3.11, and since $\tau_1 \notin D_2$, r inverts $D_{\{1,2\}}$ and so $D_{\{1,2\}}$ is elementary abelian.

As D_1 is extraspecial of order 3^9 and $D_{1,2}$ is abelian, with $\tau_1 \notin D_2$ by Lemma 3.10, it follows that $|D_{\{1,2\}}| \leq 3^4$.

Since $C_{D_1/\langle\tau_1\rangle}(\tau_2) > 1$ and is K_3K_4-invariant, using Lemma 3.10 delivers

$$1 \neq [C_{D_1}(\tau_2), r] \leq D_{\{1,2\}}.$$

Then, as r inverts $D_{\{1,2\}}$, K_3K_4 acts faithfully on $D_{\{1,2\}}$ and so $|D_{\{1,2\}}| = 3^4$ and

$$E_{12} = \langle\tau_1, \tau_2\rangle D_{\{1,2\}}$$

is elementary abelian of order 3^6 as claimed in (i).

Part (i) yields $C_{D_1}(\tau_2) \geq D_{\{1,2\}}\langle\tau_1\rangle$ has order 3^5. Since $K_2 = \langle\tau_2, \tau_2^k\rangle$ for some $k \in K_2$, $r \in Z(K_2)$ and r inverts $D_1/\langle\tau_1\rangle$, we have $C_{D_1}(\tau_2) = D_{\{1,2\}}\langle\tau_1\rangle$.

The Three Subgroups Lemma yields $[C_{D_1}(\tau_2), [D_1, \tau_2]] = 1$ and so, as $C_{D_1}(\tau_2)$ is a maximal abelian subgroup of D_1, we have

$$[D_1, \tau_2] \leq C_{D_1}(\tau_2) \leq E_{12}.$$

In particular, D_1 normalizes E_{12} and similarly so does D_2. Thus we have seen that E_{12} is normalized by $\langle D_1, D_2\rangle K_3K_4$. Finally, as e inverts τ_1 and τ_2, we have that e also normalizes $D_{\{1,2\}}$. Therefore (ii) holds.

For the proof of the remaining parts of the lemma, we return to our general notation. We have just proved that $E_{ab} = \langle\tau_a\rangle C_{D_b}(\tau_a)$.

Since $D_{\{b,c\}}$ has order 3^4 and admits $K_aK_d \cong \mathrm{SL}_2(3) \circ \mathrm{SL}_2(3)$ acting irreducibly, we have $C_{D_{\{b,c\}}}(\tau_a)$ has order 3^2, admits K_d acting faithfully and is inverted by r. Thus $C_{D_{\{b,c\}}}(\tau_a) \leq D_a$ by Lemma 3.10 and therefore $C_{D_{\{b,c\}}}(\tau_a) = D_{\{a,b,c\}}$. Similarly we can demonstrate that $D_I = C_{D_{\{a,b,c\}}}(\tau_d)$ has order 3.

Now we calculate

$$\begin{aligned} E_{ab} \cap E_{ac} &\leq C_{E_{ab}}(\tau_c) \cap C_{E_{ac}}(\tau_b) \\ &= \langle\tau_a, \tau_b\rangle C_{D_{\{a,b\}}}(\tau_c) \cap \langle\tau_a, \tau_c\rangle C_{D_{\{a,c\}}}(\tau_b) \\ &= \langle\tau_a, \tau_b\rangle D_{\{a,b,c\}} \cap \langle\tau_a, \tau_c\rangle D_{\{a,c,b\}} = \langle\tau_a\rangle D_{\{a,b,c\}} \\ &\leq E_{ab} \cap E_{ac}. \end{aligned}$$

This completes the proof of (iii).

By definition, $V_a = \langle\tau_a, D_I\rangle \leq D_a$ and so $[V_a, D_a] = \langle\tau_a\rangle$. Also, by part (i), E_{ax} centralizes V_a for all $x \in I \setminus \{a\}$. Hence

$$T_a = D_aE_a = D_aE_{ab}E_{ac}E_{ad} = D_a\langle\tau_b, \tau_c, \tau_d\rangle$$

normalizes V_a. Now $D_{\{b,c,d\}}$ has order 9 and centralizes D_I. Thus

$$D_I = C_{D_{\{b,c,d\}}}(\tau_a) = [D_{\{b,c,d\}}, \tau_a].$$

This completes the proof of (iv). □

By Lemma 3.13 (iv), for $a \in I$, V_a is centralized by E_a and $\langle D_a, D_{\{b,c,d\}}\rangle$ induces $\mathrm{SL}_2(3)$ on V_a. It follows that D_I is conjugate to $\langle \tau_a \rangle$. To build a subgroup of G isomorphic to $\mathrm{P}\Omega_8^+(3)$, we now specialize to $a = 4$ and set

$$P_4 = \langle T_4, D_{\{1,2,3\}}, r\rangle.$$

From the discussion above we have that V_4 is normal in P_4 and $O^2(P_4/C_{P_4}(V_4)) \cong \mathrm{SL}(V_4)$.

Lemma 3.14. *The following hold:*

(i) $P_4 \le N_G(E_{a4})$ *for all* $a \in I \setminus \{4\}$.

(ii) $C_{P_4}(V_4) = E_4 = O_3(P_4)$;

(iii) $P_4/E_4 \cong \mathrm{GL}_2(3)$;

(iv) $C_G(V_4)$ *is a* 3*-group.*

Proof. By Lemma 3.13 (ii), for $a \in I \setminus \{4\}$, E_{a4} is normalized by D_4 and D_a. As $D_{\{1,2,3\}} \le D_a$, we certainly have $\langle T_4, D_{\{1,2,3\}}\rangle \le N_G(E_{a4})$. Since r normalizes E_{a4}, we have proved (i).

As $E_4 = E_{14}E_{24}E_{34}$ it follows from (i) that $E_4 \trianglelefteq P_4$. As remarked above $O^2(P_4/C_{P_4}(V_4)) = \mathrm{SL}(V_4)$ and so

$$E_4 \le O_3(P_4) \le C_{P_4}(V_4).$$

Since $O^2(\mathrm{Aut}_{P_4}(V_4)) = \mathrm{SL}(V_4)$ with r inverting D_I and centralizing τ_4 we see that

$$P_4/C_{P_4}(V_4) \cong \mathrm{GL}_2(3).$$

Since $\tau_4 \in V_4$, $C_{P_4}(V_4) \le C_4$. We have already pointed out that $E_4 \trianglelefteq P_4$ and so $C_{P_4}(V_4)D_4/D_4$ normalizes $E_4D_4/D_4 = T_4/D_4$. Let $C_4^* = C_4/\langle\tau_4\rangle$. Now $N_{W_4^*}(T_4^*) = T_4^*\langle r^*\rangle$. Since r inverts D_I, $r \notin C_{P_4}(V_4)$ and so $C_{P_4}(V_4) \cap D_4W_4 \le T_4$. Let $E = E_{14}E_{24} \cap D_4$; by parts (i) and (iii) of Lemma 3.13 we have $|E| = 3^7$. If $E_{13} \cap D_4 \le E$ then $E = E_{a4}E_{b4} \cap D_4$ for all distinct $a, b \in I \setminus \{4\}$. But E is W_4-invariant by Lemma 3.13(ii), contradicting W_4 acting irreducibly on D_4^*. Therefore $|E_4 \cap D_4| \ge 3^8$. Then as $E_4 \le C_{P_4}(V_4)$ with $|D_4 : C_{D_4}(V_4)| = 3$, we conclude that (ii) and (iii) hold.

We have $C_G(V_4) \le C_4$. To prove (iv) it suffices to demonstrate that $C_{K_1K_2K_3}(V_4) = \langle \tau_1, \tau_2, \tau_3\rangle$. This follows as any non trivial subgroup of $O_2(K_1K_2K_3)$ normalized by $\langle \tau_1, \tau_2, \tau_3\rangle$ contains r and $[V_4, r] \ne 1$. □

We have all the pieces of the puzzle needed to assemble what turn out to be the minimal parabolic subgroups of $\mathrm{P}\Omega_8^+(3)$.

Notation 3.15. *For* $\{a,b,c\} = I \setminus \{4\}$, *set*

$$P_a = D_4E_{b4}E_{c4}K_a\langle e\rangle = T_4K_a\langle e\rangle$$

and

$$P_{a4} = \langle P_a, P_4\rangle.$$

Then put

$$P = \langle P_a \mid a \in I\rangle.$$

Lemma 3.16. *Assume that* $\{a,b,c\} = \{1,2,3\}$. *Then*

(i) $O_3(P_a) = D_4E_{b4}E_{c4}$ *and* $P_a/O_3(P_a) \cong \mathrm{GL}_2(3)$*;*
(ii) $O^{3'}(P_a)O^{3'}(P_b) = O^{3'}(P_b)O^{3'}(P_a)$, $O^{3'}(P_a) \neq O^{3'}(P_b)$*;*
(iii) $P_{a4}/O_3(P_{a4}) \cong \mathrm{PSL}_3(3)$, $T_4 \in \mathrm{Syl}_3(P_{a4})$*; and*
(iv) $N_{P_a}(T_4) = N_{P_b}(T_4) = N_{P_c}(T_4) = N_{P_4}(T_4) = T_4\langle e, r\rangle$, *in particular* T_4 *is a Sylow* 3*-subgroup of* P_4.

Proof. Untangling the notation given in 3.9 and 3.12, we see that

$$P_a \leq D_4W_4\langle e\rangle \leq C_4\langle e\rangle$$

and so $D_4E_{b4}E_{c4} = O_3(P_a)$ and $P_a/O_3(P_a) \cong K_a\langle e\rangle \cong \mathrm{GL}_2(3)$. So (i) is true.

In addition, (ii) follows as we have

$$O^{3'}(P_a)O^{3'}(P_b) = D_4E_{c4}K_aK_b = D_4E_{c4}K_bK_a = O^{3'}(P_b)O^{3'}(P_a).$$

And as $K_a \not\leq P_b$ we also have $O^{3'}(P_a) \neq O^{3'}(P_b)$.

Now consider part (iii). Set $V_{a4} = E_{b4} \cap E_{c4}$. Then, by Lemma 3.13 (iii), $|V_{a4}| = 3^3$ and

$$V_4 < V_{a4} = \langle \tau_4\rangle D_{bc4} \leq D_4.$$

Furthermore, by Lemmas 3.14 (i) and 3.13 (ii), P_4 and P_a both normalize E_{b4} and E_{c4} . Therefore V_{a4} is normalized by P_{a4}. Hence $P_{a4}/C_{P_{a4}}(V_{a4})$ embeds into $\mathrm{GL}_3(3)$. Since $V_{a4} \leq D_4$, $V_{a4}/\langle \tau_4\rangle$ is inverted by r, $K_a\langle e\rangle$ acts as $\mathrm{GL}_2(3)$ on $V_{a4}/\langle \tau_4\rangle$ and so

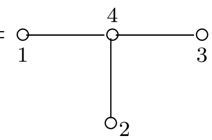
$$P_a/C_{D_4}(V_{a4})E_{b4}E_{c4} \cong 3^2 : \mathrm{GL}_2(3).$$

As e inverts τ_4 we see that e acts as an element of determinant 1 on V_{a4}. Thus P_a acts on V_{a4} as a maximal parabolic subgroup of $\mathrm{SL}_3(3)$. Since P_4 moves $\langle \tau_4\rangle$, we have that $P_{a4}/C_{P_{a4}}(V_{a4}) \cong \mathrm{SL}_3(3)$. Finally, we observe that by Lemma 3.14 (iv) $C_{P_{a4}}(V_{a4}) \leq C_G(V_4)$ is a 3-group. Then $C_{P_{a4}}(V_{a4}) = O_3(P_{a4})$ and $|T_4 : O_3(P_{a4})| = 3^3$. Hence $T_4 \in \mathrm{Syl}_3(P_{a4})$. This completes the proof of (iii).

Finally, from Lemma 3.14 we know that T_4 is a Sylow 3-subgroup of P_4 and, by Lemma 3.11, $Z(T_4) = \langle \tau_4\rangle$, so $N_G(T_4)$ normalizes $\langle \tau_4\rangle$. Therefore $\langle e, r\rangle$ is a Sylow 2-subgroup of $N_G(T_4)$ and $N_G(T_4)$ is a $\{2,3\}$-group. This forces the statement in (iv). □

Proposition 3.17. *We have* $P \cong \mathrm{P}\Omega_8^+(3)$.

Proof. We are going to apply [9]. Let Π be the Coxeter diagram

$\Pi =$ 

indexed over I and $\mathcal{D}$ be all subsets of size 2 in I containing 4. Hence $\mathcal{D}$ satisfies [9, Hypothesis 6.2(i)].

For $D = \{a, 4\} \in \mathcal{D}$, we define $P_D = P_{a4}$ as in 3.15.

Let $L = \mathrm{P}\Omega_8^+(3)$. Then the building Δ for L consists of the totally singular subspaces of the orthogonal space V for $\Omega_8^+(3)$. We let $\mathfrak{c}$ be a fixed chamber of Δ. So $\mathfrak{c} = \{V_1, V_2, V_4^1, V_4^2\}$, where V_i and V_i^j are i-dimensional totally singular subspaces of V with $V_1 \le V_2 \le V_4^1 \cap V_4^2$, the latter of dimension 3. Then, for each $D \in \mathcal{D}$, let Δ_D be the D-residue of Δ containing $\mathfrak{c}$ and set L_D the group of automorphisms of Δ_D induced by the stabiliser in L of Δ_D. Then we have

$$L_D \cong \mathrm{PSL}_3(3).$$

By Lemma 3.16 there is a surjective homomorphism ϕ_D from P_D onto L_D with kernel $O_3(P_D)$. This gives [9, Hypothesis 6.2(ii)]. For $a \in D$, we denote by P_{Da} the preimage under ϕ_D of the stabiliser in L_D of a residue of type a containing the chamber $\mathfrak{c}$ and $B_D = \bigcap_{a \in D} P_{Da}$. Thus

$$\{P_{Da} \mid a \in D\} = \{P_a \mid a \in D\}.$$

The existence of graph automorphisms of L_D implies that we may choose ϕ_D in such a way that for all $D \in \mathcal{D}$ and $a \in D$, we have $P_{Da} = P_a\phi_D$. The only 2-sets which are not a $\mathcal{D}$-set are those in $\{1, 2, 3\}$. [9, Hypothesis 6.2(v)] now follows from our Lemma 3.16 (ii). Finally [9, Hypothesis 6.2(vi)] is obvious. As $|O_3(B_D)| = 3^{12} = |O_3(L_\emptyset)|$, where $L_\emptyset$ is a Borel subgroup of L, and $C_{P_D}(O_3(P_D)) \le O_3(P_D)$ by Lemma 3.16 application of [9, Theorem 6.8] yields $O^{3'}(P) \cong \mathrm{P}\Omega_8^+(3)$. Finally, as $B_D = N_P(T_4) \le O^{3'}(P)$, the Frattini Argument gives $P = O^{3'}(P)$. This completes the proof of the proposition. □

Proposition 3.18. *We have* $G = P \cong \mathrm{P}\Omega_8^+(3)$.

Proof. From the structure of P, we have $XS \le P$. We first claim that $C_G(r)$ normalizes P. If $H = F^*(H)$, then $C_G(r) = XS \le P$ and there is nothing further to prove. So suppose that $F^*(H) < H$.

Let $T = \langle \tau_a \mid a \in I \rangle$. Then from Lemma 3.5,

$$N_{C_G(r)}(T) \cap C_G(e) \cap N_G(\langle \tau_4 \rangle) = \langle e, r, \theta \rangle$$

where $\langle\theta\rangle$ has order 3 and permutes $\{\tau_1, \tau_2, \tau_3\}$ transitively. Let θ induces $(1, 2, 3)$. We have $H = \langle\theta\rangle F^*(H)$, $\langle\theta\rangle$ normalizes D_4 and permutes $\{D_1, D_2, D_3\}$ and $\{E_{14}, E_{24}, E_{34}\}$. It follows that θ normalizes P_4 and permutes P_1, P_2, P_3. Hence θ normalizes P. Now $P\langle\theta\rangle \geq C_G(r) = XS\langle\theta\rangle$.

By [4, page 140] P has just four conjugacy classes of involutions and these classes have representatives r, j_1, j_2 and j_3. Hence by Lemma 3.2 we have that $PC_G(r)$ controls G-fusion of r in $PC_G(r)$. Therefore we may apply Theorem 2.7. As P is not soluble and not isomorphic to an alternating group the simplicity of G implies that $G = P$. □

Finally Theorem 1 follows from Proposition 3.4 and Proposition 3.18.

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