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# The Complexity of the Hamilton Cycle Problem in Hypergraphs of High Minimum Codegree\*

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## Abstract

We consider the complexity of the Hamilton cycle decision problem when restricted to  $k$ -uniform hypergraphs  $H$  of high minimum codegree  $\delta(H)$ . We show that for tight Hamilton cycles this problem is NP-hard even when restricted to  $k$ -uniform hypergraphs  $H$  with  $\delta(H) \geq \frac{n}{2} - C$ , where  $n$  is the order of  $H$  and  $C$  is a constant which depends only on  $k$ . This answers a question raised by Karpiński, Ruciński and Szymańska. Additionally we give a polynomial-time algorithm which, for a sufficiently small constant  $\varepsilon > 0$ , determines whether or not a 4-uniform hypergraph  $H$  on  $n$  vertices with  $\delta(H) \geq \frac{n}{2} - \varepsilon n$  contains a Hamilton 2-cycle. This demonstrates that some looser Hamilton cycles exhibit interestingly different behaviour compared to tight Hamilton cycles. A key part of the proof is a precise characterisation of all 4-uniform hypergraphs  $H$  on  $n$  vertices with  $\delta(H) \geq \frac{n}{2} - \varepsilon n$  which do not contain a Hamilton 2-cycle; this may be of independent interest. As an additional corollary of this characterisation, we obtain an exact Dirac-type bound for the existence of a Hamilton 2-cycle in a large 4-uniform hypergraph.

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## 1 Introduction

The study of Hamilton cycles in graphs has been a topic of great significance in graph theory, and continues to be well-studied. For example, the Hamilton cycle decision problem (given a graph, determine whether it contains a Hamilton cycle) was one of Karp's celebrated 21 NP-complete problems [9], whilst one very well-known classic result is Dirac's theorem [4], which states that any graph on  $n \geq 3$  vertices with minimum degree at least  $\frac{n}{2}$  contains a Hamilton cycle.

The problem of generalising these results to the hypergraph setting has been a highly-active area of research over the past several years (see, for example, the recent surveys by Kühn and Osthus [15], Rödl and Ruciński [16] and Zhao [21]). To describe these developments we require the following standard definitions. A  $k$ -uniform hypergraph, or  $k$ -graph  $H$  consists of a set of vertices  $V(H)$  and a set of edges  $E(H)$ , where each edge consists of  $k$  vertices. So a 2-graph is a (simple) graph. We say that a  $k$ -graph  $C$  is an  $\ell$ -cycle if its vertices can be cyclically ordered in such a way that each edge of  $C$  consists of  $k$  consecutive

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vertices, and so that each edge intersects the subsequent edge in  $\ell$  vertices. This naturally generalises the notion of a cycle in a graph, and is the most commonly-used definition of a hypergraph cycle. However, various other definitions have also been considered, such as a Berge cycle [1]. Note in particular that each edge of an  $\ell$ -cycle  $k$ -graph  $C$  has  $k - \ell$  vertices which were not contained in the previous edge, so the number of vertices of  $C$  must be divisible by  $k - \ell$ . We say that a  $k$ -graph  $H$  on  $n$  vertices contains a *Hamilton  $\ell$ -cycle* if it contains an  $n$ -vertex  $\ell$ -cycle as a subgraph; as before, this is only possible if  $k - \ell$  divides  $n$ . We refer to  $(k - 1)$ -cycles as *tight cycles*, and in the same way refer to *tight Hamilton cycles*. Given a  $k$ -graph  $H$  and a set  $S \subseteq V(H)$ , the *degree* of  $S$ , denoted  $\deg_H(S)$  (or  $\deg(S)$  when  $H$  is clear from the context), is the number of edges of  $H$  which contain  $S$  as a subset. The *minimum codegree* of  $H$ , denoted  $\delta(H)$ , is the minimum of  $\deg(S)$  taken over all sets of  $k - 1$  vertices of  $H$ , and the *maximum codegree* of  $H$ , denoted  $\Delta(H)$ , is the maximum of  $\deg(S)$  taken over all sets of  $k - 1$  vertices of  $H$ . In the graph case the maximum and minimum codegree are simply the maximum and minimum degree respectively.

An elementary reduction from the graph case demonstrates that for any  $k \geq 3$  and  $1 \leq \ell \leq k$  the Hamilton  $\ell$ -cycle decision problem (given a  $k$ -graph  $H$ , determine whether it contains a Hamilton  $\ell$ -cycle) is also NP-complete. For this reason, many authors have asked for conditions on  $H$  which render this problem tractable, or which guarantee the existence of a Hamilton  $\ell$ -cycle in  $H$ . In particular, since a Hamilton cycle in  $H$  cannot exist if  $H$  has an isolated vertex, it is natural to study minimum degree conditions on  $H$ .

### 1.1 Dirac-Type Results

The following theorem, whose various cases were proved by Rödl, Ruciński and Szemerédi [17, 18], Kühn and Osthus [14], Keevash, Kühn, Osthus and Mycroft [12], Hàn and Schacht [6], and Kühn, Osthus and Mycroft [13], is an approximate hypergraph analogue of Dirac's theorem; for any  $k$  and  $\ell$  it gives the asymptotically best-possible minimum codegree condition which guarantees the existence of a Hamilton  $\ell$ -cycle in a  $k$ -graph.

► **Theorem 1.** ([6, 12, 13, 14, 17, 18]) *For any  $k \geq 3$ ,  $1 \leq \ell \leq k - 1$  and  $\eta > 0$ , there exists  $n_0$  such that if  $n \geq n_0$  is divisible by  $k - \ell$  and  $H$  is a  $k$ -graph on  $n$  vertices with*

$$\delta(H) \geq \begin{cases} \left(\frac{1}{2} + \eta\right)n & \text{if } k - \ell \text{ divides } k, \\ \left(\frac{1}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} + \eta\right)n & \text{otherwise,} \end{cases}$$

*then  $H$  contains a Hamilton  $\ell$ -cycle.*

Simple constructions show that for any  $k$  and  $\ell$  this minimum codegree condition is best possible up to the  $\eta n$  error term. More recently the exact threshold (for large  $n$ ) has been determined in some cases: for  $k = 3, \ell = 2$  by Rödl, Ruciński and Szemerédi [19], for  $k = 3, \ell = 1$  by Czygrinow and Molla [2], and for  $k \geq 3$  and  $\ell < k/2$  by Hàn and Zhao [8]. As part of our work on the question of tractability (described in more detail in the next section), we successfully characterised all 4-graphs  $H$  with  $\delta(H) \geq \frac{n}{2} - \varepsilon n$  which do not contain a Hamilton cycle. As a straightforward consequence of this, we add to the aforementioned results the exact Dirac-type statement for the previously-open case  $k = 4, \ell = 2$ .

► **Theorem 2.** *There exists  $n_0$  such that if  $n \geq n_0$  is even and  $H$  is a 4-graph on  $n$  vertices with*

$$\delta(H) \geq \begin{cases} \frac{n}{2} - 2 & \text{if } n \text{ is divisible by } 8, \\ \frac{n}{2} - 1 & \text{otherwise,} \end{cases}$$

then  $H$  contains a Hamilton 2-cycle. Moreover, this condition is best-possible for any even  $n \geq n_0$ .

## 1.2 Tractability of the Restricted Hamilton Cycle Decision Problem

We now turn to the primary focus of this paper: minimum degree conditions which render the Hamilton cycle decision problem tractable. In the graph case, Dahlhaus, Hajnal and Karpiński [3] showed that for any fixed  $\varepsilon > 0$  this problem remains NP-complete when restricted to graphs with minimum degree at least  $(1 - \varepsilon)\frac{n}{2}$ . More recently, Karpiński, Ruciński and Szymańska [10] showed that for any  $k \geq 3$  and any fixed  $\varepsilon > 0$  the tight Hamilton cycle decision problem remains NP-complete when restricted to  $k$ -graphs with minimum codegree  $(1 - \varepsilon)\frac{n}{k}$ . They noted that, combined with Theorem 1, this left a ‘hardness gap’ of  $[\frac{n}{k}, \frac{n}{2}]$  for which the hardness of the problem remained unknown. We answer this question with the following theorem.

► **Theorem 3.** *For any  $k \geq 3$  there exists  $C$  such that the tight Hamilton cycle decision problem remains NP-complete when restricted to  $k$ -graphs  $H$  with  $\delta(H) \geq \frac{n}{2} - C$  (where  $n = |V(H)|$ ).*

Assuming that  $P \neq NP$ , Theorems 1 and 3 together imply that the minimum codegree threshold at which the tight Hamilton cycle decision problem becomes tractable is asymptotically equal to the minimum codegree threshold for the existence of a tight Hamilton cycle, mirroring the situation in the graph case. Interestingly, we can demonstrate that the Hamilton 2-cycle problem exhibits significantly different behaviour; our next theorem shows that there is a linear-size gap between the threshold at which the problem becomes tractable and at which the existence of a cycle is guaranteed.

► **Theorem 4.** *There exist a constant  $\varepsilon > 0$  and an algorithm which, given a 4-graph  $H$  on  $n$  vertices with  $\delta(H) \geq \frac{n}{2} - \varepsilon n$ , determines in time  $O(n^{25})$  whether  $H$  contains a Hamilton 2-cycle.*

A slight adaptation of the argument of Karpiński, Ruciński and Szymańska [10] mentioned above shows that for any fixed  $\varepsilon > 0$  the Hamilton 2-cycle problem remains NP-complete when restricted to 4-graphs with minimum codegree at least  $(1 - \varepsilon)\frac{n}{4}$ .

A key result in our proof of Theorem 4, which may be of independent interest, is Theorem 6, which (for sufficiently small  $\varepsilon$  and large  $n$ ) precisely characterises all 4-graphs on  $n$  vertices which satisfy  $\delta(H) \geq \frac{n}{2} - \varepsilon n$  but which do not contain a Hamilton 2-cycle. We prove this result using recently developed techniques of extremal graph theory, in particular the so-called ‘absorbing method’ of Rödl, Ruciński and Szemerédi [17]. Establishing this characterisation is the principal difficulty in the proof of Theorem 4, as then the algorithm for Theorem 4 simply checks whether this characterisation is satisfied. Likewise, Theorem 2 follows from Theorem 6 by a case analysis.

## 1.3 Discussion

In the light of Theorem 4, it would be very interesting to know which other values of  $k$  and  $\ell$  also have the property that there is a linear-size gap between the minimum codegree threshold which renders the  $k$ -graph Hamilton  $\ell$ -cycle problem tractable and the minimum codegree threshold under which the problem becomes trivial. Theorem 3 shows that this is not the case when  $\ell = k - 1$ , whilst a slight adaptation to the arguments of Karpiński,

Ruciński and Szymańska [10] demonstrates that this is also not true if  $k - \ell$  does not divide  $k$  (in which case the lower degree threshold of Theorem 1 applies); all other cases remain open.

We also note that Theorem 3 demonstrates an interesting difference between the perfect matching problem and tight Hamilton cycle problem in  $k$ -graphs. Indeed, while the unrestricted versions of both problems are NP-complete, Keevash, Knox and Mycroft [11] and Han [7] showed that the perfect matching problem can be solved in polynomial time in  $k$ -graphs  $H$  with  $\delta(H) \geq n/k$ ; complementing a previous result of Szymanska [20], who showed that for any  $\varepsilon > 0$  the problem remains NP-complete under the restriction  $\delta(H) \geq (\frac{1}{k} - \varepsilon)n$ . So, assuming  $P \neq NP$ , for any  $\frac{1}{k} \leq \alpha < \frac{1}{2}$  the two problems lie in distinct complexity classes when restricted to  $k$ -graphs with minimum codegree  $\delta(H) \geq \alpha n$ .

Finally, whilst the constant  $\varepsilon$  in Theorem 4 is quite small, we conjecture that Theorem 6 (the characterisation of 4-graphs  $H$  with  $\delta(H) \geq \frac{n}{2} - \varepsilon n$  and no Hamilton 2-cycle) is in fact valid under the weaker condition that  $\delta(H) > \frac{n}{3}$ . If true, this would imply that Theorem 4 would also hold under this weaker codegree assumption.

## 1.4 Notation

Given a set  $V$ , we write  $\binom{V}{k}$  for the set of subsets of  $V$  of size  $k$ . Also, we write  $x \ll y$  (“ $x$  is sufficiently smaller than  $y$ ”) to mean that for any  $y > 0$  there exists  $x_0 > 0$  such that for any  $x \leq x_0$  the subsequent statement holds. Similar statements with more variables are defined accordingly.

## 2 Hamilton 2-Cycles

In this section we outline the proof of Theorem 4. The key to the proof is Theorem 6, which precisely characterises all large 4-graphs  $H$  with  $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$  which do not contain a Hamilton 2-cycle. This is presented in Section 2.1. Having established this characterisation, it is fairly straightforward to exhibit a polynomial-time algorithm which tests whether a 4-graph has this property, as shown in Section 2.2. Instead, the difficult part of the proof is to prove Theorem 6; we outline how this is done in Section 2.3. Finally, in Section 2.4 we present the short deduction of Theorem 2 from Theorem 6.

### 2.1 A Characterisation of Dense 4-graphs with no Hamilton 2-Cycle.

For 4-graphs  $H$ , our characterisation considers partitions of  $V(H)$  into two parts  $A$  and  $B$ . Whenever we refer to, for example, ‘a partition  $(A, B)$  of  $V(H)$ ’, this should be interpreted as meaning a partition of  $V(H)$  into two non-empty parts  $A$  and  $B$ . Given such a partition of  $V(H)$ , we say that an edge  $e \in E(H)$  is *odd* if  $|e \cap A|$  is odd, and *even* if  $|e \cap A|$  is even. We write  $H_{\text{even}}$  for the subgraph of  $H$  consisting only of even edges of  $H$ , and similarly write  $H_{\text{odd}}$  for the subgraph of  $H$  consisting only of odd edges of  $H$ . Also, we say that a pair  $\{x, y\}$  of distinct vertices of  $H$  is a *split* pair if  $x \in A$  and  $y \in B$  or vice versa, and that  $\{x, y\}$  is an *equal* pair if  $x, y \in A$  or  $x, y \in B$ .

We define an  $\ell$ -path in a  $k$ -graph analogously to an  $\ell$ -cycle: a  $k$ -graph is an  $\ell$ -path if its vertices can be linearly ordered  $v_1, \dots, v_n$  such that every edge consists of  $k$  consecutive vertices and successive edges intersect in precisely  $\ell$  vertices. As for cycles we refer to  $(k-1)$ -paths as *tight paths*. The *length* of an  $\ell$ -path is the number of edges. Given a 4-graph  $H$ , we define the *total 2-pathlength* of  $H$  to be the maximum sum of lengths of vertex-disjoint 2-paths in  $H$ . For example,  $H$  having total 2-pathlength 3 could be achieved by 3 disjoint edges (i.e. 2-paths of length 1) in  $H$ , or a 2-path of length 3 in  $H$ , or two vertex-disjoint

2-paths in  $H$ , one of length 1 and one of length 2. Using these definitions we can now give the central definition of our characterisation.

► **Definition 5.** Let  $H$  be a 4-graph on  $n$  vertices, where  $n$  is even. We say that a partition  $(A, B)$  of  $V(H)$  is *even-good* if at least one of the following statements holds.

- (i)  $|A|$  is even or  $|A| = |B|$ .
- (ii)  $H$  contains odd edges  $e$  and  $e'$  such that either  $e \cap e' = \emptyset$  or  $e \cap e'$  is a split pair.
- (iii)  $|A| = |B| + 2$  and  $H$  contains odd edges  $e$  and  $e'$  with  $e \cap e' \in \binom{A}{2}$ .
- (iv)  $|B| = |A| + 2$  and  $H$  contains odd edges  $e$  and  $e'$  with  $e \cap e' \in \binom{B}{2}$ .

Now let  $m \in \{0, 2, 4, 6\}$  and  $d \in \{0, 2\}$  be such that  $m \equiv n \pmod{8}$  and  $d \equiv |A| - |B| \pmod{4}$ . Then we say that  $(A, B)$  is *odd-good* if at least one of the following statements holds.

- (v)  $(m, d) \in \{(0, 0), (4, 2)\}$ .
- (vi)  $(m, d) \in \{(2, 2), (6, 0)\}$  and  $H$  contains an even edge.
- (vii)  $(m, d) \in \{(4, 0), (0, 2)\}$  and  $H_{\text{even}}$  has total 2-pathlength at least two.
- (viii)  $(m, d) \in \{(6, 2), (2, 0)\}$  and either there is an edge  $e \in E(H)$  with  $|e \cap A| = |e \cap B| = 2$  or  $H_{\text{even}}$  has total 2-pathlength at least three.

A key observation is that if  $(A, B)$  is a partition of  $V(H)$  which is not even-good, then there exists a set  $X$  of at most four vertices of  $H$  such that every odd edge of  $H$  intersects  $X$ . Indeed, if  $H$  contains an odd edge  $e$ , then we may take  $X = e$ , and otherwise we may take  $X = \emptyset$ . Similarly, by choosing  $X$  to be the vertices of at most two disjoint even edges, or of a 2-path of length two in  $H_{\text{even}}$ , we find that if  $(A, B)$  is a partition of  $V(H)$  which is not odd-good, then there exists a set  $X$  of at most 8 vertices of  $H$  such that every even edge of  $H$  intersects  $X$ .

We now give our characterisation of 4-graphs of high minimum codegree with no Hamilton 2-cycle. Recall for this that any 2-cycle 4-graph has an even number of vertices.

► **Theorem 6.** *There exist  $\varepsilon, n_0 > 0$  such that the following statement holds for any even  $n \geq n_0$ . Let  $H$  be a 4-graph on  $n$  vertices with  $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$ . Then  $H$  admits a Hamilton 2-cycle if and only if every partition  $(A, B)$  of  $V(H)$  is both even-good and odd-good.*

## 2.2 The Algorithm.

Our polynomial-time algorithm for determining the existence of a Hamilton 2-cycle in a 4-graph of high codegree makes use of a special case of a result of Keevash, Knox and Mycroft [11]. This result allows us to efficiently list all partitions  $(A, B)$  of  $V(H)$  with no odd edges, or all partitions with no even edges.

► **Lemma 7.** ([11], special case of Lemma 2.2) *Let  $H$  be a 4-graph on  $n$  vertices with  $\delta(H) > \frac{n}{3}$ , and let  $x \in \{\text{even}, \text{odd}\}$ . Then there are at most 64 partitions  $(A, B)$  of  $V(H)$  for which no edge of  $H$  has parity  $x$  with respect to  $(A, B)$ . Moreover, there exists an algorithm  $\text{ListPartitions}(H, x)$  with running time  $O(n^5)$  which, given  $H$  and  $x$ , returns all such partitions.*

We now present an algorithm,  $\text{Procedure GoodPartition}(H, x)$ , which determines whether or not there exists an even-good/odd-good partition  $(A, B)$  for a 4-graph  $H$ . Note that, given a 4-graph  $H$  and a partition  $(A, B)$  of  $V(H)$ , the truth of the statements ‘ $(A, B)$  is odd-good’ and ‘ $(A, B)$  is even-good’ depend only on the values of  $n$  and  $|A|$  and whether or not  $H_{\text{odd}}$  or  $H_{\text{even}}$  contain certain subgraphs with at most 12 vertices. It follows that the validity of these statements (and therefore the condition of the ‘if’ statement in  $\text{Procedure GoodPartition}$ ) can be tested in time  $O(n^{12})$ .

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**Procedure** GoodPartition( $H, x$ )

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**Data:** A 4-graph  $H$  with vertex set  $V$  and a parity  $x \in \{\text{even}, \text{odd}\}$ .

**Result:** Determines if there is a partition  $(A, B)$  of  $V$  which is not  $x$ -good.

```

for each set  $X \subseteq V(H)$  with  $|X| = 8$  do
    Let  $V' = V \setminus X$  and  $H' = H[V']$ .
    Run Procedure ListPartitions( $H', x$ ) to obtain all partitions  $(A', B')$  of  $V'$  with no
    edges not of parity  $x$ .
    for each such partition  $(A', B')$  do
        for each partition  $(A, B)$  of  $V$  with  $A' \subseteq A$  and  $B' \subseteq B$  do
            if  $(A, B)$  is not  $x$ -good then
                State ' $(A, B)$  is not  $x$ -good', and terminate.
    State 'Every partition is  $x$ -good', and terminate.

```

---

► **Proposition 8.** Let  $H$  be a 4-graph on  $n$  vertices with  $\delta(H) > \frac{n}{3}$ , where  $n$  is even, and fix a parity  $x \in \{\text{even}, \text{odd}\}$ . Then Procedure GoodPartition( $H, x$ ) will correctly determine whether there exists a partition  $(A, B)$  of  $V(H)$  which is not  $x$ -good, with running time  $O(n^{25})$ .

**Proof.** We first establish correctness of the algorithm; for this, fix  $H$  and  $x$  as in the proposition statement. Clearly, if every partition  $(A, B)$  of  $V := V(H)$  is  $x$ -good, then GoodPartition( $H, x$ ) will output this fact. So suppose that some partition  $(A, B)$  of  $V$  is not  $x$ -good. As noted following Definition 5, we may then choose a set  $X$  of at most 8 vertices of  $H$  which is intersected by every edge of  $H$  which does not have parity  $x$ . This means that when GoodPartition( $H, x$ ) considers this set  $X$ , ListPartitions will return the partition  $(A', B')$  where  $A' = A \setminus X$  and  $B' = B \setminus X$ , and at this point GoodPartition( $H, \text{even}$ ) will return that  $(A, B)$  is not  $x$ -good, as required.

Finally we consider the running time of the algorithm. For this note that there are  $\binom{n}{8}$  choices for  $X$  in the outside 'for loop', and for each of these Procedure ListPartitions( $H', x$ ) runs in time  $O(n^5)$ . The inside 'for loops' then range over sets of size at most 64 (by Lemma 7) and  $2^8 = 256$  respectively. Finally, as noted above we may test whether a partition  $(A, B)$  is  $x$ -good in time  $O(n^{12})$ ; together these bounds combine to give the claimed running time. ◀

**Proof of Theorem 4.** Let  $n_0$  be sufficiently large and  $\varepsilon > 0$  sufficiently small for Theorem 6 to apply. Given a 4-graph  $H$  on  $n$  vertices with  $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$ , we apply the following algorithm. Firstly, if  $n$  is odd, then there can be no Hamilton 2-cycle in  $H$ , so we output this fact and terminate. Secondly, if  $n < n_0$ , then we use a brute-force approach, testing each of the at most  $n_0!$  orderings of  $V(H)$  in turn to determine whether it yields a Hamilton 2-cycle in  $H$ . We then output the appropriate answer and terminate. Finally, if  $n \geq n_0$  is even, then we first run Procedure GoodPartition( $H, \text{even}$ ), and then run Procedure GoodPartition( $H, \text{odd}$ ). If either of these procedures yields a partition  $(A, B)$  of  $V(H)$  which is not even-good or which is not odd-good, then we return that there is no Hamilton 2-cycle in  $H$ , otherwise we return that there is such a cycle. Note that in the first two cases this algorithm runs in constant time, whilst in the final case it runs in time  $O(n^{25})$  by Proposition 8. Moreover Theorem 6 ensures that this algorithm will always output the correct answer. ◀



### 2.3 Proof of Theorem 6.

We begin by establishing the forward implication of Theorem 6, expressed in the following proposition. In fact, the minimum codegree condition on  $H$  is not required for this direction.

► **Proposition 9.** If  $H$  is a 4-graph which contains a Hamilton 2-cycle, then every partition  $(A, B)$  of  $V(H)$  is both even-good and odd-good.

**Proof.** Let  $n$  be the order of  $H$ , let  $C = (v_1, v_2, \dots, v_n)$  be a Hamilton 2-cycle in  $H$  and let  $(A, B)$  be a partition of  $V(H)$ . Write  $P_i = \{v_{2i-1}, v_{2i}\}$  for each  $1 \leq i \leq \frac{n}{2}$ , so the edges of  $C$  are  $e_i := P_i \cup P_{i+1}$  for  $1 \leq i \leq \frac{n}{2}$  (with addition taken modulo  $\frac{n}{2}$ ). The key observation is that  $e_i$  is even if  $P_i$  and  $P_{i+1}$  are both split pairs or both equal pairs, and odd otherwise.

We first show that  $(A, B)$  is even-good. This holds by (ii) if  $H$  contains two disjoint odd edges, so we may assume without loss of generality that all edges of  $H$  other than  $e_1$  and  $e_{n/2}$  are even. It follows that the pairs  $P_2, P_3, \dots, P_{n/2}$  are either all split pairs or all equal pairs. In the former case, if  $P_1$  is a split pair then  $|A| = |B|$ , so (i) holds, whilst if  $P_1 \subseteq A$  then (iii) holds, and if  $P_1 \subseteq B$  then (iv) holds. In the latter case, if  $P_1$  is an equal pair then  $|A|$  is even, so (i) holds, whilst if  $P_1$  is a split pair then (ii) holds. So in all cases we find that  $(A, B)$  is even-good.

To show that  $(A, B)$  is odd-good, suppose first that 4 does not divide  $n$ , and note that by our key observation the number of even edges in  $C$  must then be odd. If  $C$  contains three or more even edges or an edge with precisely two vertices in  $A$ , then  $(A, B)$  is odd-good by (vi) and (viii), so we may assume without loss of generality that  $e_{n/2}$  is the unique even edge in  $C$  and that  $e_{n/2} \subseteq A$  or  $e_{n/2} \subseteq B$ . It follows that  $P_1, P_3, \dots, P_{n/2}$  are equal pairs and the remaining pairs are split, so  $|A| - |B| \equiv 2\lceil \frac{n}{4} \rceil \pmod{4}$ . We must therefore have  $(m, d) \in \{(2, 2), (6, 0)\}$ , and  $(A, B)$  is odd-good by (vi). On the other hand, if 4 divides  $n$ , then by our key observation the number of even edges in  $C$  is even. If this number is at least two then  $(A, B)$  is odd-good by (v) and (vii). If instead every edge of  $C$  is odd, then exactly  $\frac{n}{4}$  of the pairs  $P_i$  are equal pairs, so  $|A| - |B| \equiv \frac{n}{2} \pmod{4}$ , and  $C$  is odd-good by (v). ◀

To prove Theorem 6 it therefore suffices to prove the backwards implication. Our approach for this is motivated by the observation that if  $H$  is a 4-graph and  $(A, B)$  is a partition of  $V(H)$  which is not odd-good, then  $H$  must have very few even edges. Likewise, if  $(A, B)$  is not even-good, then  $H$  has very few odd edges. We therefore consider three cases for  $H$ : two ‘near-extremal’ cases, in which  $V(H)$  admits a partition  $(A, B)$  with few even edges or with few odd edges, and a ‘non-extremal’ case, in which there is no such partition. In the ‘non-extremal case’ we proceed by the so-called ‘absorbing’ method, introduced by Rödl, Ruciński and Szemerédi [19], in which we rely heavily on the fact that  $H$  is not ‘near-extremal’. On the other hand, in the ‘near-extremal’ cases we have significant information about the structure of  $H$  (specifically that there is a partition of  $V(H)$  with few even/odd edges). Making essential use of this structural information, we proceed by *ad hoc* methods to construct a Hamilton 2-cycle in  $H$ .

The following definition formalises our two notions of ‘near-extremal’.

► **Definition 10.** Let  $c_1, c_2 > 0$  and let  $H$  be a 4-graph on  $n$  vertices.

- (a) We say that  $H$  is  $c_1$ -even-extremal if there exists a partition  $(A, B)$  of  $V(H)$  such that  $(\frac{1}{2} - c_1)n \leq |A| \leq (\frac{1}{2} + c_1)n$  and  $H$  contains at most  $c_1 \binom{n}{4}$  odd edges.
- (b) We say that  $H$  is  $c_2$ -odd-extremal, if there exists a partition  $(A, B)$  of  $V(H)$  such that  $(\frac{1}{2} - c_2)n \leq |A| \leq (\frac{1}{2} + c_2)n$  and  $H$  contains at most  $c_2 \binom{n}{4}$  even edges.



### 2.3.1 Non-Extremal 4-Graphs

As described above, in the case when  $H$  is not near-extremal, we proceed by the ‘absorbing’ method of Rödl, Ruciński and Szemerédi [19]. To do this we establish three key lemmas. The first of these is a ‘connecting lemma’, which shows that since  $H$  is not even-extremal, we can find a constant-length 2-path connecting any two disjoint pairs of vertices. For this, we say that the *ends* of a 2-path 4-graph  $(v_1, \dots, v_n)$  are the pairs  $\{v_1, v_2\}$  and  $\{v_{n-1}, v_n\}$ .

► **Lemma 11** (Connecting lemma). *Suppose that  $\frac{1}{n} \ll \varepsilon \ll c$  and that  $H$  is a 4-graph on  $n$  vertices with  $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$  which is not  $c$ -even-extremal. Then for every two disjoint pairs  $\{a_1, a_2\}, \{b_1, b_2\} \in \binom{V}{2}$  there is a 2-path of length at most 3 whose ends are  $\{a_1, a_2\}$  and  $\{b_1, b_2\}$ .*

Loosely speaking, our proof of Lemma 11 supposes that we have pairs  $\{a_1, a_2\}$  and  $\{b_1, b_2\}$  for which no such 2-path exists. It follows that there is no pair  $\{x, y\} \in \binom{V(H)}{2}$  for which  $\{a_1, a_2, x, y\}$  and  $\{b_1, b_2, x, y\}$  are both edges of  $H$ . Combined with the minimum codegree condition of  $H$  this yields significant structural information on  $H$ , which we use to deduce that  $H$  must be  $c$ -even-extremal and so prove the lemma.

The second key lemma is an ‘absorbing lemma’, which shows that since  $H$  is neither even-extremal nor odd-extremal, we can find a short 2-path in  $H$  which can ‘absorb’ most small collections of pairs of  $H$ .

► **Lemma 12.** (Absorbing lemma) *Suppose that  $\frac{1}{n} \ll \varepsilon \ll \rho \ll \beta \ll \lambda \ll c, \mu$ . Let  $H$  be a 4-graph on  $n$  vertices with  $\delta(H) \geq \frac{n}{2} - \varepsilon n$  which is neither  $c$ -even-extremal nor  $c$ -odd-extremal. Then there is a 2-path  $P$  in  $H$  and a graph  $G$  on  $V(H)$  with the following properties.*

- (i)  $P$  has at most  $\mu n$  vertices.
- (ii) Every vertex of  $V(H) \setminus V(P)$  lies in at least  $(1 - \lambda)n$  edges of  $G$ .
- (iii) For any  $q \leq \rho n$  and any  $q$  disjoint edges  $e_1, \dots, e_q$  of  $G$  which do not intersect  $P$  there is a 2-path  $P^*$  in  $H$  with the same ends as  $P$  such that  $V(P^*) = V(P) \cup \bigcup_{j=1}^q e_j$ .

Loosely speaking, to prove Lemma 12, we first show that provided  $H$  is not  $c$ -odd-extremal, for almost every pair  $\{x, y\} \in \binom{V}{2}$  there are many 2-paths  $Q$  of length 3 which can ‘absorb’  $\{x, y\}$ , in the sense that there is a 2-path  $Q^*$  with vertex set  $V(Q) \cup \{x, y\}$  and with the same ends as  $Q$ . We take  $G$  to be the graph of such pairs. We then randomly select a linear number of 2-paths of length 3 and use Lemma 11 to connect these 2-paths into a single short 2-path  $P$  (this is where we require that  $H$  is not  $c$ -even-extremal). Next we extend  $P$  to include the small number of vertices which lie in fewer than  $(1 - \lambda)n$  edges of  $G$ , so that (ii) holds. Finally, we show that given any set of edges  $e_1, \dots, e_q$  of  $G$  as in (iii), we can match these edges to the randomly chosen paths  $Q$ , and absorb each edge into the corresponding path to obtain  $P^*$ .

Our final key lemma is a ‘path cover lemma’, which states that we can cover almost all vertices of  $H$  by a constant number of vertex-disjoint 2-paths. In fact, we do not actually need the requirement that  $H$  is not near-extremal, and can simply cite a result of Kühn, Mycroft and Osthus [13].

► **Lemma 13** (Path cover lemma [13]). *Suppose that  $\frac{1}{n} \ll \frac{1}{D} \ll \gamma \ll \eta$  and that  $H$  is a 4-graph on  $n$  vertices with  $\delta(H) \geq (\frac{1}{4} + \eta)n$ . Then  $H$  contains a set of at most  $D$  vertex-disjoint 2-paths covering all but at most  $\gamma n$  vertices of  $H$ .*

For non-extremal 4-graphs  $H$ , combining these three lemmas proves the reverse implication of Theorem 6, which we express in the following lemma.

► **Lemma 14.** *Suppose that  $\frac{1}{n} \ll \varepsilon \ll c$  and that  $n$  is even, and let  $H$  be a 4-graph of order  $n$  with  $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$ . If  $H$  is neither  $c$ -odd-extremal nor  $c$ -even-extremal, then  $H$  contains a Hamilton 2-cycle.*

**Proof sketch.** Introduce constants with  $1/n \ll 1/D, \varepsilon \ll \gamma \ll \rho \ll \beta \ll \lambda \ll c, \mu \ll 1$ , and apply Lemma 12 to obtain an absorbing 2-path  $P_0$  in  $H$  and a graph  $G$  on  $V(H)$  with the stated properties. Let  $V := V(H)$  and  $U := V(P_0)$ , and now choose uniformly at random a set  $R \subseteq V \setminus U$  of size  $\rho n$ . Next, apply Lemma 13 (with, say,  $\eta = 1/10$ ) to obtain at most  $D$  vertex-disjoint 2-paths  $P_1, \dots, P_q$  in  $H[V \setminus (U \cup R)]$  covering all but at most  $\gamma n$  vertices. By  $q$  applications of Lemma 11 we can find vertex-disjoint 2-paths  $Q_0, Q_1, \dots, Q_q$ , each of length at most 3, such that  $Q_0$  connects the end of  $P_0$  to the start of  $P_1$ ,  $Q_1$  connects the end of  $P_1$  to the start of  $P_2$ , and so forth, with  $Q_q$  connecting the end of  $P_q$  to the start of  $P_0$ . Moreover, all vertices of  $Q_i$  except those in the end of  $P_i$  or the start of  $P_{i+1}$  should be taken from  $R$ . (The random choice of  $R$  ensures that the conditions of Lemma 11 are satisfied for each application.) This yields a 2-cycle  $C = P_0 Q_0 P_1 Q_1 P_2 \dots P_q Q_q$  in  $H$  covering all vertices except the at most  $\gamma n$  vertices not covered by  $P_1, \dots, P_q$  and between  $\rho n - 3D$  and  $\rho n$  unused vertices of  $R$ . So  $X := V \setminus V(C)$  has size  $\rho n - 3D \leq |X| \leq \rho n + \gamma n$ . Furthermore,  $|X|$  is even since  $n$  and  $|V(C)|$  are both even, and our random choice of  $R$  ensures that every vertex  $x \in X$  has  $\deg_{G[X]}(x) \geq |X|/2$ . So there is a perfect matching  $e_1, \dots, e_{|X|/2}$  in  $G[X]$ ; since  $|X|/2 \leq \rho n$  we may ‘absorb’  $X$  into  $P_0$  to obtain a 2-path  $P^*$ . Replacing  $P_0$  by  $P^*$  in  $C$  gives a Hamilton 2-cycle in  $H$ . ◀

### 2.3.2 Extremal 4-Graphs

Having dealt with the ‘non-extremal’ case, it remains to deal with the two ‘near-extremal’ cases by proving the following two lemmas via an extremal case.

► **Lemma 15.** *Suppose that  $\frac{1}{n} \ll \varepsilon, c \ll 1$  and that  $n$  is even, and let  $H$  be a 4-graph of order  $n$  with  $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$ . If  $H$  is  $c$ -even-extremal and every partition of  $V(H)$  into two parts  $A$  and  $B$  is even-good, then  $H$  contains a Hamilton 2-cycle.*

► **Lemma 16.** *Suppose that  $\frac{1}{n} \ll \varepsilon, c \ll 1$  and that  $n$  is even, and let  $H$  be a 4-graph of order  $n$  with  $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$ . If  $H$  is  $c$ -odd-extremal and every partition of  $V(H)$  into two parts  $A$  and  $B$  is odd-good, then  $H$  contains a Hamilton 2-cycle.*

**Proof sketch.** As is typical of this type of argument, each lemma is proved by a long and detailed extremal case analysis, and so we limit ourselves here to a brief outline of the argument for Lemma 15 (the outline for Lemma 16 is similar with ‘even’ and ‘odd’ reversed). Let  $(A', B')$  be a partition of  $V(H)$  witnessing that  $H$  is  $c$ -even-extremal. We first observe that the bound on  $\delta(H)$  implies that  $H$  has density at least  $(\frac{1}{2} - \varepsilon)$ . Combined with the fact that  $H$  has few odd edges, this implies that almost every set  $S \subseteq V(H)$  for which  $|A' \cap S|$  is even is an edge of  $H$ . However, it is possible that a small number of vertices may lie in very few even edges, so we begin by ‘tidying up’ the partition: we move a few vertices of  $H$  from one side to the other to ensure that, for instance, every vertex of  $H$  lies in many even edges. Let  $(A, B)$  be the tidied partition. By assumption this partition  $(A, B)$  is even-good, and this fact yields some structure in  $H$  with respect to this partition (precisely what structure depends on the values of  $n$  and  $|A|$ ). For example, we might obtain two disjoint odd edges in  $H$ . We then form a short 2-path  $P$  from the given structure to satisfy the desired parity conditions, and then (using even edges only) extend  $P$  to a Hamilton 2-cycle in  $H$ . ◀

**Proof of Theorem 6.** Fix a constant  $c$  small enough for Lemmas 15 and 16. Having done so, choose  $\varepsilon$  sufficiently small for us to apply Lemma 14 with this choice of  $c$ , and  $n_0$  sufficiently

large that we may apply Lemmas 14, 15 and 16 with these choices of  $c$  and  $\varepsilon$  and any even  $n \geq n_0$ . Let  $H$  be a 4-graph on  $n$  vertices with  $\delta(H) \geq (\frac{1}{2} - \varepsilon)n$ , and suppose that every partition  $(A, B)$  of  $V(H)$  is both even-good and odd-good. If  $H$  is either  $c$ -even-extremal or  $c$ -odd-extremal then  $H$  contains a Hamilton 2-cycle by Lemma 15 or 16 respectively. On the other hand, if  $H$  is neither  $c$ -odd-extremal nor  $c$ -even-extremal then  $H$  contains a Hamilton 2-cycle by Lemma 14. This completes the proof of the backwards implication of Theorem 6; the proof of the forwards implication was Proposition 9.  $\blacktriangleleft$

## 2.4 Proof of Theorem 2

To conclude this section, we show how Theorem 2 can be deduced from Theorem 6. We begin by justifying the claim that the degree bound of Theorem 2 is best-possible. To see this, fix an even integer  $n \geq 6$ , and construct a 4-graph  $H^*$  as follows. Let  $A$  and  $B$  be disjoint sets with  $|A \cup B| = n$  such that  $|A| = \frac{n}{2} - 1$  if 8 divides  $n$  and  $|A| = \frac{n}{2}$  otherwise. Then the vertex set of  $H^*$  is  $A \cup B$ , and the edges of  $H^*$  are all sets  $e \in \binom{A \cup B}{4}$  such that  $|e \cap A|$  is odd. Then it is easily checked that  $\delta(H^*) = \frac{n}{2} - 3$  if 8 divides  $n$  and  $\frac{n}{2} - 2$  otherwise. Moreover, since  $H^*$  has no even edges, our choice of size of  $A$  implies that the partition  $(A, B)$  of  $V(H^*)$  is not odd-good. By Theorem 6 we conclude that there is no Hamilton 2-cycle in  $H^*$ .

**Proof of Theorem 2.** Choose  $\varepsilon, n_0$  as in Theorem 6. Let  $n \geq n_0$  be even and large enough that  $\frac{n}{2} - 2 \geq (\frac{1}{2} - \varepsilon)n$ , and let  $H$  be a 4-graph on  $n$  vertices which satisfies the minimum codegree condition of Theorem 2. Also let  $(A, B)$  be a partition of  $V(H)$ , and assume without loss of generality that  $|A| \leq \frac{n}{2}$ . By Theorem 6 it suffices to prove that  $(A, B)$  is even-good and odd-good. For this, note that if 8 divides  $n$  and  $|A| = \frac{n}{2}$  then  $(A, B)$  is even-good by (i) and odd-good by (v). So we may assume that if 8 divides  $n$  then  $|A| \leq \frac{n}{2} - 1$  and  $\delta(H) \geq \frac{n}{2} - 2$ , whilst otherwise we have  $|A| \leq \frac{n}{2}$  and  $\delta(H) \geq \frac{n}{2} - 1$ . Either way, we must have  $\delta(H) \geq |A| - 1$ . Also, for any distinct  $x, y, z \in V(H)$ , let  $N_B(x, y, z)$  denote the set of vertices  $w \in B$  such that  $\{x, y, z, w\} \in E(H)$ .

To see that  $(A, B)$  must be even-good, arbitrarily choose vertices  $x_1, x_2, y_1, y_2, z_1, z_2 \in A$ . Then  $|N_B(x_1, y_1, z_1)|, |N_B(x_2, y_2, z_2)| \geq \delta(H) - (|A| - 3) \geq 2$ , so we may choose distinct  $w_1, w_2 \in B$  with  $w_1 \in N_B(x_1, y_1, z_1)$  and  $w_2 \in N_B(x_2, y_2, z_2)$ . The sets  $\{x_1, y_1, z_1, w_1\}$  and  $\{x_2, y_2, z_2, w_2\}$  are then disjoint odd edges of  $H$ , so  $(A, B)$  is even-good by (ii).

We next show that  $(A, B)$  is also odd-good. For this, arbitrarily choose distinct vertices  $a_1, a_2, \dots, a_9, a'_1, \dots, a'_9 \in A$  and  $b_1, \dots, b_9 \in B$ . For any  $1 \leq i, j \leq 9$  we have  $|N_B(a_i, a'_i, b_j)| \geq \delta(H) - (|A| - 2) \geq 1$ , so there must be  $b_j^i \in B$  such that  $\{a_i, a'_i, b_j, b_j^i\}$  is an (even) edge of  $H$ . If for each  $1 \leq j \leq 9$  the vertices  $b_j^i$  for  $1 \leq i \leq 9$  are all distinct, then there is no set  $X \subseteq V(H)$  with  $|X| \leq 8$  which intersects every even edge of  $H$ . However, as observed immediately after Definition 5, such a set  $X$  must exist if  $(A, B)$  is not odd-good. We may therefore assume that  $b_j^{i'} = b_j^i$  for some  $1 \leq i, i', j \leq 9$  with  $i \neq i'$ . It follows that  $\{a_i, a'_i, b_j, b_j^i\}$  is an even edge of  $H$  with exactly two vertices in  $A$ , whilst  $(a_i, a'_i, b_j, b_j^i, a_{i'}, a'_{i'})$  is a 2-path of length 2 in  $H_{\text{even}}$ . So  $(A, B)$  is odd-good by (v), (vi), (vii) or (viii), according to the value of  $n$  modulo 8.  $\blacktriangleleft$

## 3 Tight Hamilton Cycles

Our aim in this section is to explain the principal ideas of the proof of Theorem 3, which proceeds by a series of reductions. We begin with a full proof of the case  $k = 3$ , in which case we proceed from a theorem of Garey, Johnson and Stockmeyer [5], who proved that the

Hamilton cycle problem remains NP-complete when restricted to subcubic graphs (we say that a graph  $G$  is *subcubic* if  $G$  has maximum degree  $\Delta(G) \leq 3$ ). The following proposition is an immediate corollary of that theorem.

► **Proposition 17** ([5]). The problem of determining whether a subcubic graph admits a Hamilton path is NP-complete.

The next lemma is the  $k = 3$  case of Theorem 3, which holds with  $C = 9$ .

► **Lemma 18.** *The 3-graph tight Hamilton cycle decision problem is NP-complete even when restricted to 3-graphs  $H$  on  $m$  vertices with  $\delta(H) \geq \frac{m}{2} - 9$ .*

**Proof.** Let  $G$  be a subcubic graph on  $n$  vertices, and write  $X := V(G)$ . Assume for simplicity that  $n$  is even (a very similar argument handles the case where  $n$  is odd). Fix disjoint sets  $A$  and  $B$  with  $|A| = \frac{3n}{2}$  and  $|B| = \frac{3n}{2} + 1$  such that  $X \subseteq A$ , and define a 3-graph  $H$  with vertex set  $A \cup B$  whose edges are

- (i) all sets  $e \in \binom{A \cup B}{3}$  with  $|A \cap e| \leq 1$ ,
- (ii) all sets  $e \in \binom{A \cup B}{3}$  with  $|A \cap e| = 2$  and  $A \cap e \in E(G)$  (note in particular that this requires that  $A \cap e \subseteq X$ ), and
- (iii) all sets  $e \in \binom{A}{3}$  for which no  $e' \in E(G)$  satisfies  $e' \subseteq e$ .

Observe first that  $H$  has  $m := 3n + 1$  vertices and minimum codegree  $\delta(H) \geq \frac{m}{2} - 9$ . To see this, let  $x$  and  $y$  be distinct vertices of  $H$ . If either  $x \in B$  or  $y \in B$  then  $\{x, y, z\}$  is an edge of  $H$  for any  $z \in B \setminus \{x, y\}$ , so  $\deg_H(\{x, y\}) \geq |B| - 2 = \frac{3n}{2} - 1$ . Exactly the same applies if  $x, y \in A$  and  $xy \in E(G)$ . Finally, if  $x, y \in A$  and  $xy \notin E(G)$ , then  $\{x, y, z\}$  is an edge of  $H$  for any  $z \in A \setminus \{x, y\}$  except for those  $z$  such that  $xz \in E(G)$  or  $yz \in E(G)$ . So  $\deg_H(\{x, y\}) \geq |A| - 2 - \deg_G(x) - \deg_G(y)$ ; since  $G$  is subcubic this gives  $\deg_H(\{x, y\}) \geq \frac{3n}{2} - 8 \geq \frac{m}{2} - 9$ , as claimed.

We claim that  $H$  contains a tight Hamilton cycle if and only if  $G$  contains a Hamilton path. To see this, first suppose that  $G$  contains a Hamilton path  $(x_1, \dots, x_n)$ . Enumerate the vertices of  $A \setminus X$  and  $B$  as  $a_1, a_2, \dots, a_{n/2}$  and  $b_1, b_2, \dots, b_{3n/2+1}$  respectively. Then

$$(x_1, x_2, b_1, x_3, x_4, b_2, \dots, x_{n-1}, x_n, b_{\frac{n}{2}}, b_{\frac{n}{2}+1}, a_1, b_{\frac{n}{2}+2}, b_{\frac{n}{2}+3}, a_2, \dots, a_{\frac{n}{2}}, b_{\frac{3n}{2}}, b_{\frac{3n}{2}+1})$$

is a tight Hamilton cycle in  $H$ .

Now suppose instead that  $H$  contains a tight Hamilton cycle  $C$ . Note that our construction of  $H$  ensures that there are no edges  $e, e' \in E(H)$  with  $|e \cap A| = 3$ ,  $|e' \cap A| = 2$  and  $|e \cap e'| = 2$ . Since every edge of  $C$  intersects the subsequent edge of  $C$  in precisely two vertices, and  $B \neq \emptyset$ , it follows that  $C$  cannot contain any edge  $e$  with  $|e \cap A| = 3$ . So there are at least  $\frac{n}{2}$  vertices  $a \in X$  which are succeeded in  $C$  by a vertex of  $B$ . Now let  $A_1$  be the set of vertices of  $X$  for which the subsequent vertex of  $A$  on  $C$  is in  $X$  and  $A_2$  be the set of vertices of  $X$  for which the subsequent vertex of  $A$  on  $C$  is in  $A \setminus X$ . Also let  $A_3 := A \setminus X$ , so  $A$  is the disjoint union of  $A_1$ ,  $A_2$  and  $A_3$ . By construction of  $H$ , any vertex of  $A \setminus X$  must be preceded in  $C$  by two vertices of  $B$  and succeeded in  $C$  by two vertices of  $B$ ; it follows that any vertex of  $A_2 \cup A_3$  is succeeded in  $C$  by two vertices of  $B$ , and so we obtain

$$|B| \geq (\frac{n}{2} - |A_2|) + 2(|A_2| + |A_3|) = \frac{n}{2} + |A_2| + 2|A_3| = \frac{3n}{2} + |A_2|.$$

Since  $A \setminus X$  is non-empty, we must have  $|A_2| \geq 1$ . Combined with the fact that  $|B| = \frac{3n}{2} + 1$  this implies that  $|A_2| = 1$ , and all inequalities are in fact equalities. So precisely one vertex of  $X$  is succeeded in  $C$  by two vertices of  $B$ ,  $\frac{n}{2} - 1$  vertices of  $X$  are succeeded

by one vertex of  $B$ , and the remaining  $\frac{n}{2}$  vertices of  $X$  are succeeded by a vertex of  $A$  (which must therefore be in  $X$ ). This implies that  $C$  contains a tight Hamilton path of the form  $(x_1, x_2, b_1, x_3, x_4, b_2, \dots, b_{n/2-1}, x_{n-1}, x_n)$ , where  $X = \{x_1, \dots, x_n\}$  and  $b_i \in B$  for  $1 \leq i \leq \frac{n}{2} - 1$ . By our construction of  $H$  it follows that  $(x_1, x_2, \dots, x_n)$  is a Hamilton path in  $G$ .

Altogether, this shows that any instance of the Hamilton cycle problem for subcubic graphs can be reduced to a single instance of the problem of finding a tight Hamilton cycle in a 3-graph on  $m$  vertices with  $\delta(H) \geq \frac{m}{2} - 9$ , where  $m = 3n + 1$ . Together with Proposition 17, this proves the lemma.  $\blacktriangleleft$

We conclude by outlining the steps we use to prove Theorem 3 in full generality, using the following notation. For a function  $f(n)$ , we write  $\text{HC}(k, f(n))$  (respectively  $\text{HP}(k, f(n))$ ) to denote the  $k$ -graph tight Hamilton cycle (respectively Hamilton path) decision problem restricted to  $k$ -graphs  $H$  on  $n$  vertices with minimum codegree  $\delta(H) \geq f(n)$ . On the other hand, for an integer  $D$ , we write  $\overline{\text{HC}}(k, D)$  (respectively  $\overline{\text{HP}}(k, D)$ ) to denote the  $k$ -graph tight Hamilton cycle (respectively Hamilton path) decision problem restricted to  $k$ -graphs  $H$  with maximum codegree  $\delta(H) \leq D$ . So, for example, Proposition 17 states that  $\overline{\text{HP}}(2, 3)$  is NP-complete, whilst Lemma 18 states that  $\text{HC}(3, \frac{n}{2} - 9)$  is NP-complete. We prove Theorem 3 by exhibiting the following polynomial-time reductions.

- (i) For any  $k \geq 2$  and  $D$  we give polynomial-time reductions from  $\overline{\text{HC}}(k, D)$  to  $\overline{\text{HP}}(k, D)$  and from  $\overline{\text{HP}}(k, D)$  and  $\overline{\text{HC}}(k, D)$ . These reductions are elementary and permit us the convenience of treating the tight Hamilton cycle and tight Hamilton path problems in graphs of low maximum codegree as being interchangeable.
- (ii) For any  $k \geq 2$  we give polynomial-time reductions from  $\overline{\text{HC}}(k, D)$  to  $\overline{\text{HC}}(2k-1, 2D)$  and from  $\overline{\text{HC}}(k, D)$  to  $\overline{\text{HC}}(2k, D)$ . In each case, given a  $k$ -graph  $H$  on a vertex set  $V$ , we take copies  $H_1$  and  $H_2$  of  $H$  with disjoint vertex sets  $V_1$  and  $V_2$ . For the former reduction we define a  $(2k-1)$ -graph  $H^*$  on  $V_1 \cup V_2$  whose edges are those  $(2k-1)$ -tuples which consist of an edge  $e_1$  from  $H_1$  and the copies in  $H_2$  of  $k-1$  vertices of  $e_1$ , or the same with the roles of  $H_1$  and  $H_2$  reversed. Likewise, for the latter reduction we define a  $2k$ -graph  $H^*$  on  $V_1 \cup V_2$  whose edges are those  $2k$ -tuples  $e_1 \cup e_2$  where  $e_1$  is an edge of  $H_1$ ,  $e_2$  is an edge of  $H_2$ , and  $e_2$  contains the copies of at least  $k-1$  vertices of  $e_1$ . In either case it is not too hard to show that  $H^*$  contains a tight Hamilton cycle if and only if  $H$  does, and that  $\Delta(H^*) \leq 2\Delta(H)$  in one case and  $\Delta(H^*) \leq \Delta(H)$  in the other.
- (iii) Finally, for any  $k \geq 2$  we present a polynomial-time reduction from  $\overline{\text{HP}}(k, D)$  to  $\text{HC}(2k-1, \lfloor \frac{n}{2} \rfloor - k(D+1))$  and from  $\overline{\text{HC}}(k, D)$  to  $\text{HC}(2k, \frac{n}{2} - k(D+1))$ . These are similar to the reduction given in the proof of Lemma 18, except that  $G$  is now a  $k$ -graph with  $\Delta(G) \leq D$ , and  $H$  is a  $(2k-1)$ -graph or  $2k$ -graph (according to which reduction we are presenting).

By induction on  $k$ , with Proposition 17 as the base case, the reductions of (i) and (ii) combine to prove the following theorem, which can be seen as a generalisation to  $k$ -graphs of the aforementioned theorem of Garey, Johnson and Stockmeyer.

**► Theorem 19.** *For every  $k \geq 2$  there exists  $D$  such that  $\overline{\text{HC}}(k, D)$  and  $\overline{\text{HP}}(k, D)$  are NP-complete.*

Theorem 3 follows immediately from Theorem 19 and the reductions of (iii).

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