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Causal functional calculus

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Abstract

We construct a new topology on the space of stopped paths and introduce a calculus for causal functionals on generic domains of this space. We propose a generic approach to pathwise integration without any assumption on the variation index of a path and obtain functional change of variable formulae which extend the results of Föllmer [Séminaire de probabilités 15 (1981), 143–150] and Cont and Fournié [J. Funct. Anal. 259 (2010), no. 4, 1043–1072] to a larger class of functionals, including Föllmer's pathwise integrals. We show that a class of smooth functionals possess a pathwise analogue of the martingale property. For paths that possess finite quadratic variation, our approach extends the Föllmer–Ito calculus and removes previous restriction on the time partition sequence. We introduce a foliation structure on this path space and show that harmonic functionals may be represented as pathwise integrals of closed 1-forms.

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Contents

1 | INTRODUCTION

1.1 | Motivation

Let $\pi := (\pi_n)_{n\geqslant 1}$ be a sequence of interval partitions of $[0, \infty)$ and denote Q^π the set of càdlàg paths with finite quadratic variation along π in the sense of Föllmer [14]. Then for any $f \in C^2(\mathbb{R}^d)$, the Itô formula holds pathwise along any path $x \in Q^\pi$ [14]:

$$f(x(T)) = f(x(0)) + \int_0^T \nabla f(x(t-))dx(t) + \frac{1}{2}\int_0^T \nabla^2 f(x(t)).d[x]^c(t) \tag{1}$$
$$+ \sum_{0\leqslant s\leqslant t} \Delta f(x(s)) - \nabla f(x(s-)).\Delta x(s),$$

where the second term $\int_0^T \nabla f(x(t-))dx(t)$ is a 'Föllmer integral', defined as a pointwise limit of left Riemann sums:

$$\int_0^T \nabla f(x(t-)).dx(t) := \lim_{n\to\infty} \sum_{\pi_n \ni t_i \leqslant T} \nabla f(x(t_i))(x(t_{i+1}) - x(t_i)), \tag{2}$$

without resorting to any probabilistic notion of convergence. Based on the key observation that, for any semi-martingale X, there exists a sequence of partitions π such that the sample paths of X lie almost surely in Q^π, Föllmer showed [14] that for any integrand of the form $\nabla f \circ X$, where $f \in C^2(\mathbb{R}^d)$, the pathwise integral (2) coincides with probability one with the Itô integral, thus providing a pathwise interpretation of the Itô stochastic integral.

The extension of this result to path-dependent functionals has been the focus of several recent works [1, 7, 8, 22]. In particular, a change of variable formula for a class of regular functionals of càdlàg paths was obtained in [7, Theorem 4]. Moreover, [7] (see also [1, Theorem 3.2]) establishes that, for $F \in \mathbb{C}^{1,2}(\Lambda_T)$, one may define a pathwise integral $\int_0^T \nabla_x F(t, x_{t-}).d^\pi x$ as a pointwise limit of Riemann sums as in (2).

The key idea behind these results [5–7] can be summarized as follows [5]. First, one constructs a calculus for continuous functionals on piecewise constant paths. Second, this calculus is extended to all càdlàg paths using a density argument, using piecewise-constant approximations of paths. This second step is where topology plays a role. The original construction of the functional Itô calculus was based on the uniform topology [6, 7, 12]. As is well known, piecewise constant approximation of a càdlàg path under the uniform topology requires *exact* knowledge of all points of discontinuity, which leads to a requirement [7, Remark 7] that the sequence of partitions exhausts the set $J(x)$ of discontinuity points of the path x:

$$J(x) := \{t \in [0, \infty), \quad x(t-) \neq x(t)\} \subset \liminf_{n} \pi_n. \tag{3}$$

This condition, which links the partition with the path, is not required for Föllmer's [14] results, but plays a key role in the proof of [7, Theorem 4].

The following result, whose proof is given in the Appendix, shows that this condition (3) is restrictive and need not be satisfied, even for semi-martingales:

Proposition 1.1. *There exists a semi-martingale X such that for* any *partition sequence π, $\mathbb{P}(J(X) \subset \liminf_n \pi_n) = 0$.*

A related issue is the differentiability and regularity of the pathwise integral. The Föllmer integral $\mathbb{I} : (t, x) \mapsto \int_0^t \nabla_x F.d^{\pi}x$, which is a central object in the pathwise Itô calculus, is not continuously differentiable in the sense of [7], even for $F \in \mathbb{C}^{1,2}(\Lambda_T)$.

To address these issues one needs to replace the uniform topology with another topology. Unfortunately, the usual topologies on the Skorokhod space D [21, s5] do not fit this purpose. For example, the pointwise evaluation map

$$F(x) := x(t)$$

is not J_1 continuous on D [20, VI. 2.3] and the same applies to all weaker topologies. It may thus be a lost cause to obtain a functional calculus built on top of weak topologies on D.

In this work we circumvent these obstacles by introducing a new topology on the space D of càdlàg paths. The Föllmer pathwise integral and the pathwise quadratic variation functional are shown to be continuous functionals with respect to this topology. We define a class of continuously differentiable functionals with respect to this topology and derive change of variable formulae for such functionals without requiring the restrictive condition (3). In the case of paths with finite quadratic variation along a partition sequence, our change of variable formula extends results [1, 7, 14, 18] on the Föllmer–Ito calculus and relaxes previous assumptions relating the partition sequence to the discontinuities of the underlying path. In particular we obtain a pathwise identity of Itô (Theorem 6.4) in the spirit of Beiglböck and Siorpaes' pathwise Burkholder–Davis–Gundy inequality [2].

Pathwise integration concepts and Itô-type change of variable formulae have been obtained by Cont & Perkowski [8] using an extension of Föllmer's ideas to paths with p-th order variation and by Friz and Zhang [17] using rough path theory. In contrast to these results, we define pathwise integrals as limits of (left-)Riemann sums, which naturally appear in applications, not compensated Riemann sums, and we are able to treat a greater class of functionals, notably including Föllmer integrals.

1.2 | Outline

After introducing some definitions and notations in Section 2 we prove, in Section 2.2, a new limit theorem which is useful for studying functionals involving quadratic variation. In Section 3, we introduce a new topology the space of càdlàg paths, discuss its relation with other well-known topologies and give examples of continuous functionals for this topology. In Section 4, we introduce classes of smooth causal functionals and discuss their properties. In particular, we introduce a class of functionals which are shown to satisfy a pathwise analogue of the martingale property (Theorem 5.13).

Section 5 discusses pathwise integration and functional change of variable formulae. We show in particular that pathwise integrals may be defined for class $\mathcal{M}$ functionals without any condition on the variation index (p-variation) of the underlying path. Section 6 discusses in more detail the case of functionals of càdlàg paths with finite quadratic variation and the relation of class $\mathcal{M}$ functionals to a class of path-dependent partial differential equations.

2 | PRELIMINARIES

2.1 | Notations

Denote by D_m the Skorokhod space of $\mathbb{R}^m$-valued càdlàg functions

$$t \longmapsto x(t) := (x_1(t), \dots, x_m(t))'$$

on $\mathbb{R}_+ := [0, \infty)$. Denote $\mathbb{S}_m$ (resp. BV_m) the subset of step functions (resp. locally bounded variation functions) in D_m. For $m = 1$, we will omit the subscript m. By convention, $x(0-) := x(0)$ and $\Delta x(t) := x(t) - x(t-)$. We denote by $x_t \in D_m$ (resp. $x_{t-} \in D_m$) the path $x \in D_m$ stopped at t (resp. $t-$):

$$x_t(s) = x(s \wedge t), \qquad x_{t-}(s) = x(s)1_{s<t} + x(t-)x(s)1_{s\geqslant t}.$$

We equip $(D_m, \mathfrak{d}_{\mathrm{J}_1})$ with a metric $\mathfrak{d}_{\mathrm{J}_1}$ which induces the Skorokhod (a.k.a. J_1) topology.

Let $\pi := (\pi_n)_{n\geqslant 1}$ be a fixed sequence of partitions $\pi_n = (t_0^n, \dots, t_{k_n}^n)$ of $[0, \infty)$ into intervals $0 = t_0^n < \dots < t_{k_n}^n < \infty$ such that $t_{k_n}^n \to \infty$, with vanishing mesh $|\pi_n| = \sup_{i=1..k_n} |t_i^n - t_{i-1}^n| \to 0$ on compacts. By convention, $\max(\emptyset \cap \pi_n) := 0$, $\min(\emptyset \cap \pi_n) := t_{k_n}^n$.

We denote

$$t_n' := \max\{t_i < t | t_i \in \pi_n\}, \qquad x^n := \sum_{t_i \in \pi_n} x(t_{i+1}) \mathbb{1}_{[t_i, t_{i+1})} \tag{4}$$

and by $x^{(n)}$ the (continuous) piecewise-linear approximations of x along π_n.

We denote $Q_m^\pi \subset D_m$ the subset of càdlàg paths with finite quadratic variation along π, defined as follows:

Definition 2.1 (Quadratic variation along a sequence of partitions). We say that $x \in D_m$ has finite quadratic variation along π if the sequence of step functions:

$$q_n(t) := \sum_{\pi_n \ni t_i \leqslant t} (x(t_{i+1}) - x(t_i))(x(t_{i+1}) - x(t_i))'$$

converges in the Skorokhod topology. The limit $[x]_\pi := ([x_i, x_j]_\pi)_{1\leqslant i,j\leqslant m} \in D_{m\times m}$ is called the quadratic variation of x along π.

In the sequel, we shall fix such a sequence of partitions π and drop the subscript π unless we want to emphasize the dependence on π.

As shown in [4, Theorem 3.6], Definition 2.1 is equivalent to the one given by Föllmer [14]:

Proposition 2.2 [4]. *Let $x \in D_m$, then $x \in Q_m^\pi$ if and only if $x_i, x_i + x_j \in Q^\pi$. If $x \in Q_m^\pi$, then we have the polarization identity*

$$\begin{aligned}[x_i, x_j](t) &= \frac{1}{2}\big([x_i + x_j] - [x_i] - [x_j]\big)(t) \in BV \\ &= [x_i, x_j]^c(t) + \sum_{s\leqslant t} \Delta x_i(s)\Delta x_j(s).\end{aligned} \tag{5}$$

We set $\lim_n a_n := \infty$ whenever a real sequence (a_n) does not converge. For real-valued matrices of equal dimension, we write $\langle\cdot,\cdot\rangle$ to denote the Frobenius inner product and $|\cdot|$ to denote the Frobenius norm. If f (resp. g) are $\mathbb{R}^{m\times m}$-valued functions on $[0,\infty)$, we write

$$\int_0^t f dg := \sum_{i,j} \int_0^t f_{i,j}(s-)dg_{i,j}(s) \tag{6}$$

whenever the RHS makes sense. If $x \in Q_m^\pi$ and $f \in C^2(\mathbb{R}^m)$, we write

$$\int_0^t (\nabla f \circ x)d^\pi x := \int_0^t \nabla f(x(s-))d^\pi x(s)$$

to denote the Föllmer integral [14], defined as a pointwise limit of left Riemann sums along π. The superscript π may be dropped in the sequel as π is fixed throughout.

2.2 | Quadratic Riemann sums

In this section, we focus on paths with finite quadratic variation along a sequence of partitions and extend certain limit theorems obtained in [7] for the convergence of ‘quadratic Riemann sums’ (in particular [7, Lemma 12]) to a more general setting. The main result of this section is Theorem 2.7, which is a key ingredient in the proof of change of variable formula for functionals of paths with quadratic variation.

The following result [4, Lemma 2.2] will be useful in the sequel:

Lemma 2.3. *Let v_n, v be non-negative Radon measures on $\mathbb{R}_+$ and J be the set of atoms of v. Then $v_n \to v$ vaguely on $\mathbb{R}_+$ if and only if $v_n \to v$ weakly on $[0,T]$ for e5very $T \notin J$.*

Lemma 2.4. *Let $x \in Q^\pi$, $\mu = d[x]$ be the Radon measure associated with $[x]$. For every $[0,T]$, $T_n := \max\{t_i < T | t_i \in \pi_n\}$, $T_{n+1} := \min\{t_i \geqslant T | t_i \in \pi_n\}$. Define a sequence of non-negative Radon* *measures on* $\mathbb{R}_+$ *by*

$$\mu_n([0,T]) := \sum_{t_i\in\pi_n}(x(t_{i+1})-x(t_i))^2\delta_{t_{i+1}}([0,T]) + (x(T_{n+1})-x(T_n))^2.$$

Then

(i) $\xi_n := \sum_{t_i\in\pi_n}(x(t_{i+1})-x(t_i))^2\delta_{t_i} \longrightarrow \mu$ *vaguely on* $\mathbb{R}_+$,
(ii) $\mu_n \longrightarrow \mu$ *vaguely on* $\mathbb{R}_+$.

Proof. (i) follows from [4, Theorem 2.7]. By Lemma 2.3, we may assume T to be a continuity point of $d[x]$. Let f be a continuous function on $[0,T]$. If $T=0$, then $\mu_n(\{0\})\equiv d[x](\{0\})=0$. If $T>0$, observe that $\xi_n([0,T))\longrightarrow d[x]([0,T))$ (by (i)), f is uniform continuous on $[0,T]$ and that x is right-continuous. Let $T'_{n+1} := \min\{t_i > T | t_i\in\pi_n\}$, it follows that for sufficiently large n

$$\begin{aligned}\left|\int_0^T f\,d\xi_n - \int_0^T f\,d\mu_n\right| &\leqslant \sum_{\pi_n\ni t_i<T}|f(t_i)-f(t_{i+1}\wedge T)|(x(t_{i+1})-x(t_i))^2\\ &\quad + f(T)(x(T'_{n+1})-x(T_{n+1}))^2\\ &\leqslant \sup_{t_i\in\pi_n\cap[0,T]}|f(t_i)-f(t_{i+1}\wedge T)|\xi_n([0,T))\\ &\quad +\|f\|_T(x(T'_{n+1})-x(T_{n+1}))^2 \longrightarrow 0.\end{aligned}$$

□

Lemma 2.5. *Let* $(v_n, n\geqslant 1)$ *be a sequence of non-negative Radon measures on* $\mathbb{R}_+$ *converging vaguely to a Radon measure* v *and* J *be the set of atoms of* v. *If for every* $T\in J$, *there exists a sequence* (T_n) *in* $\mathbb{R}_+$, $T_n\uparrow T$ *such that*

$$v_n(\{T_n\})\longrightarrow v(\{T\}), \tag{7}$$

then $v_n\longrightarrow v$ *weakly on* $[0,T]$ *for all* $T\geqslant 0$.

Proof. For every $T\geqslant 0$, $\tilde v_n([0,T]) := v_n([0,T]) - v_n(\{T_n\})$ and $\tilde v([0,T]) := v([0,T]) - v(\{T\})$. If $T\notin J$, the claim follows immediately from Lemma 2.3. Thus, we may assume $T\in J$. If $T=0\in J$, then $T_n\equiv 0$. Let $T>0$ and $f\in C([0,T],\|\cdot\|_\infty)$. Since $f=(f)^+-(f)^-$, we may take $f\geqslant 0$ and for sufficiently small $\epsilon>0$, we define the following extensions:

$$\overline{f}^{\epsilon}(t) := f(t)\mathbb{1}_{[0,T]}(t) + f(T)\left(1+\frac{T-t}{\epsilon}\right)\mathbb{1}_{(T,T+\epsilon]}(t),$$

$$\underline{f}^{\epsilon}(t) := f(t)\mathbb{1}_{[0,T-\epsilon]}(t) + f(T)\left(\frac{T-t}{\epsilon}\right)\mathbb{1}_{(T-\epsilon,T]}(t),$$

then $\overline{f}^{\epsilon}, \underline{f}^{\epsilon}\in C_K([0,\infty))$, $0\leqslant \underline{f}^{\epsilon}\leqslant f\mathbb{1}_{[0,T]}\leqslant \overline{f}^{\epsilon}\leqslant \|f\|_\infty$ and we have

$$\int_0^\infty \underline{f}^{\epsilon}\,d\tilde v_n \leqslant \int_0^T f\,d\tilde v_n \leqslant \int_0^\infty \overline{f}^{\epsilon}\,d\tilde v_n.$$

Since $v_n \to v$ vaguely and (7) holds, we obtain

$$0 \leqslant \limsup_n \int_0^T f d\tilde{v}_n - \liminf_n \int_0^T f d\tilde{v}_n \leqslant \int_0^\infty \overline{f}^\epsilon - \underline{f}^\epsilon d\tilde{v}$$
$$\leqslant f(T)(v([T-\epsilon, T+\epsilon]) - v(\{T\})) \xrightarrow{\epsilon} 0,$$

hence by monotone convergence

$$\lim_n \int_0^T f d\tilde{v}_n = \lim_\epsilon \int_0^\infty \underline{f}^\epsilon d\tilde{v} = \int_0^T f d\tilde{v}.$$

By (7), it follows $\lim_n \int_0^T f dv_n = \int_0^T f dv$. ☐

Lemma 2.6. *Let $(v_n, n \geqslant 1)$ be a sequence of non-negative Radon measures on $\mathbb{R}_+$ converging vaguely to a Radon measure v and J be the set of atoms of v. Let f_n, f be real-valued left-continuous functions on $\mathbb{R}_+$ and J be the set of atoms of v. If*

(i) *for every $T \in J$ there exists a sequence $(T_n) \in [0, T)$ with $T_n \uparrow T$ such that $v_n(\{T_n\}) \longrightarrow v(\{T\})$, and*
(ii) *(f_n) is locally bounded and converges pointwise to f,*

then for every $T \geqslant 0$,

$$\int_0^T f_n dv_n \longrightarrow \int_0^T f dv.$$

Proof. Let $v = v^c + v^d$ be the Lebesgue decomposition of v into an absolutely continuous part v^c and a singular (discrete) measure v^d. By (i) and Lemma 2.5, we immediately see that $(v_n - v^d) \longrightarrow v^c$ weakly for every $[0, T]$. Since v^c has no atoms, by an application of [7, Lemma 12] we have

$$\int_0^T f_n d(v_n - v^d) \longrightarrow \int_0^T f dv^c.$$

By (ii) and dominated convergence, the proof is complete. ☐

Theorem 2.7. *Let $x \in Q^\pi$, f_n, f be real-valued left-continuous functions on $\mathbb{R}_+$ such that (f_n) is locally bounded and converges pointwise to f on $\mathbb{R}_+$. Then for any $T > 0$,*

$$(i) \sum_{\pi_n \ni t_i \leqslant T} f_n(t_i)(x(t_{i+1}) - x(t_i))^2 \longrightarrow \int_0^T f d[x].$$

$$(ii) \sum_{\pi_n \ni t_i \leqslant T} f_n(t_{i+1} \wedge T)(x(t_{i+1}) - x(t_i))^2 \longrightarrow \int_0^T f d[x].$$

$$(iii) \sum_{\pi_n \ni t_i < T} f_n(t_i)(x(t_{i+1}) - x(t_i))^2 \longrightarrow \int_0^T f d[x].$$

$$(iv) \sum_{\pi_n \ni t_i < T} f_n(t_{i+1} \wedge T)(x(t_{i+1}) - x(t_i))^2 \longrightarrow \int_0^T f d[x].$$

Proof. If $T = 0$, then by (5) and that x is right-continuous and has no discontinuity at $T = 0$, the claims follow. If $T > 0$, put $T_n := \max\{t_i < T | t_i \in \pi_n\}$, $T_{n+1} := \min\{t_i \geqslant T | t_i \in \pi_n\}$, $T'_{n+1} := \min\{t_i > T | t_i \in \pi_n\}$, then $T_n \uparrow T$ and by Lemma 2.4, we observe that

$$\xi_n(\{T_n\}) = (x(T_{n+1}) - x(T_n))^2 \longrightarrow d[x](\{T\}),$$

$$\mu_n(\{T\}) = (x(T_{n+1}) - x(T_n))^2 \longrightarrow d[x](\{T\}),$$

and that

$$\sum_{\pi_n \ni t_i < T} f_n(t_i)(x(t_{i+1}) - x(t_i))^2 = \int_0^T f_n d\xi_n - f(T_{n+1})(x(T'_{n+1}) - x(T_{n+1}))^2,$$

$$\sum_{\pi_n \ni t_i \leqslant T} f_n(t_{i+1} \wedge T)(x(t_{i+1}) - x(t_i))^2 = \int_0^T f_n d\mu_n + f(T)(x(T'_{n+1}) - x(T_{n+1}))^2.$$

By the right continuity of x, Lemma 2.4 and Lemma 2.6, the proof is complete. □

As a consequence of Proposition 2.2 and Theorem 2.7 we have:

Corollary 2.8 (Multidimensional paths). *Let $x \in Q_m^\pi$, $f_n, f : \mathbb{R}_+ \mapsto \mathbb{R}^{m\times m}$ be left-continuous functions with (f_n) locally bounded and converging pointwise to f on $\mathbb{R}_+$. Then*

$$(i) \sum_{\pi_n \ni t_i \leqslant T} \langle f_n(t_i), (x(t_{i+1}) - x(t_i))(x(t_{i+1}) - x(t_i))' \rangle \longrightarrow \int_0^T f d[x],$$

$$(ii) \sum_{\pi_n \ni t_i \leqslant T} \langle f_n(t_{i+1} \wedge T), (x(t_{i+1}) - x(t_i))(x(t_{i+1}) - x(t_i))' \rangle \longrightarrow \int_0^T f d[x],$$

for every $T \geqslant 0$. In particular, the convergence also holds if the sum is replaced by $\sum_{\pi_n \ni t_i < T}$.

Remark 2.9. $t \longmapsto \int_0^t f d[x]$ is in BV and has Lebesgue decomposition:

$$\int_0^t f d[x] = \int_0^t f d[x]^c + \sum_{s \leqslant t} \langle f(s-), \Delta x(s) \Delta x(s)' \rangle.$$

3 | CONTINUOUS FUNCTIONALS

We now construct a topology on suitable subsets of

$$E := \mathbb{R}_+ \times D_m,$$

for which the Föllmer integral $x \mapsto \int_0^T \phi . d^\pi x$ will be a continuous functional of the integrator x.

3.1 | Domains for causal functionals

We are interested in *causal* (non-anticipative) functionals [5, 13], whose natural domain of definition is a set of stopped paths

$$\{(t, x_t) | t \in \mathbb{R}_+, x \in \Omega\} \subset E,$$

for a suitable set of paths $\Omega \subset D_m$, where $x_t = x(t \wedge .)$ [7].

In order to deploy our functional calculus on such functionals we require $\Omega \subset D_m$ to be closed under certain operations:

- stopping: $x \in \Omega \Longrightarrow \forall t \geqslant 0,\ x_t = x(t \wedge .) \in \Omega$;
- vertical perturbations, in order to define the vertical (Dupire) derivative:

$$x \in \Omega \Longrightarrow x_t + e\mathbb{1}_{[t,\infty)} \in \Omega;$$

- piecewise constant approximation along π.

We will call *generic* a set of paths stable under these operations:

Definition 3.1 (Generic sets of paths). A non-empty subset $\Omega \subset D_m$ is called *generic* if it satisfies:

i) Stability under piecewise constant approximation along π: For every $x \in \Omega$, $T > 0$, $\exists N \in \mathbb{N}$; $x^n_T \in \Omega, \quad \forall n \geqslant N$.
ii) Stability under vertical perturbation: For every $x \in \Omega, t \geqslant 0$, there exists a convex neighbourhood $\mathcal{U}$ of 0 such that

$$-\Delta x(t) \in \mathcal{U} \qquad \text{and} \quad x_t + e\mathbb{1}_{[t,\infty)} \in \Omega, \quad \forall e \in \mathcal{U}.$$

We will call a *domain* a set Λ of stopped paths of the form

$$\Lambda := \{(t, x_t) | t \in \mathbb{R}_+, x \in \Omega\},$$

where $\Omega \subset D_m$ is generic.

Remark 3.2. Definition 3.1(ii) implies that $-\mathcal{U}$ is a convex neighbourhood of 0 containing $\Delta x(t)$ such that

$$x_{t-} + e\mathbb{1}_{[t,\infty)} \in \Omega, \quad \forall e \in -\mathcal{U}.$$

Example 3.3. $\mathbb{S}_m$, BV_m, Q^π_m, $Q^{\pi\,+}_m$ (that is, positive paths in Q^π_m) and D_m are all generic sets. If Ω is generic, then

$$\Omega^b_a := \{x \in \Omega | a < x_i < b\}$$

for all constants a, b are all generic. Subsets of continuous paths are not generic.

Example 3.4. Let Ω be generic. Then $\Omega \cap Q^\pi_m$ is generic.

Proof. We observe $\mathbb{S}_m \subset Q^\pi_m$ and if $x \in Q^\pi_m$, then $x + \mathbb{S}_m \in Q^\pi_m$. □

On E, there already exist two well-known (product) topologies, generated by the standard topology on $\mathbb{R}_+$ and local uniform (resp. the Skorokhod J_1) topology on D_m. On a domain $\Lambda \subset E$, we define the uniform (U) and J_1 topologies as the corresponding topology induced on Λ.

Remark 3.5. Every J_1-continuous functional is U-continuous: The local uniform topology is strictly finer than the J_1 topology on D_m [20, VI].

We will now show that, if Ω is 'rich enough' to contain a path with non-zero quadratic variation as well as its piecewise-linear approximations along π, then important examples of functionals such as quadratic variation or the Föllmer integral fail to be continuous on Ω in the uniform topology. We use the following assumption:

Assumption 3.6. Ω is a generic subset and contains a path $x \in Q_m^\pi$ with $[x]_\pi$ continuous and strictly increasing, as well as its piecewise linear approximations along π:

$$\exists N \in \mathbb{N}, \forall n \geqslant N, x^{(n)} \in \Omega,$$

where $x^{(n)}$ denotes the piecewise-linear approximation of x along π_n.

Example 3.7. Q_m^π and $Q_m^{\pi+}$ satisfy Assumption 3.6, $\mathbb{S}_m$ and BV_m do not.

Lemma 3.8. *Let Ω satisfy Assumption 3.6 and $\Lambda = \{(t, x_t) | t \in \mathbb{R}_+, x \in \Omega\}$. Then the functionals*

$$F(t, x_t) := |[x](t)| \qquad G(t, x_t) := \int_0^t 2x dx$$

are not U-continuous on Λ.

Proof. If Ω satisfies Assumption 3.6, there exists $T > 0$, continuous $x, x^{(n)} \in \Omega$ such that $|[x](T)| > 0$. Since $x_T^{(n)} \longrightarrow x_T$ in the local uniform topology on $[0, \infty)$, it follows that

$$(T, x_T^{(n)}) \xrightarrow{\mathrm{U}} (T, x_T)$$

on Λ. Since $x_T^{(n)}$ is a continuous function of bounded variation on $[0, \infty)$, it follows that

$$|[x^{(n)}](T)| = 0, \quad \forall n \geqslant 1,$$

so F is not U-continuous. Using the above and the fact that $x, x^{(n)} \in Q_m^\pi$, we obtain by an application of the pathwise Itô formula [14]:

$$\begin{aligned}
&\lim_n \left| \int_0^T 2x dx - \int_0^T 2x^{(n)} dx^{(n)} \right| \\
&= \lim_n \left| |x(T)|^2 - |x(0)|^2 - tr([x](T)) - \left(|x^{(n)}(T)|^2 - |x^{(n)}(0)|^2 \right) \right| \\
&= tr([x](T)) > 0,
\end{aligned}$$

hence G is not U-continuous on Λ. □

We shall now define a new topology on a domain Λ for which these examples of functionals will be continuous.

3.2 | The π-topology

Definition 3.9 (The π-topology). For every $t \in \mathbb{R}_+, x \in \Omega$, we define $t_n' := \max\{t_i < t | t_i \in \pi_n\}$ and

$$x^n := \sum_{t_i \in \pi_n} x(t_{i+1}) \mathbb{1}_{[t_i, t_{i+1})}. \tag{8}$$

Denote $\mathfrak{X}$ the set of functionals $F : \Lambda \longmapsto \mathbb{R}$ satisfying:

1.(a) $\lim_{s \uparrow t; s \leqslant t} F(s, x_{s-}) = F(t, x_{t-})$,

(b) $\lim_{s \uparrow t; s < t} F(s, x_s) = F(t, x_{t-})$,

(c) $t_n \longrightarrow t; t_n \leqslant t_n' \Longrightarrow F(t_n, x^n_{t_n -}) \longrightarrow F(t, x_{t-})$,

(d) $t_n \longrightarrow t; t_n < t_n' \Longrightarrow F(t_n, x^n_{t_n}) \longrightarrow F(t, x_{t-})$,

2.(a) $\lim_{s \downarrow t; s \geqslant t} F(s, x_s) = F(t, x_t)$,

(b) $\lim_{s \downarrow t; s > t} F(s, x_{s-}) = F(t, x_t)$,

(c) $t_n \longrightarrow t; t_n \geqslant t_n' \Longrightarrow F(t_n, x^n_{t_n}) \longrightarrow F(t, x_t)$,

(d) $t_n \longrightarrow t; t_n > t_n' \Longrightarrow F(t_n, x^n_{t_n -}) \longrightarrow F(t, x_t)$,

for all $(t, x_t) \in \Lambda$. The initial topology generated by $\mathfrak{X}$ on Λ is called the π-topology.

We note that the definition of this topology depends on the partition sequence π.

Remark 3.10. Every U-continuous functional satisfies Definition 3.9.1(a),(b) and 2(a),(b).

Definition 3.11 (Continuous functionals). We denote $C(\Lambda)$ the set of functionals $F : \Lambda \longmapsto \mathbb{R}$ that are continuous with respect to the π-topology.

F is called *left- (resp. right-) continuous* if it satisfies property 1 (resp. property 2) in Definition 3.9.

Remark 3.12. Since

$$z_n \xrightarrow{\Lambda} z \Longleftrightarrow F(z_n) \to F(z) \quad \forall F \in \mathfrak{X},$$

we have $C(\Lambda) \subset \mathfrak{X}$ so in fact $C(\Lambda) = \mathfrak{X}$. We remark here that $C(\Lambda)$ is an algebra and that the topological space (Λ, τ_π) is Tychonoff (that is, completely regular and Hausdorff[†]).

[†] Due to the fact that $(t, x_t) \mapsto t$ & $(t, x_t) \mapsto x(t \wedge s)$ are continuous.

The following concept was introduced in [7] under the name 'predictable functional'; we redefine it here without any reference to measurability considerations:

Definition 3.13 (Strictly causal functionals). For $F : \Lambda \to \mathbb{R}^d$ denote $F_-(t, x_t) = F(t, x_{t-})$. F is *strictly causal* if $F = F_-$.

The following lemma follows from Definition 3.9.1(a) and (b) and Definition 3.9.2(a) and (b).

Lemma 3.14 (Pathwise regularity). *Let $F : \Lambda \to \mathbb{R}^d$ and $x \in \Omega$.*

(i) *If F is left-continuous, then $t \longmapsto F_-(t, x_t)$ is left-continuous and $t \longmapsto F(t, x_t)$ has left limits.*
(ii) *If F is right-continuous, then $t \longmapsto F(t, x_t)$ is right-continuous and $t \longmapsto F_-(t, x_t)$ has right limits.*
(iii) *If F is continuous, then $t \longmapsto F_-(t, x_t)$ (resp. $t \longmapsto F(t, x_t)$) is càglàd (resp. càdlàg) and its jump at time t is equal to $\Delta F(t, x_t)$.*

Example 3.15. Assume $\Omega \subset Q_m^\pi$. Then the functionals

(i) $F(t, x_t) := f(x(t))$; $\quad f \in C(\mathbb{R}^m)$,
(ii) $F(t, x_t) := f([x](t))$; $\quad f \in C(\mathbb{R}^{m\times m})$,
(iii) $F(t, x_t) := \int_0^t (f \circ x)d[x]$; $\quad f \in C(\mathbb{R}^m, \mathbb{R}^{m\times m})$,
(iv) $F(t, x_t) := \int_0^t (\nabla f \circ x)dx$; $\quad f \in C^2(\mathbb{R}^m)$,

belong to $C(\Lambda)$.

Proof. In the light of Remark 3.12, F is continuous if and only if F satisfies Definition 3.9 for all $(t, x) \in \Lambda$. Since conditions Definition 3.9.1(a),(b) and 2(a),(b) are easy to verify, we focus on Definition 3.9.1(c),(d) and 2(c),(d). (i) is trivial. For (ii), we first remark from Definition 2.1 and (5) that

$$q_n \xrightarrow{\mathrm{J}_1} [x];$$

$$\Delta q_n(t_n') = \Delta x^n(t_n')\Delta x^n(t_n')' \longrightarrow \Delta x(t)\Delta x(t)' = \Delta[x](t). \tag{9}$$

Since $[x^n](t) = q_n(t)$ and by (9), if $t_n \longrightarrow t$, the limits of $q_n(t_n)$ and $q_n(t_n-)$ are readily determined according to the rules laid down in [4, s4.2] and (ii) immediately follows from the continuity of f.

To show (iii) and (iv), it is suffice to assume $t_n \longrightarrow t$; $t_n \geqslant t_n'$ (that is, the other criteria follow similar lines of proof, see [4, s4.2]). By (9) and [4, s4.2]

$$|q_n(t_n) - q_n(t_n')| \longrightarrow 0. \tag{10}$$

A closer look at (iii), combined with Corollary 2.8, leads to

$$\begin{aligned} F(t_n, x^n_{t_n}) &= \int_0^{t_n} (f \circ x^n)d[x^n] \\ &= \sum_{\pi_n \ni t_i < t} \langle f(x(t_i)), (x(t_{i+1}) - x(t_i))(x(t_{i+1}) - x(t_i))' \rangle \longrightarrow F(t, x_t) \\ &\quad + \sum_{\pi_n \ni t_i \in (t_n', t_n]} \langle f(x(t_i)), (x(t_{i+1}) - x(t_i))(x(t_{i+1}) - x(t_i))' \rangle. \end{aligned}$$

By (10) and that $f \circ x$ is locally bounded on $\mathbb{R}_+$, we see that the absolute value of the last term is bounded by $\mathrm{const}|q_n(t_n) - q_n(t'_n)| \longrightarrow 0$.

For (iv), from the properties of the Föllmer integral [14], we first observe that

$$\begin{aligned} F(t_n, x^n_{t_n}) &= \int_0^{t_n} \nabla(f \circ x^n) dx^n \\ &= \sum_{\pi_n \ni t_i < t} \nabla f(x(t_i)) \cdot (x(t_{i+1}) - x(t_i)) \longrightarrow F(t, x_t) \\ &\quad + \sum_{\pi_n \ni t_i \in (t'_n, t_n]} \nabla f(x(t_i)) \cdot (x(t_{i+1}) - x(t_i)). \end{aligned}$$

Define $\underline{t_n} := \min\{t_i > t'_n | t_i \in \pi_n\}$, $\overline{t_n} := \min\{t_i > t_n | t_i \in \pi_n\}$ and note that $\overline{t_n} \geqslant \underline{t_n} \geqslant t$, hence

$$|f(x(\overline{t_n})) - f(x(\underline{t_n}))| \longrightarrow 0.$$

Applying a second-order Taylor expansion to f and using (10), we obtain

$$\begin{aligned} \left| \sum_{\pi_n \ni t_i \in (t'_n, t_n]} \nabla f(x(t_i)) \cdot (x(t_{i+1}) - x(t_i)) \right| &\leqslant |f(x(\overline{t_n})) - f(x(\underline{t_n}))| \\ &\quad + \mathrm{const}|q_n(t_n) - q_n(t'_n)| \longrightarrow 0. \end{aligned}$$

□

Remark 3.16. If $x \in D_m$, so are x_T and x_{T-} and the corresponding piecewise constant approximation(s) in (8) shall be denoted by $(x_T)^n$ and $(x_{T-})^n$.

The following property may be derived from [3, Lemma 12.3] and [20, VI]:

Lemma 3.17. *Let $T \geqslant 0$, $x \in D_m$, then $(x_T)^n \xrightarrow{\mathrm{J}_1} x_T$.*

Lemma 3.18. *Let $(t, x) \in \Lambda$, $t_n \longrightarrow t$ and denote $t'_n := \max\{t_i < t | t_i \in \pi_n\}$. Then*

$$\begin{aligned} &(i) && t_n \leqslant t'_n \Longrightarrow x^n_{t_n-} \xrightarrow{\mathrm{J}_1} x_{t-}, \\ &(ii) && t_n < t'_n \Longrightarrow x^n_{t_n} \xrightarrow{\mathrm{J}_1} x_{t-}, \\ &(iii) && t_n \geqslant t'_n \Longrightarrow x^n_{t_n} \xrightarrow{\mathrm{J}_1} x_t, \\ &(iv) && t_n > t'_n \Longrightarrow x^n_{t_n-} \xrightarrow{\mathrm{J}_1} x_t. \end{aligned}$$

Proof. Let $t_n \leqslant t'_n$, by Lemma 3.17, we have $(x_{t-})^n \xrightarrow{\mathrm{J}_1} (x_{t-})$. Since x is càdlàg we observe

$$\|x^n_{t_n-} - (x_{t-})^n\|_\infty \leqslant \sup_{s \in [t_n, t'_n]} |x(t_n) - x(s)| + |x(t_n) - x(t-)| \longrightarrow 0,$$

and (i) follows immediately from [20, VI.1.23]. (ii)–(iv) follow similar lines of proof. □

Theorem 3.19. *Let Ω satisfy Assumption 3.6. Then,*

(i) *every* J_1*-continuous functional is continuous,*
(ii) *there exists a continuous functional which is not* U*-continuous,*
(iii) *there exists* U*-continuous functionals which are not continuous.*

Proof. If F is J_1-continuous, then F satisfies Definition 3.9.1(a),(b) and 2(a),(b) due to Remark 3.5 and 3.10. (i) now follows immediately from Lemma 3.18. (ii) is due to Example 3.15 and Lemma 3.8.

It remains to show (iii). We first note that the U topology on Λ is metrisable, hence sequential continuity is equivalent to continuity. Let us fix a $t_0 > 0$; $t_0 \notin \cup_n \pi_n$, define

$$F(t, x_t) := |\Delta x_t(t_0)|$$

on Λ. Observe that if $x_n \xrightarrow{\mathrm{U}} x$ in D_m then it is well known that:

$$\Delta x_n(s) \longrightarrow \Delta x(s) \tag{11}$$

for $s \geqslant 0$. In particular, if $t_n \longrightarrow t$; $x_n(\cdot \wedge t_n) \xrightarrow{\mathrm{U}} x_t$ then (11) implies $\Delta x_n(\cdot \wedge t_n)(s) \longrightarrow \Delta x_t(s)$ for $s \geqslant 0$, hence F is U-continuous on Λ.

On the other hand, we take an $x \in \Omega_0$; $\Delta x(t_0) \neq 0$, it follows from our choice of t_0 that

$$F(t_0, x^n_{t_0}) = |\Delta x^n(t_0)| \equiv 0,$$

hence by Definition 3.9.2(c), F is not continuous on Λ and (iii) follows. □

So, if Ω satisfies Assumption 3.6, Theorem 3.19 and Remark 3.5 imply that

- the $\pi-$topology is strictly finer than the J_1 topology,
- the $\pi-$topology and the U topology are not comparable.

4 | SMOOTH FUNCTIONALS

The change of variable formulae in [14] make use of the concepts of *local boundedness* and the existence of a *modulus of continuity*. In this section, we shall introduce weaker notions of *boundedness* and *modulus of continuity* for causal functionals and define a corresponding notion of a $C^{1,2}$ functional on Λ, and use these notions to derive a functional change of variable formula. We then introduce $\mathcal{S}(\Lambda)$ and $\mathcal{M}(\Lambda)$, two important subspaces of $C^{1,2}(\Lambda)$.

When $\Omega \subset Q^{\pi}_m$, we will show that functionals such as quadratic variation and Föllmer integrals are not only $C^{1,2}$ but also belong to class $\mathcal{M}$, a sub-class of *infinitely differentiable* functionals. Recall the definition of Dupire's horizontal and vertical derivatives [6, 7, 12]:

Definition 4.1 (Horizontal derivative). $F : \Lambda \longmapsto \mathbb{R}$ is called differentiable in time or *horizontally differentiable* if the following limit exists for all $(t, x_t) \in \Lambda$:

$$\mathcal{D}F(t, x_t) := \lim_{h \downarrow 0} \frac{F(t+h, x_t) - F(t, x_t)}{h}.$$

Definition 4.2 (Vertical derivative). $F : \Lambda \longmapsto \mathbb{R}$ is called *vertically differentiable* if for every $(t, x_t) \in \Lambda$, the map $f : \mathcal{U}_t(x) \longmapsto \mathbb{R}$:

$$e \longmapsto F\big(t, x_t + e\mathbb{1}_{[t,\infty)}\big)$$

is differentiable at 0. $\nabla_x F(t, x_t) := \nabla_e f(0)$ is called the vertical derivative of F at $(t, x_t) \in \Lambda$.

F is called differentiable on Λ if it is vertically and horizontally differentiable at every $(t, x) \in \Lambda$. We extend the above definitions to vector-valued maps $F : \Lambda \to \mathbb{R}^{d\times n}$ whose components $F_{i,j}$ satisfy the respective conditions.

Proposition 4.3. *A causal functional $F : \Lambda \to \mathbb{R}$ is strictly causal if and only if it is vertically differentiable with vanishing vertical derivative.*

Proof. The first assertion follows from the mean value theorem. To prove the converse, let $x \in \Omega$ and put $z := x_t + e\mathbb{1}_{[t,\infty)}$ then $z_{t-} = x_{t-}$ and

$$F(t, x_t + e\mathbb{1}_{[t,\infty)}) = F(t, z_t) = F_-(t, z_t) = F_-(t, x_t) = F(t, x_t),$$

by the strict causality of F (Definition 3.13). □

Definition 4.4 (Locally bounded functional). $F : \Lambda \to \mathbb{R}$ is called *locally bounded* if for every $x \in \Omega$ and $T \geqslant 0$, there exists $n_0 \geqslant N_T(x)$ such that the family of maps

$$(t \longmapsto F(t, x^n_t), n \geqslant n_0)$$

is locally bounded on $[0, T]$.

Lemma 4.5. *Every continuous function on Λ is locally bounded.*

Proof. Let F be continuous; if F is not locally bounded, there exists $x \in \Omega$, $T \geqslant 0$, and a sub-sequence (n_k);

$$|F(t_{n_k}, x^{n_k}_{t_{n_k}})| > k, \quad \forall k \geqslant 1; \tag{12}$$

(t_{n_k}) is bounded on $[0, T]$. For ease of notation, assume $t_{n_k} \longrightarrow t \in [0, T]$ without passing through to a sub-sequence. Observe that one can always choose another sub-sequence, bounded (either above or below) by $t'_{n_k} = \max\{t_i < t | t_i \in \pi_{n_k}\}$. Since F is continuous, if $t_{n_k} < t'_{n_k}$ (resp. $t_{n_k} \geqslant t'_{n_k}$), then Definition 3.9.1(d) (resp. 2(c)) would contradict (12) as $k \uparrow \infty$. □

Lemma 4.6. *Let F be locally bounded and denote $F_-(t, x) = F(t, x_{t-})$.*

(i) *If F is left-continuous then F_- is locally bounded.*
(ii) *If F is left-continuous then $t \longmapsto F_-(t, x_t)$ is locally bounded.*
(iii) *If F is right-continuous then $t \longmapsto F(t, x_t)$ is locally bounded.*

Proof. Since F is locally bounded, there exists a constant $K > 0$ such that

$$|F(t, x^n_t)| \leqslant K$$

for all $t \leqslant T$ and all n sufficiently large. If F is left-continuous, then Definition 3.9.1(b) implies

$$K \geqslant \lim_{s\uparrow t; s<t} |F(s, x^n_s)| = |F(t, x^n_{t-})|,$$

so (i) follows. If $t_n \longrightarrow t; t_n < t'_n$, then by the left-continuity of F (that is, Definition 3.9.1(d)),

$$K \geqslant |F(t_n, x^n_{t_n})| \longrightarrow |F(t, x_{t-})|,$$

so (ii) follows. If F is right-continuous, then by Definition 3.9.2(c),

$$K \geqslant |F(t'_n, x^n_{t'_n})| \longrightarrow |F(t, x_t)|,$$

so (iii) follows. □

Definition 4.7 (Modulus of vertical continuity). We say that a function F on Λ admits a modulus of vertical continuity if for every $x \in \Omega$, $T \geqslant 0$ and $r > 0$ there exists an increasing function $\omega : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ with $\omega(0+) = 0$;

$$|F(t, x^n_{t-} + a\mathbb{1}_{[t,\infty)}) - F(t, x^n_{t-} + b\mathbb{1}_{[t,\infty)})| \leqslant \omega(|a-b|), \tag{13}$$

for all $a, b \in \mathcal{U}_{t-}(x^n) \cap \overline{B}_r(0)$, $t \leqslant T$ and sufficiently large n.

Example 4.8. Let $f \in C(\mathbb{R}_+ \times \mathbb{R}^m)$. Then $F : \Lambda \to \mathbb{R}$ defined by $F(t, x_t) := f(t, x(t))$ admits a modulus of vertical continuity.

Proof. For a given $x \in \Omega$ and $T \geqslant 0$, $r > 0$, put $\|x\|_T := \sup_{t\leqslant T} |x(t)|$, $r_0 := \alpha\|x\|_T + r$; $\alpha > 1$, then f is uniform continuous on $[0,T] \times \overline{B}_{r_0}(0)$ and a modulus of continuity of f on $[0,T] \times \overline{B}_{r_0}(0)$ is given by

$$\omega(\delta) := \sup_{|t-s|+|u-v|\leqslant\delta} \big|f(t,u) - f(s,v)\big|$$

which satisfies (13). □

Remark 4.9. If F, G admit moduli of vertical continuity, then $\alpha F + \beta G$ admits a modulus. If in addition, F_-, G_- are locally bounded, then FG admits a modulus of vertical continuity.

Lemma 4.10. *Let F be vertically differentiable and $(\nabla_x F)_-$ be locally bounded, if $\nabla_x F$ admits a modulus of vertical continuity then so does F.*

Proof. Since F is vertically differentiable and $\nabla_x F$ admits a modulus of vertical continuity ω, by the mean value theorem and the local boundedness of $(\nabla_x F)_-$, we obtain

$$|F(t, x^n_{t-} + a\mathbb{1}_{[t,\infty)}) - F(t, x^n_{t-} + b\mathbb{1}_{[t,\infty)})| \leqslant (\omega(r) + \text{const})|a-b|.$$ □

Definition 4.11 ($C^{1,2}$ functionals). We define $C^{1,2}(\Lambda)$ as the set of continuous functionals $F \in C_\pi(\Lambda)$ such that $\mathcal{D}F$, $\nabla_x F$ and $\nabla_x^2 F$ are defined on Λ and

(i) $\mathcal{D}F$ is right-continuous and locally bounded,
(ii) $(\nabla_x F)_-$ is left-continuous,
(iii) $(\nabla_x^2 F)_-$ is left-continuous, locally bounded and admits a modulus of vertical continuity.

If in addition, $(\nabla_x F)_-$ is locally bounded, then we denote $F \in C_b^{1,2}(\Lambda)$.

We now introduce two classes of functionals which, as we will observe later, play a special role in the context of stochastic analysis:

Definition 4.12 (Class $\mathcal{S}$). A continuous and differentiable functional F is of *class* $\mathcal{S}$ if $\mathcal{D}F$ is right-continuous and locally bounded, $\nabla_x F$ is left-continuous and strictly causal. We denote by $\mathcal{S}(\Lambda)$ the vector space of class $\mathcal{S}$ functionals.

Definition 4.13 (Class $\mathcal{M}$). A functional $F \in \mathcal{S}(\Lambda)$ is of *class* $\mathcal{M}$ if $\mathcal{D}F = 0$. We denote $\mathcal{M}(\Lambda)$ the set of *class* $\mathcal{M}$ functionals and $\mathcal{M}_b(\Lambda)$ the set of functionals $F \in \mathcal{M}(\Lambda)$ whose vertical derivative $\nabla_x F$ is locally bounded.

Remark 4.14. Every functional of class $\mathcal{M}$ is infinitely differentiable by Proposition 4.3.

Remarks 4.9, Lemma 4.6 and 4.10 imply that $C^{1,2}(\Lambda)$, $\mathcal{S}(\Lambda)$, $\mathcal{M}(\Lambda)$, $\mathcal{M}_b(\Lambda)$ are vector spaces; $C_b^{1,2}(\Lambda)$ is an algebra.

Lemma 4.15. *Let $\Omega \subset Q_m^\pi$. If $\phi : \Lambda \longmapsto \mathbb{R}^{m\times m}$ is such that ϕ_- is left-continuous and locally bounded, then*

$$(t, x_t) \in \Lambda \mapsto F(t, x_t) := \int_0^t \phi(s, x_{s-}) d[x](s)$$

is a continuous functional.

Proof. Since $t \longmapsto \phi(t, x_{t-})$ is left-continuous and locally bounded (Lemma 3.14(i)) and that $t \longmapsto [x_i, x_j](t)$ is in BV, càdlàg with $\Delta[x_i, x_j] \equiv \Delta x_i \Delta x_j$ (Proposition 2.2), it follows F is a finite sum of Lebesgue–Stieltjes integrals and satisfies conditions Definition 3.9.1(a),(b) and 2(a),(b). For the other conditions in Definition 3.9, it is suffice to assume $t_n \longrightarrow t; t_n \geqslant t_n'$ (that is, the other criteria follow similar lines). Define

$$\phi_n(s) := \phi(t_0, x_{t_0-}^n)\mathbb{1}_{\{0\}}(s) + \sum_{t_i \in \pi_n} \phi(t_i, x_{t_i-}^n)\mathbb{1}_{(t_i, t_{i+1}]}(s),$$

which is an $\mathbb{R}^{m\times m}$-valued left-continuous function on $\mathbb{R}_+$. By the local boundedness of ϕ_-, we see that $\exists n_0 \geqslant N(x)$; $(\phi_n)_{n\geqslant n_0}$ is locally bounded on $\mathbb{R}_+$ and converges pointwise to $s \longmapsto \phi(s, x_{s-})$ on $\mathbb{R}_+$. By Corollary 2.8(ii), we obtain

$$\begin{aligned} F(t_n, x_{t_n}^n) &= \int_0^{t_n} \phi(s, x_{s-}^n) d[x^n] \\ &= \sum_{\pi_n \ni t_i < t} \langle \phi(t_i, x_{t_i-}^n), (x(t_{i+1}) - x(t_i)(x(t_{i+1}) - x(t_i)' \rangle \longrightarrow F(t, x_t) \\ &\quad + \sum_{\pi_n \ni t_i \in (t_n', t_n]} \langle \phi(t_i, x_{t_i-}^n), (x(t_{i+1}) - x(t_i)(x(t_{i+1}) - x(t_i)' \rangle. \end{aligned}$$

Since $q_n \xrightarrow{J_1} [x]$ and by [4, s4.2], the last term is bounded by

$$\text{const}|q_n(t_n) - q_n(t'_n)| \longrightarrow 0. \qquad \square$$

As we shall see in the following examples, path-independent functionals of class $\mathcal{M}$ are simply affine functions, but in the path-dependent case this class includes many examples, in particular Föllmer integrals.

Example 4.16. Let $\mathbb{S}_m \subset \Omega$, $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m)$ and

$$F(t, x_t) := f(t, x(t)),$$

then F is of class $\mathcal{M}$ if and only if $f(t, u) = \alpha + \beta.u$ for some constants $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^m$.

Proof. For the if part: We can write $f(t, u) = \alpha + \beta \cdot u$ and hence

$$F(t, x_t) = \alpha + \beta x(t)$$

on Λ for some constants $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^m$. By Example 3.15(i) and computing the derivatives of F, we see that F is of class $\mathcal{M}$. Conversely, from Definition 4.13 and Proposition 4.3, we first obtain

(i) $\partial_t f(t, x(t)) = \mathcal{D}F(t, x_t) = 0$,
(ii) $\nabla^2 f(t, x(t)) = \nabla_x^2 F(t, x_t) = 0$,

$\forall\, t \geqslant 0, x \in \Omega$. Since $\mathbb{S}_m \subset \Omega$, we have

$$R := \{(t, x(t)) | t \in \mathbb{R}_+, x \in \Omega\} = \mathbb{R}_+ \times \mathbb{R}^m,$$

hence $\partial_t f \equiv \nabla^2 f \equiv 0$ on $\mathbb{R}_+ \times \mathbb{R}^m$. By the mean value theorem, we deduce that $\nabla f \equiv \beta$ on R, for some $\beta \in \mathbb{R}^m$. $\square$

Remark 4.17. The condition $\mathbb{S}_m \subset \Omega$ may be weakened to simply requiring that $R \subset \mathbb{R}_+ \times \mathbb{R}^m$ is convex. In this case, the converse statement holds on R.

Example 4.18 (Path-dependent examples). Let $\Omega \subset Q_m^\pi$, $\phi : \Lambda \longmapsto \mathbb{R}^{m\times m}$ such that ϕ_- is left-continuous and locally bounded, $f = (f_1, \dots, f_m) \in C^2(\mathbb{R}^m)$. Then the functionals

(i) $F(t, x_t) := \int_0^t \phi(s, x_{s-}) d[x]$,
(ii) $F(t, x_t) := \int_0^t (\nabla f \circ x) dx$,
(iii) $F(t, x_t) := \sum_{i=1}^m (\int_0^t (x_i(t) - x_i(s)) f_i(x_i(s)) dx_i(s) - \int_0^t (f_i \circ x_i) d[x_i])$

belong to $C_b^{1,2}(\Lambda)$ and (ii) and (iii) are of class $\mathcal{M}_b$.

Proof. The functional in (iii) is well defined, since

$$F(t, x_t) = \sum_i \left(x_i(t) \int_0^t f_i \circ x_i dx_i - \int_0^t x_i f_i \circ x_i dx_i - \int_0^t f_i \circ x_i d[x_i] \right). \tag{14}$$

The first two integrals in (14) are Föllmer integrals, defined as a limit of Riemann sums along π, while the last one is a Lebesgue–Stieltjes integral. Continuity of F in (i), (ii) and (iii) follows from Lemma 4.15 and Example 3.15. Since $\mathcal{D}F \equiv 0$ in all cases, let us first compute $\nabla_x^k F$ for $k = 1, 2$ and demonstrate that F possesses the required properties. In case of (i), we have

$$\nabla_x F(t, x_t) = (\phi + \phi')(t, x_{t-})\Delta x(t), \quad \nabla_x^2 F(t, x_t) = (\phi + \phi')(t, x_{t-}),$$

which are left-continuous, locally bounded and $\nabla_x^2 F$ is strictly causal, so by Proposition 4.3, Lemma 4.6(ii) and (4.10), F is $C_b^{1,2}$. In case of (ii), we obtain

$$\nabla_x F(t, x_t) = \nabla f(x(t-)),$$

which is left-continuous, locally bounded and strictly causal, hence F is of class $\mathcal{M}_b$. In case of (iii), we apply ∇_x to (14) and verify that

$$\begin{aligned}\nabla_{x_i} F(t, x_t) &= \int_0^t f_i \circ x_i dx_i - f_i(x_i(t-))\Delta x_i(t) \\ &= \left(\int f_i \circ x_i dx_i\right)(t-). \end{aligned} \tag{15}$$

Applying $f(x) := \int_0^{x_i} f_i(\lambda)d\lambda;\ x \in \mathbb{R}^m$ to (ii) and by Proposition 4.5 and Lemma 4.6(i), we see that each $\nabla_{x_i} F$ is left-continuous and locally bounded and so is $\nabla_x F$. Since $\nabla_x F$ is strictly causal, F is of class $\mathcal{M}_b$. □

5 | PATHWISE INTEGRATION AND CHANGE OF VARIABLE FORMULAE

We now discuss pathwise integration for causal functionals along paths in a generic domain. In contrast to rough integration theory [16] and the one-form approach, that is, [14], [7], and [8], we define integrals as *uncompensated* left Riemann sums, when such limits exist and form a continuous functional.

We then obtain change of variable formulae and an analogue of the classical Fundamental theorem of calculus for functionals of class $\mathcal{M}$. For paths that possess quadratic variation, we obtain a functional Föllmer-Itô formula which extends [7, Theorem 4].

In particular, we show that pathwise integral is of class $\mathcal{M}$ and that functionals of class $\mathcal{M}$ are *primitives*, that is, are representable as pathwise integrals, a fact that facilitates the *computation* of pathwise integrals, as in classical calculus.

Lemma 5.1. *Let F be a left-continuous functional, differentiable in time, if $\mathcal{D}F$ is right-continuous and locally bounded, then*

$$F(t, x_s) - F(s, x_s) = \int_s^t \mathcal{D}F(u, x_u)du, \tag{16}$$

for all $x \in \Omega$, $t \geqslant s \geqslant 0$.

Proof. Put $z := x_s \in \Omega$, then $z_t = x_s$ for $t \geqslant s$ and $z_{t-} = x_s$ for $t > s$. Define $f(t) := F(t, x_s)$ for $t \geqslant s$, then $f(t) = F(t, z_t)$ on $[s, \infty)$ and $f(t) = F(t, z_{t-})$ on (s, ∞). Since F is differentiable in time, f is right differentiable (hence right-continuous) on $[s, \infty)$ and the right derivative $f'(t)$ is $\mathcal{D}F(t, x_s)$ on $[s, \infty)$. Since F is left-continuous, it follows from Lemma 3.14 that $f(t) = F(t, z_{t-})$ is left-continuous on (s, ∞), hence we have first established that f is continuous on $[s, \infty)$. Next, we observe that

$$f'(u) = \mathcal{D}F(u, x_s) = \mathcal{D}F(u, z_u)$$

on $[s, \infty)$. The right continuity of $\mathcal{D}F$ and Lemma 3.14 implies that f' is right-continuous on $[s, \infty)$. Since $\mathcal{D}F$ is right-continuous and locally bounded, it follows from Lemma 4.6(ii) that

$$u \longrightarrow \mathcal{D}F(u, z_u)$$

is locally bounded. Hence, f' is right-continuous and bounded on $[s, T]$, hence Riemann integrable. We can conclude using a stronger version [11] of the Fundamental theorem of calculus. □

Lemma 5.2. *Let ϕ be a right-continuous and locally bounded on Λ, then*

$$\sum_{\pi_n \ni t_i \leqslant T} \int_{t_i}^{t_{i+1}} \phi(t, x^n_{t_i}) dt \longrightarrow \int_0^T \phi(t, x_t) dt,$$

for all $x \in \Omega$, $T \geqslant 0$.

Proof. Define

$$\phi_n(t) := \sum_{\pi_n \ni t_i \leqslant T} \phi(t, x^n_{t_i}) \mathbb{1}_{[t_i, t_{i+1})}(t) = \sum_{\pi_n \ni t_i \leqslant T} \phi(t, x^n_t) \mathbb{1}_{[t_i, t_{i+1})}(t).$$

By the local boundedness of ϕ, we see that $\exists n_0 \geqslant N(x)$; $(\phi_n)_{n \geqslant n_0}$ is locally bounded on $[0, T]$. Since ϕ is right-continuous, it follows from Lemma 3.14 that $t \longmapsto \phi_n(t)$ is right-continuous (hence measurable) on $[0, T]$ and from Definition 3.9.2(c) that ϕ_n converges to $t \longmapsto \phi(t, x_t)$ pointwise on $[0, T]$. and (i) follows from dominated convergence. □

Corollary 5.3. *Let ϕ be a right-continuous and locally bounded Λ, then*

$$(t, x_t) \longmapsto \int_0^t \phi(s, x_s) ds$$

is continuous.

Proof. The path $t \longmapsto \int_0^t \phi(s, x_s) ds$ is continuous. The rest follows from the local boundedness of ϕ and Lemma 5.2. □

Definition 5.4 (Pathwise integrability). Let $\phi : \Lambda \longmapsto \mathbb{R}^m$ such that ϕ_- is left-continuous. For every $x \in \Omega$, define

$$\mathbf{I}_\phi(t, x^n_t) := \sum_{\pi_n \ni t_i \leqslant t} \phi(t_i, x^n_{t_i -}) \cdot (x(t_{i+1}) - x(t_i)). \tag{17}$$

ϕ is said to be $\Lambda-$*integrable* if

- the limit $\mathbf{I}_\phi(t, x_t) := \lim_n \mathbf{I}_\phi(t, x_t^n)$ exists for each $(t, x_t) \in \Lambda$, and
- the map $\mathbf{I}_\phi : \Lambda \mapsto \mathbb{R}$ is continuous.

Note that the pathwise integral is defined as a limit of (left) Riemann sums, and not compensated Riemann sums as in rough path theory [15, 16]. One case in which such Riemann sums are known to converge is for gradients of C^2 functions along paths of finite quadratic variation:

Example 5.5. Let $\Omega = Q_m^\pi$. Then by the results of [14], for any $f \in C^2(\mathbb{R}^m)$, $\phi : \Lambda \longmapsto \mathbb{R}^m$ defined by $\phi(t, x) = \nabla_x f(t, x_t)$ is Λ-integrable and $\mathbf{I}_\phi(t, x)$ is the Föllmer integral [5]. Note that the continuity property of $\mathbf{I}_\phi$ is a consequence (and indeed, the main motivation) of the construction of the π-topology in Section 3.

Theorem 5.6. *Let $\phi : \Lambda \longmapsto \mathbb{R}^m$ such that $\phi\_$ is left-continuous and $\mathbf{I}_\phi$ the integration map defined as in (17). If for every $x \in \Omega$, $T > 0$ the sequence of step functions on* $[0, T]$

$$g_n(t) := \mathbf{I}_\phi(t, x_t^n),$$

is a Cauchy sequence in $(D[0, T], \mathfrak{d}_{J_1})$, then ϕ is Λ-integrable.

Proof. If $(g_n, n \geqslant 1)$ is a Cauchy sequence in $(D[0, T], \mathfrak{d}_{J_1})$, there exists a $G \in D$ such that $g_n \overset{J_1}{\longmapsto} G$. Hence $g_n(t) \mapsto G(t)$ for every continuity point of G on $[0, T]$. Observe that

$$\Delta g_n(t) = \begin{cases} \phi(t_i, x_{t_i-}^n) \cdot (x(t_{i+1}) - x(t_i)), & \text{if } t = t_i \in \pi_n. \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

If $\Delta G(t) > 0$, there exists [20, VI.2.1(a)] a sequence $t_n^* \to t$; $\Delta g_n(t_n^*) \to \Delta G(t)$. Using the fact that $\phi\_$ is left-continuous, x is càdlàg and (18), we see that

$$\lim_n \Delta g_n(t_n^*) = \phi(t, x_{t-}) \cdot \Delta x(t) = \lim_n \phi(t_n', x_{t_n'-}^n) \cdot \Delta x^n(t_n') = \lim_n \Delta g_n(t_n'), \tag{19}$$

else we will contradict $\Delta G(t) > 0$. Applying [20, VI.2.1(b)], we deduce that (t_n^*) must coincide with (t_n') for all n sufficiently large and by [20, VI.2.1(b.3)], we have established that

$$g_n(t) \longrightarrow G(t), \tag{20}$$

hence we can define $\mathbf{I}_\phi(t, x_t) := G(t)$ on $[0, T]$. Let $t_n'' := \min\{t_i > t_n' | t_i \in \pi_n\}$, $z := x_{t-} \in \Omega$, it follows from (17), (19) and (20) that

$$\mathbf{I}_\phi(t, x_{t-}) = \lim_n \mathbf{I}_\phi(t, z_t^n) = \lim_n \left(\mathbf{I}_\phi(t, x_t^n) - \phi(t_n', x_{t_n'-}^n) \cdot (x(t_n'') - x(t-)) \right) = G(t-),$$

hence $t \longmapsto \mathbf{I}_\phi(t, x_t)$ is càdlàg and its jump at time t is $\mathbf{I}_\phi(t, x_t) - \mathbf{I}_\phi(t, x_{t-})$. If $t_n \longrightarrow t$, the limits of $g_n(t_n)$ and $g_n(t_n-)$ are readily determined according to (19) and [20, VI.2.1(b)]. The continuity criteria in Definition 3.9 are thus satisfied. □

Proposition 5.7. *Let ϕ be Λ-integrable. Then $\mathcal{D}\mathbf{I}_\phi = 0$ and $\nabla_x \mathbf{I}_\phi = \phi_-$ on Λ.*

Proof. Let $(t,x) \in \Lambda$ and $z := x + e\mathbb{1}_{[t,\infty)} \in \Lambda$. Then

$$\begin{aligned}\mathbf{I}_\phi(t,z_t) - \mathbf{I}_\phi(t,x_t) &= \lim_n \left(\mathbf{I}_\phi(t,z_t^n) - \mathbf{I}_\phi(t,x_t^n)\right)\\ &= \lim_n \phi(t_n', z_{t_n'-}^n)\cdot e\\ &= \lim_n \phi(t_n', x_{t_n'-}^n)\cdot e = \phi(t,x_{t-})\cdot e,\end{aligned}$$

by the continuity of $\mathbf{I}_\phi$ and left-continuity of ϕ_-. □

Theorem 5.8 (Change of variable formula for class $\mathcal{S}$ functionals). *Let $F \in \mathcal{S}(\Lambda)$. Then for any $(T,x_T) \in \Lambda$, the limit*

$$\int_0^T \nabla_x F(t,x_{t-})dx := \lim_{n\to\infty} \sum_{\pi_n \ni t_i \leqslant T} \nabla_x F(t_i, x_{t_i-}^n)\cdot (x(t_{i+1}) - x(t_i)) \tag{21}$$

exists and

$$F(T,x_T) = F(0,x_0) + \int_0^T \mathcal{D}F(t,x_t)dt + \int_0^T \nabla_x F(t,x_{t-})dx.$$

Proof. See the Appendix. □

Remark 5.9. By Proposition 5.7, we see that all pathwise integrals are functionals of class $\mathcal{M}$, hence by Theorem 5.8, we can write

$$\mathbf{I}_\phi(t,x_t) = \int_0^t \phi dx. \tag{22}$$

As we shall see, the converse is also true, all integrals that may be defined by (21) are pathwise integrals in the sense of Definition 5.4:

Corollary 5.10 (Decomposition for class $\mathcal{S}$). *Let $F \in \mathcal{S}(\Lambda)$. Then $M : \Lambda \to \mathbb{R}$ defined by*

$$M(t,x_t) := F(t,x_t) - F(0.x_0) - \int_0^t \mathcal{D}F(s,x_s)ds$$

is of class $\mathcal{M}$ and $\nabla_x M = \nabla_x F$. In particular, M may be represented as a pathwise integral: there exists a Λ-integrable functional $\phi : \Lambda \to \mathbb{R}^m$ such that $M = \mathbf{I}_\phi$:

$$\forall (t,x) \in \Lambda, \qquad M(t,x) = \int_0^t \phi.dx.$$

Proof. By differentiating M, we obtain $\mathcal{D}M = 0$ and $\nabla_x M = \nabla_x F$. Continuity of M follows from Corollary 5.3 and Theorem 5.8, hence by (21), M satisfies Definition 5.4. □

In fact, all functionals of class $\mathcal{M}$ have an integral representation. We obtain as a corollary a Fundamental theorem of calculus for functionals:

Corollary 5.11.

(i) *Let ϕ be Λ-integrable. Then the map $\mathbf{I}_\phi : (t, x_t) \in \Lambda \mapsto \int_0^t \phi.dx$ is continuous, differentiable and*

$$\nabla_x \mathbf{I}_\phi = \phi_-.$$

(ii) *Let $\phi : \Lambda \to \mathbb{R}$. If $F \in \mathcal{M}(\Lambda)$ such that $\nabla_x F = \phi_-$, then ϕ is Λ-integrable and*

$$\int_0^t \phi dx = F(t, x_t) - F(0, x_0).$$

Proof. (i) is due to Proposition 5.7 and Remark 5.9. (ii) is due to (21) and Corollary 5.10. □

Example 5.12. Let $\Omega \subset Q_m^\pi$, $f_i \in C^1(\mathbb{R})$, then

$$\int_0^T \left(\int f_1 \circ x_1 dx_1, \, \dots, \int f_m \circ x_m dx_m \right)' dx$$
$$= \sum_i \left(\int_0^T (x_i(T) - x_i) f_i \circ x_i dx_i - \int_0^T f_i \circ x_i d[x_i] \right), \tag{23}$$

by an application of Corollary 5.11(ii) to the RHS of (23), Example. 4.18(iii) and (15).

An important consequence of Theorem 5.8 is to show that class $\mathcal{M}$ functionals satisfy a pathwise analogue of the *martingale* property. The concept of martingale was originally introduced to model the outcome of a *fair game* [24] across a set of outcomes. The following result, which does not make use of any probabilistic notion, shows that a class $\mathcal{M}$ functional represents the outcome of such a 'fair game', where the underlying set of outcomes is a generic subset of paths:

Theorem 5.13 (Fair game). *Let $M \in \mathcal{M}(\Lambda)$. If there exists $T > 0$ such that*

$$\forall x \in \Omega, \qquad M(T, x_T) - M(0, x_0) \geqslant 0$$

then

$$\forall x \in \Omega, \quad M(T, x_T) = M(0, x_0).$$

This result suggests that class $\mathcal{M}$ functionals may be considered pathwise analogues of martingales.

Proof. Since $\mathcal{D}M$ vanishes, by Lemma 5.1 we obtain

$$M(t, x_t) = M(t, x_t) + \int_t^T \mathcal{D}M(s, x_t) ds = M(T, x_t) \geqslant 0 \tag{24}$$

for all $t \leqslant T$, where the last inequality is due to $x_t \in \Omega$. Suppose there exists $z \in \Omega; M(T, z_T) > 0$. By Theorem 5.8 and the continuity of M, it follows

$$M(T, z_T^n) = \sum_{\pi_n \ni t_i \leqslant T} \nabla_x M(t_i, z_{t_i-}^n)(z(t_{i+1}) - z(t_i)) > 0 \tag{25}$$

for all n sufficiently large. Define $t_n^* := \min\{t_i \in \pi_n | M(t_i, z_{t_i}^n) > 0\}$, then $t_n^* \leqslant T$. By (24), (25), the left-continuity of M and the fact that $z^n \in \Omega$, we obtain

$$M(t_n^*, z_{t_n^*}^n) > M(t_n^*, z_{t_n^*-}^n) = 0,$$

hence $M(t_n^*, z_{t_n^*}^n) = \nabla_x M(t_n^*, z_{t_n^*-}^n)\Delta z(t_n^*) > 0$. Definition 3.1(ii) implies that there exists $\epsilon > 0$ such that

$$z^* := z_{t_n^*-}^n - \epsilon \Delta z(t_n^*) \mathbb{1}_{[t_n^*, \infty)} \in \Omega,$$

hence $M(t_n^*, z_{t_n^*}^*) = \nabla_x M(t_n^*, z_{t_n^*-}^n)(-\epsilon \Delta z(t_n^*)) < 0$, which contradicts (24). □

The following change of variable formula for causal functionals extends [7, Theorem 4] to $C^{1,2}(\Lambda)$, removing the condition linking the partition sequence π with the jump times of a path:

Theorem 5.14 (Change of variable formula for $C^{1,2}$ functionals). *Let $x \in \Omega \cap Q_m^\pi$. For any $F \in C^{1,2}(\Lambda)$ the following Föllmer–Itô formula holds:*

$$\begin{aligned} F(T, x_T) = F(0, x_0) &+ \int_0^T \mathcal{D}F(t, x_t)dt + \int_0^T \nabla_x F(t, x_{t-})dx \\ &+ \frac{1}{2}\int_0^T \nabla_x^2 F(t, x_{t-})d[x]^c + \sum_{t \leqslant T} (\Delta F(t, x_t) - \nabla_x F(t, x_{t-}) \cdot \Delta x(t)), \end{aligned} \tag{26}$$

where the series is absolute convergent and the pointwise limit

$$\int_0^T \nabla_x F(t, x_{t-})dx := \lim_{n\to\infty} \sum_{\pi_n \ni t_i \leqslant T} \nabla_x F(t_i, x_{t_i-}^n) \cdot (x(t_{i+1}) - x(t_i)) \tag{27}$$

exists.

Proof. See the Appendix. □

An important consequence of Theorem 5.14 is the continuity of the Föllmer integral in the π-topology:

Proposition 5.15. *Let $\Omega \subset Q_m^\pi$ and $F \in C^{1,2}(\Lambda)$. Then*

$$\begin{aligned} J : \Lambda &\longmapsto \mathbb{R} \\ (t, x) &\longmapsto J(t, x_t) := \int_0^t \nabla_x F(s, x_s)dx \end{aligned}$$

is continuous. In particular, $\nabla_x F$ is integrable and J is a pathwise integral in the sense of Definition 5.4.

Proof. We apply the functional change of variable formula (Theorem 5.14) to F. Rearranging the terms in (26) we observe that $t \longmapsto J(t, x_t)$ is càdlàg whose jump at time t is $J(t, x_t) - J(t, x_{t-})$. It remains to show that J satisfies the continuity criteria Definition 3.9.1(c),(d) and 2(c),(d). It is suffice to assume $t_n \longrightarrow t; t_n \geqslant t'_n$ (that is, the other criteria follow similarly). By (27) and that x is right-continuous, we first obtain

$$\begin{aligned} J(t_n, x^n_{t_n}) &= \int_0^{t_n} \nabla_x F(t, x^n_{t-}) dx^n \\ &= \sum_{\pi_n \ni t_i < t} \nabla_x F(t_i, x^n_{t_i-}) \cdot (x(t_{i+1}) - x(t_i)) \longrightarrow J(t, x_t) \\ &\quad + \sum_{\pi_n \ni t_i \in (t'_n, t_n]} \nabla_x F(t_i, x^n_{t_i-}) \cdot (x(t_{i+1}) - x(t_i)). \end{aligned} \tag{28}$$

We have to show that the rest term (28) vanishes as $n \uparrow \infty$. Applying (26) to the path x^n and by the local boundedness of $\mathcal{D}F$, we have

$$\begin{aligned} \left| \sum_{\pi_n \ni t_i \in (t'_n, t_n]} \nabla_x F(t_i, x^n_{t_i-}) \cdot \Delta x^n(t_i) \right| &\leqslant |F(t_n, x^n_{t_n}) - F(t'_n, x^n_{t'_n})| \\ &\quad + \text{const}|t_n - t'_n| \\ &\quad + \left| \sum_{\pi_n \ni t_i \in (t'_n, t_n]} \Delta F(t_i, x^n_{t_i}) - \nabla_x F(t_i, x^n_{t_i-}) \cdot \Delta x^n(t_i) \right|. \end{aligned}$$

Since $t_n \geqslant t'_n; t_n, t'_n \longrightarrow t$ and by the right continuity of F the first two terms vanish. Since $(\nabla_x^2 F)_-$ is locally bounded and $\nabla_x^2 F$ admits a modulus, applying a second-order Taylor expansion to the third term, we obtain

$$\left| \sum_{\pi_n \ni t_i \in (t'_n, t_n]} \Delta F(t_i, x^n_{t_i}) - \nabla_x F(t_i, x^n_{t_i-}) \cdot \Delta x^n(t_i) \right| \leqslant \text{const}|q_n(t_n) - q_n(t'_n)| \longrightarrow 0,$$

by the fact that $q_n \xrightarrow{\mathrm{J}_1} [x]$ and [4, Section 4.2]. □

6 | APPLICATION TO PATHS WITH FINITE QUADRATIC VARIATION

We now examine in more detail the case of paths of finite quadratic variation and apply the results developed in Section 5 to the case $\Omega \subset Q^\pi_m$. As we have already shown, integration and differentiation are inverse operations (Corollary 5.11). Using functionals of class $\mathcal{M}$, we show that these operations may be viewed as *isomorphisms* between certain spaces. We also obtain a pathwise identity related to Itô's isometry (Theorem 6.4).

The key objects here are functionals of class $\mathcal{M}$, which are *primitives* and may be understood as pathwise analogues of martingales (Theorem 5.13). In addition, we shall show that class $\mathcal{M}$ functionals are canonical *solutions* to path-dependent heat equations. Let us introduce the following vector spaces of integrands:

$$L(\Lambda) := \{\nabla_x F | F \in C^{1,2}(\Lambda)\}, \qquad L_b(\Lambda) := \{\nabla_x F | F \in C_b^{1,2}(\Lambda)\},$$

$$\mathcal{L}(\Lambda) := \{\nabla_x F | F \in \mathcal{M}(\Lambda)\}, \qquad \mathcal{L}_b(\Lambda) := \{\nabla_x F | F \in \mathcal{M}_b(\Lambda)\}.$$

By Proposition 5.15, the integral operator

$$\int : \phi \in L(\Lambda) \longmapsto \mathbf{I}_\phi \in \mathbb{R}^\Lambda,$$

where $\mathbf{I}_\phi$ is given by (22), is a well-defined linear operator.

Example 6.1 (Path-dependent 1-form). Let $f_i \in C^1(\mathbb{R})$, $i = 1, \dots, m$ then

$$\phi(t, x_t) := \left(\left(\int f_1 \circ x_1 dx_1\right)(t-), \dots, \left(\int f_m \circ x_m dx_m\right)(t-)\right)'$$

defines an element of $\mathcal{L}_b(\Lambda)$.

Proof. See Example 4.18(15). □

Lemma 6.2.

(i) *If $\phi \in L(\Lambda)$ then $\int \phi \in \mathcal{M}(\Lambda)$ and $\nabla_x(\int \phi) = \phi_-$.*
(ii) *If $\phi \in L_b(\Lambda)$ then $\int \phi \in \mathcal{M}_b(\Lambda)$ and $\nabla_x(\int \phi) = \phi_-$.*
(iii) *If $\phi \in \mathcal{L}(\Lambda)$ then $\int \phi \in \mathcal{M}(\Lambda)$ and $\nabla_x(\int \phi) = \phi$.*
(iv) *If $\phi \in \mathcal{L}_b(\Lambda)$ then $\int \phi \in \mathcal{M}_b(\Lambda)$ and $\nabla_x(\int \phi) = \phi$.*

Proof. It is due to Proposition 5.15 and Corollary 5.11(i). □

Corollary 6.3. *Define*

$$\mathcal{M}_0(\Lambda) := \{F \in \mathcal{M}_b(\Lambda) | F(0, x_0) \equiv 0\},$$

then the integral operator

$$\int : \mathcal{L}_b(\Lambda) \longmapsto \mathcal{M}_0(\Lambda)$$

is an isomorphism and the inverse of $\int$ is the differential operator ∇_x.

Proof. Injectivity follows from Lemma 6.2(iv). Surjectivity is due to Corollary 5.11(ii). □

We now obtain a pathwise identity of Itô,[†] and give an application. For $\phi, \psi \in \mathcal{L}_b(\Lambda)$ define $\{\phi, \psi\} \in \mathcal{L}_b(\Lambda)$ by

$$\begin{aligned} \{\phi,\psi\} &: \Lambda \mapsto \mathbb{R}^d \\ (t,x) &\to \left(\psi \int_0^{\cdot} \phi.dx + \phi \int_0^{\cdot} \psi.dx\right)(t, x_{t-}). \end{aligned}$$

Theorem 6.4. *For all $\phi, \psi \in \mathcal{L}_b(\Lambda)$, $\{\phi, \psi\} \in \mathcal{L}_b(\Lambda)$ and*

$$\left(\int \phi dx\right)\left(\int \psi dx\right) = \int \phi\psi' d[x] + \int \{\phi,\psi\} dx.$$

Proof. Recall that $C_b^{1,2}(\Lambda)$ is an algebra. Let $\phi, \psi \in \mathcal{L}_b(\Lambda)$, put $F := \int \phi dx, G := \int \psi dx$, then $F, G \in \mathcal{M}_b(\Lambda)$ by Lemma 6.2(iv). Since $\mathcal{M}_b(\Lambda) \subset C_b^{1,2}(\Lambda)$, it follows $FG \in C_b^{1,2}$. Apply the change of variable formula (Theorem 5.14) to FG; using Lemma 6.2(ii), the proof is complete. □

Corollary 6.5 (Isometry). *Let $\mathcal{E} \subset \mathcal{L}_b(\Lambda)$ be a subspace such that*

$$\forall \phi, \psi \in \mathcal{E}, \quad \{\phi, \psi\} \in \mathcal{E}$$

and denote $\mathbf{I}(\mathcal{E})$ the image of $\mathcal{E}$ under $\int$. If $\mathbb{E}$ is any positive element of the algebraic dual $C^*(\Lambda)$ such that $\mathbf{I}(\mathcal{E}) \subset ker(\mathbb{E})$, then

$$\left\langle \int \phi dx, \int \psi dx \right\rangle_{\mathbf{I}(\mathcal{E})} := \mathbb{E}\left(\int \phi dx \int \psi dx\right) = \mathbb{E}\left(\int \phi\psi' d[x]\right) =: \langle \phi, \psi \rangle_{\mathcal{E}}$$

holds for all $\phi, \psi \in \mathcal{E}$.

In particular, the bracket $\langle ., . \rangle_{\mathcal{E}}$ induces a semi-norm on $\mathcal{E}$. Denoting $\tilde{\mathcal{E}}$ the quotient space induced by the semi-norm, the integral operator

$$\begin{aligned} \tilde{\int} &: \tilde{\mathcal{E}} \longmapsto \mathbf{I}(\tilde{\mathcal{E}}) \\ \tilde{\phi} &\longmapsto \tilde{\int} \tilde{\phi} := \int \phi \end{aligned}$$

is an isometric isomorphism between the pre-Hilbert spaces $\tilde{\mathcal{E}}$ and $\mathbf{I}(\tilde{\mathcal{E}})$. The inverse of $\tilde{\int}$ is the differential operator

$$\begin{aligned} \tilde{\nabla}_x &: \mathbf{I}(\tilde{\mathcal{E}}) \longmapsto \tilde{\mathcal{E}}, \\ \tilde{F} &\longmapsto \tilde{\nabla}_x \tilde{F} := \nabla_x F. \end{aligned}$$

Proof. The result is a consequence of Corollary 6.3 and Theorem 6.4. □

We conclude with a discussion on the relation between class $\mathcal{M}(\Lambda)$ and harmonic functionals, defined as solutions to a class of path-dependent heat equations [5, Chapter 8]. Let $\Sigma : \Lambda \to S_m^+$ be a right-continuous function on Λ taking values in positive-definite symmetric $m \times m$ matrices

[†] First appeared in [19, Lemma 2].

and

$$\Omega_\Sigma := \left\{ x \in \Omega | \frac{d[x]}{dt} = \Sigma \right\} \subset \Omega$$

the set of paths with absolutely continuous quadratic variation with Lebesgue density Σ.

Definition 6.6. $F \in C^{1,2}(\Lambda)$ is called Σ-harmonic if it satisfies

$$\forall x \in \Omega_\Sigma, \quad \forall t \geqslant 0, \qquad \mathcal{D}F(t, x_t) + \frac{1}{2}\langle \nabla_x^2 F(t, x_t), \Sigma(t, x_t)\rangle = 0. \tag{29}$$

If F is Σ-harmonic, then the change of variable formula (Theorem 5.14) gives

$$F(t, x_t) = F(0, x_0) + \int_0^t \nabla_x F(s, x_{s-}) dx \tag{30}$$

for all $t \geqslant 0$ and $x \in \Omega_\Sigma$. Equality in (30) then holds on Ω_Σ. Every functional of class $\mathcal{M}$ satisfies (29), hence is Σ-harmonic for all Σ.

Theorem 6.7 (Representation of Σ-harmonic functionals). *If F is Σ-harmonic, then there exists a class $\mathcal{M}$ functional M such that*

$$M|_{\Omega_\Sigma} \equiv F.$$

In particular, M is uniquely determined by (30) on Ω_Σ.

Proof. Let $F \in C^{1,2}(\Lambda)$ be Σ-harmonic. We can define a functional $M : \Lambda \to \mathbb{R}$ by

$$M(t, x) := F(0, x_0) + \int_0^t \nabla_x F(s, x_{s-}) dx. \tag{31}$$

By Lemma 6.2(i), we see that $M \in \mathcal{M}(\Lambda)$ and $\nabla_x M = (\nabla_x F)_-$. By (30) and (31), the proof is complete. □

APPENDIX: TECHNICAL PROOFS

Proof of Proposition 1.1. For $\alpha \in \mathbb{R}_+$, define $w_\alpha(t) := 1_{[\alpha,\infty)}(t) \in D =: \Omega$, where D denotes the Skorokhod space. We assign to the collection $(w_\alpha)_{\alpha \in \mathbb{R}_+}$, a normalized Lebesgue measure

$$\mathbb{P}(\{w_\alpha | \alpha \in A\}) := \sum_{n \geqslant 1} \frac{\lambda(A \cap [0, n])}{2^{n+1}},$$

then $\mathbb{P}(\{w_\alpha | \alpha \in \mathbb{R}_+\}) = 1$ and $X_t(w) := w(t)$ is a finite variation process (that is, a semimartingale) under $\mathbb{P}$. Now let $\pi = (\pi_n)_{n \geqslant 1}$ be any sequence of time partitions and denote

$$Q_0^\pi := \{x \in Q^\pi | J(x) \subset \liminf_n \pi_n\}.$$

Since $\liminf_n \pi_n$ is countable, it follows that $\mathbb{P}(\{w_\alpha | \alpha \in \liminf_n \pi_n\}) = 0$ and therefore $\mathbb{P}(\{\omega \in \Omega | X_\cdot(\omega) \in Q_0^\pi\}) = 0$. □

Proof of Theorems 5.8 and 5.14. By the right continuity of F (Definition 3.9.2(d)), we have

$$F(T, x_T) - F(0, x_0) = \lim_n \sum_{\pi_n \ni t_i \leqslant T} F(t_{i+1}, x^n_{t_{i+1}-}) - F(t_i, x^n_{t_i-}), \tag{A.1}$$

where for all n sufficiently large, we can decompose each increment

$$\begin{aligned} &F(t_{i+1}, x^n_{t_{i+1}-}) - F(t_i, x^n_{t_i-}) \\ &\quad = F(t_{i+1}, x^n_{t_{i+1}-}) - F(t_i, x^n_{t_{i+1}-}) + F(t_i, x^n_{t_{i+1}-}) - F(t_i, x^n_{t_i-}) \\ &\quad = \underbrace{\left(F(t_{i+1}, x^n_{t_i}) - F(t_i, x^n_{t_i})\right)}_{\text{time}} + \underbrace{\left(F(t_i, x^n_{t_i}) - F(t_i, x^n_{t_i-})\right)}_{\text{space}} \end{aligned}$$

into the sum of a time ('horizontal') and a space ('vertical') increment.

Since F is left-continuous and differentiable in time, $\mathcal{D}F$ is right-continuous and locally bounded, by Lemma 5.1 each time increment may be expressed as

$$F(t_{i+1}, x^n_{t_i}) - F(t_i, x^n_{t_i}) = \int_{t_i}^{t_{i+1}} \mathcal{D}F(t, x^n_{t_i})dt.$$

By Lemma 5.2, we obtain

$$\lim_n \sum_{\pi_n \ni t_i \leqslant T} F(t_{i+1}, x^n_{t_i}) - F(t_i, x^n_{t_i}) = \int_0^T \mathcal{D}F(t, x_t)dt,$$

which in light of (A.1), implies that the sum of space increments converges to

$$\lim_n \sum_{\pi_n \ni t_i \leqslant T} \underbrace{F(t_i, x^n_{t_i}) - F(t_i, x^n_{t_i-})}_{\Delta F(t_i, x^n_{t_i})} = F(T, x_T) - F(0, x_0) - \int_0^T \mathcal{D}F(t, x_t)dt. \tag{A.2}$$

If $F \in \mathcal{S}(\Lambda)$ then $\nabla_x F$ is strictly causal and by Proposition 4.3, $\nabla_x^2 F$ is vanishing everywhere. Thus, by a second order Taylor expansion, the remainder term vanishes, so

$$F(t_i, x^n_{t_i}) - F(t_i, x^n_{t_i-}) = \nabla_x F(t_i, x^n_{t_i-}) \cdot \left(x(t_{i+1}) - x(t_i)\right)$$

and Theorem 5.8 follows. If $F \in C^{1,2}(\Lambda)$ then, by Taylor's Theorem, each space increment admits the following second-order expansion:

$$\begin{aligned} \Delta F(t_i, x^n_{t_i}) &= F\left(t_i, x^n_{t_i-} + \Delta x^n(t_i)\mathbb{1}_{[t_i,\infty)}\right) - F(t_i, x^n_{t_i-}) \\ &= \nabla_x F(t_i, x^n_{t_i-}) \cdot \Delta x^n(t_i) + \frac{1}{2}\langle \nabla_x^2 F(t_i, x^n_{t_i-}), \Delta x^n(t_i)\Delta x^n(t_i)' \rangle, \\ &\quad + R^n_{t_i}, \end{aligned} \tag{A.3}$$

where $\Delta x^n(t_i) = (x(t_{i+1}) - x(t_i))$ and

$$R^n_{t_i} = \frac{1}{2}\langle \nabla^2_x F(t_i, x^n_{t_i-} + \alpha^n_i \Delta x^n(t_i)\mathbb{1}_{[t_i,\infty)}) - \nabla^2_x F(t_i, x^n_{t_i-}), \Delta x^n(t_i)\Delta x^n(t_i)'\rangle,$$

where $\alpha^n_i \in (0,1)$. Since $x \in \Omega_2 \subset Q^\pi_m$, by Corollary 2.8 and Remark 2.9

$$\begin{aligned}\lim_n \sum_{\pi_n \ni t_i \leqslant T} \langle \nabla^2_x F(t_i, x^n_{t_i-}), \Delta x^n(t_i)\Delta x^n(t_i)'\rangle &= \int_0^T \nabla^2_x F(t, x_{t-}) d[x] \\ &= \int_0^T \nabla^2_x F(t, x_{t-}) d[x]^c + \sum_{t \leqslant T} \langle \nabla^2_x F(t, x_{t-}), \Delta x(t)\Delta x(t)'\rangle. \qquad \text{(A.4)}\end{aligned}$$

Let $\delta > 0$, $r := \sup_{t\in[0,T]} |\Delta x(t)|$, $r_\delta := \delta + \sup_{t\in[0,T+\delta]} |\Delta x(t)|$. Using a result on càdlàg functions [7, Lemma 8], we see that $|\Delta x^n(t_i)| \leqslant r_\delta$ for n sufficiently large. By Remark 3.2, we see that $\alpha^n_i \Delta x^n(t_i) \in \mathcal{U}_{t_i-}(x^n) \cap \overline{B}_{r_\delta}(0)$. Since $\nabla^2_x F$ admits a modulus of vertical continuity, it follows from Definition 4.7 that there exists a modulus of continuity ω such that

$$|R^n_{t_i}| \leqslant \frac{1}{2}\omega(r_\delta)|\Delta x^n(t_i)\Delta x^n(t_i)'|$$

for n sufficiently large, hence by an application of Corollary 2.8(i), we obtain

$$\limsup_n \sum_{\pi_n \ni t_i \leqslant T} |R^n_{t_i}| \leqslant \frac{1}{2}\omega(r_\delta) \leqslant \omega(r_\delta) tr([x](T)).$$

Send $\delta \downarrow 0$, and by the right continuity of x, we have established that

$$\limsup_n \sum_{\pi_n \ni t_i \leqslant T} |R^n_{t_i}| \leqslant \frac{1}{2}\omega(r+) tr([x](T)). \qquad \text{(A.5)}$$

Let $0 < \epsilon < r$, define the following finite sets on $[0, T]$

$$\begin{aligned} J(\epsilon) &:= \{t \leqslant T \,|\, |\Delta x(t)| > \epsilon\}, \\ J_n(\epsilon) &:= \{\pi_n \ni t_i \leqslant T \,|\, \exists t \in (t_i, t_{i+1}], |\Delta x(t)| > \epsilon\}. \end{aligned}$$

We can decompose

$$\sum_{\pi_n \ni t_i \leqslant T} R^n_{t_i} = \sum_{t_i \in J_n(\epsilon)} R^n_{t_i} + \sum_{t_i \in (J_n(\epsilon))^c} R^n_{t_i}, \qquad \text{(A.6)}$$

into two partial sums. By (A.3), the right-continuity (resp. left-continuity) of F (resp. $(\nabla_x F)_-, (\nabla^2_x F)_-$) and that x is càdlàg we obtain

$$\sum_{t_i \in J_n(\epsilon)} \left(R^n_{t_i}\right)^{\pm} \xrightarrow{n} \sum_{t \in J(\epsilon)} \Big(\Delta F(t, x_t) - \nabla_x F(t, x_{t-}) \cdot \Delta x(t)$$

$$-\frac{1}{2}\langle \nabla_x^2 F(t, x_{t-}), \Delta x(t)\Delta x(t)'\rangle\Big)^{\pm}$$
$$\leqslant \frac{1}{2}\omega(r+)tr([x](T)), \tag{A.7}$$

where the inequality follows from (A.5) and (A.6). Observe that $J(\epsilon) \uparrow J(0)$ as $\epsilon \downarrow 0$, by monotone convergence, we obtain

$$\lim_n \sum_{t_i \in J_n(\epsilon)} \left(R^n_{t_i}\right)^{\pm} \xrightarrow{\epsilon} \sum_{t \leqslant T} \Big(\Delta F(t, x_t) - \nabla_x F(t, x_{t-}) \cdot \Delta x(t)$$
$$-\frac{1}{2}\langle \nabla_x^2 F(t, x_{t-}), \Delta x(t)\Delta x(t)'\rangle\Big)^{\pm}$$
$$\leqslant \frac{1}{2}\omega(r+)tr([x](T)). \tag{A.8}$$

On the other hand, since w is monotonic, by (A.5) and (A.6), it follows that

$$\left| \limsup_n \sum_{t_i \in (J_n(\epsilon))^c} R^n_{t_i} - \liminf_n \sum_{t_i \in (J_n(\epsilon))^c} R^n_{t_i} \right| \leqslant \omega(\epsilon) tr([x](T)), \tag{A.9}$$

and by (A.2)–(A.4), (A.6), (A.7) and (A.9), so is

$$\left| \limsup_n \sum_{\pi_n \ni t_i \leqslant T} \nabla_x F^n_{t_i} \cdot \Delta x^n(t_i) - \liminf_n \sum_{\pi_n \ni t_i \leqslant T} \nabla_x F^n_{t_i} \cdot \Delta x^n(t_i) \right| \leqslant \omega(\epsilon) tr([x](T)),$$

where we have denoted $\nabla_x F^n_{t_i} := \nabla_x F(t_i, x^n_{t_i-})$. Send $\epsilon \downarrow 0$, we obtain

$$\int_0^T \nabla_x F(t, x_{t-}) dx := \lim_n \sum_{\pi_n \ni t_i \leqslant T} \nabla_x F(t_i, x^n_{t_i-}) \cdot (x(t_{i+1}) - x(t_i)). \tag{A.10}$$

Upon a second look at (A.2)–(A.4), (A.6), (A.7) and in light of (A.10), we immediately see that

$$\lim_n \sum_{t_i \in (J_n(\epsilon))^c} R^n_{t_i} =: o(\epsilon)$$

also exists and by (A.5), $|o(\epsilon)| \leqslant \frac{1}{2}\omega(\epsilon) tr([x](T)) \xrightarrow{\epsilon} 0$ which, combined with (A.6) and (A.8) implies

$$\lim_n \sum_{\pi_n \ni t_i \leqslant T} R^n_{t_i} = \sum_{t \leqslant T} \Big(\Delta F(t, x_t) - \nabla_x F(t, x_{t-}) \cdot \Delta x(t)$$
$$-\frac{1}{2}\langle \nabla_x^2 F(t, x_{t-}), \Delta x(t)\Delta x(t)'\rangle\Big). \tag{A.11}$$

In view of (A.2)–(A.4), (A.10) and (A.11), it remains to show that

$$\sum_{t\leqslant T}\Big(\Delta F(t,x_t)-\nabla_x F(t,x_{t-})\Delta x(t)-\frac{1}{2}\langle\nabla_x^2 F(t,x_{t-}),\Delta x(t)\Delta x(t)'\rangle\Big)$$
$$=\sum_{t\leqslant T}\Big(\Delta F(t,x_t)-\nabla_x F(t,x_{t-})\Delta x(t)\Big)-\frac{1}{2}\sum_{t\leqslant T}\langle\nabla_x^2 F(t,x_{t-}),\Delta x(t)\Delta x(t)'\rangle, \tag{A.12}$$

and the absolute convergence of the series. Since $(\nabla_x^2 F)_-$ is left-continuous and locally bounded, we see from Lemma 4.6(ii) that the map $t\longmapsto\nabla_x^2 F(t,x_{t-})$ is also bounded on $[0,T]$, hence by (5)

$$\frac{1}{2}\sum_{t\leqslant T}|\nabla_x^2 F(t,x_{t-})||\Delta x(t)\Delta x(t)'|\leqslant \text{const}\sum_i\Bigg(\sum_{t\leqslant T}(\Delta x_i(t))^2\Bigg)$$
$$\leqslant \text{const}\cdot tr([x](T)),$$

which, combined with (A.8) implies (A.12) and the absolute convergence of the series, hence Theorem 5.14 is proven. □

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