How Does Fairness Affect the Complexity of Gerrymandering?

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ABSTRACT

Gerrymandering is a common way to externally manipulate district-based elections where the electorate is (artificially) redistricted with an aim to favor a particular political party to win more districts in the election. Formally, given a set of m possible locations of ballot boxes and a set of n voters (with known preferences) is it possible to choose k specific locations for the ballot boxes so that the desired candidate wins in at least ℓ of them? Lewenberg et al. [AAMAS '17] and Eiben et al. [AAAI '20] studied the classical and fine-grained complexity (respectively) of the gerrymandering problem.

In recent years, the research direction of studying the algorithmic implications of introducing fairness in computational social choice has been quite active. Movtiated by this, we define two natural fairness conditions for the gerrymandering problem and design a near-optimal algorithm. Our two new conditions introduce an element of fairness of the election process by ensuring that:

- the number of voters at each ballot box is not unbounded, i.e., lies in the interval $[\mathtt{lower}, \mathtt{upper}]$ for some given parameters $\mathtt{lower}, \mathtt{upper}$
- the margin of victory at each ballot box is not unbounded, i.e., lies in the interval $[\mathtt{margin}_{\mathtt{low}}, \mathtt{margin}_{\mathtt{up}}]$ for some given parameters $\mathtt{margin}_{\mathtt{low}}, \mathtt{margin}_{\mathtt{up}}$

For the real-life implementation of redistricting, i.e., when voters are located in $\mathbb{R}^2$, we obtain the following upper and lower bounds for this fair version of the gerrymandering problem:

- There is an algorithm running in $(m+n)^{O(\sqrt{k})} \cdot |C|^{(\mathtt{upper}+\mathtt{lower}+\mathtt{margin}_{\mathtt{up}}+\mathtt{margin}_{\mathtt{low}})}$ time where C is the set of candidates participating in the election.
- Under the Exponential Time Hypothesis (ETH), we obtain an *almost tight* lower bound by ruling out algorithms running in $f(k, n, \mathtt{upper}, \mathtt{lower}) \cdot m^{o(\sqrt{k})}$ time where f is any computable function. The lower bound holds even when $\mathtt{margin}_{\mathtt{low}} = 1 = \mathtt{margin}_{\mathtt{up}}$, $k = \ell$ and there are only 2 candidates.

KEYWORDS

Fairness, Gerrymandering, Computational social choice, Elections, Manipulation, Parametrized complexity, Exponential Time Hypothesis

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1 INTRODUCTION

Elections are a fundamental process in our lives: a group of agents vote according to their individual preferences to select a final outcome from a given set of outcomes. Given the high stakes, it is highly important to preserve the sanctity of an election from manipulation by either internal or external sources. A seminal result [17, 29] shows that most standard voting rules are susceptible by (internal) manipulation: the outcome of the election can be changed significantly even if one agent votes differently from their true preference! To add to the bad news, there is some evidence [11] from economics and political science that many of the voting systems used in real life actually incentivize voters to deviate from their true preferences.

A series of highly-influential papers [2–4] initiated the study of manipulation in various different voting scenarios from the viewpoint of computational complexity: given that (internal) manipulation is possible [17, 29] how easy or hard is to actually achieve a specific outcome in a given voting system? We refer the interested reader to [6, 14–16] for more information about this active area of research in computational social choice.

Internal manipulation is typically of the form where a coalition of voters strategically vote (often different from their true preferences) to ensure the victory (or loss) of a specific candidate [6][Chapter 6]. External manipulation on the other hand asks whether an agent who is not even participating in the election can still manipulate it in a way to ensure the victory (or loss) of a specific candidate. This can be achieved in various different ways: adding or removing voters or candidates [6][Chapter 7.3], bribing voters to change their preferences [6][Chapter 7.4], redistricting in district-based elections [13, 21], etc. In this paper, we focus on election manipulation by redistricting in district based elections.

1.1 Gerrymandering

A well defined representative democracy requires well drawn districts. The political boundary of the districts should be drawn in such a manner so that the percentage of voters who vote for a particular political party in the whole state (states are divided into districts) should be represented in the outcome of the election. There is a long history of manipulating elections by redistricting political boundaries in order to favour a particular political party which in political jargon known as “Gerrymandering". The term “Gerrymandering is being coined after Elbridge Gerry, who as Governor of Massachusetts in 1812, signed a bill that created a partisan district in the Boston area whose shape was unnatural and compared to a salamander. Gerrymandering has been used in many instances to manipulate elections in real-life, e.g. by US political parties [13, 21]. Broadly speaking the process “Gerrymandering" is generally being done by the following two ways: *i*) Packing: It includes drawing of lines to include maximum number of voters of the opposing party in minimum number of districts in order to minimize the strength in most of the districts. *ii*) Cracking: It is the method by which the splitting up of the "influencing voters" of the opposition into several districts with an aim not to consolidate the supporters of the influencing voters from the opposition.

One of the natural response to such blatant manipulation via gerrymandering is to impose the restriction that each of the voter should cast their vote at the ballot box placed nearest to them. Lewenberg et al. [22] initiated the algorithmic study of election manipulation in this setting. They showed that the gerrymandering problem in this setting is NP-hard even if the number of candidates is restricted to four. Later, Eiben et al. [12] analyzed on the same setting from the parameterized complexity point of view. Inspite of restricting the voters to cast their votes in the nearest ballot box it is not completely free from gerrymandering. In order to further strengthen the election process, the notion of fairness is being introduced.

1.2 Fairness

A truly democratic country must observe its election both free and fair. Generally speaking there are eight standards that must be met to conduct a free and fair election. Apart from conducting a free and fair election, various political economists and social scientists were trying to establish an ideal voting ranking system. Finally, in 1952, US Economist Kenneth J Arrow rule out the possibility of the existence of such an ideal model which is popularly known as "Arrow's impossibility theorem" [1]. On the other hand, fairness occurring in various form in different models is a very natural concept integral to any computational social choice experiment including voting. The existence of "Arrow's impossibility theorem" theorem further increases the systematic study of different notion of fairness in various domains of computational social choice theory including political system. Fairness itself is a big word in any kind of social experiment even if we stick ourselves with democratic system, fairness is not being fully characterized. We refer the reader to earlier works [18, 23, 30] on the importance as well as characteristics of fairness in an election. To summarize, the overall goal of fairness in an election is to ensure the proportional representation of every class of stakeholders.

1.3 Our Model: Fairness to combat Gerrymandering

In this paper, we impose the following two fairness rules with a view towards preserving the sanctity of the voting mechanism:

- The number of voters at each of the ballot box is bounded i.e. $\leq$ $\texttt{upper}$ and $\geq$ $\texttt{lower}$.
- The margin on victory at each of the ballot box is also bounded i.e. $\leq \texttt{margin}_{\texttt{up}}$ and $\geq \texttt{margin}_{\texttt{low}}$.

We bound the number of voters at each of the ballot box in order to avoid the demography corresponding to each of the ballot box being skewed. On the other hand, margin of victory is an important parameter to ensure fairness in voting. It is often being noticed that the prediction of exit poll varies widely with the real outcome of the voting. One of the main reason behind this is the occurrence of different malpractices like bribery, rampant rigging at the time of voting. The "margin of victory" parameter inhibits these malpractices to some extent by increasing the chances of recounting or in some cases re-polling if it is found after the election result that the margin of victory exceeds the boundaries. This parameter is also being used to measure the number of votes that would need to change with an aim to alter a parliamentary outcome for single member preferential electorates. Various earlier works [5, 10, 25, 31] shows how to compute margin of victory for different voting rules and draw its impact in real scenario.

Here we study the impact of the above fairness rules on gerrymandering. Additionally, we also impose the standard condition that a voter cast their vote at the ballot box located nearest to her. More specifically, we study the following problem (we call it "Fair-Gerrymandering") defined as follows.

> FAIR-GERRYMANDERING-$(\mathcal{X}, \rho)$
> **Input**: A set of candidates C, a set $\mathcal{V}$ of n voters located at points in $\mathcal{X}$ whose preferences are known, a set $\mathcal{B}$ of m possible ballot box locations in $\mathcal{X}$ and a specific candidate "OUR" $\in C$
> **Parameters**: $k, \ell, m, n, \texttt{upper}, \texttt{lower}, \texttt{margin}_{\texttt{low}}, \texttt{margin}_{\texttt{up}}$
> **Assumptions**:
> - Each voter votes at the ballot box nearest to them, where distances are calculated using the metric ρ.
> - The *plurality* rule is used: each voter votes for their top-ranked candidate, and a ballot box is won by the candidate who secures most votes. There are no ties.
> - The number of voters voting a candidate at every ballot box is bounded i.e., $\leq \texttt{upper}$ and $\geq \texttt{lower}$.
> - The margin of victory at every ballot box is $\leq \texttt{margin}_{\texttt{up}}$ and $\geq \texttt{margin}_{\texttt{low}}$.
>
> **Question**: Is there a set $\mathcal{P} \subseteq \mathcal{B}$ such that $k = |\mathcal{P}|$ such that opening ballot boxes at locations in $\mathcal{P}$ then "OUR" candidate wins at least ℓ of the ballot boxes for some $\ell \leq k \leq m$.

1.4 Our Results

In this paper, we initiate study of the algorithmic complexity of the FAIR-GERRYMANDERING problem. On the algorithmic side, we obtain the following result:

THEOREM 1. *If C is the set of candidates in an election, n be the number of voters and m be the possible ballot box locations in the plane, then* FAIR-GERRYMANDERING-$(\mathbb{R}^2, \ell_2)$ *is solvable in time* $(m+n)^{O(\sqrt{k})} \cdot |C|^{(\texttt{upper}+\texttt{lower}+\texttt{margin}_{\texttt{up}}+\texttt{margin}_{\texttt{low}})}$ *where k is the number of ballot boxes for the election.*

A brute-force search for a solution will run in $m^k n^{O(1)}$. Clearly, our algorithm is efficient than an exhaustive search. We complement this algorithm with an almost-matching lower bound:

THEOREM 2. *For any $d \geq 2$, under the Exponential Time Hypothesis (ETH), the* FAIR-GERRYMANDERING-$(\mathbb{R}^d, \rho)$ *problem cannot be solved in* $f(k, n, \texttt{upper}, \texttt{lower}) \cdot m^{o(k^{1-1/d})}$ *time where f is any computable function, n is the number of voters, and k is the number of the ballot boxes opened, m is the total number of possible locations of ballot boxes and ρ is either the ℓ_∞-metric or the ℓ_q-metric for some $q \geq 1$. This lower bound holds even when there are only 2 candidates, $k = \ell$ and $\texttt{margin}_{\texttt{low}} = 1 = \texttt{margin}_{\texttt{up}}$.*

Recall that the Exponential Time Hypothesis (ETH) is a standard assumption [24] in parameterized complexity theory which states that the 3-SAT problem cannot be solved in $2^{o(N)}$ time where N is the number of variables [19, 20].

Note that since $1 \leq \texttt{lower} \leq \texttt{upper} \leq n$, the terms lower and upper are redundant in the first term of the claimed lower bound for the running time in Theorem 2. However, we have chosen to include them here for the sake of completeness so that the involvement of each of the four fairness parameters ($\texttt{lower}, \texttt{upper}, \texttt{margin}_{\texttt{low}}, \texttt{margin}_{\texttt{up}}$) in Theorem 2 is explicitly clear.

Comparison of our results & techniques to [12]:

Eiben et al. [12] studied the "vanilla" version, i.e., without any fairness constraints, of the GERRYMANDERING-$(\mathbb{R}^2, \ell_2)$ problem. Note that this "vanilla" version of the Gerrymandering problem, i.e., the GERRYMANDERING-$(\mathcal{X}, \rho)$ problem studied by [12], is a special case of the FAIR-GERRYMANDERING-$(\mathcal{X}, \rho)$ problem with the following "extreme" values of some of the parameters:

- $\texttt{margin}_{\texttt{low}} = 1$ and $\texttt{margin}_{\texttt{up}} = n$
- $\texttt{lower} = 0$ and $\texttt{upper} = n$

Eiben et al. [12] designed an $(m+n)^{O(\sqrt{k})}$ algorithm along with a lower bound of $f(k,n) \cdot m^{o(\sqrt{k})}$ under ETH. We now briefly compare our results & techniques to those of Eiben et al. [12]:

- Algorithmic result: The key idea of our algorithm lies in using the well-known separator theorem of Voronoi diagrams by Marx and Pilipczuk [26] in recursive way. The non-trivial part of the technique comes from the efficient handling of partial solutions. At each step of the recursion we combine partial solutions from the lower level. As we don't know how the final solution will look like we might have cut a district several time into smaller pieces. We maintain possible solutions for all such pieces to compute the final district partitioning, ensuring the fairness criteria.
- Lower bound: Our reduction is similar to that of [12] for the "vanilla" version of Gerrymandering, but reducing from the $(k \times k)$-GRID-TILING-$\geq$ version of the problem (instead of $(k \times k)$-GRID-TILING as was done by [12]) helps us to simplify some of the arguments. Further we able to generalize our reduction which works well for any ℓ_q^d where $1 \leq q \leq \infty$ and d is arbitrary. This reduction is from the d-dimensional $\geq$-CSP problem which has been recently used to show lower bounds for various problems in computational geometry [8, 9].

  Note that if we set $k = \ell$ and $\texttt{margin}_{\texttt{low}} = 1 = \texttt{margin}_{\texttt{up}}$ in the reduction, it implies that our desired candidate has to win each ballot box by exactly one vote. For the other two parameters, we only need to use the naive bounds $1 \leq \texttt{lower} \leq \texttt{upper} \leq n$.

2 PRELIMINARIES & NOTATION

We mostly follow the denotions and notations from [7]. Given a set of voters $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ and a set of candidates C, the preference ranking of voter v_i is a total order (i.e., transitive, complete, reflexive, and antisymmetric relation) $\prec_i$ over C. To represent that the voter v_i prefers candidate c_1 over c_2 we write $c_1 \prec_i c_2$. Let $\pi(C)$ be the set of preference rankings for the set of candidate C. For the sake of completeness we assume $\pi(C)$ also contains a null string σ_0. We denote the preference rankings of n voters by $\sigma^n = (\sigma_1, \sigma_2, \ldots \sigma_n)$. When k out of n voters participate in a voting we write it as σ^k which has $n-k$ null strings corresponds to the voters not participating. A voting rule is a function $\texttt{r}\colon (\pi(C))^n \to C$. An election $\mathcal{E}$ is comprised of a set of voters $\mathcal{V}$, set of candidates C, a preferential ranking $\prec_i$ and a voting rule $\texttt{r}$. We consider elections where voting rule only considers the number of voters in each preferential ranking. We define a function $\phi_{\texttt{r}}\colon C \times (\pi(C))^n \to \mathbb{Z}^+ \cup \{0\}$ called vote aggregator. If c_i is a *winner* in an election $\mathcal{E}$ within a set $S = \{v'_1, v'_2, \ldots v'_p\}$ of p candidates then $\phi_{\texttt{r}}(c_i, \sigma^n) > \phi_{\texttt{r}}(c_j, \sigma^n)$ for every $c_j (j \neq i) \in S$. Depending on the voting rule the function $\phi_{\texttt{r}}$ can be defined in many ways such as Borda scoring rule. When $\texttt{r}$ is understood form the context we use ϕ instead of $\phi_{\texttt{r}}$ in remaining part of the paper.

All vectors considered in this paper have length d. If $\mathbf{a}$ is a vector then for each $i \in [d]$ its i^{th}-coordinate is denoted by $\mathbf{a}[i]$. Addition and subtraction of vectors is denoted by $\oplus$ and $\ominus$ respectively. The unit i^{th} vector is denoted by $\mathbf{e}_i$ and has $\mathbf{e}_i[i] = 1$ and $\mathbf{e}_i[j] = 0$ for each $j \neq i$. The set $\{1, 2, \ldots, n\}$ is denoted by $[n]$.

3 ALGORITHM FOR FAIR GERRYMANDERING IN $(\mathbb{R}^2, \ell_2)$

Our algorithm works as follows: Consider the Voronoi diagram of S. A Voronoi diagram in the plane can be viewed as a 3-regular planar graph with k-faces and $O(k)$ many vertices each of which coresponds to a corner point of a voronoi cell. It is known that there exists a polygon Γ for any 3-regular planar graph, such that it goes alternately through $O(k)$ vertices and faces and there are at most $\frac{2}{3}k$ faces strictly inside Γ and at most $\frac{2}{3}k$ faces strictly outside Γ [27]. Our algorithm guesses all possible variants of Γ and for each Γ it recurse into smaller subproblems defined inside and outside of Γ. If the size of the subproblem at a recursive step reaches below certain threshold, algorithm exhaustively searches for a solution. The algorithm returns YES if all the "small" subproblems of size below the threshold gets satisfied and computes a solution from the recursion tree. If one of the "small" subproblems doesn't satisfy its conditions the algorithm returns NO. The depth of the recursion tree is $O(\log k)$. More formally, we restate our main algorithmic result as follows.

THEOREM 1. *If C is the set of candidates in an election, n be the number of voters and m be the possible ballot box locations in the plane, then* FAIR-GERRYMANDERING-$(\mathbb{R}^2, \ell_2)$ *is solvable in time* $(m+n)^{O(\sqrt{k})} \cdot |C|^{(\texttt{upper}+\texttt{lower}+\texttt{margin}_{\texttt{up}}+\texttt{margin}_{\texttt{low}})}$ *where k is the number of ballot boxes for the election.*

Recall that the "vanilla" version, i.e., the GERRYMANDERING-$(\mathcal{X}, \rho)$ problem, is a special case of the FAIR-GERRYMANDERING-$(\mathcal{X}, \rho)$ problem with the parameters $\texttt{margin}_{\texttt{low}} = 1$, $\texttt{margin}_{\texttt{up}} = n$, $\texttt{lower} = 0$ and $\texttt{upper} = n$. Our algorithm also computes GERRYMANDERING-$(\mathcal{X}, \rho)$ in a little worse running time than the algorithm in [12].

Before describing the details of our algorithm, first we state the result by Marx and Pilipczuk about the small balanced separators in Voronoi diagram which we use for recursive partition.

LEMMA 3. *[27] Let S be a set of k points on the plane. Consider the Voronoi diagram of S, and the planar graph G associated with it. There is a polygon Γ which has length $O(\sqrt{k})$ and its vertices* *alternate between elements of S and vertices of G and each segment of Γ lies inside a face of G. At most $\frac{2}{3}k$ faces lie strictly inside Γ, and at most $\frac{2}{3}k$ strictly outside.*

Note that since Γ passes through $O(k)$ faces, the number of faces which lie non-strictly inside (outside) Γ is bounded by $\frac{3}{4}k$ when k is greater than γ, where γ is some constant. Note that, we refer this constant as the constant of Lemma 3 later in our proof.

Our algorithm takes $(C, \mathcal{V}, \mathcal{B}, \ell, k,$ "OUR", $\texttt{upper}, \texttt{lower}, \texttt{margin}_{\texttt{up}}, \texttt{margin}_{\texttt{low}})$ as an input and invokes a subroutine. The subroutine first finds an equilateral triangle T such that $\mathcal{V}$ and $\mathcal{B}$ are completely inside T. Then it mirrors each vertex of the triangle against the side which is non-adjacent to it. The resulting set is denoted by $\mathcal{T}$. Consider the Voronoi diagram of $\mathcal{B} \cup \mathcal{T}$. Outerfaces of the Voronoi diagram are precisely the three cells corresponding to $\mathcal{T}$ which are disjoint from $\mathcal{B}$ and $\mathcal{V}$. The subroutine constructs a polygon Δ in the following way: begin from a point of $\mathcal{T}$ and move to the next clockwise point of $\mathcal{T}$ by a sequence to two segments having their common point on the boundary of the corresponding outerfaces; repeat this process until the polygon returns to the starting point.

We define *states* which are instances of the problem. A state is defined as $R = (\mathcal{V}, \mathcal{B}, \ell, k, \mathcal{F}, v, \Delta, x, y, w, z)$ where

- $\mathcal{F} \subset \mathcal{B}$ is a subset of ballot boxes such that for any $f \in \mathcal{F}$, $v(\sigma, f) \in \mathbb{Z}^+ \cup \{0\}$ and $v(\sigma, f) \leq |\mathcal{V}|$
- For every $f \in \mathcal{F}$ define $\Sigma_f = \{\sigma \colon v(\sigma, f) \neq 0\}$.
- Δ is a collection of segments form the boundary of the region containing $\mathcal{V}$ and $\mathcal{B}$.
- Parameters $x, y, w, z \in \mathbb{Z}$ and $y \leq x$, $w \leq x$, $z \leq w$.
- each segment $\delta \in \Delta$ has exactly one endpoint in $\mathcal{F}$, denoted by $\delta_{\mathcal{F}}$.

Our algorithm recurse over states. The algorithm decides whether a given state is *valid* or not. It further returns a particular subset $\mathcal{S} \subset \mathcal{B} \setminus \mathcal{F}$ satisfying Definition 1.

Definition 1 (Valid State)**.** *A state is valid if there exists a subset $\mathcal{S} \subset \mathcal{B} \setminus \mathcal{F}$ such that $|S| + |\mathcal{F}| = k$ and where $i \in \{1, 2, \ldots n\}$, in the election with voters $\mathcal{V}$ and ballot boxes $\mathcal{S} \cup \mathcal{F}$, there exists $r \geq \ell$ ballot boxes $s_1, s_2, \ldots, s_r \in \mathcal{S} \setminus \mathcal{F}$ such that*

- *vote aggregator function evaluated on target candidate "OUR" i.e., $\phi($"OUR"$, \sigma^{\xi_i, n}) > \phi(c_j, \sigma^{\xi_i, n}) \forall c_j \in C \setminus \{$"OUR"$\}$ and $i \in \{1, 2, \ldots, r\}$ where $\xi_i = |\Sigma_{s_i}|$, and*
- *$\phi($"OUR"$, \sigma^{\xi_i}) - \phi(c_j, \sigma^{\xi_i}) \leq w$, $\forall c_j \in C \setminus \{$"OUR"$\}$ and $i \in \{1, 2, \ldots, r\}$ where $\xi_i = |\Sigma_{s_i}|$ and $\forall c_j \in C$, and*
- *$\phi($"OUR"$, \sigma^{\xi_i}) - \phi(c_j, \sigma^{\xi_i}) \geq z$, $\forall c_j \in C \setminus \{$"OUR"$\}$ and $i \in \{1, 2, \ldots, r\}$ where $\xi_i = |\Sigma_{s_i}|$ and $\forall c_j \in C$, and*
- *$\phi(c_j, \sigma^{\xi_i}) \leq x \; \forall c_j \in C$ and $\phi(c_j, \sigma^{\xi_i}) \geq y \; \forall c_j \in C$, and*
- *in the Voronoi diagram of $\mathcal{S} \cup \mathcal{F}$, each segment δ of Δ lies completely inside the cell corresponding to the ballot box $\delta_{\mathcal{F}}$*

The initial state is defined as $R_0 = (\mathcal{V}, \mathcal{B} \cup \mathcal{T}, \ell, k + 3, \mathcal{F}, v, \Delta, \texttt{upper}, \texttt{lower}, \texttt{margin}_{\texttt{up}}, \texttt{margin}_{\texttt{low}})$ where $v(f, \sigma) = 0$ for any $f \in \mathcal{T}$ and $\sigma \in \pi(C)$. Since for any $v \in \mathcal{V}$ any $b \in \mathcal{B}$ is closer to v than any $f \in \mathcal{T}$, R_0 is valid if and only if $(C, \mathcal{V}, \mathcal{B}, \ell, k,$ "OUR", $\texttt{upper}, \texttt{lower}, \texttt{margin}_{\texttt{up}}, \texttt{margin}_{\texttt{low}})$ is an YES-instance.

Recursive Step On a given state $R = (\mathcal{V}, \mathcal{B}, \ell, k, \mathcal{F}, v, \Delta, \texttt{upper}, \texttt{lower}, \texttt{margin}_{\texttt{up}}, \texttt{margin}_{\texttt{low}})$ if $k - |\mathcal{F}| \leq \gamma$, where γ is the constant mentioned after the statement of Lemma 3, our algorithm tries all possible $\mathcal{S} \subset \mathcal{B}$ of size $k - |\mathcal{F}|$ and for each set check the conditions of Definition 1. Since $\gamma = O(1)$, the whole procedure takes polynomial time. If $k = |\mathcal{F}| > \gamma$ the algorithm tries all possible polygons Γ of the form Lemma 3. Since a vertex of a Voronoi diagram constructed over any subset of $\mathcal{B}$ is equidistant from three elements from this subset and is uniquely determined by these three elements, there are at most $|\mathcal{B}|^3$ potential locations for a Voronoi diagram vertex. Therefore there are at most $|\mathcal{B}|^{4\alpha\sqrt{k}}$ variants for Γ. The algorithm considers only these Γ which do not go out of the region defined by Δ.

For each possible Γ, the algorithm splits the instance into two parts. Let Q be the set of ballot boxes on Γ. Let $\mathcal{V}_1$ ($\mathcal{V}_2$) be the set of voters inside (outside) of Γ, $\mathcal{B}_1$ ($\mathcal{B}_2$) be the set of possible ballot boxes inside (outside) of Γ. In other words $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Let $\mathcal{F}_i = Q \cup (\mathcal{F} \cap \mathcal{B}_i)$, for $i \in \{1, 2\}$ and Δ_i be the set of segments $\delta \in \Delta$ such that $\delta_{\mathcal{F}} \in \mathcal{F}_i$. It defines the preference counts v_1 and v_2 as follows:

- for ballot boxes $f \in \mathcal{F} \setminus Q$ the value $v_i(\sigma, f) = v(\sigma, f)$ where $i \in \{1, 2\}$, the value $x_i = x$, $y_i = y$ and $w_i = w$ where $i \in \{1, 2\}$
- for ballot boxes $f \in \mathcal{F} \cap Q$ it guesses a split $v(\sigma, f)$ into $v_1(\sigma, f)$, $v_2(\sigma, f)$; x into x_i's; y into y_i's; z into z_i's and w into w_i's such that $x \geq x_1 + x_2$, $y \leq y_1 + y_2$, $w \geq w_1 + w_2$ and $z \leq z_1 + z_2$
- for ballot boxes $f \in Q \setminus \mathcal{F}$ it additionally guesses the values of $v_1(\sigma, f)$, $v_2(\sigma, f)$, x_1, x_2, y_1, y_2 and w_1, w_2 such that $x \geq x_1 + x_2$, $y \leq y_1 + y_2$, $w \geq w_1 + w_2$ and $z \leq z_1 + z_2$

Finally, the algorithm guesses how $k + |Q \setminus \mathcal{F}|$ can be split into two parts k_1 and k_2 such that each of $k_i \leq \frac{3}{4}k$ where $i \in \{1, 2\}$. It also guesses how $\ell - w$ can be split between ℓ_1 and ℓ_2, where w is the number of boxes in $Q \setminus \mathcal{F}$ where target candidate "OUR" wins if voters from V_1 and V_2 vote according to v_1 and v_2. Next, it recurs on the state $R_1 = (\mathcal{V}_1, \mathcal{B}_1, \ell, k, \mathcal{F}_1, v_1, \Delta, x_1, y_1, w_1, z_1)$ and $R_2 = (\mathcal{V}_2, \mathcal{B}_2, \ell, k, \mathcal{F}_2, v_2, \Delta, x_2, y_2, w_2, z_2)$. If both R_1 and R_2 are reported to be valid states, it returns that R is valid and take $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup (Q \setminus \mathcal{F})$. Otherwise, it continues to the next choice of Γ. If for all choices of Γ it does not succeed, the algorithm returns that R is not valid.

Correctness We prove the correctness of the algorithm by applying induction on k. In the base case $k - |\mathcal{F}| \leq \gamma$ the algorithm tries all possible ways to select the ballot boxes and then checks the conditions. In the case $k - |\mathcal{F}| > \gamma$, first assume that the algorithm reports a given state $R = (\mathcal{V}, \mathcal{B}, \ell, k, \mathcal{F}, v, \Delta, x, y, w, z)$ as valid. So for some Γ, two splitting states $R_1 = (\mathcal{V}_1, \mathcal{B}_1, \ell, k, \mathcal{F}_1, v_1, \Delta, x_1, y_1, w_1, z_1)$ and $R_2 = (\mathcal{V}_2, \mathcal{B}_2, \ell, k, \mathcal{F}_2, v_2, \Delta, x_2, y_2, w_2, z_2)$ reported to be valid states by the algorithm. By the induction hypothesis these two states are valid since both k_1 and k_2 are at most $\frac{3}{4}k$. Consider sets $\mathcal{S}_1 \subset \mathcal{B}_1 \setminus \mathcal{F}_1$ and $\mathcal{S}_2 \subset \mathcal{B}_2 \setminus \mathcal{F}_2$ returned by the recursive calls. By induction $\mathcal{S}_1$ and $\mathcal{S}_2$ also satisfy Definition 1 on the validity of R_1 and R_2 , respectively. As before, denote $\mathcal{S}_1 \cup \mathcal{S}_2 \cup (Q \setminus \mathcal{F})$ by $\mathcal{S}$. We claim that $\mathcal{S}$ satisfies Definition 1 for R, and therefore R is valid. By construction of R_1 and R_2, $k = k_1 + k_2 - |Q \setminus \mathcal{F}|$, and since $\mathcal{S}_1, \mathcal{S}_2$, and $Q \setminus \mathcal{F}$ are pairwise disjoint, the size of S is indeed equal to $k - |\mathcal{F}|$. We use the Claim 1 from [12] to show that for voters in each part, the box where they vote is the same as the election with boxes $\mathcal{S}_i \cup \mathcal{F}_i$ and in the election with boxes $\mathcal{S} \cup \mathcal{F}$.

Claim 1. *[12] For $i \in \{1,2\}$ and any voter $v \in \mathcal{V}_i$, the closest box to v in $\mathcal{S} \cup \mathcal{F}$ is also the closest box in $\mathcal{S}_i \cup \mathcal{F}_i$.*

Next, we show that in the election with boxes $\mathcal{S} \cup \mathcal{F}$ the target candidate "OUR" wins in at least ℓ ballot boxes of $\mathcal{S}$ and that the votes in districts of $\mathcal{F}$ are distributed according to v.

By Claim 1, for each $i \in \{1,2\}$ and for each box in $\mathcal{S}_i$, the voters who votes in the election with ballot boxes $\mathcal{S} \cup \mathcal{F}$ and the election with ballot boxes $\mathcal{S}_i \cup \mathcal{F}_i$ are exactly same. This implies the target candidate "OUR" wins in exactly $\ell_1 + \ell_2$ boxes out of the boxes in S_1 and S_2. For each $i \in \{1,2\}$ and each ballot box $b \in \mathcal{F}_i \setminus Q$, the set of voters who votes in the ballot box b remains same in both the instance of the election, with boxes $\mathcal{S} \cup \mathcal{F}$ and with boxes $\mathcal{S}_i \cup \mathcal{F}_i$. Since $v(b,\sigma) = v_i(b,\sigma)$ for any $\sigma \in \pi(C)$, the vote distribution on $\mathcal{F}_i \setminus Q$ is as desired. This also implies $\phi(c, \sigma^{\xi}) \le \texttt{upper}$ for all $c \in C$ and $\phi(c, \sigma^{\xi}) \ge \texttt{lower}$, where $\xi = |\Sigma_b|$. If c^* wins the ballot box b then $\phi(c^*, \sigma^{\xi}) - \phi(c^*, \sigma^{\xi}) \le \texttt{margin}_{\texttt{up}}$ and $\phi(c^*, \sigma^{\xi}) - \phi(c^*, \sigma^{\xi}) \ge \texttt{margin}_{\texttt{low}}$ for all $c \in C$.

For each ballot box $q \in Q$ and each preference $\sigma \in \pi(C)$, $v(q,\sigma) = v_1(q,\sigma) + v_2(q,\sigma)$ by construction. Since voters from $\mathcal{V}_1$ and $\mathcal{V}_2$ who vote in ballot box q are exactly preserved, $v(q,\sigma)$ is indeed equal to the number of voters with preference list σ who vote in the ballot box q. This additionally implies $\phi(c, \sigma^{\zeta}) \le \texttt{upper}$ for all $c \in C$ and $\phi(c, \sigma^{\zeta}) \ge \texttt{lower}$ where $\zeta = |\Sigma_q|$. If c^* wins the ballot box q then $\phi(c^*, \sigma^{\zeta}) - \phi(c^*, \sigma^{\zeta}) \le \texttt{margin}_{\texttt{up}}$ and $\phi(c^*, \sigma^{\zeta}) - \phi(c^*, \sigma^{\zeta}) \ge \texttt{margin}_{\texttt{low}}$ for all $c \in C$. This finishes the proof that R is valid and S satisfies Definition 1.

Finally, we show that for each $\delta \in \Delta$, δ lies inside the cell of $\delta_{\mathcal{F}}$ in the Voronoi diagram of $\mathcal{S} \cup \mathcal{F}$. If $\delta_{\mathcal{F}} \in Q$, δ lies inside the cell of $\delta_{\mathcal{F}}$ in the Voronoi diagram of $\mathcal{S}_i \cup \mathcal{F}_i$ for $i \in \{1,2\}$. This is because R_1 and R_2 are valid. As a consequence, no other point in $\mathcal{S} \cup \mathcal{F}$ is closer to each point on δ than $\delta_{\mathcal{F}}$. If $\delta_{\mathcal{F}} \notin Q$, δ is completely inside or outside of Γ, and the same argument as in Claim 1 shows that for any point on δ the closest ballot box is preserved, then it has to be $\delta_{\mathcal{F}}$, since δ is in Δ_i for some $i \in \{1,2\}$, and R_i is valid.

Conversely, assume that R is valid. We show that the algorithm correctly outputs YES. Consider a particular $\mathcal{S}$ from Definition 1. By Lemma 3 there exists a polygon Γ of length $O(\sqrt{k})$ for the Voronoi diagram of $\mathcal{S} \cup \mathcal{F}$ that is a balanced separator for $\mathcal{S} \cup \mathcal{F}$. Since the algorithm tries all polygons of this form, it eventually find Γ as well, or report that R is valid earlier. When the algorithm considers the polygon Γ, in one of the instances the algorithm guesses the correct values of k_1, k_2, ℓ_1, ℓ_2, v_1, v_2, x_1, x_2, y_1, y_2, z_1, z_2 and w_1, w_2 on the ballot boxes of $Q \setminus \mathcal{F}$, according to the elections on $\mathcal{S} \cup \mathcal{F}$. The new states R_1 and R_2 are valid since the elections on $\mathcal{S} \cap \mathcal{F}$ exactly induce conditions on R_i. There exists a valid selection of ballot boxes $\mathcal{S}_i = (\mathcal{S} \cup \mathcal{F}) \cap \mathcal{B}_i \setminus Q$, for each $i \in \{1,2\}$. Therefore the recursive call finds R_i valid by induction. That finishes the proof.

Running Time The running time analysis of our algorithm is similar to the running time analysis of the algorithm in [12]. For the sake of completeness we included the proof. Let $T(k)$ be the worst-case running time of our algorithm where k be the size of the solution that the algorithm received as a parameter of the input state. In the case $\gamma \le k$, the algorithm does an exhaustive search in polynomial time. That implies for each of the candidates the parameters $\texttt{upper}$, $\texttt{lower}$, $\texttt{margin}_{\texttt{up}}$ and $\texttt{margin}_{\texttt{low}}$ can take at most the maximum of its value and we have to check instances for all such possible combinations of the parameters. So $T(k) = (m+n)^{\gamma+\texttt{upper}+\texttt{lower}+\texttt{margin}_{\texttt{up}}+\texttt{margin}_{\texttt{low}}}$. If $k > \gamma$, we try at most $m^{4\alpha\sqrt{k}}$ polygons Γ. For each of the polygons, we spent at most $k\ell n^{2\alpha\sqrt{k}+\texttt{upper}+\texttt{lower}+\texttt{margin}_{\texttt{up}}+\texttt{margin}_{\texttt{low}}}$ time to guess the parameters of two new instances $k_1, \ell_1, v_1, x_1, y_1, w_1$ and $k_2, \ell_2, v_2, x_2, v_2, w_2$. We run the algorithm on the two instances recursively and in both in both instances the value of the parameter is bounded by $\frac{3k}{4}$. So we obtain the following recurrence relation:

- $T(k) \le (m+n)^{\gamma} \cdot |C|^{(\texttt{upper}+\texttt{lower}+\texttt{margin}_{\texttt{up}}+\texttt{margin}_{\texttt{low}})}$ when $k \le \gamma$
- $T(k) \le (m+n)^{F\sqrt{k}} \cdot |C|^{(\texttt{upper}+\texttt{lower}+\texttt{margin}_{\texttt{up}}+\texttt{margin}_{\texttt{low}})}$. $T(\frac{3}{4}k)$ (for some constant F) when $k > \gamma$

Solving this recurrence relation, we obtain $T(k) \le (m+n)^{O(\sqrt{k})} \cdot |C|^{(\texttt{upper}+\texttt{lower}+\texttt{margin}_{\texttt{up}}+\texttt{margin}_{\texttt{low}})}$.

4 LOWER BOUND FOR FAIR GERRYMANDERING IN $(\mathbb{R}^d, \rho)$

The goal of this section is to prove Theorem 2 which is restated below:

THEOREM 2. *For any $d \ge 2$, under the Exponential Time Hypothesis (ETH), the* FAIR-GERRYMANDERING-$(\mathbb{R}^d, \rho)$ *problem cannot be solved in $f(k, n, \texttt{upper}, \texttt{lower}) \cdot m^{o(k^{1-1/d})}$ time where f is any computable function, n is the number of voters, and k is the number of the ballot boxes opened, m is the total number of possible locations of ballot boxes and ρ is either the ℓ_∞-metric or the ℓ_q-metric for some $q \ge 1$. This lower bound holds even when there are only 2 candidates, $k = \ell$ and $\texttt{margin}_{\texttt{low}} = 1 = \texttt{margin}_{\texttt{up}}$.*

We prove Theorem 2 via a reduction from the d-dimensional $\ge$-CSP problem for which a lower bound was shown in [28].

The rest of this section is organized as follows: Section 4.1 introduces the framework of$\ge$-CSP along with the relevant definitions. The construction describing the reduction from $\ge$-CSP to FAIR-GERRYMANDERING-$(\mathbb{R}^d, \ell_2)$ is outlined in Section 4.2. The two directions showing the correctness of the aforementioned reduction are described in Section B.1 and Section B.2 respectively. Finally, everything is tied together in Section 4.4 to complete the proof of Theorem 2.

4.1 Hardness of the d-dimensional $\ge$-CSP problem

In this section, which closely follows [28], we give a brief introduction and necessary definitions of the d-dimensional $\ge$-CSP framework before stating the formal theorem (Theorem 4) that will be used in our reduction.

Constraint Satisfaction Problems (CSPs) are a general way to represent several important problems in theoretical computer science. In this paper, we only deal with binary CSPs:

Definition 2 (binary CSPs). *An instance of a binary constraint satisfaction problem (CSP) is a triple $\mathcal{I} = (V, D, C)$ where V is a set of variables, D is a domain of values and C is a set of constraints. There are two types of constraints:*

- Unary constraints*: For some $v \in V$ there is a unary constraint $\langle v, R_v \rangle$ where $R_v \subseteq D$.*

- Binary constraints*: For some $u, v \in V$ there is a binary constraint $\langle (u,v), R_{u,v}\rangle$ where $R_{u,v} \subseteq D \times D$.*

Solving a given CSP instance $\mathcal{I} = (V, D, C)$ is to check whether there exists a satisfying assignment for it, i.e., a function $f : V \to D$ such that all the constraints are satisfied. For a binary CSP, a satisfying assignment f has the property that for each unary constraint $\langle v, R_v\rangle$ we have $f(v) \in R_v$ and for each binary constraint $\langle (u,v), R_{u,v}\rangle$ we have $\big(f(u), f(v)\big) \in R_{u,v}$.

The constraint graph of a CSP instance $\mathcal{I} = (V, D, C)$ is an undirected graph $G_{\mathcal{I}}$ whose vertex set is V and the adjacency relation is defined as follows: two vertices $u, v \in V$ are adjacent in $G_{\mathcal{I}}$ if there is a constraint in $\mathcal{I}$ which contains both u and v. The size $|\mathcal{I}|$ of a binary CSP $\mathcal{I} = (V, D, C)$ is the combined size of the variables, domain and the constraints. With appropriate preprocessing (for example, combining different constraints on the same variables) we can assume that $|\mathcal{I}| = \big(|V| + |D|\big)^{O(1)}$. Marx and Sidiropoulos [28] observed that considering binary CSPs whose primal graph is a subgraph of the d-dimensional grid is useful in showing lower bounds for geometric problems in d-dimensions.

Definition 3 (grids). *The d-dimensional grid $\mathrm{R}[\kappa, d]$ is an undirected graph with vertex set $[\kappa]^d$ and two vertices $\mathbf{a}, \mathbf{b}$ are adjacent if and only if $\sum_{i=1}^{d} \big|\mathbf{a}[i] - \mathbf{b}[i]\big| = 1$.*

Definition 4 (geometric CSPs). *A d-dimensional geometric $\geq$-CSP $\mathcal{I} = (V, D, C)$ is a binary CSP whose*

- *set of variables V is a subset of $\mathrm{R}[\kappa, d]$ for some $\kappa \geq 1$*
- *domain is $[N]^d$ for some integer $N \geq 1$*
- *constraint graph $G_{\mathcal{I}}$ is an induced subgraph of $\mathrm{R}[\kappa, d]$*
- *binary constraints are of the following type: if $\mathbf{a}, \mathbf{a}' \in V$ such that $\mathbf{a}' = \mathbf{a} \oplus \mathbf{e}_i$ for some $i \in [d]$ then there is a binary constraint $\big\langle (\mathbf{a}, \mathbf{a}'), R_{\mathbf{a},\mathbf{a}'}\big\rangle$ with $R_{\mathbf{a},\mathbf{a}'} = \Big\{\big((x_1, x_2, \ldots, x_d), (y_1, y_2, \ldots, y_d)\big) \mid x_i \geq y_i\Big\}$*

Since the constraint graph $G_{\mathcal{I}}$ is an induced subgraph of $\mathrm{R}[\kappa, d]$, it follows that the set of unary constraints of a d-dimensional geometric $\geq$-CSP is sufficient to completely define it. We now formally state the result of Marx and Sidiropoulos [28] which gives a lower bound on the complexity of checking whether a given d-dimensional geometric $\geq$-CSP has a satisfying assignment or not.

THEOREM 4. *Theorem 2.10 in [28] If for some fixed $d \geq 2$, there is an $f(|V|) \cdot |\mathcal{I}|^{o(|V|^{1-1/d})}$ time algorithm for solving a d-dimensional geometric $\geq$-CSP $\mathcal{I}$ for some computable function f, then the Exponential Time Hypothesis (ETH) fails.*

Remark 1. *The problem defined in [28] is actually d-dimensional geometric $\leq$-CSP which has $\leq$-constraints instead of the $\geq$-constraints in the d-dimensional geometric $\leq$-CSP problem. However, it is easy to see that d-dimensional geometric $\leq$-CSP and d-dimensional geometric $\geq$-CSP are equivalent by the following operation: replace each unary constraint $(x_1, x_2, \ldots, x_{d-1}, x_d) \in R_{\mathbf{a}}$ by $(N+1-x_1, N+1-x_2, \ldots, N+1-x_{d-1}, N+1-x_d)$ for each $\mathbf{a} \in V$.*

4.2 Construction of the FAIR-GERRYMANDERING-$(\mathbb{R}^d, \ell_2)$ instance

Let $\mathcal{I} = (V, D, C)$ be an instance of d-dimensional geometric $\geq$-CSP. From this, we will now construct an instance $\mathcal{U} = \big(C = \{\text{“OUR"}, \text{“OTHER"}\}, \mathcal{V}, \mathcal{B}, k = |V|, \ell = |V|, \mathtt{margin}_{\mathtt{low}} = 1 = \mathtt{margin}_{\mathtt{up}}, \text{“OUR"}\big)$ of FAIR-GERRYMANDERING-$(\mathbb{R}^d, \ell_2)$ such that $\mathcal{I}$ is satisfiable if and only if $\mathcal{U}$ has a solution (see Figure 1 for an illustration).

Fix the following three values

$$\epsilon = \frac{1}{4}; \quad D = 2d \cdot N^2; \quad C = 2D + (N-1) \tag{1}$$

$$\text{We assume } d, N \geq 2 \text{ and so Equation 1} \Rightarrow D \geq 8N \tag{2}$$

4.2.1 Adding internal vertices: For each $\mathbf{a} \in V$ we define a set of points denoted by INTERNAL$_1(\mathbf{a})$ as follows:

- The $\mathtt{temp\text{-}origin}(\mathbf{a})$ is $1^d = (1, 1, \ldots, 1)$.
- For each $\mathbf{x} \in R_{\mathbf{a}} \subseteq [N]^d$ we add a point located at $\mathbf{x} = \mathtt{temp\text{-}origin}(\mathbf{a}) \oplus \big(\mathbf{x} \ominus 1^d\big)$.

We now perform the following three operations (in order) to obtain our final set of points $\mathcal{P}$:

4.2.2 Mirroring: For each $i \in [\kappa]$ we define the function $\mathtt{flip}_i : [N] \to [N]$ as follows: for each $q \in [N]$

$$\mathtt{flip}_i(q) = \begin{cases} N+1-q & \text{if } i \text{ is even} \\ q & \text{if } i \text{ is odd} \end{cases} \tag{3}$$

Observation 1. *Note that $\forall\ i \in [\kappa]$ and $\forall\ q \in [N]$ we have $\mathtt{flip}_i\big(\mathtt{flip}_i(q)\big) = q$.*

For each $\mathbf{a} \in V$, we make “mirroring" changes to all the points of INTERNAL$_1(\mathbf{a})$ as follows: If $\mathbf{x} \in$ INTERNAL$_1(\mathbf{a})$ then we replace it with $\mathbf{y}$ such that $\mathbf{y}[i] = \mathtt{flip}_{\mathbf{a}[i]}\big(\mathbf{x}[i]\big)$ for each $i \in [d]$. We call this set of points as INTERNAL$_2(\mathbf{a})$.

4.2.3 Translation: We now fix the location of the origin of each grids by translation as follows: for each $\mathbf{a} \in V$ set

$$\begin{aligned} \mathtt{orig}(\mathbf{a}) &= \mathtt{temp\text{-}origin}(\mathbf{a}) \oplus C \cdot \big(\mathbf{a} \ominus 1^d\big) \\ &= 1^d \oplus C \cdot \big(\mathbf{a} \ominus 1^d\big) \end{aligned} \tag{4}$$

Note that this also shifts all points of INTERNAL$_2(\mathbf{a})$ accordingly: each point $\mathbf{y} \in$ INTERNAL$_2(\mathbf{a})$ is shifted to the point $\mathtt{orig}(\mathbf{a}) \oplus \big(\mathbf{y} \ominus 1^d\big) = C \cdot \big(\mathbf{a} \ominus 1^d\big) \oplus \mathbf{y}$. We denote this new set of points by INTERNAL$_3(\mathbf{a})$.

We are now ready to add the voters. For this, we start by fixing a bijective function $\phi : V \to \big[|V|\big]$.

4.2.4 Adding “OUR" voters at origin points. For each $\mathbf{a} \in V$, we define

$$\text{INTERNAL}_4(\mathbf{a}) = \text{INTERNAL}_3(\mathbf{a}) \bigcup \{\mathtt{orig}(\mathbf{a})\} \tag{5}$$

For each $\mathbf{a} \in V$ we place $1 + 2d \cdot (2d+1)^{\phi(\mathbf{a})}$ “OUR" voters at the point $\mathtt{origin}(a)$.

4.2.5 Adding “OTHER" voters at border points: For each $\mathbf{a} \in V$, we define a set of “border" points by adding points corresponding to the adjacencies in $G_{\mathcal{I}}$. Since $G_{\mathcal{I}}$ is an induced subgraph of the d-dimensional grid $\mathrm{R}[\kappa, d]$, an edge $\mathbf{b}$-$\mathbf{b}'$ in $G_{\mathcal{I}}$ is of the following form: there exists $j \in [d]$ such that $\mathbf{b}' = \mathbf{b} \oplus \mathbf{e}_j$.

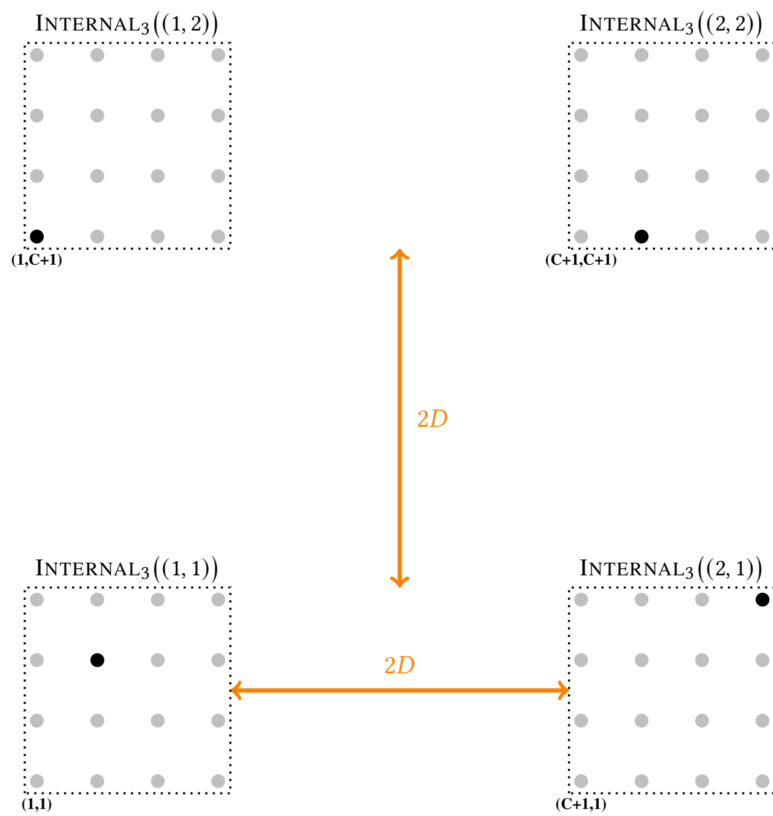

Figure 1: An illustration of the construction (Section 4.2) of the GERRYMANDERING-$(\mathbb{R}^d, \ell_2)$ instance when $\kappa = 2 = d, N = 4$ and $V = \mathbf{R}[\kappa, 2] = \{(1,1), (1,2), (2,1), (2,2)\}$. For each $a \in V$, we first add (Section 4.2.1) the point $(x, y) \in [4] \times [4]$ to INTERNAL$_1(a)$ if $(x, y) \in R_a$.
The second step is the mirroring (Section 4.2.2) to obtain the set of points INTERNAL$_2(a)$ for each $a \in V$. For example, the x-coordinate of every point in INTERNAL$_1\big((2,1)\big)$ is flipped while the y-coordinate stays the same: the black vertex shows how if $(1,4) \in$ INTERNAL$_1\big((2,1)\big)$ then it gets flipped to $(4,4)$.
The third and final step is the translation (Section 4.2.3) which moves the $[4] \times [4]$ grids corresponding to the different $a \in V$ according to Equation 4.
Note that the gray points show all possible locations of the $[4] \times [4]$ grid for the points of INTERNAL$_3$ each $a \in V$. The actual points are shown in black color: for each of the four grids we show one point. For $a = (1,1)$ we show the location of the point which gets placed due to $(2,3) \in R_{(1,1)}$. For $a = (2,1)$ we show the location of the point which gets placed due to $(1,4) \in R_{(2,1)}$. For $a = (1,2)$ we show the location of the point which gets placed due to $(1,4) \in R_{(1,2)}$. For $a = (2,2)$ we show the location of the point which gets placed due to $(3,4) \in R_{(2,2)}$.

For each $a \in V$ and each $i \in [d]$, we have two cases depending on the parity of $a[i]$:

(1) $\underline{a[i] \text{ is odd}}$:
 (i) If a and $(a \oplus \mathbf{e}_i)$ form an edge in $G_{\mathcal{I}}$ then introduce $(2d+1)^{\phi(a)}$ "OTHER" voters at the point $\mathtt{mid}_a^{+i} := \mathtt{orig}(a) \oplus \mathbf{e}_i \cdot \big((N-1) + (D-\epsilon)\big)$. See Figure 2 (Appendix A) for an illustration.
 (ii) Otherwise if a and $(a \oplus \mathbf{e}_i)$ do not form an edge in $G_{\mathcal{I}}$ then introduce $(2d+1)^{\phi(a)}$ "OTHER" voters at the point $\mathtt{mid}_a^{+i} := \mathtt{orig}(a) \oplus \mathbf{e}_i \cdot \big((N-1) + (D-2N)\big)$. See Figure 3 (Appendix A) for an illustration.
 (iii) If a and $(a \ominus \mathbf{e}_i)$ form an edge in $G_{\mathcal{I}}$ then introduce $(2d+1)^{\phi(a)}$ "OTHER" voters at the point $\mathtt{mid}_a^{-i} := \mathtt{orig}(a) \ominus \mathbf{e}_i \cdot (D - \epsilon)$.
 (iv) Otherwise if a and $(a \ominus \mathbf{e}_i)$ do not form an edge in $G_{\mathcal{I}}$ then introduce $(2d+1)^{\phi(a)}$ "OTHER" voters at the point $\mathtt{mid}_a^{-i} := \mathtt{orig}(a) \ominus \mathbf{e}_i \cdot (D - 2N)$.

(2) $\underline{a[i] \text{ is even}}$:
 (i) If a and $(a \oplus \mathbf{e}_i)$ form an edge in $G_{\mathcal{I}}$ then introduce $(2d+1)^{\phi(a)}$ "OTHER" voters at the point $\mathtt{mid}_a^{+i} := \mathtt{orig}(a) \oplus \mathbf{e}_i \cdot \big((N-1) + (D+\epsilon-N)\big)$.
 (ii) Otherwise if a and $(a \oplus \mathbf{e}_i)$ do not form an edge in $G_{\mathcal{I}}$ then introduce $(2d+1)^{\phi(a)}$ "OTHER" voters at the point

$$\mathtt{mid}_{\mathtt{a}}^{+i} := \mathtt{orig}(\mathtt{a}) \oplus \mathbf{e}_i \cdot \big((N-1)+(D-2N)\big).$$

(iii) If a and $(\mathtt{a} \ominus \mathbf{e}_i)$ form an edge in $G_{\mathcal{I}}$ then introduce $(2d+1)^{\phi(\mathtt{a})}$ "OTHER" voters at the point $\mathtt{mid}_{\mathtt{a}}^{-i} := \mathtt{orig}(\mathtt{a}) \ominus \mathbf{e}_i \cdot (D+\epsilon-N)$.

(iv) Otherwise if a and $(\mathtt{a} \ominus \mathbf{e}_i)$ do not form an edge in $G_{\mathcal{I}}$ then introduce $(2d+1)^{\phi(\mathtt{a})}$ "OTHER" voters at the point $\mathtt{mid}_{\mathtt{a}}^{-i} := \mathtt{orig}(\mathtt{a}) \ominus \mathbf{e}_i \cdot (D-2N)$.

For each $\mathtt{a} \in V$, we define

$$\textsc{Border}(\mathtt{a}) = \bigcup_{i=1}^{d} \left\{ \mathtt{mid}_{\mathtt{a}}^{+i}, \mathtt{mid}_{\mathtt{a}}^{-i} \right\} \tag{6}$$

Hence, it follows that $\big|\textsc{Border}(\mathtt{a})\big| = 2d$ for each $\mathtt{a} \in V$.

Observation 2. *For each* $\mathtt{a} \in V$ *we have*

- *There are exactly* $1 + 2d \cdot (2d+1)^{\phi(\mathtt{a})}$ *"OUR" voters at the point* $\mathtt{orig}(a)$
- *There are exactly* $2d \cdot (2d+1)^{\phi(\mathtt{a})}$ *"OTHER" voters combined at the points in* $\textsc{Border}(a)$

Hence the total number of voters is given by

$$\begin{aligned} n := & \sum_{\mathtt{a} \in V} \left(1 + 4d \cdot (2d+1)^{\phi(a)}\right) \\ & \leq |V| + 4d \cdot |V| \cdot (2d+1)^{|V|} = \tau(|V|, d) \end{aligned} \tag{7}$$

for some function τ where we have used the fact that $\phi : V \to \big[|V|\big]$.

The set of possible locations of ballot boxes is given by $\mathcal{B} = \bigcup_{\mathtt{a} \in V} \textsc{Internal}_3(\mathtt{a})$. Hence, from Section 4.2.1-Section 4.2.3 we have that

$$m := |\mathcal{B}| \leq C \tag{8}$$

We set the total number of ballot boxes to be allocated and won to both be equal to $|V|$, i.e.

$$k = |V| = \ell \tag{9}$$

Finally, for each $\mathtt{a} \in V$ we use the notation

$$\mathcal{P}(\mathtt{a}) := \textsc{Internal}_4(\mathtt{a}) \bigcup \textsc{Border}(\mathtt{a}) \tag{10}$$

This completes the construction of the instance $\mathcal{U} = \big(C = \{\text{"OUR"}, \text{"OTHER"}\}, \mathcal{V}, \mathcal{B}, k = |V|, \ell = |V|, \text{"OUR"}\big)$ of the Fair-Gerrymandering-$(\mathbb{R}^d, \ell_2)$ problem from an instance $\mathcal{I} = (V, D, C)$ of d-dimensional geometric $\geq$-CSP.

4.3 Fair-Gerrymandering-$(\mathbb{R}^d, \ell_2)$ has a solution $\Leftrightarrow$ d-dimensional geometric $\geq$-CSP is satisfiable

The proof of the forward and reverse directions is deferred to Section B.1 and Section B.2 respectively.

4.4 Proof of Theorem 2

Finally, we are ready to prove Theorem 2 which is restated below:

Theorem 2. *For any* $d \geq 2$*, under the Exponential Time Hypothesis (ETH), the* Fair-Gerrymandering-$(\mathbb{R}^d, \rho)$ *problem cannot be solved in* $f(k, n, \mathtt{upper}, \mathtt{lower}) \cdot m^{o(k^{1-1/d})}$ *time where* f *is any computable function,* n *is the number of voters, and* k *is the number of the ballot boxes opened,* m *is the total number of possible locations of ballot boxes and* ρ *is either the* ℓ_∞*-metric or the* ℓ_q*-metric for some* $q \geq 1$*. This lower bound holds even when there are only 2 candidates,* $k = \ell$ *and* $\mathtt{margin}_{low} = 1 = \mathtt{margin}_{up}$*.*

Proof. Fix any $d \geq 2$. Given an instance $\mathcal{I} = (V, D, C)$ of a d-dimensional $\geq$-CSP, we obtained a reduction (described in Section 4.2) to build an instance of Fair-Gerrymandering-$(\mathbb{R}^d, \ell_2)$ given by $\mathcal{U} = \big(C = \{\text{"OUR"}, \text{"OTHER"}\}, \mathcal{V}, \mathcal{B}, k = |V|, \ell = |V|, \text{"OUR"}\big)$. The correctness of this reduction follows from the two directions shown in Section B.1 and Section B.2: $\mathcal{I} = (V, D, C)$ has a satisfying assignment if and only if $\mathcal{U}$ has a solution.

Theorem 4 states that assuming the Exponential Time Hypothesis (ETH) there is no computable function f such that instances $\mathcal{I} = (V, D, C)$ of d-dimensional geometric $\geq$-CSP can be solved in $f(|V|) \cdot |\mathcal{I}|^{o(|V|^{1-1/d})}$ time. The reduction from Section 4.2 converts an instance of $\mathcal{I} = (V, D, C)$ of d-dimensional geometric $\geq$-CSP in $|\mathcal{I}|^{O(1)}$ time into an equivalent instance $\mathcal{U} = \big(C = \{\text{"OUR"}, \text{"OTHER"}\}, \mathcal{V}, \mathcal{B}, k = |V|, \ell = |V|, \text{"OUR"}\big)$ of Fair-Gerrymandering-$(\mathbb{R}^d, \ell_2)$ such that

- $k = |V|$ (from Equation 9)
- $n = |C| = \tau(|V|, d)$ for some function τ (from Equation 7)
- $m = |\mathcal{B}| \leq |C| \leq |I|$ (from Equation 8)

Hence, it follows that assuming the Exponential Time Hypothesis (ETH) there is no computable function f such that the Fair-Gerrymandering-$(\mathbb{R}^d, \ell_2)$ problem can be solved in $f(k, n) \cdot m^{o(k^{1-1/d})}$ time where n is the number of voters, m is the total number of possible ballot box locations and k is the size of the actual ballot boxes which are opened. Note that our reduction (similar to that of [12]) has only two candidates: "OUR" and "OTHER". Our lower bound extends to the ℓ_∞-metric with exactly the same construction: in fact some of the proofs are simpler for ℓ_∞ as compared to ℓ_2. The only minor change needed to make the lower bound work for the ℓ_q-metric (for $q \geq 1$) is to change the value of D in Equation 1 to $2dN^q$ instead of $2dN^2$. ◻

Remark 2. *Our lower bound is stated for the case when* $k = \ell$*, but can easily be made to also work when* $\ell < k$ *by placing some dummy voters far away.*

5 CONCLUDING REMARKS

In this paper, we have studied the Gerrymandering problem in the presence of some newly introduced natural fairness conditions from algorithm and computational hardness point of view. Our algorithm is almost near-optimal and our hardness framework which works well for any arbitrary dimension can be of independent interest in proving ETH based hardness for other various "nearest-neighbor" type of problems. We hope this will lead to more algorithmic work on fairness in elections and more generally in computational social choice.

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A MISSING FIGURES

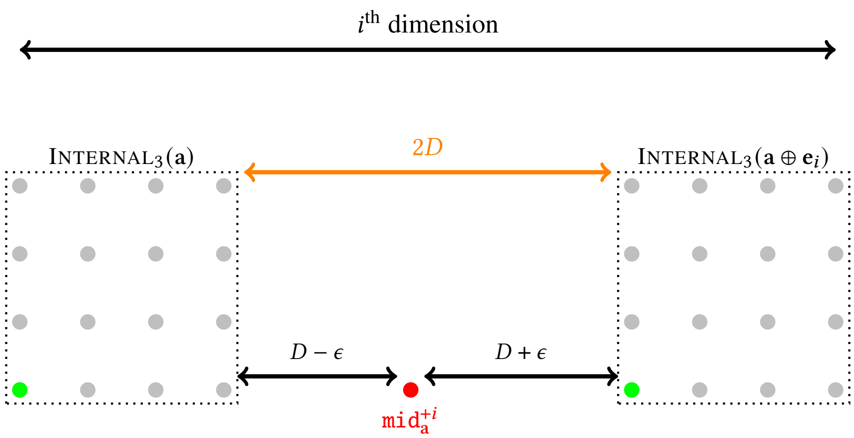

Figure 2: This figure illustrates how the location of the point $\mathtt{mid}_{\mathrm{a}}^{+i}$ is decided when $\mathrm{a}[i]$ is odd and the two vertices $\mathrm{a}, \mathrm{a} \oplus \mathrm{e}_i$ *form* an edge in $G_{\mathcal{I}}$. This corresponds to Case 1(i) from Section 4.2.5. The left hand side shows the points of INTERNAL(a) with the point $\mathtt{orig}(\mathrm{a})$ highlighted in green color. The right hand side shows the points of INTERNAL$(\mathrm{a} \oplus \mathrm{e}_i)$ with the point $\mathtt{orig}(\mathrm{a} \oplus \mathrm{e}_i)$ highlighted in green color. The point $\mathtt{mid}_{\mathrm{a}}^{+i}$ is then placed at $\mathtt{orig}(\mathrm{a}) \oplus \mathrm{e}_i \cdot \big((N-1) + (D-\epsilon)\big)$.

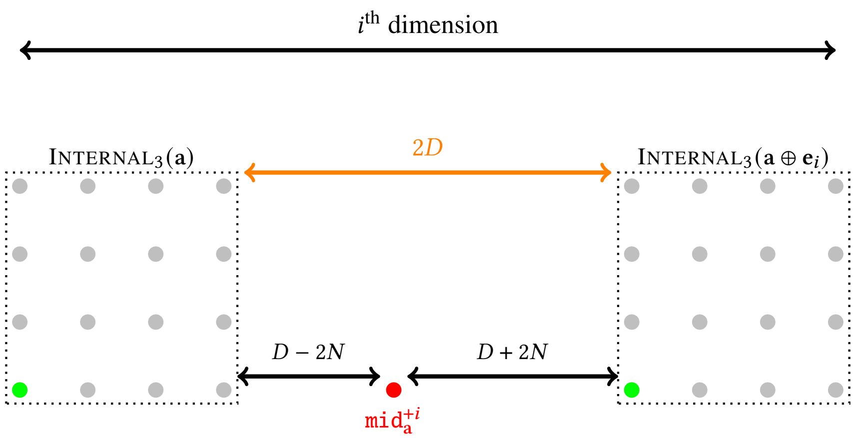

Figure 3: This figure illustrates how the location of the point $\mathtt{mid}_{\mathrm{a}}^{+i}$ is decided when $\mathrm{a}[i]$ is odd and the two vertices $\mathrm{a}, \mathrm{a} \oplus \mathrm{e}_i$ *do not* form an edge in $G_{\mathcal{I}}$. This corresponds to Case 1(ii) from Section 4.2.5. The left hand side shows the points of INTERNAL(a) with the point $\mathtt{orig}(\mathrm{a})$ highlighted in green color. The right hand side shows the points of INTERNAL$(\mathrm{a} \oplus \mathrm{e}_i)$ with the point $\mathtt{orig}(\mathrm{a} \oplus \mathrm{e}_i)$ highlighted in green color. The point $\mathtt{mid}_{\mathrm{a}}^{+i}$ is then placed at $\mathtt{orig}(\mathrm{a}) \oplus \mathrm{e}_i \cdot \big((N-1) + (D-2N)\big)$.

B MISSING PROOFS FROM SECTION 4

B.1 FAIR-GERRYMANDERING-$(\mathbb{R}^d, \ell_2)$ has a solution $\Rightarrow$ d-dimensional geometric $\geq$-CSP is satisfiable

Suppose that the instance $\mathcal{U} = \big(C = \{\text{"OUR"}, \text{"OTHER"}\}, \mathcal{V}, \mathcal{B}, k = |V|, \ell = |V|, \text{"OUR"}\big)$ has a solution. In this section we show that the d-dimensional geometric $\geq$-CSP $\mathcal{I} = (V, D, C)$ has a satisfying assignment $f : V \to D$.

Let $\mathcal{P}'$ be the set of $|V|$ ballot box locations opened in the solution to the instance $\mathcal{U}$. Since the set of ballot boxes $\mathcal{P}'$ provides a solution to the instance $\mathcal{U}$ with $k = |V| = \ell$ it follows that "OUR" wins each of the ballot boxes in $\mathcal{P}'$. The next claim shows that the margin of victory for "OUR" over "OTHER" in each ballot box of $\mathcal{P}'$ is exactly one vote.

Claim 2. *Each of the ballot boxes in $\mathcal{P}'$ is won by "OUR" by exactly one vote.*

PROOF. Since the instance $\mathcal{U}$ of FAIR-GERRYMANDERING-$(\mathbb{R}^d, \ell_2)$ has a solution it follows that each of the $k = |V|$ ballot boxes is won by "OUR" candidate by at least one vote. Since $\texttt{margin}_{\texttt{low}} = 1 = \texttt{margin}_{\texttt{up}}$, it follows that each of the $k = |V|$ ballot boxes is won by "OUR" candidate by exactly one vote. ◻

Claim 3. *If* $\mathbf{a}, \mathbf{a}' \in V$ *are such that* $\phi(\mathbf{a}') > \phi(\mathbf{a})$ *then the number of "OTHER" voters at any point in* BORDER$(\mathbf{a}')$ *is more than the sum of "OTHER" voters over all points in* BORDER$(\mathbf{a})$*, i.e.,* $(2d+1)^{\phi(\mathbf{a}')} > 2d \cdot (2d+1)^{\phi(\mathbf{a})}$.

PROOF. By Section 4.2.5, the number of "OTHER" voters at any point in BORDER$(\mathbf{a}')$ is $(2d+1)^{\phi(\mathbf{a}')}$ and the total number of "OTHER" voters over all points in BORDER(a) is $2d \cdot (2d+1)^{\phi(\mathbf{a})}$. Since $\phi : V \to \big[|V|\big]$ is a bijection and $\phi(\mathbf{a}') > \phi(\mathbf{a})$ we have $(2d+1)^{\phi(\mathbf{a}')} \geq (2d+1)^{\phi(\mathbf{a})+1} = (2d+1)\cdot(2d+1)^{\phi(\mathbf{a})} > 2d\cdot(2d+1)^{\phi(\mathbf{a})}$ since $(2d+1) > 2d$. ◻

Claim 4. *For each* $\mathbf{a} \in V$ *the number of "OTHER" voters at any point in* BORDER$(\mathbf{a})$ *is more than the sum of "OTHER" voters over all points from the set* $\bigcup_{\phi(\mathbf{x})<\phi(\mathbf{a})}$ BORDER$(\mathbf{x})$*, i.e.,* $(2d+1)^{\phi(\mathbf{a})} > \sum_{\mathbf{x}\,:\,\phi(\mathbf{x})<\phi(\mathbf{a})} 2d \cdot (2d+1)^{\phi(\mathbf{x})}$.

PROOF. By Section 4.2.5, the number of "OTHER" voters at any point in BORDER$(\mathbf{a})$ is $(2d+1)^{\phi(\mathbf{a})}$. By Section 4.2.5 and the fact that $\phi : V \to \big[|V|\big]$ is a bijection, it follows that the total number of "OTHER" voters over all points from the set $\bigcup_{\phi(\mathbf{x})<\phi(\mathbf{a})}$ BORDER$(\mathbf{x})$ is $\sum_{i=1}^{\phi(\mathbf{a})-1} 2d\cdot(2d+1)^i$. Hence, we have $\sum_{\mathbf{x}\,:\,\phi(\mathbf{x})<\phi(\mathbf{a})} 2d\cdot(2d+1)^{\phi(\mathbf{x})} = 2d\cdot\sum_{i=1}^{\phi(\mathbf{a})-1}(2d+1)^i = 2d\cdot(2d+1)\cdot\sum_{i=0}^{\phi(\mathbf{a})-2}(2d+1)^i = 2d\cdot(2d+1)\cdot\frac{(2d+1)^{\phi(\mathbf{a})-1}-1}{(2d+1)-1} = (2d+1)\cdot\big((2d+1)^{\phi(\mathbf{a})-1}-1\big) < (2d+1)^{\phi(\mathbf{a})}$. ◻

Now we show that any solution to $\mathcal{P}'$ opens exactly one ballot box from among the locations given by the points from INTERNAL$_3(\mathbf{a})$ for each $\mathbf{a} \in V$.

LEMMA 5. *For each* $\mathbf{a} \in V$*, we have* $\big|\mathcal{P}' \cap \text{INTERNAL}_3(\mathbf{a})\big| = 1$.

PROOF. Since $|\mathcal{P}'| = |V|$ and the set of possible locations of ballot boxes is given by $\mathcal{B} = \bigcup_{\mathbf{a}\in V}$ INTERNAL$_3(\mathbf{a})$, it follows that $\mathcal{P}' = \bigcup_{\mathbf{a}\in V}\big(\mathcal{P}' \cap \text{INTERNAL}_3(\mathbf{a})\big)$. Observe that to prove the lemma it is enough to show that $\big|\mathcal{P}' \cap \text{INTERNAL}_3(\mathbf{a})\big| \geq 1$ for each $\mathbf{a} \in V$. This is because then we would have $|V| = |\mathcal{P}'| = \sum_{\mathbf{a}\in V}\big|\mathcal{P}' \cap \text{INTERNAL}_3(\mathbf{a})\big| \geq |V|$ which implies that $\big|\mathcal{P}' \cap \text{INTERNAL}_3(\mathbf{a})\big| = 1$ for each $\mathbf{a} \in V$.

It remains to show that the existence of a variable $\mathbf{a}' \in V$ such that $\big|\mathcal{P}' \cap \text{INTERNAL}_3(\mathbf{a}')\big| = 0$ leads to a contradiction. In this case since $|\mathcal{P}'| = |V|$ there must exist a variable $\mathbf{b} \in V$ such that $|\mathcal{P}' \cap \text{INTERNAL}_3(\mathbf{b})| \geq 2$. Let $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r\} \subseteq \text{INTERNAL}_3(\mathbf{b})$ be the set of ballot boxes opened in $\mathcal{P}'$ for some $r \geq 2$. Without loss of generality, let $\mathbf{b}_1$ be the closest[1] ballot box to $\texttt{origin}(\mathbf{b})$ from the set $\{b_1, b_2, \ldots, b_r\}$. This means that the $1 + 2d\cdot(2d+1)^{\phi(\mathbf{b})}$ "OUR" voters located at $\texttt{origin}(\mathbf{b})$ vote at the ballot box located at $\mathbf{b}_1$. Since there are exactly $|V|$ locations where "OUR" voters are placed, viz. the set of points $\{\texttt{origin}(\mathbf{a}) \;:\; \mathbf{a} \in V\}$, and we need to win $|\mathcal{P}'| = |V|$ ballot boxes it follows that each ballot box is voted at by exactly one location having "OUR" voters. Therefore, the total number of votes for "OUR" at the ballot box $\mathbf{b}_1$ is exactly $1 + 2d\cdot(2d+1)^{\phi(\mathbf{b})}$. By Claim 2, it follows that

$$\text{The number of votes for "OTHER" at } \mathbf{b}_1 \text{ is exactly } 2d\cdot(2d+1)^{\phi(\mathbf{b})} \tag{11}$$

We now show that the "OTHER" voters from at least one of the points in BORDER$(\mathbf{b})$ do not vote at $\mathbf{b}_1$. Since $\mathbf{b}_1 \neq \mathbf{b}_2$ there exists $\ell \in [d]$ such that $\mathbf{b}_1[\ell] \neq \mathbf{b}_2[\ell]$. Without loss of generality, let $\mathbf{b}_2[\ell] > \mathbf{b}_1[\ell]$. Let $\mathbf{b}_1 = \texttt{orig}(\mathbf{b}) + \mathbf{t}_1$ and $\mathbf{b}_2 = \texttt{orig}(\mathbf{b}) + \mathbf{t}_2$. Since $\mathbf{b}_1, \mathbf{b}_2 \in \text{INTERNAL}_3(\mathbf{b})$ it follows that $(N-1) \geq \mathbf{t}_1[j], \mathbf{t}_2[j] \geq 0$ for each $j \in [d]$. Since $\mathbf{b}_2[\ell] > \mathbf{b}_1[\ell]$ we have $\big(\mathbf{t}_2[\ell] - \mathbf{t}_1[\ell]\big) \geq 1$.

Claim 5. $\texttt{dist}\big(\mathbf{b}_2, \texttt{mid}_{\mathbf{b}}^{+\ell}\big) < \texttt{dist}\big(\mathbf{b}_1, \texttt{mid}_{\mathbf{b}}^{+\ell}\big)$

PROOF. There are two cases to consider depending on whether or not $\mathbf{b}$ and $\mathbf{b} \oplus \mathbf{e}_\ell$ form an edge in $G_{\mathcal{I}}$:

(1) $\mathbf{b}$ and $\mathbf{b} \oplus \mathbf{e}_\ell$ form an edge in $G_{\mathcal{I}}$: We have two subcases depending on the parity of $\mathbf{b}[\ell]$

1(i) $\mathbf{b}[\ell]$ is odd: From Section 4.2.5, we have

$$
\begin{aligned}
&\texttt{dist}\big(\mathbf{b}_2, \texttt{mid}_{\mathbf{b}}^{+\ell}\big)^2 \\
&= \Big((N-1) + (D-\epsilon) - \mathbf{t}_2[\ell]\Big)^2 + \sum_{j=1, j\neq\ell}^{d} \mathbf{t}_2[j]^2 \\
&< \Big((N-1) + (D-\epsilon) - \mathbf{t}_2[\ell]\Big)^2 + dN^2 \qquad (\text{since } N > \mathbf{t}_2[j]\ \forall j \in [d]) \\
&< \Big((N-1) + (D-\epsilon) - \mathbf{t}_1[\ell]\Big)^2 \qquad (\text{from Equation 1, Equation 2 \& } \big(\mathbf{t}_2[\ell] - \mathbf{t}_1[\ell]\big) \geq 1) \\
&\leq \Big((N-1) + (D-\epsilon) - \mathbf{t}_1[\ell]\Big)^2 + \sum_{j=1, j\neq\ell}^{d} \mathbf{t}_1[j]^2 \\
&= \texttt{dist}\big(\mathbf{b}_1, \texttt{mid}_{\mathbf{b}}^{+\ell}\big)^2
\end{aligned}
$$

[1] If there are two or more ballot boxes which are closest to $\texttt{origin}(\mathbf{b})$ then we can perturb location of $\texttt{origin}(\mathbf{b})$ very slightly to ensure that there is exactly one ballot box from $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r\}$ which is closest to it

1(ii) $\underline{\mathbf{b}[\ell] \text{ is even}}$: From Section 4.2.5, we have

$$\begin{aligned}
&\mathtt{dist}(\mathbf{b}_2, \mathtt{mid}_{\mathbf{b}}^{+\ell})^2 \\
&= \left((D+\epsilon-1) - \mathbf{t}_2[\ell]\right)^2 + \sum_{j=1, j\neq\ell}^{d} \mathbf{t}_2[j]^2 \\
&< \left((D+\epsilon-1) - \mathbf{t}_2[\ell]\right)^2 + dN^2 \\
&\qquad\qquad (\text{since } N > \mathbf{t}_2[j] \text{ for each } j \in [d]) \\
&< \left((D+\epsilon-1) - \mathbf{t}_1[\ell]\right)^2 \\
&\qquad\qquad (\text{from Equation 1, Equation 2 and } (\mathbf{t}_2[\ell] - \mathbf{t}_1[\ell]) \geq 1) \\
&\leq \left((D+\epsilon-1) - \mathbf{t}_1[\ell]\right)^2 + \sum_{j=1, j\neq\ell}^{d} \mathbf{t}_1[j]^2 \\
&= \mathtt{dist}(\mathbf{b}_1, \mathtt{mid}_{\mathbf{b}}^{+\ell})^2
\end{aligned}$$

(2) $\underline{\mathbf{b} \text{ and } \mathbf{b} \oplus \mathbf{e}_\ell \text{ do not form an edge in } G_{\mathcal{I}}}$: We have two subcases depending on the parity of $\mathbf{b}[\ell]$

2(i) $\underline{\mathbf{b}[\ell] \text{ is odd}}$: From Section 4.2.5, we have

$$\begin{aligned}
&\mathtt{dist}(\mathbf{b}_2, \mathtt{mid}_{\mathbf{b}}^{+\ell})^2 \\
&= \left((D-N-1) - \mathbf{t}_2[\ell]\right)^2 + \Big(\sum_{j=1, j\neq\ell}^{d} \mathbf{t}_2[j]^2\Big) \\
&< \left((D-N-1) - \mathbf{t}_2[\ell]\right)^2 + dN^2 \\
&\qquad\qquad (\text{since } N > \mathbf{t}_2[j] \text{ for each } j \in [d]) \\
&< \left((D-N-1) - \mathbf{t}_1[\ell]\right)^2 \\
&\qquad\qquad (\text{from Equation 1, Equation 2 \& } (\mathbf{t}_2[\ell] - \mathbf{t}_1[\ell]) \geq 1) \\
&\leq \left((D-N-1) - \mathbf{t}_1[\ell]\right)^2 + \Big(\sum_{j=1, j\neq\ell}^{d} \mathbf{t}_1[j]^2\Big) \\
&= \mathtt{dist}(\mathbf{b}_1, \mathtt{mid}_{\mathbf{b}}^{+\ell})^2
\end{aligned}$$

2(ii) $\underline{\mathbf{b}[\ell] \text{ is even}}$: The argument for this case is exactly the same as in Case 2(i).

This concludes the proof of Claim 5. □

From Claim 5, there is at least one point in BORDER$(\mathbf{b})$ such that the $(2d+1)^{\phi(\mathbf{b})}$ "OTHER" voters located at this point do not vote at $\mathbf{b}_1$. We now calculate how many "OTHER" voters could have voted at the ballot box $\mathbf{b}_1$:

- From Claim 5, the "OTHER" voters from at most $(2d-1)$ of the points in BORDER$(\mathbf{b})$ vote at the ballot box $\mathbf{b}_1$.
- From Claim 3 and Equation 11, there is no point in $\bigcup_{\mathbf{b}' : \phi(\mathbf{b}') > \phi(\mathbf{b})}$ BORDER$(\mathbf{b}')$ such that the "OTHER" voters at this point can vote at the ballot box $\mathbf{b}_1$.
- From Claim 4, the total number of "OTHER" voters located at points in the set $\bigcup_{\mathbf{x} : \phi(\mathbf{x}) < \phi(\mathbf{b})}$ BORDER(x) is $< (2d+1)^{\phi(\mathbf{b})}$.

Since "OTHER" voters are located only at the points from the set $\bigcup_{\mathbf{a} \in V}$ BORDER$(\mathbf{a})$, we have that the maximum number of "OTHER" votes at the ballot box $\mathbf{b}_1$ is $< (2d-1) \cdot (2d+1)^{\phi(\mathbf{b})} + (2d+1)^{\phi(\mathbf{b})} = 2d \cdot (2d+1)^{\phi(\mathbf{b})}$. This contradicts Equation 11, and concludes the proof of Lemma 5. □

Since $|\mathcal{P}'| = |V|$, from Lemma 5 it follows that

$$\begin{aligned}
&\forall\, \mathbf{a} \in V,\ \exists\, \beta(\mathbf{a}) \in \text{INTERNAL}_3(\mathbf{a}) \\
&\text{such that } \left(\mathcal{P}' \cap \mathcal{P}(\mathbf{a})\right) = \{\beta(a)\}
\end{aligned} \tag{12}$$

The next claim gives a lower bound on the distance between internal points corresponding to different variables.

Claim 6. *Let* $\mathbf{a}, \mathbf{a}' \in V$ *such that* $\mathbf{a} \neq \mathbf{a}'$. *For any* $\mathbf{q} \in \text{INTERNAL}_4(\mathbf{a})$ *and any* $\mathbf{s} \in \text{INTERNAL}_4(\mathbf{a}')$ *we have* $\mathtt{dist}(\mathbf{q}, \mathbf{s}) \geq 2D$.

PROOF. Since $\mathbf{a} \neq \mathbf{a}'$ there exists some $j \in [d]$ such that $\mathbf{a}[j] \neq \mathbf{a}'[j]$. Let $\mathbf{q} = \mathtt{orig}(\mathbf{a}) \oplus \mathbf{q}'$ and $\mathbf{s} = \mathtt{orig}(\mathbf{a}') \oplus \mathbf{s}'$. Then it follows from Section 4.2.1, Equation 4 and Equation 5 that for each $i \in [d]$ we have $0 \leq \mathbf{q}'[i], \mathbf{s}'[i] \leq (N-1)$.

$$\begin{aligned}
&\mathtt{dist}(\mathbf{q}, \mathbf{s}) = \mathtt{dist}(\mathtt{orig}(\mathbf{a}) \oplus \mathbf{q}', \mathtt{orig}(\mathbf{a}') \oplus \mathbf{s}') \\
&\geq \left| C \cdot (\mathbf{a}[j] - \mathbf{a}'[j]) + (\mathbf{q}'[j] - \mathbf{s}'[j]) \right| \\
&\qquad\qquad (\text{only considering the } j^{\text{th}}\text{-coordinate}) \\
&\geq \left| C \cdot (\mathbf{a}[j] - \mathbf{a}'[j]) \right| - \left| (\mathbf{q}'[j] - \mathbf{s}'[j]) \right| \\
&\qquad\qquad (\text{by triangle inequality}) \\
&\geq C - (N-1) \\
&\qquad\qquad (\text{since } \mathbf{a}[j] \neq \mathbf{a}'[j] \text{ and } 0 \leq \mathbf{q}'[j], \mathbf{s}'[j] \leq (N-1)) \\
&= 2D \qquad\qquad (\text{from Equation 1})
\end{aligned}$$

□

LEMMA 6. *For each* $\mathbf{a} \in V$, *the exact set of voters who vote at the ballot box* $\beta(\mathbf{a})$ *are*

- *the "OUR" voters at location* $\mathtt{origin}(\mathbf{a})$, *and*
- *the "OTHER" voters at each of the 2d points in* BORDER$(\mathbf{a})$

PROOF. Fix $\mathbf{a} \in V$. We first show that the only point from $\bigcup_{\mathbf{b} \in V} \mathtt{origin}(\mathbf{b})$ that votes at $\beta(\mathbf{a})$ is $\mathtt{origin}(\mathbf{a})$. Since $\beta(\mathbf{a}) \in$ INTERNAL$_3(\mathbf{a})$, Section 4.2.1- Section 4.2.3 and Equation 1 together imply that $\mathtt{dist}(\mathtt{origin}(\mathbf{a}), \beta(\mathbf{a})) \leq \sqrt{dN^2} = \sqrt{D/2} \leq D$. If $\mathbf{a}' \neq \mathbf{a}$ then Claim 6 implies that $\mathtt{dist}(\mathtt{origin}(\mathbf{a}), \beta(\mathbf{a}')) \geq 2D$ since $\beta(\mathbf{a}') \in$ INTERNAL$_3(\mathbf{a}') \subseteq$ INTERNAL$_4(\mathbf{a}')$ and $\mathtt{origin}(\mathbf{a}) \in$ INTERNAL$_4(\mathbf{a})$. Hence, for each $\mathbf{x} \in V$ the set of "OUR" voters who vote at the ballot box $\beta(\mathbf{x})$ is exactly those located at $\mathtt{origin}(\mathbf{x})$. Observation 2 implies that the total number of "OUR" votes at the ballot box $\beta(\mathbf{a})$ is exactly $1 + 2d \cdot (2d+1)^{\phi(\mathbf{a})}$. Claim 2 implies that

$$\begin{aligned}
&\text{the number of "OTHER" votes at } \beta(\mathbf{a}) \\
&\text{is exactly } 2d \cdot (2d+1)^{\phi(\mathbf{a})}
\end{aligned} \tag{13}$$

By Claim 3, no point from BORDER$(\mathbf{a}')$ can vote at $\beta(\mathbf{a})$ if $\phi(\mathbf{a}') > \phi(\beta)$. We now claim that all points from BORDER$(\mathbf{a})$ must vote at $\beta(\mathbf{a})$: suppose to the contrary that at least one of the $2d$ points from BORDER$(\mathbf{a})$ does not vote at $\beta(\mathbf{a})$. Note that (Section 4.2.5) each of the $2d$ points in BORDER$(\mathbf{a})$ has exactly $(2d+1)^{\phi(\mathbf{a})}$ "OTHER" voters, i.e., we need at least $(2d+1)^{\phi(\mathbf{a})}$ "OTHER" voters from outside BORDER$(\mathbf{a})$ to vote at $\beta(\mathbf{a})$. However, Claim 4 implies that the sum of all "OTHER" voters from the set of points $\bigcup_{\mathbf{x} : \phi(\mathbf{x}) < \phi(\mathbf{a})}$ BORDER$(\mathbf{x})$ is $< (2d+1)^{\phi(\mathbf{a})}$ leading to a contradiction. Hence, each of the $2d$ points from BORDER$(\mathbf{a})$ votes at $\beta(\mathbf{a})$ which gives a total of $2d \cdot (2d+1)^{\phi(\mathbf{a})}$ "OTHER" votes at $\beta(\mathbf{a})$. By Equation 13, it follows that the exact set of "OTHER" voters who vote at $\beta(\mathbf{a})$ are those located at points from the set $\text{BORDER}(\mathbf{a})$. □

We now construct a satisfying assignment for the instance $\mathcal{I} = (V, D, C)$ of d-dimensional geometric $\geq$-CSP.

LEMMA 7. *The instance $\mathcal{I} = (V, D, C)$ of d-dimensional geometric $\geq$-CSP has a satisfying assignment.*

PROOF. For each $\mathbf{a} \in V$ let

$$\gamma(\mathbf{a}) = \beta(\mathbf{a}) \ominus C \cdot (\mathbf{a} - 1^d) = \beta(a) \ominus \mathtt{orig}(\mathbf{a}) \oplus 1^d \tag{14}$$

where have used Equation 4. We claim that the function $f : V \longrightarrow D$ given by $f(a)[i] = \mathtt{flip}_{\mathbf{a}[i]}(\gamma(\mathbf{a})[i])$ for each $i \in [d]$ is a satisfying assignment for the instance $\mathcal{I} = (V, D, C)$ of d-dimensional geometric $\geq$-CSP.

First we show that f satisfies each unary constraint. For $\mathbf{a} \in V$ we have

$\beta(\mathbf{a}) \in \text{INTERNAL}_3(\mathbf{a})$ (from Equation 12)
$\Rightarrow \gamma(\mathbf{a}) \in \text{INTERNAL}_2(\mathbf{a})$ (from Section 4.2.3 and Equation 14)
$\Rightarrow f(\mathbf{a}) \in \text{INTERNAL}_1(\mathbf{a})$ (from Section 4.2.2 & Observation 1)
$\Rightarrow f(\mathbf{a}) \in R_{\mathbf{a}}$ (from Section 4.2.1)

Next we show that f satisfies each binary constraint. By Definition 4, every binary constraint in C has the following structure: there exists a variable $\mathbf{b} \in V$ and an index $i \in [d]$ such that the binary constraint is $\langle(\mathbf{b}, \mathbf{b} \oplus \mathbf{e}_i), R_{\mathbf{b},\mathbf{b}\oplus\mathbf{e}_i}\rangle$ where $R_{\mathbf{b},\mathbf{b}\oplus\mathbf{e}_i} = \Big\{\big((x_1, x_2, \ldots, x_d), (y_1, y_2, \ldots, y_d)\big) \subseteq R_{\mathbf{b}} \times R_{\mathbf{b}\oplus\mathbf{e}_i} \mid x_i \geq y_i\Big\}$. If $f(\mathbf{b})[i] \geq f(\mathbf{b}\oplus\mathbf{e}_i)[i]$ then the binary constraint $\langle(\mathbf{b}, \mathbf{b}\oplus\mathbf{e}_i), R_{\mathbf{b},\mathbf{b}\oplus\mathbf{e}_i}\rangle$ is satisfied. We have two cases depending on the parity of $\mathbf{b}[i]$:

(1) $\mathbf{b}[i]$ is odd: By Lemma 6, the "OTHER" voters at $\mathtt{mid}^{+i}_{\mathbf{b}}$ vote at the ballot box $\beta(\mathbf{b})$ instead of the ballot box $\beta(\mathbf{b} \oplus \beta_i)$. Hence, we have

$$\mathtt{dist}\big(\mathtt{mid}^{+i}_{\mathbf{b}}, \beta(\mathbf{b})\big) < \mathtt{dist}\big(\mathtt{mid}^{+i}_{\mathbf{b}}, \beta(\mathbf{b} \oplus \mathbf{e}_i)\big) \tag{15}$$

Recall from Section 4.2.5 that $\mathtt{mid}^{+i}_{\mathbf{b}} := \mathtt{orig}(\mathbf{b}) \oplus \mathbf{e}_i \cdot \big((N-1) + (D-\epsilon)\big)$. From Equation 4, we get that $\mathtt{orig}(\mathbf{b} \oplus \mathbf{e}_i) = \mathtt{orig}(\mathbf{b}) \oplus \mathbf{e}_i \cdot (N - 1 + 2D)$. Hence, Equation 14 and Equation 15 together imply that

$$\begin{aligned}&\Big(N - \gamma(\mathbf{b})[i]) + (D-\epsilon)\Big)^2 + \sum_{j=1, j\neq i}^{d} \Big(\gamma(\mathbf{b})[j] - 1\Big)^2 \\ &< \Big(\big(\gamma(\mathbf{b}\oplus\mathbf{e}_i)[i] - 1\big) + (D+\epsilon)\Big)^2 \\ &+ \sum_{j=1, j\neq i}^{d} \Big(\gamma(\mathbf{b}\oplus\mathbf{e}_i)[j] - 1\Big)^2\end{aligned} \tag{16}$$

Since $\mathbf{b}[i]$ is odd, by Equation 3 we have $f(\mathbf{b})[i] = \gamma(\mathbf{b})[i]$ and $f(\mathbf{b} \oplus \mathbf{e}_i)[i] = (N+1) - \gamma(\mathbf{b} \oplus \mathbf{e}_i)[i]$. Equation 16 can be rewritten as

$$\begin{aligned}&\Big(N - f(\mathbf{b})[i]) + (D-\epsilon)\Big)^2 + \sum_{j=1, j\neq i}^{d} \Big(\gamma(\mathbf{b})[j] - 1\Big)^2 \\ &< \Big(\big(N - f(\mathbf{b}\oplus\mathbf{e}_i)[i]\big) + (D+\epsilon)\Big)^2 \\ &+ \sum_{j=1, j\neq i}^{d} \Big(\gamma(\mathbf{b}\oplus\mathbf{e}_i)[j] - 1\Big)^2\end{aligned} \tag{17}$$

We now claim that $f(\mathbf{b})[i] \geq f(\mathbf{b} \oplus \mathbf{e}_i)[i]$. Suppose to the contrary that $f(\mathbf{b})[i] < f(\mathbf{b} \oplus \mathbf{e}_i)[i]$. Since $f(\mathbf{b})[i], f(\mathbf{b} \oplus \mathbf{e}_i)[i] \in [N]$ it follows that $\big(f(\mathbf{b} \oplus \mathbf{e}_i)[i] - f(\mathbf{b})[i]\big) \geq 1$. Then from Equation 17 we have

$$\begin{aligned}&(2d-2)N^2 = (d-1)N^2 + (d-1)N^2 \\ &> \sum_{j=1, j\neq i}^{d} \big(\gamma(\mathbf{b}\oplus\mathbf{e}_i)[j] - 1\big)^2 - \sum_{j=1, j\neq i}^{d} \big(\gamma(\mathbf{b})[j] - 1\big)^2 \\ &\qquad \text{(since } \gamma(\mathbf{b})[j], \gamma(\mathbf{b}\oplus\mathbf{e}_i)[j] \in [N] \text{ for each } j \in [d]) \\ &> \Big((N - f(\mathbf{b})[i]) + (D-\epsilon)\Big)^2 \\ &\quad - \Big((N - f(\mathbf{b}\oplus\mathbf{e}_i)[i]) + (D+\epsilon)\Big)^2 \\ &= \Big(2D + (N - f(\mathbf{b})[i]) + (N - f(\mathbf{b}')[i])\Big) \\ &\quad \times \big(f(\mathbf{b}\oplus\mathbf{e}_i)[i] - f(\mathbf{b})[i] - 2\epsilon\big) \\ &\geq 2D \cdot \frac{1}{2} = D\end{aligned}$$

which is a contradiction since $D = 2d \cdot N^2$ (Equation 1). To derive the last line we have used $1 \leq f(\mathbf{b})[i], f(\mathbf{b} \oplus \mathbf{e}_i)[i] \leq N$ and $\big(f(\mathbf{b} \oplus \mathbf{e}_i)[i] - f(\mathbf{b})[i]\big) \geq 1$ and $\epsilon = \frac{1}{4}$.

(2) $\mathbf{b}[i]$ is even: By Lemma 6, the "OTHER" voters at $\mathtt{mid}^{-i}_{\mathbf{b}\oplus\mathbf{e}_i}$ vote at the ballot box $\beta(\mathbf{b} \oplus \mathbf{e}_i)$ instead of the ballot box $\beta(\mathbf{b})$. Hence, we have

$$\mathtt{dist}\big(\mathtt{mid}^{-i}_{\mathbf{b}\oplus\mathbf{e}_i}, \beta(\mathbf{b} \oplus \mathbf{e}_i)\big) < \mathtt{dist}\big(\mathtt{mid}^{-i}_{\mathbf{b}\oplus\mathbf{e}_i}, \beta(\mathbf{b})\big) \tag{18}$$

Recall from Section 4.2.5 that $\mathtt{mid}^{-i}_{\mathbf{b}\oplus\mathbf{e}_i} := \mathtt{orig}(\mathbf{b} \oplus \mathbf{e}_i) \ominus \mathbf{e}_i \cdot (D - \epsilon)$. From Equation 4, we get that $\mathtt{orig}(\mathbf{b}) = \mathtt{orig}(\mathbf{b} \oplus \mathbf{e}_i) \ominus \mathbf{e}_i \cdot (N - 1 + 2D)$. Hence, Equation 14 and Equation 18 together imply that

$$\begin{aligned}&\Big(\big(\gamma(\mathbf{b}\oplus\mathbf{e}_i)[i] - 1\big) \\ &+ (D-\epsilon)\Big)^2 + \sum_{j=1, j\neq i}^{d} \Big(\gamma(\mathbf{b}\oplus\mathbf{e}_i)[j] - 1\Big)^2 \\ &< \Big(\big(N - \gamma(\mathbf{b})[i]\big) + (D+\epsilon)\Big)^2 + \sum_{j=1, j\neq i}^{d} \Big(\gamma(\mathbf{b})[j] - 1\Big)^2\end{aligned} \tag{19}$$

Since $\mathbf{b}[i]$ is even, by Equation 3 we have $f(\mathbf{b})[i] = N + 1 - \gamma(\mathbf{b})[i]$ and $f(\mathbf{b} \oplus \mathbf{e}_i)[i] = \gamma(\mathbf{b} \oplus \mathbf{e}_i)[i]$. Equation 19 can be rewritten as

$$\begin{aligned}&\Big(\big(f(\mathbf{b}\oplus\mathbf{e}_i)[i] - 1\big) + (D-\epsilon)\Big)^2 \\ &+ \sum_{j=1, j\neq i}^{d} \Big(\gamma(\mathbf{b}\oplus\mathbf{e}_i)[j] - 1\Big)^2 \\ &< \Big(\big(f(\mathbf{b})[i] - 1\big) + (D+\epsilon)\Big)^2 + \sum_{j=1, j\neq i}^{d} \Big(\gamma(\mathbf{b})[j] - 1\Big)^2\end{aligned} \tag{20}$$

We now claim that $f(\mathbf{b})[i] \geq f(\mathbf{b} \oplus \mathbf{e}_i)[i]$. Suppose to the contrary that $f(\mathbf{b})[i] < f(\mathbf{b} \oplus \mathbf{e}_i)[i]$. Since $f(\mathbf{b})[i], f(\mathbf{b} \oplus \mathbf{e}_i)[i] \in [N]$ it follows that $\big(f(\mathbf{b} \oplus \mathbf{e}_i)[i] - f(\mathbf{b})[i]\big) \geq 1$.

Then from Equation 20 we have

$$
\begin{aligned}
&(2d-2)N^2 = (d-1)N^2 + (d-1)N^2 \\
&\geq \sum_{j=1, j\neq i}^{d} \left(\gamma(\mathbf{b})[j]-1\right)^2 - \sum_{j=1, j\neq i}^{d} \left(\gamma(\mathbf{b}\oplus \mathbf{e}_i)[j]-1\right)^2 \\
&\qquad \text{(since } \gamma(\mathbf{b})[j], \gamma(\mathbf{b}\oplus \mathbf{e}_i)[j] \in [N] \text{ for each } j \in [d]) \\
&> \Big((f(\mathbf{b}\oplus \mathbf{e}_i)[i]-1) + (D-\epsilon) \Big)^2 \\
&\quad - \Big((f(\mathbf{b})[i]-1) + (D+\epsilon) \Big)^2 \\
&= \Big(2D + (f(\mathbf{b})[i]-1) + (f(\mathbf{b}\oplus \mathbf{e}_i)[i]-1) \Big) \\
&\quad \times \left(f(\mathbf{b}\oplus \mathbf{e}_i)[i] - f(\mathbf{b})[i] - 2\epsilon \right) \\
&\geq 2D \cdot \frac{1}{2} = D
\end{aligned}
$$

which is a contradiction since $D = 2d \cdot N^2$ (Equation 1). To derive the last line we have used $1 \leq f(\mathbf{b})[i], f(\mathbf{b}\oplus \mathbf{e}_i)[i] \leq N$ and $\left(f(\mathbf{b}\oplus \mathbf{e}_i)[i] - f(\mathbf{b})[i]\right) \geq 1$ and $\epsilon = \frac{1}{4}$.

This concludes the proof of Lemma 7. □

B.2 d-dimensional geometric $\geq$-CSP is satisfiable $\Rightarrow$ FAIR-GERRYMANDERING-$(\mathbb{R}^d, \ell_2)$ has a solution

Suppose that the d-dimensional geometric $\geq$-CSP $\mathcal{I} = (V, D, C)$ has a satisfying assignment $f : V \to D$. In this section we show that the instance $\mathcal{U} = (C = \{\text{"OUR"}, \text{"OTHER"}\}, \mathcal{V}, \mathcal{B}, k = |V|, \ell = |V|, \text{"OUR"})$ has a solution.

Since $f : V \to D$ is a satisfying assignment for $\mathcal{I}$,

$$
\begin{aligned}
&\forall\, \mathbf{a} \in V,\ \text{we have } f(\mathbf{a}) \in R_{\mathbf{a}} && (21)\\
&\forall\, \mathbf{a} \in V, \forall\, i \in [d] \text{ if } \mathbf{a}\text{-}(\mathbf{a}\oplus \mathbf{e}_i) \text{ is an edge in } G_{\mathcal{I}} \\
&\text{then } f(\mathbf{a})[i] \geq f(\mathbf{a}\oplus \mathbf{e}_i)[i] && (22)
\end{aligned}
$$

We now construct a set of locations $\mathcal{P}''$ where the ballot boxes are opened. For each $\mathbf{a} \in V$, we define the following two vectors let

$$ g(\mathbf{a}) \in \mathbb{R}^d \text{ such that } g(\mathbf{a})[i] = \texttt{flip}_{\mathbf{a}[i]}\left(f(\mathbf{a})[i]\right) \tag{23} $$

$$ h(\mathbf{a}) = \left(\texttt{orig}(\mathbf{a}) \ominus 1^d\right) \oplus g(\mathbf{a}) = C \cdot (\mathbf{a} \ominus 1^d) \oplus g(\mathbf{a}) \tag{24} $$

Let $\mathcal{P}'' = \{h(\mathbf{a}) \mid \mathbf{a} \in V\}$. The next lemma shows that $h(\mathbf{a}) \in \text{INTERNAL}_3(\mathbf{a})$ for each $\mathbf{a} \in V$.

LEMMA 8. *$|\mathcal{P}'' \cap \text{INTERNAL}_3(\mathbf{a})| = 1$ for each $\mathbf{a} \in V$.*

PROOF. We prove the lemma by showing that $h(\mathbf{a}) \in \text{INTERNAL}_3(\mathbf{a})$ for each $\mathbf{a} \in V$. Fix any $\mathbf{b} \in V$. Then we have

$$
\begin{aligned}
& f(\mathbf{b}) \in R_{\mathbf{b}} && \text{(from Equation 21)}\\
\Rightarrow\ & f(\mathbf{b}) \in \text{INTERNAL}_1(\mathbf{b}) && \text{(from Section 4.2.1)}\\
\Rightarrow\ & g(\mathbf{b}) \in \text{INTERNAL}_2(\mathbf{b}) && \text{(from Section 4.2.2 and Equation 23)}\\
\Rightarrow\ & h(\mathbf{b}) \in \text{INTERNAL}_3(\mathbf{b}) && \text{(from Section 4.2.3 and Equation 24)}
\end{aligned}
$$

□

Opening a ballot box at each point from the set $\mathcal{P}''$ is feasible from Lemma 8 and the fact that $\mathcal{B} = \bigcup_{\mathbf{a}\in V} \text{INTERNAL}_3(\mathbf{a})$. Since $|\mathcal{P}''| = |V|$ we have opened exactly $k = |V|$ ballot boxes. It remains to show that each ballot box in $\mathcal{P}''$ is won by "OUR". The proof strategy is as follows: for each $\mathbf{a} \in V$ we will show that the "OUR" voters located at the point $\texttt{origin}(\mathbf{a})$ and the "OTHER" voters located at **each of the** $2d$ points in BORDER$(\mathbf{a})$ vote at the ballot box opened at $h(\mathbf{a})$. By Observation 2, this would imply that the ballot box opened at $h(\mathbf{a})$ is won by "OUR".

LEMMA 9. *For each $\mathbf{a} \in V$, the following set of voters vote at the ballot box $h(\mathbf{a})$*

- *the "OUR" voters at $\texttt{origin}(\mathbf{a})$, and*
- *the "OTHER" voters at each of the $2d$ points in BORDER$(\mathbf{a})$.*

PROOF. Fix any variable $\mathbf{a} \in V$. We will prove the lemma by showing that for any point $\mathbf{p} \in \left(\text{BORDER}(\mathbf{a}) \bigcup \texttt{origin}(\mathbf{a})\right)$ and any $\mathbf{a}' \in V$ such that $\mathbf{a}' \neq \mathbf{a}$, we have $\texttt{dist}(\mathbf{p}, h(\mathbf{a})) < \texttt{dist}(\mathbf{p}, h(\mathbf{a}'))$. Let $h(\mathbf{a}) = \texttt{orig}(\mathbf{a}) + \mathbf{t}$ and $h(\mathbf{a}') = \texttt{orig}(\mathbf{a}') + \mathbf{t}'$. By Equation 24, we have that

$$ \mathbf{t} = g(\mathbf{a}) \ominus 1^d \quad \text{and} \quad \mathbf{t}' = g(\mathbf{a}') \ominus 1^d \tag{25} $$

If $\mathbf{p} = \texttt{origin}(\mathbf{a})$, then Section 4.2.1- Section 4.2.3 imply that $(N-1) \geq \left(\left|\mathbf{p}[\ell] - h(\mathbf{a})[\ell]\right|\right) \geq 0$ for each $\ell \in [d]$. Hence, Equation 1 and Equation 2 imply that $\texttt{dist}(\mathbf{p}, h(\mathbf{a})) \leq \sqrt{dN^2} < 2D$. Since $\mathbf{a} \neq \mathbf{a}', \mathbf{p} \in \text{INTERNAL}_4(\mathbf{a})$ and $h(\mathbf{a}') \in \text{INTERNAL}_4(\mathbf{a}')$, Claim 6 implies $\texttt{dist}(\mathbf{p}, h(\mathbf{a}')) \geq 2D$. Hence, we have $\texttt{dist}(\mathbf{p}, h(\mathbf{a}')) \geq 2D > \texttt{dist}(\mathbf{p}, h(\mathbf{a}))$.

Henceforth, we assume that $\mathbf{p} \in \text{BORDER}(\mathbf{a})$. By Equation 6, there exists $i \in [d]$ such that $\mathbf{p} \in \left(\texttt{mid}_{\mathbf{a}}^{+i} \bigcup \texttt{mid}_{\mathbf{a}}^{-i}\right)$. We argue the case when $\mathbf{p} = \texttt{mid}_{\mathbf{a}}^{+i}$: the case when $\mathbf{p} = \texttt{mid}_{\mathbf{a}}^{-i}$ is analogous. There are four cases to consider depending on the parity of $\mathbf{a}[i]$ and whether or not the two points $\mathbf{a}$ and $(\mathbf{a}\oplus \mathbf{e}_i)$ form an edge in $G_{\mathcal{I}}$:

(1) $\underline{\mathbf{a}[i] \text{ is odd and } \mathbf{a}\text{-}(\mathbf{a}\oplus \mathbf{e}_i) \text{ form an edge in } G_{\mathcal{I}}}$: By Section 4.2.5, it follows that $\mathbf{p} = \texttt{mid}_{\mathbf{a}}^{+i} = \texttt{orig}(\mathbf{a}) \oplus \mathbf{e}_i \cdot \left((N-1) + (D-\epsilon)\right)$. Then from Equation 1 and Equation 2 we have

$$
\begin{aligned}
&\texttt{dist}\left((\mathbf{p}, h(\mathbf{a})\right) \\
&= \sqrt{\Big((D-\epsilon) + (N-1) - t[i]\Big)^2 + \sum_{j=1, j\neq i}^{d} t[j]^2} \\
&< \sqrt{(D+N)^2 + dN^2} < 2D && (26)
\end{aligned}
$$

where we have used the bounds $0 \leq \mathbf{t}[\ell] \leq (N-1)$ for each $\ell \in [d]$.

If there exists $j \in [d]$ such that $j \neq i$ and $\mathbf{a}[j] \neq \mathbf{a}'[j]$, then

$$
\begin{aligned}
&\texttt{dist}(\mathbf{p}, h(\mathbf{a}')) = \texttt{dist}\left(\texttt{mid}_{\mathbf{a}}^{+i}, \texttt{orig}(\mathbf{a}') \oplus \mathbf{t}'\right) \\
&\geq \left| C \cdot (\mathbf{a}'[j] - \mathbf{a}[j]) + \mathbf{t}'[j] \right| \\
&\qquad \text{(only counting along } j^{\text{th}}\text{-coordinate)} \\
&\geq \left| C \cdot (\mathbf{a}[j] - \mathbf{a}'[j]) \right| - \left| \mathbf{t}'[j] \right| && \text{(by triangle inequality)} \\
&\geq C - (N-1) && \text{(since } \mathbf{a}[j] \neq \mathbf{a}'[j] \text{ and } 0 \leq \mathbf{t}'[j] \leq (N-1)) \\
&= 2D && \text{(from Equation 1)} \\
&> \texttt{dist}\left((\mathbf{p}, h(\mathbf{a})\right) && \text{(from Equation 26)}
\end{aligned}
$$

Hence, we can assume that $\mathbf{a}[j] = \mathbf{a}'[j]$ for each $j \in [d]$ such that $j \neq i$. We have three subcases now:

- $\underline{\mathbf{a}'[i] \leq \mathbf{a}[i] - 1}$: In this subcase we have

$$\begin{aligned}
&\mathtt{dist}\left(\mathbf{p}, h(\mathbf{a}')\right)\\
&\geq C \cdot \left(\mathbf{a}[i] - \mathbf{a}'[i]\right) + \left((D-\epsilon) + (N-1) - \mathbf{t}'[i]\right) && \text{(only counting along } i^{\text{th}}\text{-coordinate)}\\
&\geq C + \left(D - \epsilon\right) && \text{(since } 0 \leq \mathbf{t}'[i] \leq N-1 \text{ and } \left(\mathbf{a}[i] - \mathbf{a}'[i]\right) \geq 1)\\
&\geq 2D && \text{(from Equation 1 and Equation 2)}\\
&> \mathtt{dist}\left((\mathbf{p}, h(\mathbf{a})\right) && \text{(from Equation 26)}
\end{aligned}$$

- $\underline{\mathbf{a}'[i] \geq \mathbf{a}[i] + 2}$: In this subcase we have

$$\begin{aligned}
&\mathtt{dist}\left(\mathbf{p}, h(\mathbf{a}')\right)\\
&\geq C \cdot \left(\mathbf{a}'[i] - \mathbf{a}[i]\right) + \left(\mathbf{t}'[i] - (D-\epsilon) - (N-1)\right) && \text{(only counting along } i^{\text{th}}\text{-coordinate)}\\
&\geq 2C - \left(D + N\right) && \text{(since } \left(\mathbf{a}'[i] - \mathbf{a}[i]\right) \geq 2 \text{ and } 0 \leq \mathbf{t}'[i] \text{ and } \epsilon = \tfrac{1}{4})\\
&\geq 2D && \text{(from Equation 1 and Equation 2)}\\
&> \mathtt{dist}\left((\mathbf{p}, h(\mathbf{a})\right) && \text{(from Equation 26)}
\end{aligned}$$

- $\underline{\mathbf{a}'[i] = \mathbf{a}[i] + 1}$: In the last remaining subcase we have $\mathbf{a}' = \mathbf{a} \oplus \mathbf{e}_i$. Hence, by Equation 22 we have that $f(\mathbf{a})[i] \geq f(\mathbf{a}')[i]$. Since $\mathbf{a}[i]$ is odd, by Equation 23 and Equation 3 we have $g(\mathbf{a})[i] = \mathtt{flip}_{\mathbf{a}[i]}\left(f(\mathbf{a})[i]\right) = f(\mathbf{a})[i]$. Since $\mathbf{a}'[i] = \mathbf{a}[i] + 1$ is even, by Equation 23 and Equation 3 we have $g(\mathbf{a}')[i] = \mathtt{flip}_{\mathbf{a}'[i]}\left(f(\mathbf{a}')[i]\right) = N + 1 - f(\mathbf{a}')[i]$. Therefore, $f(\mathbf{a})[i] \geq f(\mathbf{a}')[i]$ implies that $g(\mathbf{a})[i] \geq N + 1 - g(\mathbf{a}')[i]$. From Equation 25 we can conclude that $\mathbf{t}[i] \geq N + 1 - \mathbf{t}'[i]$. Then we have

$$\begin{aligned}
&\mathtt{dist}\left(\mathbf{p}, h(\mathbf{a})\right)^2\\
&= \left((D-\epsilon) + (N-1) - \mathbf{t}[i]\right)^2 + \left(\sum_{j=1, j\neq i}^{d} \mathbf{t}[j]^2\right) && \text{(since } \mathbf{p} = \mathtt{mid}_{\mathbf{a}}^{+i})\\
&\leq \left((D-\epsilon) + (N-1) - \mathbf{t}[i]\right)^2 + dN^2 && \text{(since } \mathbf{t}[j] \leq N-1 \text{ for each } j \in [d])\\
&< \left((D+\epsilon) + \mathbf{t}'[i]\right)^2 && \text{(from Equation 1, Equation 2 and } \left(\mathbf{t}'[i] + \mathbf{t}[i]\right) \geq N+1)\\
&\leq \left((D+\epsilon) + \mathbf{t}'[i]\right)^2 + \left(\sum_{j=1, j\neq i}^{d} \mathbf{t}'[j]^2\right)\\
&= \mathtt{dist}\left(\mathbf{p}, h(\mathbf{a}')\right)^2
\end{aligned}$$

(2) $\underline{\mathbf{a}[i] \text{ is odd and } \mathbf{a}\text{-}(\mathbf{a} \oplus \mathbf{e}_i) \text{ do not form an edge in } G_{\mathcal{I}}}$: By Section 4.2.5, it follows that $\mathbf{p} = \mathtt{mid}_{\mathbf{a}}^{+i} = \mathtt{orig}(\mathbf{a}) \oplus \mathbf{e}_i \cdot \left((N-1) + (D-2N)\right)$. Then from Equation 1 and Equation 2 we have

$$\begin{aligned}
&\mathtt{dist}\left((\mathbf{p}, h(\mathbf{a})\right)\\
&= \sqrt{\left((D-2N) + (N-1) - t[i]\right)^2 + \sum_{j=1, j\neq i}^{d} t[j]^2}\\
&< \sqrt{(D-N)^2 + dN^2} < 2D - N && (27)
\end{aligned}$$

where we have used the bounds $0 \leq \mathbf{t}[\ell] \leq (N-1)$ for each $\ell \in [d]$.

If there exists $j \in [d]$ such that $j \neq i$ and $\mathbf{a}[j] \neq \mathbf{a}'[j]$, then

$$\begin{aligned}
&\mathtt{dist}\left(\mathbf{p}, h(\mathbf{a}')\right) = \mathtt{dist}\left(\mathtt{mid}_{\mathbf{a}}^{+i}, \mathtt{orig}(\mathbf{a}') \oplus \mathbf{t}'\right)\\
&\geq \left|C \cdot \left(\mathbf{a}'[j] - \mathbf{a}[j]\right) + \mathbf{t}'[j]\right| && \text{(only counting along } j^{\text{th}}\text{-coordinate)}\\
&\geq \left|C \cdot \left(\mathbf{a}[j] - \mathbf{a}'[j]\right)\right| - \left|\mathbf{t}'[j]\right| && \text{(by triangle inequality)}\\
&\geq C - (N-1) && \text{(since } \mathbf{a}[j] \neq \mathbf{a}'[j] \text{ and } 0 \leq \mathbf{t}'[j] \leq (N-1))\\
&= 2D && \text{(from Equation 1)}\\
&> \mathtt{dist}\left((\mathbf{p}, h(\mathbf{a})\right) && \text{(from Equation 27)}
\end{aligned}$$

Hence, we can assume that $\mathbf{a}[j] = \mathbf{a}'[j]$ for each $j \in [d]$ such that $j \neq i$. We have three subcases now:

- $\underline{\mathbf{a}'[i] \leq \mathbf{a}[i] - 1}$: In this subcase, we have

$$\begin{aligned}
&\mathtt{dist}\left(\mathbf{p}, h(\mathbf{a}')\right)\\
&\geq C \cdot \left(\mathbf{a}[i] - \mathbf{a}'[i]\right) + \left((D-2N) + (N-1) - \mathbf{t}'[i]\right) && \text{(only counting along } i^{\text{th}}\text{-coordinate)}\\
&\geq C + \left(D - 2N\right) && \text{(since } 0 \leq \mathbf{t}'[i] \leq N-1 \text{ and } \left(\mathbf{a}[i] - \mathbf{a}'[i]\right) \geq 1)\\
&\geq 2D && \text{(from Equation 1 and Equation 2)}\\
&> \mathtt{dist}\left((\mathbf{p}, h(\mathbf{a})\right) && \text{(from Equation 27)}
\end{aligned}$$

- $\underline{\mathbf{a}'[i] \geq \mathbf{a}[i] + 2}$: In this subcase, we have

$$\begin{aligned}
&\mathtt{dist}\left(\mathbf{p}, h(\mathbf{a}')\right)\\
&\geq C \cdot \left(\mathbf{a}'[i] - \mathbf{a}[i]\right) + \left(\mathbf{t}'[i] - (D-2N) - (N-1)\right) && \text{(only counting along } i^{\text{th}}\text{-coordinate)}\\
&\geq 2C - \left(D - N\right) && \text{(since } \left(\mathbf{a}'[i] - \mathbf{a}[i]\right) \geq 2 \text{ and } 0 \leq \mathbf{t}'[i])\\
&\geq 2D && \text{(from Equation 1 and Equation 2)}\\
&> \mathtt{dist}\left((\mathbf{p}, h(\mathbf{a})\right) && \text{(from Equation 27)}
\end{aligned}$$

- $\underline{\mathbf{a}'[i] = \mathbf{a}[i] + 1}$: In the last remaining subcase, we have $\mathbf{a}' = \mathbf{a} \oplus \mathbf{e}_i$.

$$\begin{aligned}
&\mathtt{dist}\left(\mathbf{p}, h(\mathbf{a})\right)^2\\
&= \left((D-2N) + (N-1) - \mathbf{t}[i]\right)^2 + \sum_{j=1, j\neq i}^{d} \mathbf{t}[j]^2\\
&\leq \left((D-2N) + (N-1) - \mathbf{t}[i]\right)^2 + dN^2 && \text{(since } \mathbf{t}[j] \leq N-1 \text{ for each } j \in [d])\\
&< \left((D+2N)\right)^2 && \text{(from Equation 1, Equation 2 and since } 0 \leq \mathbf{t}[i] \leq N-1)\\
&\leq \left((D+2N) + \mathbf{t}'[i]\right)^2 + \sum_{j=1, j\neq i}^{d} \mathbf{t}'[j]^2 && \text{(since } \mathbf{t}'[i] \geq 0)\\
&= \mathtt{dist}\left(\mathbf{p}, h(\mathbf{a}')\right)^2
\end{aligned}$$

(3) $\underline{\mathbf{a}[i] \text{ is even and } \mathbf{a}\text{-}(\mathbf{a} \oplus \mathbf{e}_i) \text{ form an edge in } G_{\mathcal{I}}}$: By Section 4.2.5, it follows that $\mathbf{p} = \mathtt{mid}_{\mathbf{a}}^{+i} = \mathtt{orig}(\mathbf{a}) \oplus \mathbf{e}_i \cdot \left((N-1) + (D+\epsilon-N)\right)$.

Then from Equation 1 and Equation 2 we have

$$\texttt{dist}\big((\mathbf{p}, h(\mathbf{a})\big) = \sqrt{\Big(D+\epsilon-1-t[i]\Big)^2 + \sum_{j=1, j\neq i}^{d} t[j]^2} \leq \sqrt{D^2+dN^2} < 2D \tag{28}$$

where we have used the bounds $0 \leq \mathbf{t}[\ell] \leq (N-1)$ for each $\ell \in [d]$.

If there exists $j \in [d]$ such that $j \neq i$ and $\mathbf{a}[j] \neq \mathbf{a}'[j]$, then

$$\begin{aligned}
&\texttt{dist}\big(\mathbf{p}, h(\mathbf{a}')\big) = \texttt{dist}\big(\texttt{plus}_{\mathbf{a}}^{+i}, \texttt{orig}(\mathbf{a}') \oplus \mathbf{t}'\big)\\
&\geq \Big|C \cdot \big(\mathbf{a}'[j] - \mathbf{a}[j]\big) + \mathbf{t}'[j]\Big| && \text{(only counting along } j^{\text{th}}\text{-coordinate)}\\
&\geq \Big|C \cdot \big(\mathbf{a}[j] - \mathbf{a}'[j]\big)\Big| - \big|\mathbf{t}'[j]\big| && \text{(by triangle inequality)}\\
&\geq C - (N-1) && \text{(since } \mathbf{a}[j] \neq \mathbf{a}'[j] \text{ and } 0 \leq \mathbf{t}'[j] \leq (N-1))\\
&= 2D && \text{(from Equation 1)}\\
&> \texttt{dist}\big((\mathbf{p}, h(\mathbf{a})\big) && \text{(from Equation 28)}
\end{aligned}$$

Hence, we can assume that $\mathbf{a}[j] = \mathbf{a}'[j]$ for each $j \in [d]$ such that $j \neq i$. We have three subcases now:

- $\underline{\mathbf{a}'[i] \leq \mathbf{a}[i] - 1}$: In this subcase, we have

$$\begin{aligned}
&\texttt{dist}\big(\mathbf{p}, h(\mathbf{a}')\big)\\
&\geq C \cdot \big(\mathbf{a}[i] - \mathbf{a}'[i]\big) + \big(D+\epsilon-1-\mathbf{t}'[i]\big) && \text{(only counting along } i^{\text{th}}\text{-coordinate)}\\
&\geq C + \big(D+\epsilon-N\big) && \text{(since } \big(\mathbf{a}[i]-\mathbf{a}'[i]\big) \geq 1 \text{ \& } \mathbf{t}'[i] \leq N-1)\\
&> 2D && \text{(from Equation 1 and Equation 2)}\\
&> \texttt{dist}\big((\mathbf{p}, h(\mathbf{a})\big) && \text{(from Equation 28)}
\end{aligned}$$

- $\underline{\mathbf{a}'[i] \geq \mathbf{a}[i] + 2}$: In this subcase, we have

$$\begin{aligned}
&\texttt{dist}\big(\mathbf{p}, h(\mathbf{a}')\big)\\
&\geq C \cdot \big(\mathbf{a}'[i] - \mathbf{a}[i]\big) + \big(\mathbf{t}'[i] - (D+\epsilon-1)\big) && \text{(only counting along } i^{\text{th}}\text{-coordinate)}\\
&\geq 2C - D && \text{(since } \big(\mathbf{a}'[i]-\mathbf{a}[i]\big) \geq 2 \text{ and } 0 \leq \mathbf{t}'[i] \text{ and } \epsilon = \tfrac{1}{4})\\
&> 2D && \text{(from Equation 1 and Equation 2)}\\
&> \texttt{dist}\big((\mathbf{p}, h(\mathbf{a})\big) && \text{(from Equation 28)}
\end{aligned}$$

- $\underline{\mathbf{a}'[i] = \mathbf{a}[i] + 1}$: In the last remaining subcase, we have $\underline{\mathbf{a}' = \mathbf{a} \oplus \mathbf{e}_i}$.

$$\begin{aligned}
&\texttt{dist}\big(\mathbf{p}, h(\mathbf{a})\big)^2 = \big(D+\epsilon-1-\mathbf{t}[i]\big)^2 + \sum_{j=1, j\neq i}^{d} \mathbf{t}[j]^2 && \text{(since } \mathbf{p} = \texttt{mid}_{\mathbf{a}}^{+i})\\
&< \big(D+\epsilon-1-\mathbf{t}[i]\big)^2 + dN^2 && \text{(since } \mathbf{t}[j] \leq (N-1) \text{ for each } j \in [d])\\
&< (D+N-1)^2 && \text{(from Equation 1, Equation 2 and since } \mathbf{t}[i] \geq 0)\\
&< \big(D-\epsilon+N+\mathbf{t}'[i]\big)^2 && \text{(since } \epsilon = \tfrac{1}{4} \text{ and } \mathbf{t}'[i] \geq 0)\\
&\leq \big(D-\epsilon+N+\mathbf{t}'[i]\big)^2 + \sum_{j=1, j\neq i}^{d} \mathbf{t}'[j]^2\\
&= \texttt{dist}\big(\mathbf{p}, h(\mathbf{a}')\big)^2
\end{aligned}$$

(4) $\underline{\mathbf{a}[i] \text{ is even and } \mathbf{a}\text{-}(\mathbf{a} \oplus \mathbf{e}_i) \text{ do not form an edge in } G_{\mathcal{I}}}$: By Section 4.2.5, it follows that $\mathbf{p} = \texttt{mid}_{\mathbf{a}}^{+i} = \texttt{orig}(\mathbf{a}) \oplus \mathbf{e}_i \cdot \big((N-1) + (D-2N)\big)$. Then from Equation 1 and Equation 2 we have

$$\texttt{dist}\big((\mathbf{p}, h(\mathbf{a})\big) = \sqrt{\Big(D-N-1-t[i]\Big)^2 + \sum_{j=1, j\neq i}^{d} t[j]^2} \leq \sqrt{D^2+dN^2} < 2D - N < 2D \tag{29}$$

where we have used the bounds $0 \leq \mathbf{t}[\ell] \leq (N-1)$ for each $\ell \in [d]$.

If there exists $j \in [d]$ such that $j \neq i$ and $\mathbf{a}[j] \neq \mathbf{a}'[j]$, then

$$\begin{aligned}
&\texttt{dist}\big(\mathbf{p}, h(\mathbf{a}')\big) = \texttt{dist}\big(\texttt{plus}_{\mathbf{a}}^{+i}, \texttt{orig}(\mathbf{a}') \oplus \mathbf{t}'\big)\\
&\geq \Big|C \cdot \big(\mathbf{a}'[j] - \mathbf{a}[j]\big) + \mathbf{t}'[j]\Big| && \text{(only counting along } j^{\text{th}}\text{-coordinate)}\\
&\geq \Big|C \cdot \big(\mathbf{a}[j] - \mathbf{a}'[j]\big)\Big| - \big|\mathbf{t}'[j]\big| && \text{(by triangle inequality)}\\
&\geq C - (N-1) && \text{(since } \mathbf{a}[j] \neq \mathbf{a}'[j] \text{ and } 0 \leq \mathbf{t}'[j] \leq (N-1))\\
&= 2D && \text{(from Equation 1)}\\
&> \texttt{dist}\big((\mathbf{p}, h(\mathbf{a})\big) && \text{(from Equation 29)}
\end{aligned}$$

Hence, we can assume that $\mathbf{a}[j] = \mathbf{a}'[j]$ for each $j \in [d]$ such that $j \neq i$. We have three subcases now:

- $\underline{\mathbf{a}'[i] \leq \mathbf{a}[i] - 1}$: In this subcase we have

$$\begin{aligned}
&\texttt{dist}\big(\mathbf{p}, h(\mathbf{a}')\big)\\
&\geq C \cdot \big(\mathbf{a}[i] - \mathbf{a}'[i]\big) + \big(D-N-1-\mathbf{t}'[i]\big) && \text{(only counting along } i^{\text{th}}\text{-coordinate)}\\
&\geq C + \big(D-N-1-\mathbf{t}'[i]\big) && \text{(since } \big(\mathbf{a}[i]-\mathbf{a}'[i]\big) \geq 1)\\
&> 2D - N && \text{(from Equation 1, Equation 2 and since } \mathbf{t}'[i] \leq N-1)\\
&> \texttt{dist}\big((\mathbf{p}, h(\mathbf{a})\big) && \text{(from Equation 29)}
\end{aligned}$$

- $\underline{\mathbf{a}'[i] \geq \mathbf{a}[i] + 2}$: In this subcase we have

$$\begin{aligned}
&\mathtt{dist}\big(\mathbf{p}, h(\mathbf{a}')\big)\\
&\geq C \cdot \big(\mathbf{a}'[i] - \mathbf{a}[i]\big) + \big(\mathbf{t}'[i] - (D - N - 1)\big) && \text{(only counting along } i^{\text{th}}\text{-coordinate)}\\
&\geq 2C - (D - N - 1) && \text{(since } \big(\mathbf{a}'[i] - \mathbf{a}[i]\big) \geq 2 \text{ and } 0 \leq \mathbf{t}'[i]\text{)}\\
&> 2D - N && \text{(from Equation 1 and Equation 2)}\\
&> \mathtt{dist}\big((\mathbf{p}, h(\mathbf{a})\big) && \text{(from Equation 29)}
\end{aligned}$$

- $\underline{\mathbf{a}'[i] = \mathbf{a}[i] + 1}$: In the last remaining subcase we have $\underline{\mathbf{a}' = \mathbf{a} \oplus \mathbf{e}_i}$.

$$\begin{aligned}
&\mathtt{dist}\big(\mathbf{p}, h(\mathbf{a})\big)^2\\
&= \big(D - N - 1 - \mathbf{t}[i]\big)^2 + \sum_{j=1, j\neq i}^{d} \mathbf{t}[j]^2 && \text{(since } \mathbf{p} = \mathtt{mid}_{\mathbf{a}}^{+i}\text{)}\\
&< \big(D - N - 1 - \mathbf{t}[i]\big)^2 + dN^2 && \text{(since } \mathbf{t}[j] \leq (N-1) \text{ for each } j \in [d]\text{)}\\
&\leq (D + 2N)^2 && \text{(from Equation 1, Equation 2 and since } 0 \leq \mathbf{t}[i] \leq N - 1\text{)}\\
&\leq \big(D + 2N + \mathbf{t}'[i]\big)^2 && \text{(since } \mathbf{t}'[i] \geq 0\text{)}\\
&\leq \big(D + 2N + \mathbf{t}'[i]\big)^2 + \sum_{j=1, j\neq i}^{d} \mathbf{t}'[j]^2\\
&= \mathtt{dist}\big(\mathbf{p}, h(\mathbf{a}')\big)^2
\end{aligned}$$

This concludes the proof of Lemma 9. □

From Lemma 9 and Observation 2, it follows that for each $\mathbf{a} \in V$ the number of "OUR" voters at the ballot box $h(\mathbf{a})$ is exactly $1 + 2d \cdot (2d+1)^{\phi(\mathbf{a})}$ and the number of "OTHER" voters at the ballot box $h(\mathbf{a})$ is exactly $2d \cdot (2d+1)^{\phi(\mathbf{a})}$, i.e., "OUR" wins the ballot box $h(\mathbf{a})$. Therefore, each of the ballot boxes from the set $\mathcal{P}''$ is won by "OUR", and since $|\mathcal{P}''| = |V|$ it follows that $\mathcal{P}''$ is a solution for the instance $\mathcal{U} = \big(C = \{\text{"OUR"}, \text{"OTHER"}\}, \mathcal{V}, \mathcal{B}, k = |V|, \ell = |V|, \text{"OUR"}\big)$ of FAIR-GERRYMANDERING-$(\mathbb{R}^d, \ell_2)$.