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Baar, Wouter; Stella, Leonardo; Bauso, Dario

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Evolutionary dynamics on a regular networked structured and unstructured multi-population

Wouter Baar[1] | **Leonardo Stella**[2] | **Dario Bauso**[1,3]

[1]Jan C. Willems Center for Systems and Control, ENTEG, Faculty of Science and Engineering, University of Groningen, Groningen, The Netherlands

[2]School of Computer Science, College of Engineering and Physical Sciences, University of Birmingham, Birmingham, UK

[3]Dipartimento di Ingegneria, University of Palermo, Palermo, Italy

Correspondence
Leonardo Stella, School of Computer Science, College of Engineering and Physical Sciences, University of Birmingham, Birmingham, UK.
Email: l.stella@bham.ac.uk

Abstract

In this paper, we study collective decision-making in a multi-population framework, where groups of individuals represent whole populations that interact by means of a regular network. Each group consists of a number of players and every player can choose between two options. A group is characterized by three variables, the first two denoting the fractions of individuals committed to each option and the last one representing the fraction of players not committed to either option. The state of every population is influenced by the state of neighboring groups. The contribution of this work is the following. First, we find explicit expressions for all the equilibrium points of the proposed system, and show that these represent equilibrium points where populations reach consensus, namely, where all populations have the same states. We also derive a sufficient condition for local asymptotic stability as well as exponential asymptotic stability. Then, we study a structured model where every population is now assumed to represent a structured complex network. We conclude the paper by providing a set of simulations that corroborate the theoretical findings.

KEYWORDS

asymptotic stability, collective decision-making, complex networks, consensus, multi-agent systems, networked systems

1 | INTRODUCTION

In this paper, we take inspiration from a biologically-inspired model, where a swarm of honeybees carries out the nest-site selection process. The aim of this process is to reach consensus on one of two available options.[1-4] In the traditional formulation of the model, it is assumed that the two options have equal value in order to show the effectiveness of the cross-inhibitory signalling used by honeybees to reach consensus. Motivated by this problem, we develop a multi-population framework that considers more than one group of individuals interacting through a network. In recent years, bio-inspired collective decision-making has seen increasing interest, including cross-disciplinary research works that shed some light on the interactions and strategic aspects through the lenses of game theory and network theory.[5-7] In these works, a group of individuals is considered and every player within the group can choose between two equally-favorable options, option 1 or option 2, or can choose to remain uncommitted. Rather than studying the individual behavior of each player, we are interested in quantifying how many players choose a certain strategy. To do this, and in accordance with the literature, we provide a mean-field approximation by taking the population size very large.

In recent years, the principles underpinning collective decision-making in honeybees have been extensively studied within cross-disciplinary research involving behavioral ecology, psychology and neurosciences. Two predominant behavioral traits were found during the large number of experiments investigating the collective decision-making process in honeybees: the so-called waggle dance, performed by scout bees to share information about available nest sites in order to recruit other bees; and the cross-inhibitory stop signal, which is used to prevent other bees to advertise for the competing options.[4] In particular, the most studied case is where the swarm has to choose between options with near-equal values, and a critical value of the stop signal was found to prevent deadlocks.[4,5] While a seminal study on the decision-making in house-hunting honeybees with more than two options was conducted in 1997 and 1998 and reported in Reference 8, recent research,[9] has extended the previous model to the best-of-*N* case, analysing the symmetric and asymmetric models. Recent applications of the decision-making in honeybee swarms include multi-agent decision-making and network design.[10,11]

Motivated by the networked evolutionary models that arise in decision-making processes,[12-17] we extend the collective decision-making inspired by honeybee swarms to a multi-population setting, where groups of individuals need to decide between two options, for example, an election where the options represent a left or right political party, and the cross-inhibitory signals represent smear campaigns by the opposing party. Recent applications of the proposed networked model as well as similar multi-population models include the study of virus propagation models, where a disease infects a population and spreads over different groups of individuals susceptible to the virus.[18] When two viruses compete to infect a population of susceptible individuals, we are in the presence of a bi-virus model.[19]

An element of novelty of this work in comparison with the existing literature is the study of evolutionary dynamics on a multi-population, as opposed to a single population. This networked multi-population setting is captured by a graph, where each node represents a population. Every population is characterized by three variables, the fractions of players that are committed to option 1, those committed to option 2 and the uncommitted individuals. These fractions depend on the state of the neighboring populations to account for individuals that are completely susceptible to opinions coming from outside as in the popular work by Hegselmann and Krause.[12] Alternatively, one can consider a different degree of stubbornness and susceptibility in opinion dynamics, but this is not the focus of our article. We assume throughout the paper that the graph of the networked multi-population is unweighted and regular, by which we mean that every node has the same amount of neighbors.

Highlights of contributions. The following is a list of contributions to the preliminary version of this work that appeared as a conference proceeding.[20] First, we have generalized and improved some of the main results by finding explicit expressions for all the equilibrium points of the proposed system as well as showing that these equilibria represent consensus equilibrium points, namely, equilibrium points where all populations have the same states. Second, we derive a sufficient condition for local and exponential asymptotic stability. Finally, we investigate the model where the interactions of each population are captured by an additional structure that takes the form of a complex network.

This paper is organized as in the following. In Section 2, we present the formulation of our multi-population model. We study the existence of the equilibrium points and carry out the stability analysis. In Sections 3 and 4, we model the interactions internal to each population via a complex network and carry out the local stability analysis for the general system as well as in the case of a symmetric equilibrium. Section 5 presents a set of simulations to corroborate the theoretical results. Finally, in Section 6, we draw conclusions and discuss possible future directions of research.

NOTATION

We use the following notations throughout the paper. A networked multi-population can be represented by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in which each node i in the set $\mathcal{V}$ represents a population, and each edge $\{i,j\} \in \mathcal{E}$ represents a link between two nodes and, hence, populations. The symbol $\mathcal{N}_i$ indicates the neighbor set of any node i. The set of feasible states Δ is defined by $\Delta = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n | x_i \geq 0, y_i \geq 0, x_i + y_i \leq 1\}$.

2 | THE UNSTRUCTURED NETWORKED MODEL

In this section, we develop and study a collective decision-making model on a networked multi-population. To do so, we first formalize a networked multi-population as follows.

Definition 1 (Networked multi-population). A networked multi-population is a set of populations that interact by means of a network topology.

In a networked multi-population, the dynamics of a population are influenced by neighboring populations. The model where there is no structure on a single population i is henceforth referred to as *unstructured networked model*. Consider a multi-population that consists of multiple groups of players, where it is assumed that players within a group can all interact with one another. The evolutionary dynamics are as follows. Every player can spontaneously decide to commit to an option (parametrized by γ), or can decide to commit to an option by means of imitation (parametrized by r), because we assume the individuals exhibit crowd-seeking behaviors, and thus are attracted to the option that has the most committed players. Players can also decide to withdraw from their chosen state, either by spontaneously giving up their commitment (parametrized by α), or by being lured to become uncommitted by other players. We then also talk about cross-inhibitory signals that are sent to opposing players to attract them to become uncommitted (parametrized by σ). Finally, when modeling the dynamics of such systems, let the states of the model be x, y, z, denoting the fraction of individuals committed to option 1, the fraction of individuals committed to option 2 and the fraction of uncommitted individuals, respectively. We introduce a subscript i to denote the i^{th} population. Note that we have $x_i + y_i + z_i = 1$ at all times, or equivalently, $z_i = 1 - x_i - y_i$. This enables us to describe the dynamics solely in terms of x_i and y_i:

$$\begin{aligned} \dot{x}_i &= \left(\gamma + r\sum_{j\in\mathcal{N}_i} x_j\right)(1 - x_i - y_i) - x_i\left(\alpha + \sigma\sum_{j\in\mathcal{N}_i} y_j\right), \\ \dot{y}_i &= \left(\gamma + r\sum_{j\in\mathcal{N}_i} y_j\right)(1 - x_i - y_i) - y_i\left(\alpha + \sigma\sum_{j\in\mathcal{N}_i} x_j\right), \end{aligned} \tag{1}$$

and of course $\dot{z}_i = -\dot{x}_i - \dot{y}_i$. These dynamics and the role of the parameters are summarized in Figure 1. It is worth noting that the imitation parameter r and the cross-inhibitory parameter σ determine the influence of the neighboring populations on the dynamics. The dynamics in population i thus depend on neighboring populations $j \in \mathcal{N}_i$, where $\mathcal{N}_i$ is the set of neighbor nodes to i, that is, $\mathcal{N}_i = \{j \in \mathcal{V} | \{i, j\} \in \mathcal{E}\}$. Throughout, it is assumed that we consider a regular and unweighted graph, so the cardinality of the neighbor set, namely, $d := |\mathcal{N}_i|$, is the same for every node and is denoted by parameter d. The motivation for this choice can be found in relevant literature on decision-making processes via evolutionary dynamics.[17,21,22] But it also serves the purposes of making the problem more tractable, allowing us to derive analytic expressions for the equilibria of the system. The following result ensures the well-posedness of our problem, by proving that the set of feasible states Δ is positively invariant.

Lemma 1. *For any initial condition* $(x(0), y(0)) \in \Delta$, *the system state remains in this set*, $(x(t), y(t)) \in \Delta$, *for all* $t > 0$.

Proof. This is a corollary from Nagumo's Theorem.[23] To see this, we compute the direction of the vector field of the points on the boundary of Δ, and show that they point inwards. Consider any node i. Recall that parameters $\gamma, r, \alpha, \sigma$ are all positive-valued, and assume that $x_j, y_j \geq 0$ for all $j \neq i$. We now show that $\Delta_i = \{(x_i, y_i) | x_i \geq 0, y_i \geq 0, x_i + y_i \leq 1\}$ is a positively invariant set. The boundary of Δ_i is

$$\begin{aligned} \partial\Delta_i &= \{(x_i, y_i) \in \Delta_i \,|\, y_i = 0\} \cup \{(x_i, y_i) \in \Delta_i | x_i = 0\} \\ &\quad \cup \{(x_i, y_i) \in \Delta_i \,|\, x_i + y_i = 1\} \\ &=: \partial\Delta_i^1 \cup \partial\Delta_i^2 \cup \partial\Delta_i^3. \end{aligned}$$

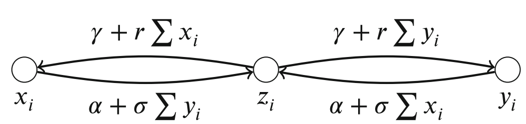

FIGURE 1 The evolutionary dynamics at the node level.

For the points in $\partial\Delta_i^1$, we have that

$$\dot{y}_i = \left(\gamma + r\sum_{j\in\mathcal{N}_i} a_{ij}y_j\right)(1-x_i) \geq 0,$$

so these points are moving upwards inside Δ_i. For points in $\partial\Delta_i^2$ we have that they point inside Δ_i as well, since

$$\dot{x}_i = \left(\gamma + r\sum_{j\in\mathcal{N}_i} a_{ij}x_j\right)(1-y_i) \geq 0.$$

And finally, for points in $\partial\Delta_i^3$ we have that

$$\dot{x}_i = -x_i\left(\alpha + \sigma\sum_{j\in\mathcal{N}_i} a_{ij}y_j\right) \leq 0,$$
$$\dot{y}_i = -y_i\left(\alpha + \sigma\sum_{j\in\mathcal{N}_i} a_{ij}x_j\right) \leq 0,$$

which points inward Δ_i as well. To conclude, we see that for any point on the boundary of Δ, the vector field points inward Δ. As a consequence of Nagumo's Theorem, we can conclude that the set Δ is positively invariant. ■

We are now ready to establish the following result.

Theorem 1. *The equilibrium points for* (1) *are* $(x^*, y^*, z^*) = (\xi\mathbb{1}_n, \mu\mathbb{1}_n, \zeta\mathbb{1}_n)$, *with the equilibrium values* ξ, μ, ζ *given by*

- *Case 1. If* $\xi = \mu$,

$$\xi = \mu = \frac{-(2\gamma - rd + \alpha) + \sqrt{(2\gamma - rd + \alpha)^2 + 4\gamma(2rd + \sigma d)}}{2(2rd + \sigma d)}, \quad \zeta = 1 - 2\xi. \tag{2}$$

- *Case 2. If* $\zeta = \frac{\alpha}{dr}$, *let* $\rho = \frac{-\left(\frac{\alpha}{r}-d\right)+\sqrt{\left(\frac{\alpha}{r}-d\right)^2-\frac{4\gamma\alpha}{r\sigma}}}{2d}$ *and* $\tau = \frac{-\left(\frac{\alpha}{r}-d\right)-\sqrt{\left(\frac{\alpha}{r}-d\right)^2-\frac{4\gamma\alpha}{r\sigma}}}{2d}$. *Then we have two equilibria,*

$$\begin{array}{llll} \text{either} & \xi = \rho, & \mu = \tau, & \zeta = \frac{\alpha}{rd}, \\ \text{or} & \xi = \tau, & \mu = \rho, & \zeta = \frac{\alpha}{rd}. \end{array}$$

Proof. The system dynamics for node i are described by Equation (1). We remark that the equations for $\dot{x}_i$ and $\dot{y}_i$ are the same for any node i, and hence we limit ourselves to investigate the dynamics of a generic node i. The analytical result that we obtain for the generic node i holds true for all the other nodes. In particular, the converging value for the dynamics of node i is the same for all nodes. Then, we can drop the index i and denote the common value for x_i by ξ, the common value for y_i by μ, and the value for z_i by $\zeta = 1 - \xi - \mu$. The equilibrium is thus given by $(x^*, y^*, z^*) = (\xi\mathbb{1}_n, \mu\mathbb{1}_n, \zeta\mathbb{1}_n)$, with $\zeta = 1 - \xi - \mu$. We will now find explicit expressions for the equilibrium values. To this purpose, we set $\dot{x}_i = 0$ and $\dot{y}_i = 0$, and the equations in (1) reduce to

$$\begin{aligned} 0 = \dot{x}_i &= (\gamma + rd\xi)\zeta - \xi(\alpha + \sigma d\mu), \\ 0 = \dot{y}_i &= (\gamma + rd\mu)\zeta - \mu(\alpha + \sigma d\xi). \end{aligned} \tag{3}$$

From (3), it follows that

$$0 = \dot{x}_i - \dot{y}_i = rd(\xi - \mu)\zeta - \xi\alpha + \mu\alpha = (rd\zeta - \alpha)(\xi - \mu).$$

From this expression we obtain that either $\xi = \mu$ or $\zeta = \frac{\alpha}{rd}$, hence, the need to distinguish two cases. Note that the two cases are not mutually exclusive. In Case 1, if $\xi = \mu$, the first equation of (3) reduces to

$$\begin{aligned} 0 = \dot{x}_i &= (\gamma + rd\xi)(1 - 2\xi) - \xi(\alpha + \sigma d\xi) \\ &= \gamma - 2\gamma\xi + rd\xi - 2rd\xi^2 - \alpha\xi - \sigma d\xi^2 \\ &= (2rd + \sigma d)\xi^2 + (2\gamma - rd + \alpha)\xi - \gamma. \end{aligned}$$

This is a quadratic equation in ξ and the solution is

$$\xi = \frac{-(2\gamma - rd + \alpha) \pm \sqrt{(2\gamma - rd + \alpha)^2 + 4\gamma(2rd + \sigma d)}}{2(2rd + \sigma d)}.$$

In general, one obtains two solutions to a quadratic equation, but the solution with a minus sign in front of the square root cannot be an equilibrium since the value of ξ would be negative. This equilibrium is not in the set of feasible states Δ, and because of Lemma 1, the system dynamics can never converge to it. This completes Case 1 since $\mu = \xi$ and $\zeta = 1 - 2\xi$. We now turn our attention to Case 2. If $\zeta = \frac{\alpha}{rd}$, given that $\mu = 1 - \xi - \frac{\alpha}{rd}$, we obtain

$$\begin{aligned} 0 = \dot{x}_i &= (\gamma + rd\xi)\frac{\alpha}{rd} - \xi\left(\alpha + \sigma d\left(1 - \xi - \frac{\alpha}{rd}\right)\right) \\ &= d\xi^2 + \left(\frac{\alpha}{r} - d\right)\xi + \frac{\gamma\alpha}{rd\sigma}. \end{aligned}$$

This is a quadratic equation in ξ and has two solutions, $\xi = \rho$ and $\xi = \tau$, where ρ and τ are given by

$$\rho = \frac{-\left(\frac{\alpha}{r} - d\right) + \sqrt{\left(\frac{\alpha}{r} - d\right)^2 - \frac{4\gamma\alpha}{r\sigma}}}{2d}, \quad \tau = \frac{-\left(\frac{\alpha}{r} - d\right) - \sqrt{\left(\frac{\alpha}{r} - d\right)^2 - \frac{4\gamma\alpha}{r\sigma}}}{2d}.$$

Contrary to Case 1, now both solutions are feasible solutions since they can be within the interval $[0, 1]$. This means that we have found two equilibria, either $\xi = \rho$ and $\mu = 1 - \xi - \frac{\alpha}{rd} = 1 - \rho - \frac{\alpha}{rd}$, or $\xi = \tau$ and $\mu = 1 - \tau - \frac{\alpha}{rd}$. It should be noted that

$$\begin{aligned} \rho &= \frac{-\left(\frac{\alpha}{r} - d\right) + \sqrt{\left(\frac{\alpha}{r} - d\right)^2 - \frac{4\gamma\alpha}{r\sigma}}}{2d} = -\frac{\alpha}{2rd} + \frac{1}{2} + \frac{\sqrt{\left(\frac{\alpha}{r} - d\right)^2 - \frac{4\gamma\alpha}{r\sigma}}}{2d} \\ &= 1 - \frac{1}{2} + \frac{\sqrt{\left(\frac{\alpha}{r} - d\right)^2 - \frac{4\gamma\alpha}{r\sigma}}}{2d} + \frac{\alpha}{2rd} - \frac{\alpha}{rd} = 1 - \left(\frac{-\left(\frac{\alpha}{r} - d\right) - \sqrt{\left(\frac{\alpha}{r} - d\right)^2 - \frac{4\gamma\alpha}{r\sigma}}}{2d}\right) - \frac{\alpha}{rd} \\ &= 1 - \tau - \frac{\alpha}{rd}. \end{aligned}$$

Or equivalently, $\tau = 1 - \rho - \frac{\alpha}{rd}$. The two equilibria are thus simply given by

$$\begin{aligned} \xi = \rho, \quad \mu = \tau, \quad \zeta = \tfrac{\alpha}{rd}, \\ \xi = \tau, \quad \mu = \rho, \quad \zeta = \tfrac{\alpha}{rd}. \end{aligned}$$

The two equilibria in Case 2 are thus symmetric in the sense that the values for option 1 and option 2 are simply swapped. This completes the proof. ∎

We refer to the equilibria established in Theorem 1 as *consensus equilibria*. Our notion of consensus refers to the fact that the fractions of committed and uncommitted players, ξ, μ and ζ, are the same in every population. In the following remarks we provide insight into the asymptotic cases in which either the spontaneous uncommitment or the spontaneous commitment are dominant.

Remark 1. Note that the equilibria in Case 2 can be disregarded if $\alpha > rd$, since the equilibrium values would then be outside the set of feasible states Δ. Indeed, if the effect of the spontaneous uncommitment is dominant, that is, α is significantly larger than the other system parameters γ, r and σ, we reach a symmetric equilibrium in which the values of committing to an option, ξ and μ, are close to zero. To see this, consider the following

$$\lim_{\alpha\to\infty}\xi = \lim_{\alpha\to\infty}\frac{-(2\gamma - rd + \alpha) + \sqrt{(2\gamma - rd + \alpha)^2 + 4\gamma(2rd + \sigma d)}}{2(2rd + \sigma d)}.$$

Multiplying the latter expression by $\frac{-(2\gamma-rd+\alpha)-\sqrt{(2\gamma-rd+\alpha)^2+4\gamma(2rd+\sigma d)}}{-(2\gamma-rd+\alpha)-\sqrt{(2\gamma-rd+\alpha)^2+4\gamma(2rd+\sigma d)}}$, one obtains

$$\begin{aligned}\lim_{\alpha\to\infty}\xi &= \lim_{\alpha\to\infty}\frac{(2\gamma - rd + \alpha)^2 - ((2\gamma - rd + \alpha)^2 + 4\gamma(2rd + \sigma d))^2}{-2(2rd + \sigma d)\cdot\left((2\gamma - rd + \alpha) + \sqrt{(2\gamma - rd + \alpha)^2 + 4\gamma(2rd + \sigma d)}\right)}\\ &\overset{(*)}{=} \lim_{\alpha\to\infty}\frac{4\gamma(2rd + \sigma d)^2}{-2(2rd + \sigma d)\cdot\left((2\gamma - rd + \alpha) + \sqrt{(2\gamma - rd + \alpha)^2 + 4\gamma(2rd + \sigma d)}\right)} = 0.\end{aligned}$$

This result is also intuitive if one considers the dynamics as represented in Figure 1. If α is the dominating factor, committed players have a very strong tendency to become uncommitted, and as a result the fractions of committed individuals $x_i(t)$ and $y_i(t)$ tend to go to zero.

Similarly to the previous remark, we can also investigate the effect of parameter γ on the dynamics.

Remark 2. From Figure 1 it is intuitive that if the spontaneous commitment is dominant, every agent will commit to an option. As both options are equally favourable, as it is the case in most literature on collective decision-making in honeybee swarms,[24] we expect that the final distribution of the agents is $x_i = 0.5, y_i = 0.5, z_i = 0$. This intuitive line of reasoning can be corroborated by the following analytical derivation. Starting from the expression indicated by (*) in the previous remark, and dividing both numerator and denominator by γ, we obtain:

$$\lim_{\gamma\to\infty}\xi = \lim_{\gamma\to\infty}\frac{4(2rd + \sigma d)^2}{-2(2rd + \sigma d)\cdot\left(\left(2 - \frac{rd}{\gamma} + \frac{\alpha}{\gamma}\right) + \sqrt{\left(2 - \frac{rd}{\gamma} + \frac{\alpha}{\gamma}\right)^2 + \frac{4}{\gamma}(2rd + \sigma d)}\right)} = \frac{1}{2}.$$

For the consensus equilibrium established in Theorem 1, we have the following sufficient condition for exponential stability. We show that two inequalities must hold on the strength of the cross-inhibitory signals, and these conditions can be checked a priori.

Theorem 2. *The consensus equilibrium $x^* = (\xi\mathbb{1}_n, \mu\mathbb{1}_n, \zeta\mathbb{1}_n)$ that is reached under dynamics (1) is locally exponentially stable if the following inequalities on the cross-inhibitory signal σ hold*:

$$\sigma > \frac{\alpha - rd(1 - \xi - \mu)}{d(1 - \mu)} \text{ and } \sigma > \frac{\alpha - rd(1 - \xi - \mu)}{d(1 - \xi)}.$$

Proof. We compute the Jacobian and evaluate it at the equilibrium $x^* = (\xi\mathbb{1}_n, \mu\mathbb{1}_n, \zeta\mathbb{1}_n)$. Consider the dynamics described by (1), then the partial derivatives of $\dot{x}_i$ are given by the following expressions

$$\begin{aligned}\frac{\partial\dot{x}_i}{\partial x_i} &= -\gamma - r\sum_{j\in\mathcal{N}_i}x_j\\ &\quad -\alpha - \sigma\sum_{j\in\mathcal{N}_i}y_j,\end{aligned} \qquad \frac{\partial\dot{x}_i}{\partial y_i} = -\gamma - r\sum_{j\in\mathcal{N}_i}x_j,$$

$$\frac{\partial\dot{x}_i}{\partial x_j} = \begin{cases}0 & \text{if } j\notin\mathcal{N}_i,\\ rz_i & \text{if } j\in\mathcal{N}_i,\end{cases} \qquad \frac{\partial\dot{x}_i}{\partial y_j} = \begin{cases}0 & \text{if } j\notin\mathcal{N}_i,\\ -\sigma x_i & \text{if } j\in\mathcal{N}_i.\end{cases}$$

The partial derivatives of $\dot{y}_i$ are obtained in a similar fashion. We evaluate the expressions of the partial derivatives at the equilibrium $x^* = (\xi \mathbb{1}_n, \mu \mathbb{1}_n, \zeta \mathbb{1}_n)$ and we make use of the fact that $1 - \xi - \mu = \zeta$. This gives the following Jacobian

$$
\begin{aligned}
J &= \left[\frac{\partial(\dot{x}_1, \dots, \dot{x}_n, \dot{y}_1, \dots, \dot{y}_n)}{\partial(x_1, \dots, x_n, y_1, \dots, y_n)}\right]\Bigg|_{x=x^*} \\
&= \left[\begin{array}{c|c} \frac{\partial(\dot{x}_1, \dots, \dot{x}_n)}{\partial(x_1, \dots, x_n)} & \frac{\partial(\dot{x}_1, \dots, \dot{x}_n)}{\partial(y_1, \dots, y_n)} \\ \hline \frac{\partial(\dot{y}_1, \dots, \dot{y}_n)}{\partial(x_1, \dots, x_n)} & \frac{\partial(\dot{y}_1, \dots, \dot{y}_n)}{\partial(y_1, \dots, y_n)} \end{array}\right]\Bigg|_{x=x^*} \\
&= \left[\begin{array}{c|c} (r - r\xi - r\mu)A- & -\sigma A - (\gamma + rd\xi)I_n \\ (\gamma + rd\xi + \alpha + \sigma d\mu)I_n & \\ \hline & (r - r\xi - r\mu)A- \\ -\sigma A - (\gamma + rd\mu)I_n & (\gamma + rd\mu + \alpha + \sigma d\xi)I_n \end{array}\right],
\end{aligned}
$$

where A denotes the adjacency matrix of the network. To prove exponential stability the eigenvalues of the above matrix should be contained in the open left half of the complex plane. Computing the eigenvalues of the above $2n \times 2n$ Jacobian matrix is rather difficult, however, we can find an estimate using the Gershgorin circle theorem. Gershgorin circle theorem states that each eigenvalue λ_i is contained in a circle around J_{ii} with radius $\sum_{i \neq j} |J_{ij}|$. Indeed, we note that all diagonal entries J_{ii} are strictly negative, so if we require that the radius of the circle does not cross the imaginary axis we know that the eigenvalue has a strictly negative real part. Using this, we find a sufficient condition for stability. So for $i = 1, \dots, n$ we require

$$
\begin{aligned}
-J_{ii} &> \sum_{j \neq i} |J_{ij}| \\
\gamma + rd\xi + \alpha + \sigma d\mu &> rd\zeta + \sigma d + \gamma + rd\xi \\
\alpha + \sigma d\mu &> rd\zeta + \sigma d \\
\sigma &> \frac{rd\zeta - \alpha}{d\mu - d} = \frac{\alpha - rd\zeta}{d(1 - \mu)}.
\end{aligned}
$$

Analogously, for $i = n + 1, \dots, 2n$ we must have $-J_{ii} > \sum_{j \neq i} |J_{ij}|$, and similar computations show that this gives the second constraint. This completes the proof. ■

3 | A STRUCTURED NETWORKED MULTI-POPULATION

In the previous section, the internal structure of the population was not considered. In the remainder of this paper we assume that every population is characterized by a structured environment,[24,25] in accordance with the definition that follows. The structured environment is given by a complex network. A complex network is a network on a large number of nodes, where the node degree distribution follows a power law.

Definition 2 (Structured networked multi-population). A structured networked multi-population is a networked multi-population in which each population possesses an internal structure given by a complex network.

More precisely, each node i of the networked multi-population now represents a structured population given by a complex network. Here, the key idea is that individuals from a population with a high number of connections with other players (within the same population) can have different inclinations to commit to option 1 or 2 from a player with a low connectivity.

Formally, we let $P(k)$ be the distribution of the node degrees, for $k = 1, 2, \dots, k_{\max}$. Then we denote x_i^k as the portion of the i^{th} population with k connections, that are committed to the first option. Similarly, for y_i^k and z_i^k, and we have $z_i^k = 1 - x_i^k - y_i^k$. We also refer to the pair (x_i^k, y_i^k) (or the triple (x_i^k, y_i^k, z_i^k)) as the k^{th} cluster of the i^{th} population. It is noteworthy that the full state of the networked population has as state vectors

$$\left(x_1^1,\ x_1^2,\ \dots,\ x_1^{k_{\max}},\ \dots,\ x_n^1,\ x_n^2,\ \dots,\ x_n^{k_{\max}}\right)^T,$$
$$\left(y_1^1,\ y_1^2,\ \dots,\ y_1^{k_{\max}},\ \dots,\ y_n^1,\ y_n^2,\ \dots,\ y_n^{k_{\max}}\right)^T,$$

where we assumed that for every complex network, the degree distribution $P(k)$ is the same, and hence the maximal node degree $k_{\max}$ is also equal for every population. Let us now consider the dynamics for a single cluster k within a single complex network i of the multi-population. In other words, we will now come up with dynamics for x_i^k and y_i^k. We introduce $\psi_k = \frac{k}{k_{\max}}$ as a parameter that captures the connectivity of a cluster. We have that ψ_k is close to zero if k is very small (no connectivity) and ψ_k is close to one if k is close to $k_{\max}$ (full connectivity). For the discrete random variable k, with probabilities $P(k)$, we denote the mean by $\langle k\rangle$ with

$$\langle k\rangle = \sum_{k=1}^{k_{\max}} kP(k).$$

Let us introduce θ_i^x and θ_i^y as the probabilities that a randomly chosen link will point to a player that uses strategy x or y in group i, respectively.[26] These new variables capture the first moment. They are defined as

$$\theta_i^x = \frac{1}{\langle k\rangle}\sum_{k=1}^{k_{\max}} kP(k)x_i^k, \quad \theta_i^y = \frac{1}{\langle k\rangle}\sum_{k=1}^{k_{\max}} kP(k)y_i^k.$$

The dynamics of cluster k in group i are now given by

$$\begin{aligned}\dot{x}_i^k &= (1 - x_i^k - y_i^k)\left(r\psi_k\sum_{j\in\mathcal{N}_i}\theta_j^x + \gamma\right) - x_i^k\left(\alpha + \sigma\psi_k\sum_{j\in\mathcal{N}_i}\theta_j^y\right),\\ \dot{y}_i^k &= (1 - x_i^k - y_i^k)\left(r\psi_k\sum_{j\in\mathcal{N}_i}\theta_j^y + \gamma\right) - y_i^k\left(\alpha + \sigma\psi_k\sum_{j\in\mathcal{N}_i}\theta_j^x\right).\end{aligned} \tag{4}$$

Note that these dynamics are normalized, in the sense that $x_i^k + y_i^k + z_i^k = 1$ for every $k \in \{1, 2, \dots, k_{\max}\}$ and for every $i \in \{1, 2, \dots, n\}$. We made this normalization because $z_i^k = 1 - x_i^k - y_i^k$ holds true, and, as a result this variable became redundant. These dynamics are similar to the model in (1) except that we now sum over the multiple θ parameters, and the connectivity parameter ψ_k now plays a role as well, since we are looking at a multi-population of complex networks. By aggregating over the differential equations of $\dot{x}_i^k$ and $\dot{y}_i^k$ using $\frac{1}{\langle k\rangle}\sum_k kP(k)$, we obtain dynamics for $\dot{\theta}_i^x$ and $\dot{\theta}_i^y$, which are given by

$$\begin{aligned}\dot{\theta}_i^x &= (-\gamma - \alpha)\theta_i^x - \gamma\theta_i^y + \frac{r\Psi_i^z}{k_{\max}}\sum_{j\in\mathcal{N}_i}\theta_j^x - \frac{\sigma\Psi_i^x}{k_{\max}}\sum_{j\in\mathcal{N}_i}\theta_j^y + \gamma,\\ \dot{\theta}_i^y &= -\gamma\theta_i^x + (-\gamma - \alpha)\theta_i^y - \frac{\sigma\Psi_i^y}{k_{\max}}\sum_{j\in\mathcal{N}_i}\theta_j^x + \frac{r\Psi_i^z}{k_{\max}}\sum_{j\in\mathcal{N}_i}\theta_j^y + \gamma,\end{aligned}$$

where we have simplified our notation by introducing

$$\begin{aligned}V &= \textstyle\sum_k k^2P(k), & \Psi_i^x &= \frac{1}{\langle k\rangle}\textstyle\sum_k k^2P(k)x_i^k,\\ \Psi_i^y &= \frac{1}{\langle k\rangle}\textstyle\sum_k k^2P(k)y_i^k, & \Psi_i^z &= \frac{1}{\langle k\rangle}\textstyle\sum_k k^2P(k)z_i^k.\end{aligned}$$

Note that we have $V = \langle k\rangle\left(\Psi_i^x + \Psi_i^y + \Psi_i^z\right)$. These variables capture the second moment. We can write the dynamics for θ_i^x and θ_i^y for $i = 1, 2, \dots, n$ in vector form, by introducing

$$\theta^x = (\theta^x_1,\ \theta^x_2,\ \dots,\ \theta^x_n),\quad \theta^y = (\theta^y_1, \theta^y_2,\ \dots,\ \theta^y_n).$$

We then have

$$\begin{pmatrix} \dot\theta^x \\ \dot\theta^y \end{pmatrix} = \begin{bmatrix} (-\gamma-\alpha)I + \frac{r}{k_{\max}}\Psi^z A & -\gamma I - \frac{\sigma}{k_{\max}}\Psi^x A \\ -\gamma I - \frac{\sigma}{k_{\max}}\Psi^y A & (-\gamma-\alpha)I + \frac{r}{k_{\max}}\Psi^z A \end{bmatrix} \begin{pmatrix} \theta^x \\ \theta^y \end{pmatrix} + \begin{pmatrix} \gamma\mathbb{1} \\ \gamma\mathbb{1} \end{pmatrix}, \tag{5}$$

where we use A to denote the adjacency matrix of the multi-population, and where we use Ψ^x, Ψ^y and Ψ^z to denote

$$\begin{aligned} \Psi^x &= \mathrm{diag}(\Psi^x_1, \Psi^x_2, \dots, \Psi^x_n), \\ \Psi^y &= \mathrm{diag}(\Psi^y_1, \Psi^y_2, \dots, \Psi^y_n), \\ \Psi^z &= \mathrm{diag}(\Psi^z_1, \Psi^z_2, \dots, \Psi^z_n). \end{aligned}$$

The equilibria of the above model are in general hard to find, but the equilibrium $(\theta^{x*}, \theta^{y*})$ satisfies at least the following expression

$$\begin{pmatrix} \theta^{x*} \\ \theta^{y*} \end{pmatrix} = \begin{bmatrix} (\gamma+\alpha)I - \frac{r}{k_{\max}}\Psi^z A & \gamma I + \frac{\sigma}{k_{\max}}\Psi^x A \\ \gamma I + \frac{\sigma}{k_{\max}}\Psi^y A & (\gamma+\alpha)I - \frac{r}{k_{\max}}\Psi^z A \end{bmatrix}^{-1} \begin{pmatrix} \gamma\mathbb{1} \\ \gamma\mathbb{1} \end{pmatrix},$$

where the matrix is nonsingular if it is diagonally dominant. Computation of an equilibrium analytically is a difficult task, and is beyond the scope of this paper. Suppose for now, an equilibrium $(\theta^{x*}, \theta^{y*})$ is found. Then we can conclude the following result on stability.

Theorem 3. *Let the structured networked multi-population be characterized by a regular and unweighted graph with node degree d and with adjacency matrix $A = (a_{ij})$. Then, an equilibrium $(\theta^{x*}, \theta^{y*})$ of (5) is locally asymptotically stable if the following pair of inequalities is satisfied:*

$$\begin{aligned} \alpha &> \frac{r}{k_{\max}}\Psi^z_* d + \frac{\sigma}{k_{\max}}\Psi^x_* d, \\ \alpha &> \frac{\sigma}{k_{\max}}\Psi^y_* d + \frac{r}{k_{\max}}\Psi^z_* d, \end{aligned}$$

where $\Psi^x_*, \Psi^y_*, \Psi^z_*$ denote the second moments $\Psi^x_i, \Psi^y_i, \Psi^z_i$, since the latter are the same for every population i.

It should be noted that the equilibrium values play a role in the computation of Ψ^x_*, Ψ^y_* and Ψ^z_*.

Proof. The result follows directly from applying Greshgorin's circle theorem to the affine system of (5). Computing the eigenvalues of the system matrix explicitly is difficult, however, they can be estimated by using Gershgorin's circle theorem. For every row of the system, we consider the following point: requiring that it is larger in magnitude than the sum of the off-diagonal entries entails that the eigenvalues are contained in the open left-half complex plane. This is true since the diagonal entry is strictly negative. So for $i = 1, 2, \dots, n$ we have that

$$\begin{aligned} |-\gamma-\alpha| &> \left|\frac{r}{k_{\max}}\Psi^z_i \sum_{j=1}^n a_{ij}\right| + |-\gamma| + \left|-\frac{\sigma}{k_{\max}}\Psi^x_i \sum_{j=1}^n a_{ij}\right| \\ \alpha &> \frac{r}{k_{\max}}\Psi^z_i \sum_{j=1}^n a_{ij} + \frac{\sigma}{k_{\max}}\Psi^x_i \sum_{j=1}^n a_{ij} \\ \alpha &> \frac{r}{k_{\max}}\Psi^z_i d + \frac{\sigma}{k_{\max}}\Psi^x_i d, \end{aligned}$$

since $\sum_{j=1}^n a_{ij} = |\mathcal{N}_i| = d$ for all i. Similarly, for rows $i = n+1, \dots, 2n$ we require that

$$\alpha > \frac{r}{k_{\max}}\Psi^z_i d + \frac{\sigma}{k_{\max}}\Psi^y_i d.$$

Furthermore, we note that since we consider a regular graph, we reach a consensus equilibrium, that is, the values for x_i^k and y_i^k are the same for every $1 \leq i \leq n$, as in Theorem 1. Finally, we have

$$\begin{aligned}
\Psi_i^x &= \frac{1}{\langle k \rangle} \sum_k k^2 P(k) x_i^k = \frac{1}{\langle k \rangle} \sum_k k^2 P(k) \xi^k = \Psi_*^x, \\
\Psi_i^y &= \frac{1}{\langle k \rangle} \sum_k k^2 P(k) y_i^k = \frac{1}{\langle k \rangle} \sum_k k^2 P(k) \mu^k = \Psi_*^y, \\
\Psi_i^z &= \frac{1}{\langle k \rangle} \sum_k k^2 P(k) z_i^k = \frac{1}{\langle k \rangle} \sum_k k^2 P(K) \zeta^k = \Psi_*^z.
\end{aligned}$$

This completes the proof. ∎

In the special case of reaching a symmetric equilibrium as in Case 1 of Theorem 1, we have the following corollary.

Corollary 1. *Let the underlying network of the multi-population be a regular and unweighted graph, with node degree d. Then, any symmetric equilibrium of* (5) *is locally asymptotically stable if the following condition on the cross-inhibitory signal holds*

$$\sigma < 2r - \frac{rV}{< k > \Psi_*} + \frac{\alpha k_{\max}}{\Psi_* d}, \tag{6}$$

where $\Psi_* := \Psi_*^x = \Psi_*^y$.

Proof. Since we have a symmetric equilibrium, $\Psi_*^x = \Psi_*^y =: \Psi_*$. Furthermore this means $\Psi_*^z = \frac{V}{<k>} - \Psi_*^x - \Psi_*^y = \frac{V}{<k>} - 2\Psi_*$. Starting from Theorem 3 and performing some algebraic manipulations, we have

$$\begin{aligned}
\alpha &> \frac{r}{k_{\max}} \Psi_*^z d + \frac{\sigma}{k_{\max}} \Psi_*^x d \\
\alpha &> \frac{r}{k_{\max}} \left(\frac{V}{< k >} - 2\Psi_* \right) d + \frac{\sigma}{k_{\max}} \Psi_* d \\
\frac{-\sigma}{k_{\max}} \Psi_* d &> \frac{r}{k_{\max}} \left(\frac{V}{< k >} - 2\Psi_* \right) d - \alpha \\
\sigma &< \frac{-r}{\Psi_*} \left(\frac{V}{< k >} - 2\Psi_* \right) + \frac{\alpha k_{\max}}{\Psi_* d} = 2r - \frac{rV}{< k > \Psi_*} + \frac{\alpha k_{\max}}{\Psi_* d},
\end{aligned}$$

and the proof is completed. ∎

It should be stressed that this result is in agreement with Theorem 3.[6] Interestingly enough, it is worth pointing out that our result comes from applying Greshgorin's circle theorem, while the work by Stella and Bauso[6] analyses a 2×2 matrix via its trace and determinant. We have thus derived a similar result using a different methodology.

4 | SYMMETRIC EQUILIBRIUM IN STRUCTURED ENVIRONMENT

Throughout this section, we assume that the equilibrium we reached is a symmetric equilibrium, cf. Case 1 in Theorem 2. In such case, we have that $x_i^k = y_i^k = \xi_i^k = \mu_i^k$. Since this is a consensus equilibrium as described before, this value is in fact equal to ξ^k, where the equilibrium point can still depend on the complex network node degree k. We remark that for these symmetric consensus equilibria, $\theta_i^x = \theta_i^y = \theta$. Hence,

$$\sum_{j \in \mathcal{N}_i} \theta_j^x = \sum_{j \in \mathcal{N}_i} \theta_j^y = d\theta,$$

where d denotes the degree of the regular network. The dynamics for x_i^k and y_i^k are now described by

$$\dot{x}_i^k = (1 - x_i^k - y_i^k)(r\psi_k d\theta + \gamma) - x_i^k(\alpha + \sigma\psi_k d\theta),$$
$$\dot{y}_i^k = (1 - x_i^k - y_i^k)(r\psi_k d\theta + \gamma) - y_i^k(\alpha + \sigma\psi_k d\theta).$$

Equivalently, we can write these dynamics in matrix form as

$$\begin{pmatrix} \dot{x}_i^k \\ \dot{y}_i^k \end{pmatrix} = \begin{bmatrix} -(r+\sigma)\psi_k d\theta - \alpha - \gamma & -r\psi_k d\theta - \gamma \\ -r\psi_k d\theta - \gamma & -(r+\sigma)\psi_k d\theta - \alpha - \gamma \end{bmatrix} \begin{pmatrix} x_i^k \\ y_i^k \end{pmatrix} + \begin{pmatrix} r\psi_k d\theta + \gamma \\ r\psi_k d\theta + \gamma \end{pmatrix}. \tag{7}$$

The remainder of this section focuses on the study of the above system. By direct computation, one can show that the above affine system is stable by computing the eigenvalues of the system matrix.

Theorem 4. *The above system has eigenvalues $\lambda_1 = -\sigma\psi_k d\theta - \alpha$ and $\lambda_2 = -(2r+\sigma)\psi_k d\theta - 2\gamma - \alpha$, which are strictly negative and, therefore, the system is asymptotically stable. Furthermore, convergence is faster as ψ_k increases.*

Next, we study the equilibrium of system (7).

Theorem 5. *For system (7), a symmetric equilibrium is given by*

$$x_i^{k*} = y_i^{k*} = x^* = \frac{r\psi_k d\theta + \gamma}{(2r+\sigma)\psi_k d\theta + 2\gamma + \alpha}.$$

Proof. As the dynamics of system (7) are of the form $\dot{x} = Ax + c$, the equilibrium is simply given by $x^* = -A^{-1}c$. We remark that the determinant of A is given by

$$\det(A) = \Delta = 2(r\psi_k d\theta + \gamma)(\sigma\psi_k d\theta + \alpha) + (\sigma\psi_k d\theta + \alpha)^2,$$

which is nonzero as the system parameters are all positive. So matrix A is always nonsingular. Performing straightforward computations to obtain $-A^{-1}c$ yields the desired result. ■

One can show that if $\frac{\alpha}{\gamma} > \frac{\sigma}{r}$, we have that $\frac{\partial x_i^{k*}}{\partial \psi_k}(\psi_k) > 0$ for all ψ_k, thus the equilibrium x_i^{k*} has a higher value in its x component, since the connectivity ψ_k is increasing. In such case, the fraction of uncommitted individuals (within a cluster) is decreasing.

5 | NUMERICAL STUDIES

In this section, we provide two sets of simulations to support our results. The first set of simulations is carried out to show an application of Theorem 1. We look at a regular and unweighted network given by the Buckminster Fuller geodesic dome, which is a regular graph on 60 nodes where each node has degree 3, see the top-right corner of Figure 2. The outcome of this set of simulations is shown in Figure 2. The values of the parameters are set as follows: $\gamma = 0.2$, $\alpha = 0.4$, $r = 0.3$ and $\sigma = 0.4$.

The trajectories of x_i are depicted in dashed blue lines, while the trajectories of y_i are shown in solid red lines. We observe that we reach a consensus equilibrium with $x_i = \xi = 0.222$ and $y_i = \mu = 0.333$. We note that $\zeta = 1 - \xi - \mu = 0.444$. This is equal to $\frac{\alpha}{rd}$. Computing the values of ξ and μ using the formulas in Case 2 of Theorem 1 yields the same values as the results obtained from the simulation.

Next, we perform a second set of simulations on the networked multi-population where the interactions within each population are modeled by means of a structured environment. We assume that every node of the networked multi-population is a complex network, and this complex network is clustered based on the connectivity. As every population i is now a complex network, cluster k in population i is simply the portion of nodes with k connections. We have thus $k_{\max}$ clusters, where the final cluster has maximum connectivity equal to $k_{\max}$. We assume that every population of the

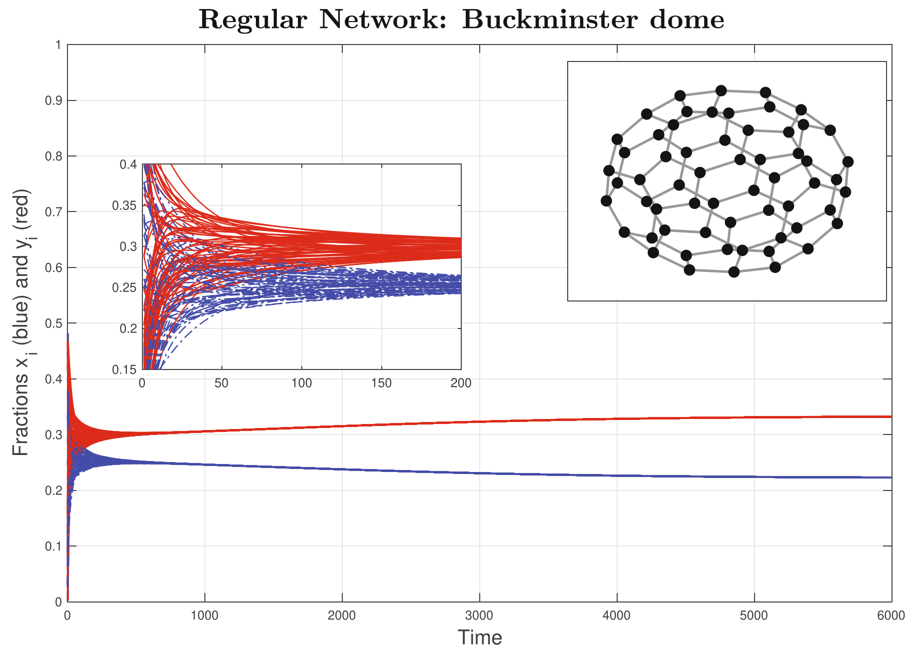

FIGURE 2 Dynamics on the regular networked Buckminster dome.

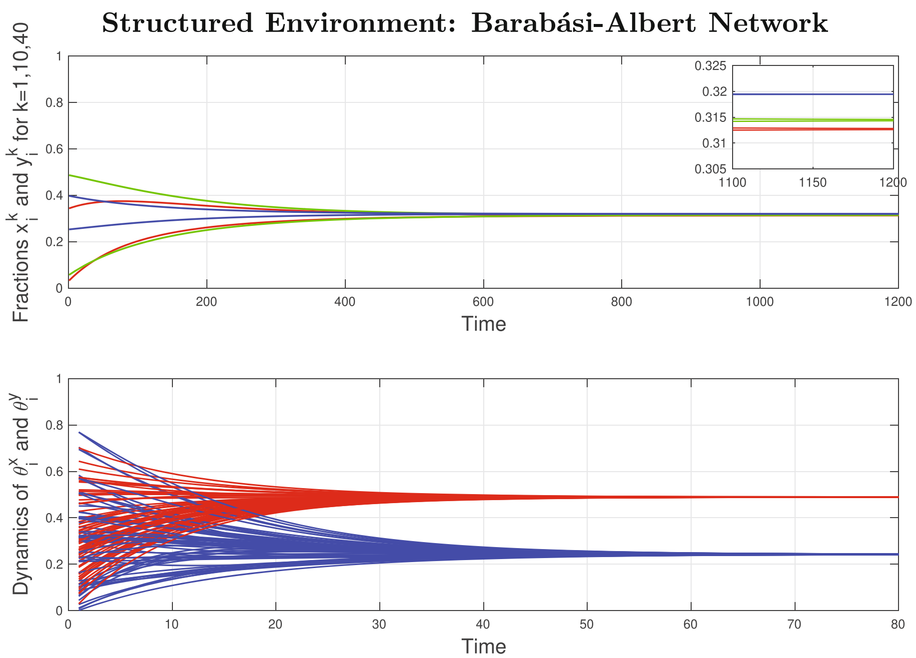

FIGURE 3 Evolutionary dynamics for different clusters of a population, with connectivity $k = 1$ (red), $k = 10$ (green) and $k = 40$ (blue), respectively (top). Dynamics of θ_i^x and θ_i^y for all $i = 1, 2, \dots, 60$ (bottom).

networked multi-population has the same probability distribution of the complex network, and this discrete distribution follows a power law distribution.

This complex network was constructed using the Barabási-Albert model, and thus the complex network is scale-free. We consider the networked multi-population where the overall network is given by the regular graph of the Buckminster dome as before. So, we have in fact an interconnected network of 60 complex networks, where each complex network is scale free and follows the degree distribution of the Barabási-Albert model.

We set the system parameters to $\gamma = 0.5, \alpha = 0.6, r = 0.4$ and $\sigma = 0.3$. Then, we simulate the normalized dynamics of each cluster according to Equation (4). At every iteration, for every population, we compute the θ_i^x and θ_i^y. We have plotted the dynamics for different clusters $k = 1$, $k = 10$, and $k = 40$ of an arbitrary population of the multi-population. The results are given in Figure 3 (top). Here, the normalized fractions x_i^1 and y_i^1 are plotted in red, x_i^{10} and y_i^{10} are plotted in green, and x_i^{40} and y_i^{40} are plotted in blue. We note that the six trajectories converge to three values: x_i^1 and y_i^1 converge to 0.3126, x_i^{10} and y_i^{10} converge to 0.3145 and x_i^{40} and y_i^{40} converge to 0.3194. We remark that for every cluster, we reach a symmetric equilibrium since x_i^k and y_i^k converge to the same value. We note that the value of the components of this equilibrium increases as the connectivity increases, which is in accordance with Theorem 6.

Finally, we perform a simulation on the model described by (5). We consider again the multi-population of the Buckminster dome, using the same values of σ, r, α and γ as in the previous simulation. As starting condition, we take the steady state values of x_i^{k*} and y_i^{k*} from the previous simulation, so that we can compute the second moments Ψ_i^x, Ψ_i^y and Ψ_i^z. Having fixed these, we now consider the dynamics given by (5). The time evolution of θ_i^x and θ_i^y is given in Figure 3 (bottom), where θ_i^x is plotted in red and θ_i^y is plotted in blue.

We observe that the θ_i^x and θ_i^y reach a consensus equilibrium. It is worth to highlight that for these values of the system, condition (6) is satisfied and we have reached asymptotic stability, in accordance with Corollary 5.

6 | CONCLUSION

In this paper, we have studied bio-inspired evolutionary dynamics on a multi-population, where the structure is captured by a regular network. The dynamics of an individual of each group not only change as a result of the interactions within its own group, but they also take into account the states of neighboring populations. First, we have given a description of the average behavior of each population, where the states of the system represent the fractions of the population committed to each option. We have shown that the steady state is a consensus equilibrium, and a sufficient condition for stability of this equilibrium in terms of a lower bound on the cross-inhibitory signals has been given. Secondly, we have studied the situation where a structured environment is considered. The individuals in each population now interact by means of a complex network. Within a group, it is then possible to cluster the group based on the internal connectivity of a player, and we have developed a model for the dynamics of each cluster of each population. By aggregating the equations over all clusters of a single population, we have found a description in terms of second moments. We have analyzed this model and we have presented the results in terms of stability of the equilibria. The paper is concluded by carrying out two sets of simulations to validate our theoretical results. In our future research, we plan to study the impact of a general network of a multi-population, where the condition that the underlying network is regular is dropped. We also want to investigate the heterogeneous case, where the system parameters are different for each group. Finally, we find the case of asymmetric parameters of interest as well.

AUTHOR CONTRIBUTIONS

Wouter Baar, Leonardo Stella, and Dario Bauso designed the research, Wouter Baar performed the research and Wouter Baar and Leonardo Stella wrote the paper.

CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID

Leonardo Stella https://orcid.org/0000-0002-2670-9873
Dario Bauso https://orcid.org/0000-0001-9713-677X

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