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Moonshine at Landau-Ginzburg points

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ABSTRACT: We formulate a conjecture predicting unexpected relationships among the coefficients of the elliptic expansions of Klein's modular j-function around $j = 0$ and $j = 1728$. Our conjecture is inspired by recent developments in mirror symmetry, in particular by work of Tu [Tu19] computing categorical enumerative invariants of matrix factorization categories and by work of Li-Shen-Zhou [LSZ20] computing FJRW invariants of elliptic curves.

Since this paper was announced a proof of the main conjecture has been claimed by Hong-Mertens-Ono-Zhang [HMOZ21].

1. The conjecture

1.1. The Monstrous Moonshine conjecture describes a surprising relationship, discovered in the late 1970s, between the coefficients of the Fourier expansion of Klein's j-function around the cusp

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20235856256q^4 + \cdots$$

and dimensions of irreducible representations of the Monster group. Fourier expansions of other modular forms around the cusp are critically important in number theory and algebraic geometry. In particular such expansions appear directly in computations of Gromov-Witten invariants of elliptic curves [Dij95].

1.2. In this note we study the elliptic expansion of the j-function around the hexagonal point $j = 0$ and the square point $j = 1728$, instead of around the cusp $j = \infty$. At $j = 0$ the elliptic curve is the Fermat cubic, cut out in $\mathbb{P}^2$ by $x^3 + y^3 + z^3 = 0$, while at $j = 1728$ it is given by $x^4 + y^4 + z^2 = 0$ in the weighted projective space $\mathbb{P}^2_{1,1,2}$.

From an enumerative geometry perspective the fact that we work around the hexagonal and square points instead of around the cusp suggests that we are working with Fan-Jarvis-Ruan-Witten (FJRW) invariants instead of Gromov-Witten invariants. See (2.7) for details.

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1.3. Let $\mathbb{H}$ and $\mathbb{D}$ denote the upper half plane and the unit disk in the complex plane, respectively. Fix $\tau_* = e^{\pi \mathtt{i}/3}$ or $\tau_* = \mathtt{i}$ as the points* in $\mathbb{H}$ around which to carry out the expansion.

The uniformizing map S around τ_* is the map

$$S : \mathbb{H} \to \mathbb{D}, \qquad S(\tau) = \frac{\tau - \tau_*}{\tau - \bar{\tau}_*},$$

with inverse

$$S^{-1} : \mathbb{D} \to \mathbb{H}, \qquad S^{-1}(w) = \frac{\tau_* - \bar{\tau}_* w}{1 - w}.$$

The *elliptic* expansion of j around τ_* is simply the Taylor expansion of $j \circ S^{-1}$ around $w = 0$. Its coefficients are closely related [Zag08, Proposition 17] to the values of the higher modular derivatives $\partial^n j(\tau_*)$,

$$j\left(S^{-1}(w)\right) = \sum_{n=0}^{\infty} \frac{(4\pi \operatorname{Im} \tau_*)^n \partial^n j(\tau_*)}{n!} w^n.$$

1.4. The values of the higher modular derivatives of j can be computed term-by-term by a well-known recursive procedure. The results are rational multiples of products of powers of the Chowla-Selberg period† Ω and of π.

Let $s(w) = 2\pi\Omega^2 \cdot S(w)$ denote the rescaling of S by the factor $2\pi\Omega^2$. Then around $\tau_* = \exp(\pi \mathtt{i}/3)$ we have

$$j\left(s^{-1}(w)\right) = 13824w^3 - 39744w^6 + \frac{1920024}{35}w^9 - \frac{1736613}{35}w^{12} + \cdots,$$

while around $\tau_* = \mathtt{i}$ we have

$$j\left(s^{-1}(w)\right) = 1728 + 20736w^2 + 105984w^4 + \frac{1594112}{5}w^6 + \frac{3398656}{5}w^8 + \cdots.$$

1.5. The following power series have been introduced independently by Shen-Zhou [SZ18, (3.41), (3.45)] in their study of the LG/CY correspondence for elliptic curves, and by Tu [Tu19, Section 4] in his study of categorical Saito theory of Fermat cubics:

$$g(t) = \sum_{n=0}^{\infty} (-1)^n \frac{((3n-2)!!!)^3}{(3n)!} t^{3n},$$

$$h(t) = \sum_{n=0}^{\infty} (-1)^n \frac{((3n-1)!!!)^3}{(3n+1)!} t^{3n+1}.$$

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*Any other point in the $\mathrm{SL}(2,\mathbb{Z})$ orbit of τ_* works equally well, with only minor changes in the constants below.

†The exact value of Ω is unimportant, but in this case $\Omega = 1/\sqrt{6\pi}\left(\Gamma(1/3)/\Gamma(2/3)\right)^{3/2}$ for the hexagonal point and $\Omega = 1/\sqrt{8\pi}\left(\Gamma(1/4)/\Gamma(3/4)\right)$ for the square point.

In both cases it was argued that the ratio $h(t)/g(t)$ gives a flat coordinate on the moduli space of versal deformations $x^3 + y^3 + z^3 + 3txyz = 0$ of the Fermat cubic.

Similarly, for the elliptic quartic we introduce the two power series below

$$g(t) = \sum_{n=0}^{\infty} \frac{((4n-3)!!!!)^2}{(2n)!} t^{2n},$$
$$h(t) = \sum_{n=0}^{\infty} \frac{((4n-1)!!!!)^2}{(2n+1)!} t^{2n+1}.$$

Even though the notation g, h appears overloaded, it should be evident from context which power series we refer to.

Our main result is the following conjecture.

1.6. Conjecture. *(a) Around the hexagonal point the elliptic expansion of the j-function satisfies*

$$\begin{aligned} j\left(s^{-1}\left(\frac{h(t)}{g(t)}\right)\right) &= 27t^3\left(\frac{8-t^3}{1+t^3}\right)^3 \\ &= 13824t^3 - 46656t^6 + 99144t^9 - 171315t^{12} + 263169t^{15} - \cdots . \end{aligned}$$

(b) Around the square point the elliptic expansion of the j-function satisfies

$$\begin{aligned} j\left(s^{-1}\left(\frac{h(t)}{g(t)}\right)\right) &= (192 + 256t^2)\left(\frac{3+4t^2}{1-4t^2}\right)^2 \\ &= 1728 + 20736t^2 + 147456t^4 + 851968t^6 + 4456448t^8 + \cdots . \end{aligned}$$

1.7. Notes. It is remarkable that the coefficients in the above power series are all integers, despite $j(s^{-1}(w))$ only having rational coefficients. (For the expansion at the hexagonal point the integrality of the coefficients follows from [SZ18].) Our attempts to find other modular forms with this integrality property, using other combinations of the Eisenstein modular forms E_2, E_4, and E_6 have been unsuccessful.

The validity of the formulas above has been verified by computer up to t^{24}. A recent proof of Conjecture 1.6 was announced, after our paper was made public, in [HMOZ21].

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2. Mirror symmetry origin of the conjecture

2.1. The original statement of mirror symmetry is formulated as the equality of two power series associated to a pair $(X, \check{X})$ of mirror symmetric families of Calabi-Yau varieties. These two power series are

(a) the generating series, in a formal variable Q, of the enumerative invariants of the family X (the A-model potential);

(b) the Taylor expansion of a Hodge-theoretic function (the period) on the moduli space of complex structures M^{cx} of the mirror family $\check{X}$, with respect to a flat coordinate q on this moduli space (the B-model potential).

In order to compare the two power series, the variables q and Q are identified via an invertible map ψ called the mirror map.

In physics the formal variable Q is viewed as a flat coordinate on the (ill-defined mathematically) complexified Kähler moduli space $M^{\mathsf{Käh}}$, and the mirror map is interpreted as an isomorphism

$$\psi : M^{\mathsf{cx}} \to M^{\mathsf{Käh}}$$

between germs of M^{cx} and $M^{\mathsf{Käh}}$ around special points. Traditionally these special points are the large complex limit point and the large volume point, respectively.

2.2. The original mirror symmetry computation of [COGP91] follows this pattern. It predicts a formula for the generating series of genus zero Gromov-Witten invariants of the quintic X, by equating it to the expansion of a period (solution of the Picard-Fuchs equation) for the family of mirror quintics $\check{X}$. The equality of the two sides allows one to calculate the genus zero Gromov-Witten invariants, by expanding the period map of the family $\check{X}$ with respect to a certain flat coordinate on the moduli space of complex structures of mirror quintics.

As another example consider a two-torus X (elliptic curve with arbitrary choice of complex structure). The $g = 1, n = 1$ Gromov-Witten invariant of degree $d \geq 1$ with insertion the Poincaré dual class of a point counts in this case the number of isogenies of degree d to a fixed elliptic curve. As such it satisfies

$$\langle [\mathsf{pt}]^{\mathsf{PD}} \rangle_{1,1}^{X,d} = \sum_{k|d} k = \sigma_1(d),$$

and hence the generating series of these invariants (including the $d = 0$ case) is $-\frac{1}{24}E_2(Q)$ where E_2 denotes the quasi-modular Eisenstein form of weight two (see [Dij95] for the original derivation of this calculation). The main result of ([CT17]) is that this equals the expansion in $q = \exp(2\pi \mathrm{i}\tau)$, around $q = 0$, of the function of categorical enumerative $(1,1)$ invariants for the corresponding family $\check{X}$ of mirror elliptic curves.

2.3. Implicit in the above calculation for elliptic curves are the two facts that

(a) q is the flat coordinate, around the cusp, on the moduli space of elliptic curves;

(b) the mirror map ψ for elliptic curves identifies q with Q.

The main intuition behind Conjecture 1.6 is a similar set of assumptions, but for the flat coordinates around the hexagonal or square points instead of around the cusp. Below we will give precise conjectural descriptions of the flat coordinates q and Q around the hexagonal point $\check{F} \in M^{\mathsf{cx}}$ and its mirror $F \in M^{\mathsf{K\ddot{a}h}}$. The analysis for the square point is entirely similar.

2.4. To understand these flat coordinates we need good descriptions of $M^{\mathsf{K\ddot{a}h}}$ and M^{cx} around F and $\check{F}$. We will review first the classical situation (around the cusp) described in the work of Polishchuk-Zaslow [PZ98].

Polishchuk-Zaslow take the space $M^{\mathsf{K\ddot{a}h}}$ on a two-torus to be the quotient of $\mathbb{H}$, with coordinate ρ, by $\rho \sim \rho + 1$. For each $\rho \in M^{\mathsf{K\ddot{a}h}}$ they construct a Fukaya category $\mathcal{F}^0(X^\rho)$ on the two-torus X^ρ endowed with this structure. The quotient above is precisely the same as the neighborhood of the cusp on the moduli space M^{cx} of complex structures on a two-torus[‡]. For Polishchuk-Zaslow the mirror map is simply the identity $\tau \leftrightarrow \rho$: the complex elliptic curve $\check{X}^\tau$ with modular parameter τ corresponds to the two-torus X^ρ with complexified Kähler structure $\rho = \tau$.

2.5. Even without explicitly constructing $M^{\mathsf{K\ddot{a}h}}$ as a moduli space of geometric objects, we could have understood its structure around the large volume limit point through mirror symmetry. Indeed, we could have simply taken $M^{\mathsf{K\ddot{a}h}}$ to *be* the neighborhood of the large complex structure limit point in M^{cx}, a space we understand. With this point of view the mirror map is always the identity.

2.6. We would like to understand a similar picture around the hexagonal point $\check{F} \in M^{\mathsf{cx}}$. Even though the results of Polishchuk-Zaslow do not extend to $\check{F}$, we can still conjecture that there is a larger moduli space $M^{\mathsf{K\ddot{a}h}}$ of "extended Kähler structures" (which no longer parametrizes just classical Kähler classes as before) and a point $F \in M^{\mathsf{K\ddot{a}h}}$ such that F corresponds under mirror symmetry to $\check{F}$. Then the point of view in (2.5) allows

[‡]We ignore the stack structure of M^{cx}, which only adds an extra $\mathbb{Z}/2\mathbb{Z}$ stabilizer.

us to understand the local structure of $M^{\mathsf{K\ddot{a}h}}$ around F: it should be the same as M^{cx} around $\check{F}$.

The germ of M^{cx} around $\check{F}$ is the quotient of $\mathbb{H}$ by

$$\tau \sim \frac{\tau - 1}{\tau},$$

exhibiting the germ of $\mathbb{H}$ around τ_* as a triple cover of M^{cx} branched over $\check{F}$. We will *define* the germ of $M^{\mathsf{K\ddot{a}h}}$ around F to be the quotient of $\mathbb{H}$ (with coordinate ρ) by $\rho \sim (\rho - 1)/\rho$. We think of $\rho \in \mathbb{H}$ as giving an (extended type) "complexified Kähler class" on the two torus, and write X^ρ for this (fictitious) symplectic geometry object. We emphasize that we do not attempt to give a rigorous mathematical definition of X^ρ, though it would be natural to suggest that the non-commutative geometric object associated to it should be the Fukaya-Seidel category one sees at this point. Despite this, the mirror map is, as before, $\tau \leftrightarrow \rho$.

2.7. The natural question is then what is the flat coordinate on $M^{\mathsf{K\ddot{a}h}}$ (as defined above) around F. We conjecture that this flat coordinate is $Q = s(\rho)^3$. The justification for this comes from work of Li-Shen-Zhou [LSZ20], where the authors suggest that the natural way to interpret the generating series of FJRW invariants for two-tori as a function of ρ is via the map s (with a different rescaling from ours). It would be natural to guess from their work that $s(\rho)$ is the flat coordinate. However, since ρ is only defined up to the equivalence $\rho \sim (\rho - 1)/\rho$, the equality

$$s\left(\frac{\rho - 1}{\rho}\right)^3 = s(\rho)^3$$

implies that Q descends[§] to a coordinate on $M^{\mathsf{K\ddot{a}h}}$, which we conjecture to be the flat coordinate around F.

2.8. In the B-model we have seen ([SZ18], [Tu19]) that $h(t)/g(t)$ gives a flat coordinate on the base $\mathbb{A}^1_t$ of the Hesse pencil of elliptic curves,

$$E_t : \quad x^3 + y^3 + z^3 + 3txyz = 0.$$

In particular, Tu's work was motivated by a study of categories of graded matrix factorizations, but via Orlov's correspondence [Orl06] these are equivalent to the derived categories of the above elliptic curves.

Again, $h(t)/g(t)$ does not give a coordinate on M^{cx} because locally $\mathbb{A}^1_t$ is a branched triple cover of M^{cx} around $\check{F}$. Its replacement $q = \big(h(t)/g(t)\big)^3$ does descend to a coordinate on M^{cx} around $\check{F}$, and we conjecture it is flat.

[§]This is not the only modification of $s(\tau)$ that descends to a coordinate on $M^{\mathsf{K\ddot{a}h}}$, which in general will not be flat. The same issue appears in the B-model.

2.9. By our construction of $M^{\mathsf{K\ddot{a}h}}$ the mirror map ψ is the identity, so the mirror of the complex curve $\check{X}^\tau$ with modular parameter τ is the symplectic object X^ρ with $\rho = \psi(\tau) = \tau$. (Despite being equal we prefer to keep ρ and τ distinct since they represent different geometric objects.)

Flat coordinates are unique up to multiplication by a scalar when the moduli spaces $M^{\mathsf{K\ddot{a}h}}$ and M^{cx} are one-dimensional. (The rescaling factor $2\pi\Omega^2$ in (1.4) was chosen so that this constant equals one.) It follows that the flat coordinates of X^ρ and $\check{X}^\tau$ are equal for $\rho = \tau$.

Consider a Hesse elliptic curve E_t for some value of t. It can be written as $\check{X}^\tau$ for some (non-unique) modular parameter $\tau \in \mathbb{H}$. The mirror of this curve is X^ρ for $\rho = \tau$. (We think of $\rho \in M^{\mathsf{K\ddot{a}h}}$, so the ambiguity in τ disappears.) It follows that

$$\left(\frac{h(t)}{g(t)}\right)^3 = q(\check{X}^\tau) = Q(X^\rho) = s(\rho)^3,$$

or, using the fact that s is invertible,

$$s^{-1}\left(\frac{h(t)}{g(t)}\right) \sim \rho$$

where $\sim$ is the equivalence relation used to define M^{cx} in (2.5). Applying the j-function to both sides and noting that it is $\sim$-invariant we get

$$j\left(s^{-1}\left(\frac{h(t)}{g(t)}\right)\right) = j(\rho) = j(E_t).$$

For the Hesse pencil the j-function can be computed easily [AD09] and the result is

$$j(E_t) = 27t^3\left(\frac{8-t^3}{1+t^3}\right)^3.$$

This is the statement of the conjecture.

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