Tomographic Fourier Extension Identities for Submanifolds of R^n

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TOMOGRAPHIC FOURIER EXTENSION IDENTITIES FOR SUBMANIFOLDS OF $\mathbb{R}^n$

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Abstract. We establish identities for the composition $T_{k,n}(|\widehat{gd\sigma}|^2)$, where $g \mapsto \widehat{gd\sigma}$ is the Fourier extension operator associated with a general smooth k-dimensional submanifold of $\mathbb{R}^n$, and $T_{k,n}$ is the k-plane transform. Several connections to problems in Fourier restriction theory are presented.

1. Introduction

The purpose of this article is to establish identities involving expressions of the form

$$T_{k,n}(|\widehat{gd\sigma}|^2), \tag{1}$$

where $T_{k,n}$ denotes the *k-plane transform* on $\mathbb{R}^n$ and

$$\widehat{gd\sigma}(x) = \int_S g(\xi)e^{-2\pi i x\cdot\xi} d\sigma(\xi)$$

is the *Fourier extension operator* associated with a general smooth k-dimensional submanifold S of $\mathbb{R}^n$; here $d\sigma$ denotes surface measure on S. Many problems in harmonic analysis and its applications call for an understanding of Fourier extension operators, and we refer to [17] for further context. The particular interest in compositions of the form (1) stems from two very simple observations. The first is that they involve L^2 norms on affine subspaces, allowing for the application of L^2 methods (such as Plancherel's theorem). The second is the well-known fact that the k-plane transform is invertible on L^2, raising the prospect that a variety of expressions involving $|\widehat{gd\sigma}|^2$ may be understood via the composition (1). This idea may be traced back at least as far as Planchon and Vega [15] (see also [18]), and has motivated several works recently – see for example [7], [11], [4], [12].

In this work we place particular emphasis on the *generality of the submanifold* S, allowing us to bring the underlying geometric features of the tomographic data $T_{k,n}(|\widehat{gd\sigma}|^2)$ to the fore. Our results build on the work of the first and second authors in [12] in the particular case where S is the unit sphere. In addition to clarifying the underlying geometry in that work, the generality of our results leads to richer connections with contemporary problems in Fourier restriction theory.

In what follows we use the standard parametrisation of the grassman manifold $\mathfrak{M}_{k,n}$ of k-planes in $\mathbb{R}^n$ by a k-dimensional subspace π and a translation parameter $y \in \pi^\perp$. This allows the k-plane transform to be written as

$$T_{k,n}f(\pi, y) = \int_{\mathbb{R}^n} f(x)\delta_{\pi+\{y\}}(x)dx := \int f(x+y)d\lambda_\pi(x),$$

where λ_π is Lebesgue measure on π.

2. Identities and applications

Our main result here consists of two simple formulae for $T_{k,n}(|\widehat{gd\sigma}|^2)(\pi, y)$ that hold under the assumption that S and π satisfy a certain transversality condition. Notably, we see that $T_{k,n}$ is unable to distinguish $|\widehat{gd\sigma}|^2$ from $T^*_{n-k,n}\mu$ for a large family of measures μ on $\mathfrak{M}_{n-k,n}$, again provided a

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suitable transversality condition is satisfied. Here $T^*_{n-k,n}$ denotes the adjoint $(n-k)$-plane transform on $\mathbb{R}^n$,

$$T^*_{n-k,n}\mu(x) = \int_{\mathfrak{M}_{n-k,n}} \delta_{\theta+\{z\}}(x)d\mu(\theta,z), \tag{2}$$

which is of course a superposition of Dirac masses on $(n-k)$-planes. This may be interpreted as an explicit manifestation of the close relationship between Fourier restriction and Kakeya-type problems, and we refer to [12] and our forthcoming applications for more on this perspective. The family of measures μ with this property, which of course depends on the function g, is constructed via measures ν on the tangent bundle

$$TS = \{(\xi, y) : \xi \in S, y \in T_\xi S\},$$

and specifically those whose pushforwards ν_S under the natural projection map $TS \ni (\xi, y) \mapsto \xi \in S$ are given by

$$d\nu_S(\xi) = |g(\xi)|^2 d\sigma(\xi). \tag{3}$$

Perhaps the simplest such measure ν is $d\nu(y,\xi) = \delta_0(y)|g(\xi)|^2 dyd\sigma(\xi)$. Finally, the family of measures μ that we seek consists of the pushforwards of such ν under the (Gauss) map

$$TS \ni (\xi, y) \mapsto ((T_\xi S)^\perp, y) \in \mathfrak{M}_{n-k,n},$$

so that

$$\int_{\mathfrak{M}_{n-k,n}} \varphi d\mu = \int_{TS} \varphi((T_\xi S)^\perp, y)d\nu(\xi, y).$$

We note that the support of μ is contained in the set of translates of the normal planes to S.

Theorem 1. *Suppose that S is a k-dimensional C^1 submanifold of $\mathbb{R}^n$ and π is a k-dimensional subspace of $\mathbb{R}^n$ for which*

$$T_\xi S \cap \pi^\perp = \{0\} \;\; \textit{for all} \;\; \xi \in S, \tag{T}$$

and

$$\langle \xi - \eta\rangle \cap \pi^\perp = \{0\} \;\; \textit{for all} \;\; \xi, \eta \in S. \tag{GT}$$

Then for any measure μ satisfying the above conditions,

$$T_{k,n}(|\widehat{gd\sigma}|^2)(\pi, y) = \int_S \frac{|g(\xi)|^2}{|(T_\xi S)^\perp \wedge \pi|} d\sigma(\xi) = T_{k,n}T^*_{n-k,n}(\mu)(\pi, y). \tag{4}$$

Before turning to the context of Theorem 1 we make some clarifying remarks. Regarding the wedge product in (4), given any two transverse subspaces V, W of $\mathbb{R}^n$ of complementary dimensions ℓ and m, we define $|V \wedge W|$ to be $|v_1 \wedge \cdots \wedge v_\ell \wedge w_1 \wedge \cdots \wedge w_m|$, where $\{v_j\}$, $\{w_j\}$ are orthonormal bases of V and W respectively. Equivalently,

$$|V \wedge W| := \int_V \int_U e^{-\pi|u+v|^2} dudv,$$

which of course has the advantage of being explicitly well-defined.

The conditions (T) and (GT) are transversality conditions relating S and π, and both are necessary for (4) to hold – a fact that we establish in Section 4. The condition (GT) guarantees that S intersects any translate of $\pi^\perp$ in at most one point, allowing it to be viewed as a graph of a function ϕ over π. The condition (T) stipulates that all tangent spaces to S meet $\pi^\perp$ transversely, and ensures that this function ϕ is C^1. Specifically, if $U \subseteq \pi$ is the orthogonal projection of S onto π, and $u \in U$, then $\phi(u)$ is the unique element of the set $S \cap (\{u\} + \pi^\perp) - \{u\}$. By construction, $\phi : U \to \pi^\perp$, and

$$S = \{u + \phi(u) : u \in U\}. \tag{5}$$

The conditions (T) and (GT) are closely related. The local condition (T) may be viewed as a limiting (or infinitesimal) form of the global condition (GT) as η approaches ξ. Under various assumptions on S one of these conditions may be seen to imply the other. For example, if $k = 1$ and S is connected (making S a curve), an application of the mean value theorem reveals that (T) $\Longrightarrow$ (GT). In general, however, neither of these conditions implies the other, even if S is connected. Helical surfaces provide simple examples for which (T) is satisfied while (GT) is not. On the other hand, curves in the plane with a point of inflection can satisfy (GT) but not (T).

Of course if S has dimension or codimension 1, then Theorem 1 involves the *Radon transform* $R := T_{n-1,n}$ and *X-ray transform* $X := T_{1,n}$ respectively. Specifically, for $k = 1$ the identity (4) becomes

$$X(|\widehat{gd\sigma}|^2)(\omega, v) = \int_S \frac{|g(\xi)|^2}{|\tau(\xi) \cdot \omega|} d\sigma(\xi) = XR^*\mu(\omega, v),$$

where $\tau(\xi)$ denotes a unit tangent to S. Here we are identifying one-dimensional subspaces π with vectors $\omega \in \mathbb{S}^{n-1}$. On the other hand, for $k = n-1$ the identity (4) becomes

$$R(|\widehat{gd\sigma}|^2)(\omega, t) = \int_S \frac{|g(\xi)|^2}{|v(\xi) \cdot \omega|} d\sigma(\xi) = RX^*\mu(\omega, t),$$

where $v(\xi)$ denotes a unit normal to S. Here we have indulged a slightly more serious abuse of notation by reparametrising hyperplanes by a normal vector $\omega \in \mathbb{S}^{n-1}$ and a distance t from the origin.

The first identity in (4), and in particular its independence of the translation parameter $y \in \pi^\perp$, may be interpreted as a *conservation law*, generalising certain energy conservation properties of dispersive PDE, such as the time-dependent Schrödinger equation. This perspective is best understood in the setting of *parametrised extension operators*, where the submanifold S is parametrised by a C^1 injective map $\Sigma : U \to \mathbb{R}^n$. Here U is a subset of $\mathbb{R}^k$, and the (parametrised) extension operator associated to Σ is given by

$$Ef(x) = \int_U e^{-2\pi i x\cdot\Sigma(\xi)} f(\xi) d\xi,$$

where $x \in \mathbb{R}^n$. A simple computation reveals that $Ef = \widehat{gd\sigma}$, where f and g are related by

$$f(\xi) = \left|\frac{\partial\Sigma}{\partial\xi_1} \wedge \cdots \wedge \frac{\partial\Sigma}{\partial\xi_k}\right| g(\Sigma(\xi)),$$

and the first of the two identities in (4) becomes

$$T_{k,n}(|Ef|^2)(\pi, y) = \int_U \frac{|f(\xi)|^2}{\left|\frac{\partial\Sigma}{\partial\xi_1} \wedge \cdots \wedge \frac{\partial\Sigma}{\partial\xi_k} \wedge \pi^\perp\right|} d\xi. \tag{6}$$

In the particular case where $k = n-1$, $U = \mathbb{R}^{n-1}$, $\Sigma(\xi) = (\xi, |\xi|^2)$, and $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, then $u(x,t) := Ef(x,t)$ solves the *Schrödinger equation* $i\partial_t u = \Delta_x u$ with initial data $u(x,0) = \widehat{f}(x)$. Taking π to be purely spatial – that is the horizontal hyperplane at height t – the identity (6) reduces to the classical energy conservation law $\|u(\cdot,t)\|_{L^2_x} = \|u(\cdot,0)\|_{L^2_x}$.

We conclude this section with three applications that serve to further contextualise Theorem 1.

2.1. **Application 1: Restriction-Brascamp–Lieb inequalities.** Theorem 1, combined with a well-known theorem of Barthe [3], is easily seen to imply the endpoint case of the *restriction-Brascamp–Lieb inequality* ([8], [19], [6]), in the so-called rank-1 case. The restriction-Brascamp–Lieb inequality is a broad generalisation of the multilinear restriction inequality of [10], involving a collection of extension operators associated with submanifolds of various dimensions. Concretely, suppose that for each $1 \le j \le m$, $\Sigma_j : U_j \to \mathbb{R}^n$ is a smooth parametrisation of a n_j-dimensional submanifold S_j of $\mathbb{R}^n$ by a neighbourhood U_j of the origin in $\mathbb{R}^{n_j}$, and let

$$E_j g_j(\xi) := \int_{U_j} e^{-2\pi i\xi\cdot\Sigma_j(x)} g_j(x) dx$$

be the associated (parametrised) extension operator. In this setting it is natural to conjecture that if the Brascamp–Lieb constant $\mathrm{BL}(\mathbf{L}, \mathbf{p})$ is finite for the linear maps $L_j := (\mathrm{d}\Sigma_j(0))^* : \mathbb{R}^n \to \mathbb{R}^{n_j}$, then provided the neighbourhoods U_j of 0 are chosen to be small enough, the inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} |E_j g_j|^{2p_j} \lesssim \prod_{j=1}^{m} \|g_j\|_{L^2(U_j)}^{2p_j} \tag{7}$$

holds for all $g_j \in L^2(U_j)$, $1 \leq j \leq m$. We note that the weaker local inequality

$$\int_{B(0,R)} \prod_{j=1}^{m} |E_j g_j|^{2p_j} \leq C_\varepsilon R^\varepsilon \prod_{j=1}^{m} \|g_j\|_{L^2(U_j)}^{2p_j},$$

involving an arbitrary $\varepsilon > 0$ loss was established in [8] (see also [5] and [19] where the power loss is reduced to polylogarithmic). Here we make a modest contribution to this problem using a simple instance of Theorem 1.

Corollary 2. *The global inequality* (7) *holds whenever* $n_1 = \cdots = n_m = 1$.

Proof. We begin by observing that the linear map $L_j := (\mathrm{d}\Sigma_j(0))^* : \mathbb{R}^n \to \mathbb{R}$ satisfies $L_j x = \langle x, v_j \rangle$, where v_j is a tangent vector to S_j at the point $\Sigma_j(0)$. By Barthe's finiteness characterisation [3], the Brascamp–Lieb constant $\mathrm{BL}(\mathbf{L}, \mathbf{p})$ is finite if and only if $\mathbf{p}$ belongs to the convex hull of the points $\mathbf{p}^J$, where J denotes a subset of $[1, m] := \{1, \ldots, m\}$ of cardinality n, the exponent $p_j^J = \mathbf{1}_J(j)$, and J is such that the set $\{v_j : j \in J\}$ forms a basis for $\mathbb{R}^n$. Here $\mathbf{1}_J : [0, m] \to \{0, 1\}$ denotes the indicator function of J. Now, since $\mathrm{BL}(\mathbf{L}, \mathbf{p})$ is finite, there are nonnegative scalars λ_J such that

$$\mathbf{p} = \sum_J \lambda_J \mathbf{p}^J \quad \text{and} \quad \sum_J \lambda_J = 1.$$

By the m-linear Hölder inequality it follows that

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} |E_j g_j|^{2p_j} = \int_{\mathbb{R}^n} \prod_J \Big(\prod_{j=1}^{m} |E_j g_j|^{2p_j^J} \Big)^{\lambda_J} = \int_{\mathbb{R}^n} \prod_J \Big(\prod_{j \in J} |E_j g_j|^2 \Big)^{\lambda_J} \leq \prod_J \Big\| \prod_{j \in J} E_j g_j \Big\|_2^{2\lambda_J},$$

and so it remains to show that

$$\Big\| \prod_{j \in J} E_j g_j \Big\|_2 \lesssim \prod_{j \in J} \|g_j\|_2 \tag{8}$$

for each J. Provided the neighbourhoods U_j are chosen small enough, this elementary inequality is a straightforward consequence of Theorem 1 (in its parametrised form (6)) applied to the cartesian product

$$S := \prod_{j \in J} S_j \subset (\mathbb{R}^n)^n \cong \mathbb{R}^{n^2}$$

with the (diagonal) subspace $\pi = \{(x_1, \ldots, x_n) \in (\mathbb{R}^n)^n : x_1 = \cdots = x_n\}$. It is also instructive to provide a direct proof of (8) as it merely requires a change of variables and Plancherel's theorem. □

2.2. **Application 2: Weighted L^2 norm identities.** Theorem 1 sheds a little light on a variant of a longstanding conjecture of Stein [16] concerning the manner in which Kakeya-type maximal operators might control Fourier extension operators. This was initially considered for the extension operator for the sphere, and we refer to [9] and the references there for some context and results. In the setting of rather general submanifolds of $\mathbb{R}^n$ this (somewhat tentative) conjecture arose in discussions between the first author and Tony Carbery some years ago, and states that if S is a smooth (compact) k-dimensional submanifold of $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |\widehat{g d\sigma}(x)|^2 w(x) dx \lesssim \int_S |g(\xi)|^2 \sup_{y \in T_\xi S} T_{n-k,n} w((T_\xi S)^\perp, y) d\sigma(\xi). \tag{9}$$

A manifestly weaker form of (9), similarly generalising a conjecture attributed to Mizohata and Takeuchi – see [2] – is the claim that

$$\int_{\mathbb{R}^n} |\widehat{g d\sigma}(x)|^2 w(x) dx \lesssim \|T_{n-k,n} w\|_\infty \|g\|_2^2, \tag{10}$$

where the L^∞ norm should be interpreted as a supremum rather than an essential supremum. We refer to [9] and [1] for further discussion.

In the particular case that w lies in the image of $T_{k,n}^*$, and satisfies a certain transversality condition, the conjectural inequality (9) follows from Theorem 1, where it is in fact *an identity.*

Corollary 3. *If $w = T_{k,n}^* u$ for some $u : \mathfrak{M}_{k,n} \to [0,\infty)$ whose support is transverse to S in the sense of* (T) and (GT), then

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}(x)|^2 w(x)dx = \int_S |g(\xi)|^2 \sup_{y\in T_\xi S} T_{n-k,n}w((T_\xi S)^\perp, y)d\sigma(\xi). \tag{11}$$

A similar observation was made in the case $S = \mathbb{S}^{n-1}$ in [12]; see also [13] for some related results.

Proof. Observe first that by an application of (4),

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}|^2 w = \langle T_{k,n}(|\widehat{gd\sigma}|^2), u\rangle = \langle T_{k,n}T_{n-k,n}^*\mu, u\rangle = \int_{\mathfrak{M}_{k,n}} T_{n-k,n}w\, d\mu.$$

Here μ may be any admissible measure on $\mathfrak{M}_{n-k,n}$ – that is, the pushforward of a measure ν on TS satisfying the marginal condition (3); for example $d\nu(\xi,y) = |g(\xi)|^2\delta_0(y)dyd\sigma(\xi)$. By the transversality hypotheses on the support of u, it follows that $T_{n-k,n}w(\pi,y)$ is independent of y, and so

$$\begin{aligned}\int_{\mathbb{R}^n} |\widehat{gd\sigma}|^2 w &= \int_{\mathfrak{M}_{k,n}} \sup_{z\in\pi^\perp} (T_{n-k,n}w(\pi,z))\, d\mu(\pi,y)\\ &= \int_{TS} \sup_{z\in T_\xi S} \big(T_{n-k,n}w((T_\xi S)^\perp, z)\big)\, d\nu(\xi,y)\\ &= \int_S |g(\xi)|^2 \sup_{y\in T_\xi S} T_{n-k,n}w((T_\xi S)^\perp, y)d\sigma(\xi),\end{aligned}$$

by the definition of μ and the marginal condition (3). □

2.3. **Application 3: Convolution identities for extension operators.** Theorem 1 is quickly seen to imply the following multilinear extension identity of Iliopoulou and the first author in [11].

Corollary 4. *Suppose $S_1,\dots,S_n$ are smooth codimension-1 submanifolds of $\mathbb{R}^n$. Suppose further that the volume form $v_1(\xi_1)\wedge\cdots\wedge v_n(\xi_n)$ is nonvanishing, where $v_j(\xi_j)$ is a unit normal to S_j for each $\xi_j \in S_j$ and $1 \le j \le n$. Then,*

$$|\widehat{g_1d\sigma_1}|^2 * \cdots * |\widehat{g_nd\sigma_n}|^2 \equiv \int_{S_1\times\cdots\times S_n} \frac{|g_1(\xi_1)|^2\cdots|g_n(\xi_n)|^2}{|v_1(\xi_1)\wedge\cdots\wedge v_n(\xi_n)|} d\sigma_1(\xi_1)\cdots d\sigma_n(\xi_n).$$

Proof. By modulation invariance it suffices to prove the claimed identity at the origin. Observe that

$$|\widehat{g_1d\sigma_1}|^2 * \cdots * |\widehat{g_nd\sigma_n}|^2(0) = n^{-n/2}T_{n(n-1),n^2}(|\widehat{gd\sigma_S}|^2)(\pi,0),$$

where $S = S_1\times\cdots\times S_n$, $g = g_1\otimes\cdots\otimes g_n$ and

$$\pi = \Big\{x = (x_1,\dots,x_n) \in (\mathbb{R}^n)^n : \sum_{j=1}^n x_j = 0\Big\}.$$

Finally, a straightforward linear algebra argument reveals that

$$|(T_\xi S)^\perp \wedge \pi| = n^{-n/2}|v_1(\xi_1)\wedge\cdots\wedge v_n(\xi_n)|,$$

allowing the desired conclusion to follow from Theorem 1. □

As may be expected Corollary 4 may be generalised to submanifolds $S_1,\dots,S_m$ or varying dimensions, provided certain natural arithmetic and transversality conditions are satisfied.

3. The proof of Theorem 1

We begin with the first identity in (4), and use (5) to write

$$\widehat{gd\sigma}(x) = \int_U e^{ix\cdot(\Sigma(u))} g(\Sigma(u)) J(u) d\lambda_\pi(u),$$

where $J(u) = \left|\frac{\partial\Sigma}{du_1} \wedge \cdots \wedge \frac{\partial\Sigma}{\partial u_k}\right|$ and $\Sigma(u) = u + \phi(u)$. Consequently, by Plancherel's theorem on π,

$$\begin{aligned}
T_{k,n}(|\widehat{gd\sigma}|^2)(\pi, y) &= \int_\pi |\widehat{gd\sigma}(x+y)|^2 d\lambda_\pi(x) \\
&= \int_\pi \left|\int_U e^{-2\pi i(x\cdot u + y\cdot\phi(u))} g(u+\phi(u)) J(u) du\right|^2 d\lambda_\pi(x) \\
&= \int_\pi \left|e^{-2\pi i y\cdot\phi(u)} g(u+\phi(u)) J(u)\right|^2 d\lambda_\pi(u) \\
&= \int_\pi \left|g(u+\phi(u)) J(u)^{1/2}\right|^2 J(u) d\lambda_\pi(u) \\
&= \int_S |g(\xi)|^2 J(u(\xi)) d\sigma(\xi),
\end{aligned}$$

where $u(\xi)$ is the orthogonal projection of $\xi \in S$ onto π. Since $|(T_{\Sigma(u)}S)^\perp \wedge \pi| = |T_{\Sigma(u)}S \wedge \pi^\perp|$ it therefore remains to show that

$$J(u) = \frac{1}{|T_{\Sigma(u)}S \wedge \pi^\perp|}. \tag{12}$$

In order to establish (12) we may, by the rotation-invariance of the statement of Theorem 1, assume that $\pi = \langle e_1, \ldots, e_k\rangle$, where $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$. We observe first that since $\phi : \pi \to \pi^\perp$,

$$\left|\frac{\partial\Sigma}{\partial u_1} \wedge \cdots \wedge \frac{\partial\Sigma}{\partial u_k} \wedge e_{k+1} \wedge \cdots \wedge e_n\right| = 1. \tag{13}$$

Next we construct orthogonal (as opposed to orthonormal) $v_1, \ldots, v_k \in T_{\Sigma(u)}S$ from $\frac{\partial\Sigma}{\partial u_1}, \ldots, \frac{\partial\Sigma}{\partial u_k}$ by the Gram–Schmidt process, and observe that

$$v_1 \wedge \cdots \wedge v_k = \frac{\partial\Sigma}{\partial u_1} \wedge \cdots \wedge \frac{\partial\Sigma}{\partial u_k}.$$

Consequently,

$$|v_1| \cdots |v_k| = \left|\frac{\partial\Sigma}{\partial u_1} \wedge \cdots \wedge \frac{\partial\Sigma}{\partial u_k}\right|,$$

and so by (13),

$$|T_{\Sigma(u)}S \wedge \pi^\perp| = \left|\frac{v_1}{|v_1|} \wedge \cdots \wedge \frac{v_k}{|v_k|} \wedge e_{k+1} \wedge \cdots \wedge e_n\right| = \frac{1}{J(u)}.$$

Turning to the second identity in (4), by (2) we have

$$\begin{aligned}
T_{k,n}T^*_{n-k,n}\mu(\pi, y) &= \int_{\mathbb{R}^n} T^*_{n-k,n}\mu(x)\delta_{\pi+\{y\}}(x)dx \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathfrak{M}_{n-k,n}} \delta_{\theta+\{z\}}(x) d\mu(\theta, z)\right) \delta_{\pi+\{y\}}(x)dx \\
&= \int_{\mathfrak{M}_{n-k,n}} \left(\int_{\mathbb{R}^n} \delta_{\pi+\{y\}}(x)\delta_{\theta+\{z\}}(x)dx\right) d\mu(\theta, z).
\end{aligned}$$

Recalling that the transverse subspaces π and θ are k-dimensional and $(n-k)$-dimensional respectively, an affine change of variables reveals that

$$\int_{\mathbb{R}^n} \delta_{\pi+\{y\}}(x)\delta_{\theta+\{z\}}(x)dx = \frac{1}{|\theta\wedge\pi|} \tag{14}$$

for all $y \in \pi^\perp$ and $z \in \theta^\perp$. Hence,

$$\begin{aligned}
T_{k,n}T^*_{n-k,n}\mu(\pi,y) &= \int_{\mathfrak{M}_{n-k,n}} \frac{d\mu(\theta,z)}{|\theta\wedge\pi|}\\
&= \int_{TS} \frac{d\nu(\xi,y')}{|(T_\xi S)^\perp\wedge\pi|}\\
&= \int_{S} \frac{d\nu_S(\xi)}{|(T_\xi S)^\perp\wedge\pi|}\\
&= \int_{S} \frac{|g(\xi)|^2}{|(T_\xi S)^\perp\wedge\pi|}d\sigma(\xi),
\end{aligned}$$

where in the final line we have used (3).

4. Optimality

While the hypothesis (T) is clearly necessary for finiteness in (4), the necessity of the global hypothesis (GT) is rather less aparent. Here we establish that if the data $T_k(|\widehat{gd\sigma}|^2)(\pi,y)$ is independent of y, as claimed by (4), then (GT) must hold. To this end we fix a k-dimensional C^1 submanifold S and a k-dimensional subspace π, and assume that (GT) fails, that is, there exist $\xi_0,\eta_0\in S$ such that $\xi_0-\eta_0\in\pi^\perp\backslash\{0\}$. We then define $g_0: S\to\mathbb{R}$ by $g_0=\chi_{C_\varepsilon(\xi_0)}+\chi_{C_\varepsilon(\eta_0)}$, where $C_\varepsilon(a):=S\cap\{\xi\in\mathbb{R}^n : |\xi-a|<\varepsilon\}$, and $\varepsilon>0$ will be chosen later. It will suffice to show that the directional derivative

$$(\xi_0-\eta_0)\cdot\nabla_y\big[T_{k,n}(|\widehat{g_0d\sigma}|^2)(\pi,\cdot)\big](0) = (\xi_0-\eta_0)\cdot\nabla_y\Big[\int_\pi|\widehat{g_0d\sigma}(x+y)|^2d\lambda_\pi(x)\Big]|_{y=0} \tag{15}$$

does not vanish for some $\varepsilon>0$. We remark that

$$\int_\pi |\widehat{gd\sigma}(x+y)|^2d\lambda_\pi(x)<\infty \tag{16}$$

for all $y\in\pi^\perp$ and all $g\in L^2(S)$ thanks to (T). In particular, (16) holds for $g=g_0$ and $g(\xi)=(\xi_0-\eta_0)\cdot\xi g_0(\xi)$, from which it follows that

$$\int_\pi \big|(\xi_0-\eta_0)\cdot\nabla_{\mathbb{R}^n}\big(|\widehat{g_0d\sigma}|^2\big)(x)\big|d\lambda_\pi(x)<\infty. \tag{17}$$

This justifies an interchange of derivative and integral in (15), reducing matters to establishing that

$$\int_\pi (\xi_0-\eta_0)\cdot\nabla_{\mathbb{R}^n}\big(|\widehat{g_0d\sigma}|^2\big)(x)d\lambda_\pi(x)\neq 0 \tag{18}$$

for some $\varepsilon>0$. By the dominated convergence theorem, (17) also ensures that

$$(\xi_0-\eta_0)\cdot\nabla_y\Big[\int_\pi|\widehat{g_0d\sigma}(x+y)|^2\,d\lambda_\pi(x)\Big]|_{y=0} = \lim_{R\to\infty}\int_\pi(\xi_0-\eta_0)\cdot\nabla_{\mathbb{R}^n}\big(|\widehat{g_0d\sigma}|^2\big)(x)\gamma_R(x)d\lambda_\pi(x), \tag{19}$$

where $\gamma_R(x)=\gamma(x/R)$ for some Schwartz function γ on π chosen to have compact Fourier support and satisfy $\gamma(0)=1$. For sufficiently large R, by Fubini's theorem we have

$$\begin{aligned}
&\int_\pi(\xi_0-\eta_0)\cdot\nabla_{\mathbb{R}^n}\big(|\widehat{g_0d\sigma}|^2\big)(x)\gamma_R(x)d\lambda_\pi(x)\\
&\qquad = -2\pi i\int_S\int_S(\xi_0-\eta_0)\cdot(\xi-\eta)g_0(\xi)g_0(\eta)\widehat{\gamma}_R(P_\pi(\xi-\eta))d\sigma(\xi)d\sigma(\eta).
\end{aligned}$$

We now claim that

$$\int_S\int_S(\xi_0-\eta_0)\cdot(\xi-\eta)g_0(\xi)g_0(\eta)\widehat{\gamma}_R(P_\pi(\xi-\eta))d\sigma(\xi)d\sigma(\eta)$$

is bounded away from zero for sufficiently large R, provided ε is chosen small enough. By our choice of g_0 we now write

$$\int_S\int_S(\xi_0-\eta_0)\cdot(\xi-\eta)g_0(\xi)g_0(\eta)\widehat{\gamma}_R(P_\pi(\xi-\eta))d\sigma(\xi)d\sigma(\eta)=2I_{\xi_0,\eta_0}(R)+I'_{\xi_0,\eta_0}(R),$$

where

$$I_{\xi_0,\eta_0}(R):=\int_S\int_S(\xi_0-\eta_0)\cdot(\xi-\eta)\chi_{C_\varepsilon(\xi_0)}(\xi)\chi_{C_\varepsilon(\eta_0)}(\eta)\widehat{\gamma}_R(P_\pi(\xi-\eta))d\sigma(\xi)d\sigma(\eta).$$

By (T) we may choose $\varepsilon>0$ so that (GT) holds on $C_\varepsilon(\xi_0)$ and $C_\varepsilon(\eta_0)$. In particular, for ζ_0 being ξ_0 or η_0, we have

$$\int_{C_\varepsilon(\zeta_0)}\varphi(\xi)d\sigma(\xi)=\int_{U_{\varepsilon,\zeta_0}}\varphi(u+\phi_{\varepsilon,\zeta_0}(u))J_{\varepsilon,\zeta_0}(u)du,$$

for some open $U_{\varepsilon,\zeta_0}\subset\pi$ and $\phi_{\varepsilon,\zeta_0}:U_{\varepsilon,\zeta_0}\to\pi^\perp$, and any test function φ. Note that the Jacobians are bounded from above and below. Since $\xi_0-\eta_0\in\pi^\perp$, it is straightforward to verify that $|I'_{\xi_0,\eta_0}(R)|\lesssim_{\varepsilon,\xi_0,\eta_0}R^{-1}$. Hence it suffices to show

$$|I_{\xi_0,\eta_0}(R)|>0 \tag{20}$$

uniformly in sufficiently large R. Since ε is small, $(\xi_0-\eta_0)\cdot(\xi-\eta)\sim|\xi_0-\eta_0|^2$ for all $\xi\in C_\varepsilon(\xi_0)$ and all $\eta\in C_\varepsilon(\eta_0)$, and hence, using the fact that $\xi_0-\eta_0\in\pi^\perp$,

$$\begin{aligned}I_{\xi_0,\eta_0}(R)&\sim|\xi_0-\eta_0|^2\int_{C_\varepsilon(\xi_0)}\int_{C_\varepsilon(\eta_0)}\widehat{\gamma}_R(P_\pi(\xi-\eta))d\sigma(\xi)d\sigma(\eta)\\&=|\xi_0-\eta_0|^2\int_{C_\varepsilon(\xi_0)-\xi_0}\int_{C_\varepsilon(\eta_0)-\eta_0}\widehat{\gamma}_R(P_\pi(\xi-\eta))d\sigma(\xi)d\sigma(\eta),\end{aligned}$$

where $d\sigma$ continues to denote surface measure. If $u_0\in U_{\varepsilon,\xi_0}$ is such that $\xi_0=u_0+\phi_{\varepsilon,\xi_0}(u_0)$, and $v_0\in U_{\varepsilon,\eta_0}$ is such that $\eta_0=v_0+\phi_{\varepsilon,\eta_0}(v_0)$, then

$$C_\varepsilon(\xi_0)-\xi_0=\{u'+\phi_{\varepsilon,\xi_0}(u'+u_0)-\phi_{\varepsilon,\xi_0}(u_0):u'\in U'_{\varepsilon,\xi_0}\},$$

where $U'_{\varepsilon,\xi_0}:=U_{\varepsilon,\xi_0}-u_0$, and similarly

$$C_\varepsilon(\eta_0)-\eta_0=\{v'+\phi_{\varepsilon,\eta_0}(v'+v_0)-\phi_{\varepsilon,\eta_0}(v_0):v'\in U'_{\varepsilon,\eta_0}\},\quad U'_{\varepsilon,\eta_0}:=U_{\varepsilon,\eta_0}-v_0.$$

Since $\phi_{\varepsilon,\xi_0}(u'+u_0)-\phi_{\varepsilon,\xi_0}(u_0)$ and $\phi_{\varepsilon,\eta_0}(v'+v_0)-\phi_{\varepsilon,\eta_0}(v_0)$ belong to $\pi^\perp$,

$$P_\pi(\xi-\eta)=P_\pi\big(u'+\phi_{\varepsilon,\xi_0}(u'+u_0)-\phi_{\varepsilon,\xi_0}(u_0)-(v'+\phi_{\varepsilon,\eta_0}(v'+v_0)-\phi_{\varepsilon,\eta_0}(v_0))\big)=u'-v'$$

for all $\xi\in C_\varepsilon(\xi_0)$ and $\eta\in C_\varepsilon(\eta_0)$. Hence,

$$I_{\xi_0,\eta_0}(R)\sim|\xi_0-\eta_0|^2\int_{U'_{\varepsilon,\xi_0}}\int_{U'_{\varepsilon,\eta_0}}\widehat{\gamma}_R(u'-v')du'dv'.$$

Since U'_{ε,ξ_0} and U'_{ε,η_0} contain the origin,

$$I_{\xi_0,\eta_0}(R)\gtrsim|\xi_0-\eta_0|^2\int_{U'_{\varepsilon,\xi_0}\cap U'_{\varepsilon,\eta_0}}\int_{U'_{\varepsilon,\xi_0}\cap U'_{\varepsilon,\eta_0}}\widehat{\gamma}_R(u'-v')du'dv'\sim_\varepsilon|\xi_0-\eta_0|^2,$$

from which (20) follows.

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