Ramsey equivalence for asymmetric pairs of graphs

Boyadzhiyska, Simona; Clemens, Dennis; Gupta, Pranshu; Rollin, Jonathan

DOI:
10.1137/23M1558239

RAMSEY EQUIVALENCE FOR ASYMMETRIC PAIRS OF GRAPHS*

SIMONA BOYADZHIYSKA[†], DENNIS CLEMENS[‡], PRANSHU GUPTA[§], AND JONATHAN ROLLIN[¶]

Abstract. A graph F is Ramsey for a pair of graphs (G, H) if any red/blue-coloring of the edges of F yields a copy of G with all edges colored red or a copy of H with all edges colored blue. Two pairs of graphs are called Ramsey equivalent if they have the same collection of Ramsey graphs. The symmetric setting, that is, the case $G = H$, received considerable attention. This led to the open question whether there are connected graphs G and G' such that (G, G) and (G', G') are Ramsey equivalent. We make progress on the asymmetric version of this question and identify several non-trivial families of Ramsey equivalent pairs of connected graphs.

Certain pairs of stars provide a first, albeit trivial, example of Ramsey equivalent pairs of connected graphs. Our first result characterizes all Ramsey equivalent pairs of stars. The rest of the paper focuses on pairs of the form (T, K_t), where T is a tree and K_t is a complete graph. We show that, if T belongs to a certain family of trees, including all non-trivial stars, then (T, K_t) is Ramsey equivalent to a family of pairs of the form (T, H), where H is obtained from K_t by attaching disjoint smaller cliques to some of its vertices. In addition, we establish that for (T, H) to be Ramsey equivalent to (T, K_t), H must have roughly this form. On the other hand, we prove that for many other trees T, including all odd-diameter trees, (T, K_t) is not equivalent to any such pair, not even to the pair $(T, K_t \cdot K_2)$, where $K_t \cdot K_2$ is a complete graph K_t with a single edge attached.

Key words. Graph Ramsey Theory, Ramsey Equivalence

MSC codes. 05D10

1. Introduction.

1.1. Ramsey equivalence. We say that a graph F is *Ramsey* for another graph H, if any red/blue-coloring of the edges of F yields a copy of H all of whose edges have the same color, that is, a *monochromatic* copy of H; we write $\mathcal{R}(H)$ for the set of all Ramsey graphs for H. The seminal result of Ramsey [24] establishes that $\mathcal{R}(H) \neq \emptyset$ for any graph H. Characterizing the graphs in $\mathcal{R}(H)$ exactly is a hard task accomplished for only few small graphs so far (see for example [4, 8]). A natural next problem then is to investigate the properties of the graphs belonging to $\mathcal{R}(H)$ for a given H.

The most prominently studied property is the smallest number of vertices among all graphs in $\mathcal{R}(H)$ for a given H. This quantity is called the *Ramsey number* of H. Ramsey numbers have proven to be notoriously hard to compute in many cases and have attracted a lot of attention over the years; see e.g. the survey [11] and the references therein. In the 1970s researchers initiated the study of other graph parameters in the context of Ramsey graphs, like the number of edges, the clique

*

Funding: The first author was supported by the Deutsche Forschungsgemeinschaft (DFG) Graduiertenkolleg "Facets of Complexity" (GRK 2434).

[†]School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK (s.s.boyadzhiyska@bham.ac.uk). This research was done when the author was affiliated with the Institut für Mathematik, Freie Universität Berlin, 14195 Berlin, Germany.

[‡]Hamburg University of Technology, Institute of Mathematics, Hamburg, Germany (dennis.clemens@tuhh.de).

[§]Faculty of Computer Science and Mathematics, University of Passau, Passau, Germany (pranshu.gupta@uni-passau.de). This research was done when the author was affiliated with the Hamburg University of Technology, Institute of Mathematics, Hamburg, Germany.

[¶]FernUniversität in Hagen, Fakultät für Mathematik und Informatik, Hagen, Germany (jonathan.rollin@fernuni-hagen.de).

number, and the minimum degree (see for example [8, 13, 14]). A natural question to ask when studying problems of this type is: how do these values change when we modify the target graph H slightly? More generally, how does the collection of Ramsey graphs change? This question motivated Szabó, Zumstein, and Zürcher [28] to define the notion of Ramsey equivalence.

Definition 1.1. Two graphs H and H' are *Ramsey equivalent*, denoted $H \sim H'$, if $\mathcal{R}(H) = \mathcal{R}(H')$.

It is not difficult to show that Ramsey equivalent pairs of graphs exist: for instance, the graph obtained by adding an isolated vertex to the clique K_t for $t \geq 3$ is Ramsey equivalent to K_t. Szabó, Zumstein, and Zürcher [28] found further examples of disconnected graphs that are Ramsey equivalent to the clique K_t (see also [3, 15] for further results in this direction) and asked whether there exist any *connected* such graphs. Perhaps surprisingly, several years later Fox, Grinshpun, Liebenau, Person, and Szabó [15] settled this question in the negative: they showed that no connected graph is Ramsey equivalent to K_t. In light of this result, they raised the following question:

Question 1.2 ([15]). *Is there a pair of non-isomorphic connected graphs that are Ramsey equivalent?*

The above question is wide open. Not much is known even in the special case where the two graphs differ only by a pendent edge. It was shown by Clemens, Liebenau, and Reding [10] that no pair of 3-connected graphs can be Ramsey equivalent. A result from Grinsphpun's PhD thesis [16, Lemma 2.6.3.] allows us to show non-equivalence for further pairs consisting of a graph H and the graph H with a pendent edge. Further evidence that the answer to Question 1.2 might be negative is provided for example in [1, 27].

In this paper, we study Ramsey equivalence in the asymmetric setting and explore a variant of Question 1.2. We begin by defining the necessary notions. We say that a graph F is *Ramsey* for a pair of graphs (G, H), and write $F \to (G, H)$, if, for every red/blue-coloring of the edges of F there exists a red copy of G, that is, a copy of G with all edges colored red, or a blue copy of H, defined similarly. We denote the collection of all Ramsey graphs for (G, H) by $\mathcal{R}(G, H)$. We call a graph $F \in \mathcal{R}(G, H)$ *Ramsey-minimal* for (G, H) if no proper subgraph of F is Ramsey for (G, H), and we denote the corresponding collection by $\mathcal{M}(G, H)$.

Definition 1.3. We call two pairs of graphs (G, H) and (G', H') *Ramsey equivalent*, denoted $(G, H) \sim (G', H')$, if $\mathcal{R}(G, H) = \mathcal{R}(G', H')$.

Note that the collection of Ramsey graphs for (G, H) is uniquely determined by those graphs that are Ramsey-minimal for (G, H), and hence two pairs of graphs (G, H) and (G', H') are Ramsey equivalent if and only if $\mathcal{M}(G, H) = \mathcal{M}(G', H')$. For any two pairs of graphs that are not Ramsey equivalent there is a graph that is Ramsey for one pair but not for the other. We say that such a graph *distinguishes* the two pairs.

Our goal is to explore the notion of Ramsey equivalence for asymmetric pairs of connected graphs and in particular the asymmetric version of Question 1.2. Previously known results allow us to exclude some potential candidates. Let $\omega(G)$ denote the clique number of a graph G, defined as the largest integer n such that K_n is a subgraph of G. A famous result of Nešetřil and Rödl [21] establishes that, for every graph G, there is a Ramsey graph for G that has the same clique number as G.

Hence, the disjoint union of G and H has a Ramsey graph F with clique number $\max\{\omega(G), \omega(H)\}$ and this graph F is also a Ramsey graph for (G, H). This gives the following statement which we shall use several times in our proofs.

THEOREM 1.4 ([21]). *Each pair (G, H) of graphs has a Ramsey graph with clique number equal to* $\max\{\omega(G), \omega(H)\}$.

This result implies $(G, H) \not\sim (G', H')$ if $\max\{\omega(G), \omega(H)\} \neq \max\{\omega(G'), \omega(H')\}$. As a second example, Savery [27, Section 3.1] proved that $(G, H) \not\sim (G', H')$ for all graphs G, H, G', and H' with $\chi(G) + \chi(H) \neq \chi(G') + \chi(H')$, where $\chi(G)$ denotes the chromatic number of G.

It turns out, however, that the asymmetric version of Question 1.2 has an affirmative answer. Let $K_{1,s}$ denote a star with s edges. Through a simple application of Petersen's Theorem [23], Burr, Erdős, Faudree, Rousseau, and Schelp [6] showed that, for any odd integers r, $s \geq 1$, the only Ramsey-minimal graph for the pair of stars $(K_{1,r}, K_{1,s})$ is the star $K_{1,r+s-1}$. Thus, any odd integers $r, s, r', s' \geq 1$ with $r + s = r' + s'$ satisfy $(K_{1,r}, K_{1,s}) \sim (K_{1,r'}, K_{1,s'})$. This example is perhaps not very satisfying, as pairs of odd stars have only a single Ramsey-minimal graph. It is then interesting to ask whether there are any Ramsey equivalent pairs of connected graphs with a larger, maybe even an infinite number of Ramsey-minimal graphs. Our main result shows that the answer is yes, exhibiting an infinite family of Ramsey equivalent pairs of connected graphs of the form $(T, K_t) \sim (T, K_t \cdot K_2)$, where T is a certain kind of tree.

1.2. Results. In light of the discussion in the previous paragraph, one might ask whether there exist any other pairs of stars that are Ramsey equivalent. In our first result, we answer this question negatively. Note that $\mathcal{M}(K_{1,r}, K_{1,s})$ is infinite whenever rs is even [6].

THEOREM 1.5. *Let a, b, x, y be positive integers with $\{a, b\} \neq \{x, y\}$. Then $(K_{1,a}, K_{1,b}) \sim (K_{1,x}, K_{1,y})$ if and only if $a + b = x + y$ and a, b, x, and y are odd.*

Note that each pair of stars has a Ramsey graph that is a star. This star distinguishes the pair of stars from any pair of connected graphs that involves a graph that is not a star.

We next study Ramsey equivalence for pairs of the form (T, K_t), where T is a tree and $t \geq 3$. Note that in the case where T is a single vertex or edge the collection of Ramsey graphs is trivial, as $\mathcal{M}(K_1, K_t) = \{K_1\}$ and $\mathcal{M}(K_2, K_t) = \{K_t\}$. From now on, unless otherwise specified, we will assume that T has at least two edges. It was shown by Łuczak [19] that in this case $\mathcal{M}(T, K_t)$ is infinite. Perhaps surprisingly, we find non-trivial Ramsey equivalent pairs in this setting. To describe some of those pairs, we need the following definitions. For integers $a \geq 1$, $b \geq 2$, and $t \geq 3$ with $a \leq t$, let $K_t \cdot aK_b$ denote the graph consisting of a copy of K_t and a pairwise vertex-disjoint copies of K_b, each sharing exactly one vertex with the copy of K_t (see Figure 1.1 left for an example). We call a tree T an *(s-)suitable caterpillar*, if T consists of a path P on three vertices and up to $3s - 1$ further vertices of degree 1 such that the endpoints of P are of degree exactly $s + 1$ in T and the middle vertex of P is of degree at most $s + 1$ in T (see Figure 1.1 right).

THEOREM 1.6.

(a) For all integers $s \geq 2$ and $t \geq 3$, we have $(K_{1,s}, K_t) \sim (K_{1,s}, K_t \cdot K_2)$.

(b) Let $a \geq 1$ and $b \geq 2$ be integers, and let T be a star with at least two edges or a suitable caterpillar. For any large enough t, we have $(T, K_t) \sim (T, K_t \cdot aK_b)$.

FIG. 1.1. *The graph $K_6 \cdot 2K_3$ (left) and the largest 3-suitable caterpillar (right).*

Observe that the first part of the above theorem holds for each $t \geq 3$, while we need a sufficiently large t to prove the second part. We do not know whether the statement is true for small values of t.

We complement the equivalence result above by proving Ramsey non-equivalence for several other families of pairs of trees and cliques. Theorem 1.4 shows that we may restrict our attention to pairs (G, H) with $\max\{\omega(G), \omega(H)\} = t$, since otherwise $(T, K_t) \not\sim (G, H)$. Before we state the result, we again need some definitions. The length of a path is its number of edges. The diameter $\operatorname{diam}(T)$ of a tree T is the length of its longest path. Trees of even diameter contain a unique central vertex, that is, a vertex that is the middle vertex in each longest path. Let $\mathcal{T}$ denote the class of all trees T of diameter at least three such that:

- if $\operatorname{diam}(T)$ is even, the neighbors of the central vertex of T are of degree at most two, and
- if $\operatorname{diam}(T) = 4$, the central vertex is of degree at least 3.

Note in particular that this class contains all trees of odd diameter.

Theorem 1.6 above shows that for some trees T and large t the Ramsey graphs for (T, K_t) do not change when we attach certain disjoint pendent graphs at the vertices of K_t. The first part of the following theorem states that this behavior does not generalize to trees from the family $\mathcal{T}$ defined above in a strong sense: for each tree $T \in \mathcal{T}$ and each $t \geq 3$, we have $(T, K_t) \not\sim (T, K_t \cdot K_2)$. The second part shows that, if $(T, K_t) \sim (T, H)$ for some connected graph $H \neq K_t$, the graph H must consist of a copy of K_t with some disjoint pendent graphs, each attached to a different vertex of the clique. Note that this is precisely the form that the graph H takes in Theorem 1.6(b); therefore, part (b) of the next theorem demonstrates that this equivalence result is best possible in a certain sense. Finally, the third part of the theorem below considers modifications to the first component of the pair, namely T, and shows that T cannot be replaced by *any* other connected graph G if the second component of the pair stays unchanged. We emphasize that certain pairs (T, K_t) and (G, K_t) cannot be distinguished by, for instance, their Ramsey numbers alone: for example, Keevash, Long, and Skokan [18] showed that when $\ell = \Omega\left(\frac{\log t}{\log \log t}\right)$ the Ramsey numbers of (C_ℓ, K_t) and (T_ℓ, K_t) are the same, where C_ℓ and T_ℓ denote a cycle and a tree on ℓ vertices, respectively.

THEOREM 1.7. *Let $t \geq 3$ be an integer.*
(a) For any $T \in \mathcal{T}$ and any connected graph $H \neq K_t$, we have $(T, K_t) \not\sim (T, H)$.
(b) For any tree T and any graph H that contains a copy K of K_t and a cycle with vertices from both $V(K)$ and $V(H) \setminus V(K)$, we have $(T, K_t) \not\sim (T, H)$.
(c) For any tree T and any connected graph $G \neq T$, we have $(T, K_t) \not\sim (G, K_t)$.

As we will see in Section 4, our construction actually allows us to prove the statement from the first part of the theorem above for a larger class of trees. Since our results do not lead to a complete characterization of those trees T for which

$(T, K_t) \sim (T, K_t \cdot K_2)$, we choose to state the simpler, albeit somewhat weaker, result here. As a specific example, note that our results imply that for a path P and for sufficiently large t we have $(P, K_t) \sim (P, K_t \cdot K_2)$ if and only if P has two or four edges.

In this paper, we study what pairs of connected graphs (G, H) can be Ramsey equivalent to pairs of the form (T, K_t). We focus on the two cases $G = T$ and $H = K_t$. It would be interesting to know whether there are any pairs (G, H) of connected graphs with $G \subsetneq T$ and $K_t \subsetneq H$ that are Ramsey equivalent to (T, K_t).

1.3. Notation. Given a graph G, we denote its vertex set and its edge set by $V(G)$ and $E(G)$, respectively. For a set $X \subseteq V(G)$ we write $G - X$ for the graph obtained from G by removing the vertices in X and all their incident edges; for a single vertex $x \in V(G)$, we write $G - x = G - \{x\}$; similarly for a subgraph F of G we let $G - F = G - V(F)$. For a set $Y \subseteq E(G)$, we write $G - Y$ for the graph obtained from G by removing the edges in Y; for a single edge $e \in E(G)$, we write $G - e = G - \{e\}$. Throughout the paper unless otherwise specified a coloring is meant to be an edge-coloring of the given graph G. As we always call the two colors red and blue, we use red/blue-coloring and 2-coloring as synonyms of each other. Given any two graphs H_1 and H_2, we say that a 2-coloring is (H_1, H_2)-free, if there is no red copy of H_1 and no blue copy of H_2.

1.4. Organization of the paper. In Section 2, we prove Theorem 1.5. Section 3 contains the proof of our main equivalence result, namely Theorem 1.6, and in Section 4 we prove Theorem 1.7 on Ramsey non-equivalent pairs.

2. Pairs of Stars. In this section, we prove Theorem 1.5. We note that this theorem can be deduced from Theorem 1 in [22]. However, the calculations are tedious and for completeness we present explicit constructions here when we want to show that there exist graphs that are Ramsey for certain pairs of stars and not Ramsey for other pairs of stars.

Observe that, given positive integers a and b, an $(a + b - 2)$-regular graph F is a Ramsey graph for a pair $(K_{1,a}, K_{1,b})$ if and only if $E(F)$ cannot be decomposed into an $(a - 1)$-regular subgraph and a $(b - 1)$-regular subgraph. A k-regular spanning subgraph is also called a *k-factor*. Our results rely on the rich theory on factors. Specifically we need the following fact.

LEMMA 2.1. *Let p, q and r be integers, with p and q being odd. Further, assume that $p < q \leq r$ if r is odd, and that $p < q \leq r/2$ if r is even. There is an r-regular graph that has a q-factor and no p-factor.*

In order to prove the above lemma, we apply a theorem due to Belck [2] (a special case of the well-known f-factor theorem of Tutte [29]), which provides a necessary and sufficient condition for the existence of k-factors in regular graphs. For a graph G and a set $D \subseteq V(G)$, we call a component C of $G - D$ an *odd component* with respect to D if $|V(C)|$ is odd, and we let $q_G(D)$ denote the number of such components. We use the following corollary of Theorem IV from [2].

THEOREM 2.2 ([2]). *Let G be a graph and let $p > 0$ be an odd integer. If there exists a set $D \subseteq V(G)$ such that $p|D| < q_G(D)$, then G has no p-factor.*

Proof of Lemma 2.1. Given p, q and r as described in the statement, we aim to construct an r-regular graph F that has a q-factor and no p-factor. The graph F will be constructed in three steps.

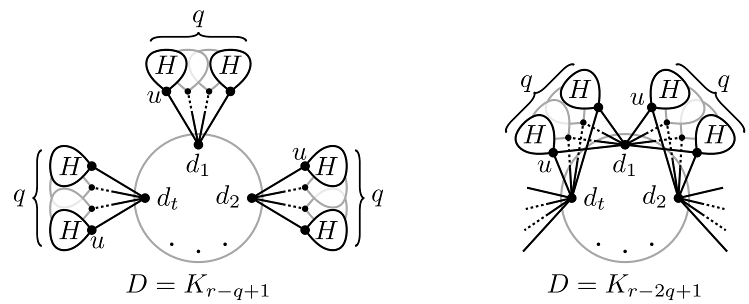

FIG. 2.1. *A construction of an r-regular graph with a q-factor and no p-factor for odd r (left) and even r (right).*

In the first step, we find an r-regular graph G with an even number of vertices that has a q-factor G_q and a matching M_G with $\lfloor (r-1)/2 \rfloor$ edges which contains exactly $(q-1)/2$ edges of G_q. To this end, define G to be the graph obtained by taking $2r(r-q+1)$ copies $Q_{i,j}$ of K_{q+1}, with $i \in [r-q+1]$ and $j \in [2r]$, and adding a perfect matching between any two copies $Q_{i_1,j}$ and $Q_{i_2,j}$ with $i_1 \neq i_2$ and $j \in [2r]$. Then G has an even number of vertices and is r-regular. Moreover, the subgraph G_q that consists of all $Q_{i,j}$ is a q-factor. The matching M_G can be found by taking $\frac{q-1}{2}$ independent edges from $Q_{1,1}$ and one edge from every matching between $Q_{1,j}$ and $Q_{2,j}$ with $2 \le j \le \lfloor \frac{r-q+2}{2} \rfloor$. Set $M_q := M_G \cap E(G_q)$.

For the second step, let H, respectively H_q, denote the graphs obtained from G, respectively G_q, by adding a new vertex u and replacing every edge $vw \in M_G$, respectively $vw \in M_q$, by the edges uv and uw. Then H has an odd number of vertices, u is of degree $2\lfloor (r-1)/2 \rfloor$ in H, and all other vertices are of degree r. Moreover, H_q is a spanning subgraph of H in which u is of degree $q-1$ and all other vertices are of degree q.

For the third step, we consider two cases depending on the parity of r.

Case 1: r is odd. In this case u has degree $r-1$ in the graph H. Let $t = r-q+1$, and let $F = F(q,r)$ denote the graph obtained from a copy of K_t with vertex set $D = \{d_j : j \in [t]\}$ and qt vertex disjoint copies $H^1, \dots, H^{qt}$ of the graph H as follows: For each $i \in [qt]$, let u^i denote the copy of u in H^i. We partition the set $\{u^i : i \in [qt]\}$ into t sets $U_1, U_2, \dots, U_t$ each of size q and, for each $j \in [t]$, add an edge between d_j and each vertex in U_j. An illustration of the construction is given in Figure 2.1 (left). Then F is r-regular. Moreover, F has a q-factor, given by the subgraph consisting of all copies of H_q (coming from the H^i with $i \in [qt]$) and all edges between D and the copies of u.

It thus remains to show that F does not admit a p-factor. This follows from Theorem 2.2. Indeed, the odd components of $F-D$ are exactly the $q(r-q+1)$ copies of H, and therefore

$$p|D| - q_G(D) = p|D| - q(r-q+1) = (p-q)(r-q+1) < 0\,,$$

since $p < q \le r$ by assumption.

Case 2: r is even. In this case u has degree $r-2$ in the graph H. Moreover, by assumption we have $q \le r/2$. Let $t = r-2q+1$, and let $F = F(q,r)$ denote the graph obtained from a copy of K_t with vertex set $D = \{d_1, d_2, \dots, d_t\}$ and qt vertex disjoint copies $H^1, \dots, H^{qt}$ of the graph H as follows: For each $i \in [qt]$, let u^i denote the copy of u in H^i. We partition the set $\{u^i : i \in [qt]\}$ into t sets $U_1, U_2, \ldots, U_t$ each of size q and, for each $j \in [t]$, add an edge between d_j and each vertex in $U_j \cup U_{j+1}$, where $U_{t+1} := U_1$. This is illustrated in Figure 2.1 (right).

Then the graph F is r-regular. Moreover, it contains a q-factor consisting of all copies of H_q (coming from the H^i with $i \in [qt]$) and all edges between d_j and U_j for every $j \in [t]$. Furthermore, by Theorem 2.2, F does not have a p-factor. Indeed, the odd components of $F - D$ are exactly the qt copies of H, and therefore

$$p|D| - q_G(D) = p|D| - qt = (p-q)t < 0\,,$$

since $p < q$ and $q \leq r/2$ and hence $t \geq 1$ by assumption. □

Proof of Theorem 1.5. First observe that $K_{1,a+b-1}$ is Ramsey for $(K_{1,x}, K_{1,y})$ if and only if $x+y \leq a+b$. This shows that $(K_{1,a}, K_{1,b}) \not\sim (K_{1,x}, K_{1,y})$ when $a+b \neq x+y$. For the remainder of the proof assume that $a+b = x+y$.

As discussed in the introduction, if a and b are both odd, then $K_{1,a+b-1}$ is the unique minimal Ramsey graph for $(K_{1,a}, K_{1,b})$ [6]. So $(K_{1,a}, K_{1,b}) \sim (K_{1,x}, K_{1,y})$ if a, b, x, y are all odd. It remains to consider the case where at least one of a, b, x, and y is even and find a distinguishing graph, that is, a graph that is Ramsey for one of the pairs of stars and not Ramsey for the other pair. Without loss of generality, assume that a is the largest even number in $\{a, b, x, y\}$. Let $r = a+b-2 = x+y-2$. Recall that an r-regular graph is a Ramsey for $(K_{1,a}, K_{1,b})$ if and only if it has no $(a-1)$-factor. We consider several cases.

Case 1: xy is odd. Then each Ramsey graph for $(K_{1,x}, K_{1,y})$ contains $K_{1,a+b-1}$, as remarked above, and hence no graph of maximum degree at most r is a Ramsey graph for $(K_{1,x}, K_{1,y})$. Consider an r-regular graph on an odd number of vertices, which exists since r is even. Since $(a-1)$ is odd, this graph does not have an $(a-1)$-factor and is therefore Ramsey for $(K_{1,a}, K_{1,b})$ but not Ramsey for $(K_{1,x}, K_{1,y})$. Thus $(K_{1,a}, K_{1,b}) \not\sim (K_{1,x}, K_{1,y})$.

Case 2: xy is even. We may assume that x is the larger even number in $\{x, y\}$. Then $a > x$, since $a+b = x+y$, $\{a, b\} \neq \{x, y\}$, and a is the largest even number in $\{a, b, x, y\}$. We again distinguish two cases.

Case 2.1: b is odd. Then r is odd. Setting $q = a-1$ and $p = x-1$, we know that p and q are odd, and $p < q \leq r$. Hence, using Lemma 2.1, we find an r-regular graph F that has an $(a-1)$-factor and no $(x-1)$-factor. Thus $F \not\to (K_{1,a}, K_{1,b})$ and $F \to (K_{1,x}, K_{1,y})$. Hence $(K_{1,a}, K_{1,b}) \not\sim (K_{1,x}, K_{1,y})$.

Case 2.1: b is even. Then r and y are even. As we have $a > x \geq y$ and $a+b-2 = r = x+y-2$, we obtain $b < y \leq \frac{r+2}{2}$. Setting $q = y-1$ and $p = b-1$, we know that p and q are odd, and $p < q \leq \frac{r}{2}$. Hence, using Lemma 2.1, we find an r-regular graph F that has a $(y-1)$-factor and no $(b-1)$-factor. Thus $F \not\to (K_{1,x}, K_{1,y})$ and $F \to (K_{1,a}, K_{1,b})$. Hence $(K_{1,a}, K_{1,b}) \not\sim (K_{1,x}, K_{1,y})$. □

3. Equivalence results for trees and cliques.

Proof of Theorem 1.6(a). If $F \to (K_{1,s}, K_t \cdot K_2)$, then also $F \to (K_{1,s}, K_t)$. It suffices to show that, if $F \not\to (K_{1,s}, K_t \cdot K_2)$, then also $F \not\to (K_{1,s}, K_t)$.

Let F denote a graph that is not Ramsey for $(K_{1,s}, K_t \cdot K_2)$ and let c be a $(K_{1,s}, K_t \cdot K_2)$-free coloring of F that minimizes the number of blue copies of K_t among all such colorings. We claim that c has no blue copies of K_t and hence $F \not\to (K_{1,s}, K_t)$.

For a contradiction, assume that there exists a blue copy of K_t under c. We will show that we can recolor certain edges to obtain a $(K_{1,s}, K_t \cdot K_2)$-free coloring of F with fewer blue copies of K_t than under c, which gives a contradiction. The high-level

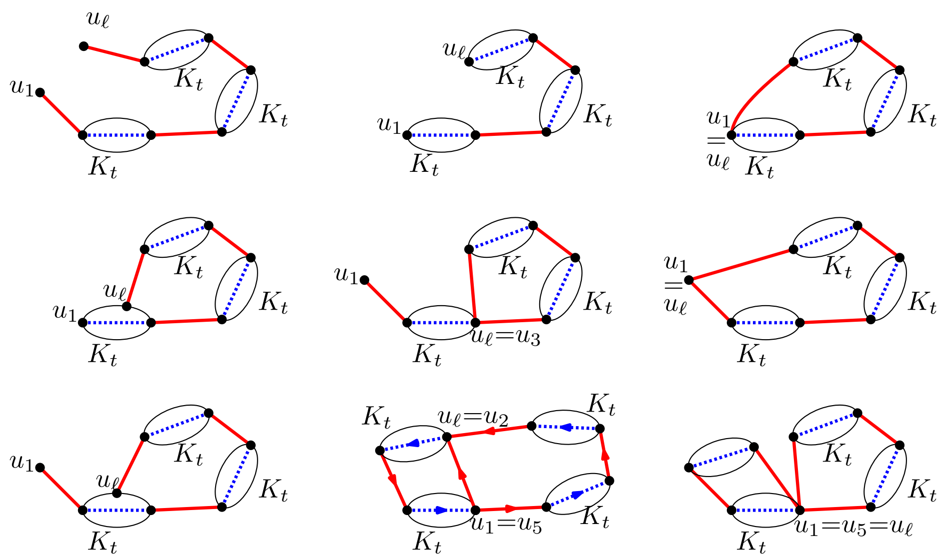

FIG. 3.1. *Several examples of feasible walks (with solid lines representing red edges and dotted lines representing blue edges).*

idea of the recoloring procedure is to choose an arbitrary blue copy of K_t and switch the color of one of its edges from blue to red. If this produces red copies of $K_{1,s}$ (there could be one at each endpoint), we recolor one edge from each such star from red to blue. We shall see that this does not create new blue copies of K_t, but it might create blue copies of $K_t \cdot K_2$. We proceed greedily, picking edges from blue copies of K_t and red copies $K_{1,s}$ and switching their colors, until we reach a desired $(K_{1,s}, K_t \cdot K_2)$-free coloring of F. Note that the edges picked during the procedure eventually can be arranged into a walk with alternating colors.

We provide the detailed arguments next. First observe that the blue copies of K_t must be pairwise disjoint and that there are no blue edges leaving any of these copies, that is, the copies of K_t form isolated components in the blue subgraph of F under c. A walk in F is a subgraph of F formed by a sequence $u_1, \dots, u_\ell$ of (not necessarily distinct) vertices of F with edges u_iu_{i+1} for all $i \in [\ell - 1]$. We call the vertices u_1 and u_ℓ the endpoints of W, even if $u_1 = u_\ell$. If W is a walk in F and K is a blue copy of K_t in F under c, we say that K is visited (by W) if W contains an edge of K; otherwise K is unvisited (by W). A walk W is *feasible* if it satisfies the following properties:

i) Each edge of F occurs at most once in W.
ii) W contains at least one blue edge.
iii) The edges of W are alternately colored red and blue, that is, $c(u_iu_{i+1}) \neq c(u_{i+1}u_{i+2})$ for $i \in [\ell - 2]$.
iv) Each blue edge in W is contained in a blue copy of K_t, and each blue copy of K_t has at most one edge in W.
v) No vertex of W is contained in an unvisited blue copy of K_t in F.
vi) An endpoint of W that is not incident to any red edge in W is not incident to any red edge in F.

Refer to Figure 3.1 for an illustration.

We first observe that feasible walks exist and can be found with the following greedy procedure: Start with an arbitrary edge e belonging to a blue copy of K_t. This satisfies the first five properties above but not necessarily the last. For each endpoint of e that is incident to some red edge in F, the walk then follows one such red edge. Now the first four properties are still satisfied, but property v) might become invalid (for the current endpoints), while property vi) becomes valid. If (in either direction) the walk has reached a so far unvisited blue copy of K_t, the walk follows an arbitrary blue edge in this copy. In this way, the procedure continues in both directions, extending W so that the first four properties are satisfied in each step, until the latter two conditions are satisfied as well. Observe that the procedure is guaranteed to terminate, since each edge of F occurs at most once in W.

We next make some observations about the structure of W. If W repeats a vertex, this vertex must be u_1 or u_ℓ. Indeed, if $u_i = u_j$ for some $1 < i < j < \ell$, then u_i has degree at least four in W by property i) and, since W is alternating by property iii), u_i is incident to at least two blue edges in W. But this is not possible since the blue copies of K_t are disjoint and W traverses at most one edge from each such copy by iv). In other words, W must be a red/blue-alternating path, except possibly the edges u_1u_2 and $u_{\ell-1}u_\ell$. Therefore, each vertex of W that is not an endpoint is incident to exactly one red and one blue edge in W; each of the endpoints may be incident to at most three red edges in W but again to at most one blue edge in W.

We now choose a feasible walk W with the smallest number of red edges. We obtain a new coloring $\tilde{c}$ by switching the colors of the edges in W. We claim that the coloring $\tilde{c}$ contains

1. no red copy of $K_{1,s}$,
2. fewer blue copies of K_t than c, and
3. no blue copy of $K_t \cdot K_2$.

By the definition of c, this leads to the desired contradiction.

To prove (1), first note that the switch does not change the number of red edges incident to vertices not in W. Now consider a vertex u of W. By our earlier observation, u is incident to at most one blue edge in W. If u is also incident to a red edge in W, then the switch does not increase the total number of red edges incident to u. If there is no red edge incident to u in W, then by property vi), we know that there are no red edges incident to u in F under c; hence u is only incident to one red edge under $\tilde{c}$. Therefore, there is no red copy of $K_{1,s}$ under $\tilde{c}$ since $s \geq 2$.

We prove (2) now. By properties ii) and iv), W contains at least one edge belonging to a copy of K_t which is blue under c. So after switching colors, this copy of K_t contains a red edge. Therefore, the only way for (2) to fail is that we create a new blue copy of K_t by switching the colors along W. So, consider any edge uv in W whose color switched from red to blue and such that uv is contained in a copy K of K_t. We aim to show that K is not monochromatic blue under $\tilde{c}$. To do so, we choose an arbitrary vertex $x \in V(K) \setminus \{u, v\}$, which exists as $t \geq 3$. What we will see is that either ux or vx is red under $\tilde{c}$, which will prove the claim. Assume that the statement is false. We distinguish three cases depending on the colors of ux and vx under c.

Case 1: Assume ux and vx are both red under c. Then, by assumption, both of these edges and uv must have switched colors and hence belong to W. This however contradicts the above observation that at most two vertices in W are incident to two red edges of W under c.

Case 2: Assume ux and vx are both blue under c. Since W is alternating and contains at least one blue edge, at least one of u and v, say u, must be contained in a blue copy K' of K_t under c. But then K' together with the edge ux (if $x \notin V(K')$)

or the edge vx (if $x \in V(K')$) forms a blue copy of $K_t \cdot K_2$ under c, a contradiction to the choice of c.

Case 3: Assume ux is red and vx is blue under c (the case where ux is blue and vx is red is similar). Then vx did not switch colors, i.e., $vx \notin W$, and both ux and uv switched colors, i.e., $ux, uv \in W$. Therefore, u is incident to two red edges of W under the coloring c, and hence must be an endpoint of W as observed above. So, one of v and x, say x, is not an endpoint of W and is therefore contained in a blue copy K' of K_t under c. Then $v \in V(K')$, since vx is blue and c does not contain a blue copy of $K_t \cdot K_2$. Since W is alternating, since x is not an endpoint of W, and since the blue copies of K_t are disjoint, the walk W contains an edge xy from K'. Since $vx \notin W$, we have $y \neq v$. By property iv) of W and again since blue copies of K_t under c are disjoint, it follows that v is not incident to any blue edge in W and must therefore be an endpoint. Removing v from W, and hence the red edge uv, yields a feasible walk W with fewer red edges, a contradiction to the choice of W.

It remains to check property (3). Assume that there is a blue copy of $K_t \cdot K_2$ under $\tilde{c}$, with blue copy K of K_t and pendent blue edge f. As we have already seen, the switching of colors does not create new blue copies of K_t. Hence, K is blue under c and all edges intersecting K in exactly one vertex are red under c, since c does not contain a blue copy of $K_t \cdot K_2$. This means that f is red under c and hence $f \in W$. By property v) and K is disjoint from all other blue copies of K_t under c, the walk W contains an edge from K. The color of this edge is then switched from blue to red, a contradiction. Altogether we see that $\tilde{c}$ is $(K_{1,s}, K_t)$-free and hence $F \not\to (K_{1,s}, K_t)$. □

To prove Theorem 1.6(b) we first prove Proposition 3.2 below, which states that for certain graphs G, which we call k-woven, we have $(G, K_t) \sim (G, K_t \cdot aK_b)$ for any integers $a \geq 1$ and $b \geq 2$ and a sufficiently large integer t. We will then prove Theorem 1.6(b) by showing that stars on at least two edges and suitable caterpillars are k-woven for appropriately chosen k.

Definition 3.1. We call a graph G *k-woven* if, for each graph F that contains an edge uv which is contained in all copies of G in F, there is a set $Y_{uv} \subseteq E(F) \setminus \{uv\}$ such that the following holds: Y_{uv} consists of at most k edges incident to u and at most k edges incident to v, and each copy of G in F contains an edge from Y_{uv}. In other words, Y_{uv} is a set of edges from F of size at most $2k$ whose removal yields a graph with no copies of G that still contains the edge uv.

As a simple example, it is not difficult to check that stars with at least two edges are 1-woven. Indeed, if a graph F has an edge uv that is contained in each copy of some star $K_{1,s}$ in F, then u and v are of degree at most s in F (and all other vertices are of degree at most $s-1$). So any set Y_{uv} consisting of one edge incident to u and one edge incident to v in $G - uv$ satisfies the condition stated above, that is, each copy of $K_{1,s}$ in F contains an edge from Y_{uv}. We demonstrate the utility of k-woven graphs in Proposition 3.2 below.

The motivation behind the definition of k-woven graphs is again a recoloring procedure. Given a k-woven graph G and a $(G, K_t \cdot aK_b)$-free coloring, we want to recolor some edges to obtain a (G, K_t)-free coloring. To get rid of the blue copies of K_t, we pick a certain set M of edges from all these copies (forming a matching) and switch their color to red. This "destroys" the blue copies of K_t, but might create undesired red copies of G instead. We then want to get rid of these red copies of G by switching the color of some of their edges to blue. Each red copy of G contains an edge uv from M. Of course, we do not want to switch the color of uv back to blue, so

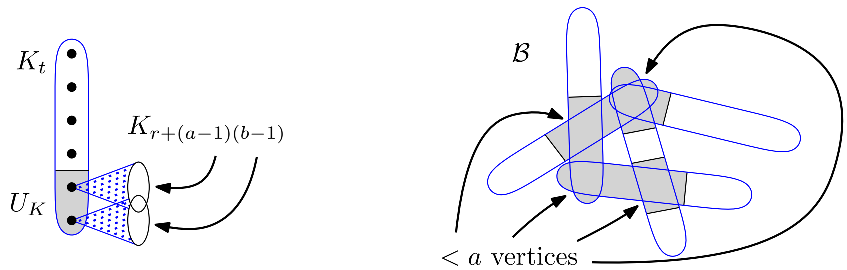

FIG. 3.2. *Left: The set U_K (gray background) in a blue copy K of K_t under φ_1. Right: A set $\mathcal{B}$ of blue copies of K_t under φ_1 with pairwise intersection of size less than a. The intersections are contained in the respective sets U_K by Claim 3.3.*

we choose from the red edges incident to the endpoints of uv. Specifically, this will be the set Y_{uv} mentioned in the definition of k-woven graphs. The key property we need is, that Y_{uv} contains only a bounded number of edges (at most k at each endpoint) and still "hits" all copies of G containing uv. By this, we "destroy" the red copies of G while we avoid creating new blue copies of K_t, as t is large.

PROPOSITION 3.2. *Let G be a k-woven graph and let $a \geq 1$ and $b \geq 2$ be integers. If t is sufficiently large, then $(G, K_t) \sim (G, K_t \cdot aK_b)$.*

Proof. Clearly, each Ramsey graph for $(G, K_t \cdot aK_b)$ is also a Ramsey graph for (G, K_t). Let $r = r(G, K_{b-1})$, let $t \geq 4k + 2(r + (a-1)(b-1)) + (a-1)$, and consider a graph F with $F \not\to (G, K_t \cdot aK_b)$. We shall show that $F \not\to (G, K_t)$. Let φ_1 denote a $(G, K_t \cdot aK_b)$-free coloring of F. Our goal is to recolor several edges to obtain a (G, K_t)-free coloring of F. For each blue copy K of K_t in F, let $U_K \subseteq V(K)$ denote the set of vertices u in K such that there are at least $r + (a-1)(b-1)$ blue edges between u and $F - K$ whose endpoints induce a complete graph in $F - K$. See Figure 3.2 (left).

Let $\mathcal{B}$ denote a maximal set of blue copies of K_t in F such that any two copies of K_t in $\mathcal{B}$ intersect in fewer than a vertices. See Figure 3.2 (right). We first make several general observations.

CLAIM 3.3. *For any two copies K, $K' \in \mathcal{B}$, we have $V(K) \cap V(K') \subseteq U_K \cap U_{K'}$ and hence $(V(K) \setminus U_K) \cap (V(K') \setminus U_{K'}) = \emptyset$.*

Proof. For each vertex $u \in V(K) \cap V(K')$ the number of blue edges between u and $K - K'$, as well as between u and $K' - K$, is at least $t - (a-1) \geq r + (a-1)(b-1)$. Since these blue neighborhoods induce a complete graph, we have $u \in U_K$ and $u \in U_{K'}$. ∎

CLAIM 3.4. *For every $K \in \mathcal{B}$, we have $|U_K| \leq a - 1$ and hence $|V(K) \setminus U_K| \geq 2$.*

Proof. Under φ_1, each vertex in U_K has a blue neighborhood of size at least $r + (a-1)(b-1)$ in $F - K$ that induces a complete graph. As there are no red copies of G under φ_1, by the definition of Ramsey number, we iteratively find $\min\{a, |U_K|\}$ vertex-disjoint blue copies of K_{b-1} in the blue neighborhood of U_K, one for up to $\min\{a, |U_K|\}$ vertices in U_K. So $|U_K| \leq a - 1$, as there is no blue copy of $K_t \cdot aK_b$ under φ_1. This shows that $|V(K) \setminus U_K| \geq t - a + 1 \geq 2$, as required. ∎

We shall now recolor some edges contained in or incident to cliques in $\mathcal{B}$ to obtain an (G, K_t)-free coloring of F. Choose a maximum matching in each clique $K - U_K$ for $K \in \mathcal{B}$ (note that these cliques are vertex-disjoint by Claim 3.3; see Figure 3.3), and let M be the union of these matchings. Let φ_2 denote the coloring obtained

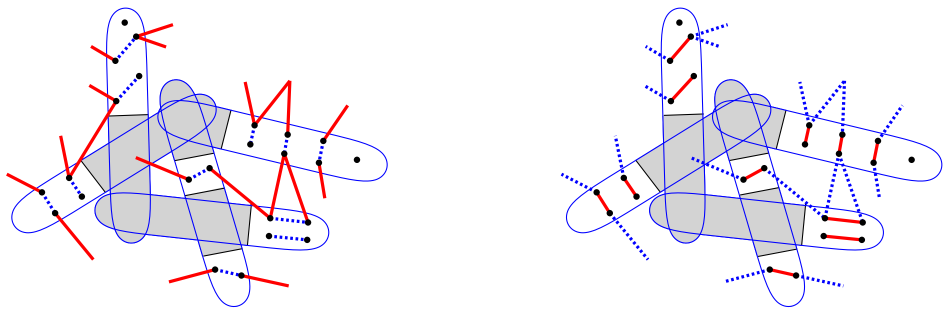

FIG. 3.3. *Left: A largest blue (dotted) matching M in $\cup_{K\in\mathcal{B}}(V(K)\setminus U_K)$ with some pendent red (solid) edges under ϕ_1. Right: The final coloring φ_3 is obtained by switching colors in M (from blue to red) and at most k further incident edges at each vertex in M (from red to blue).*

from φ_1 by switching the color of each edge in M from blue to red. Each red copy of G under φ_2 contains an edge from M. Let $u_1v_1,\dots,u_{|M|}v_{|M|}$ denote the edges of M in an arbitrary order. We shall use the fact that G is k-woven to find sets $Y_1,\dots,Y_{|M|}\subseteq E(F)\setminus M$ such that each set Y_i consists of at most k red edges incident to u_i and at most k red edges incident to v_i and such that each red copy of G in F under φ_2 contains an edge from Y_i for some $i\in[|M|]$. To do so consider the subgraph F_1 of F formed by all red copies of G under φ_2 containing u_1v_1 and not containing u_jv_j for $j>1$. Then $F_1-u_1v_1$ contains no copy of G (as it contains no edge from M), and hence, since G is k-woven, there is a desired set $Y_1\subseteq E(F_1)\setminus\{u_1v_1\}$. For $i>1$ we proceed iteratively. Having chosen $Y_1,\dots,Y_{i-1}$, let F_i denote the subgraph of F formed by all red copies of G under φ_2 containing u_iv_i, not containing any edge from Y_j for $j<i$, and not containing u_jv_j for $j>i$. We claim that $F_i-u_iv_i$ contains no copy of G. For a copy G' of G in F consider the largest j such that G' contains the edge u_jv_j. If $j<i$, then G' contains an edge from $Y_{j'}$, for some $j'\le j$, and hence is not contained in F_i. If $j\ge i$, then u_jv_j is not contained in $F_i-u_iv_i$. In both cases, G' is not contained in $F_i-u_iv_i$. Hence, since G is k-woven, there is a desired set $Y_i\subseteq E(F_i)\setminus\{u_iv_i\}$.

Now let φ_3 denote the coloring obtained from φ_2 by switching the color of each edge in $\cup_{1\le i\le|M|}Y_i$ from red to blue (see Figure 3.3). Then there are no red copies of G under φ_3. Indeed, each red copy G' of G under φ_2 contains an edge from Y_i for some i. We shall prove that there are no blue copies of K_t under φ_3. Let K' denote a copy of K_t in F.

First suppose that $K'\in\mathcal{B}$. By Claim 3.4, we have $|V(K')\setminus U_{K'}|\ge 2$, and hence K' contains a red edge under φ_3 from $E(K')\cap E(M)$.

We may assume then that $K'\notin\mathcal{B}$. If for each $K\in\mathcal{B}$ we have $V(K)\cap V(K')\subseteq U_K$, then $|V(K)\cap V(K')|<a$ by Claim 3.4. By the maximality of $\mathcal{B}$, K' contains a red edge under φ_1. This edge is red under φ_3, since only edges incident to M switched colors from red to blue and the edges in K' are not incident to M (as $V(K)\cap V(K')\subseteq U_K$ for each $K\in\mathcal{B}$ here). So K' is not blue in this case.

If there is $K\in\mathcal{B}$ with $|V(K)\cap V(K')|>\lceil\frac{t-|U_K|}{2}\rceil+|U_K|$, then K' contains an edge from $M\cap K$ due to the maximality of M. This edge is red under φ_3 and so K' is not blue.

If neither of the two previous cases holds, then let $V=\cup_{K\in\mathcal{B}}(V(K')\cap(V(K)\setminus U_K))$. By assumption $|V|\ge 1$ (since we are not in the first case). Each vertex $v\in V$ is contained in exactly one $K\in\mathcal{B}$ and, since we are not in the second case, the number of edges between v and $K' - K$ is at least $t - \lceil \frac{t-|U_K|}{2} \rceil - |U_K| = \lfloor \frac{t-|U_K|}{2} \rfloor$. Since $v \notin U_K$, fewer than $r+(a-1)(b-1)$ of those edges are colored blue under φ_1 (as their endpoints induce a complete subgraph of K'). Together, this means that the number of red edges under φ_1 incident to v in K' is at least $\lfloor \frac{t-|U_K|}{2} \rfloor - r - (a-1)(b-1) + 1 \geq 2k+1$, using the fact that $|U_K| \leq a-1$ by Claim 3.4. In total there are at least $(2k+1)|V|/2 > k|V|$ red edges in K' under φ_1. To obtain φ_3, at most $k|V|$ edges in K' switched their color from red to blue (as the union of $Y_1, \ldots, Y_{|M|}$ is a collection of stars with distinct centers and at most k edges each). This shows that at least one edge in K' is red under φ_3.

Altogether there are no red copies of G and no blue copies of K_t under φ_3 and hence $F \not\to (G, K_t)$. □

Proof of Theorem 1.6(b). We shall prove that for any integers $a \geq 1$ and $b \geq 2$ and any tree T that is either a star with at least two edges or a suitable caterpillar, there is a sufficiently large t such that $(T, K_t) \sim (T, K_t \cdot aK_b)$. By Proposition 3.2, it suffices to show that stars and suitable caterpillars are k-woven for some k. As mentioned above, it is not difficult to check that stars with at least two edges are 1-woven. We now focus on caterpillars, and claim that every s-suitable caterpillar is $2(s+1)^2$-woven.

Let T be an s-suitable caterpillar, that is, T consists of a path abc and s leaves adjacent to a, s leaves adjacent to c, and $s' < s$ leaves adjacent to b. We shall prove that T is k-woven for $k = 2(s+1)^2$. Let F be a graph with an edge uv that is contained in all copies of T in F, and let $F' = F - uv$. We need to find a set $Y_{uv} \subseteq E(F')$ consisting of at most k edges incident to u and at most k edges incident to v such that Y_{uv} contains an edge from each copy of T in F.

First suppose that there is a copy T_0 of T in F in which u is a leaf. The neighbor of u in T_0 is v, since T_0 contains uv by assumption. Then v is of degree at most $|V(T)| - 2 = 1 + 2s + s' \leq k$ in F', since otherwise uv can be replaced in T_0 by some edge vw in F' to form a copy of T entirely in F', which does not exist by assumption. Let Y_v consist of all edges in F' incident to v. If each copy of T in F contains an edge from Y_v, then we can choose $Y_{uv} = Y_v$ as our desired edge set. Otherwise, there is a copy of T in F containing no edges from Y_v. In such a copy of T the vertex v is a leaf, since uv is the only edge incident to v not in Y_v. Similarly as above, u is of degree at most k in F', and hence we can choose Y_{uv} to consist of all edges incident to u and all edges incident to v in F'.

It remains to consider the case where neither u nor v is a leaf in any copy of T in F. By the symmetry of T, we may assume that in each copy of T in F the edge ab corresponds to uv, where a corresponds to either u or v. Let N_u and N_v denote the set of neighbors of u and v in F', respectively, that are of degree at least $s+1$ in F' (see Figure 3.4 left). In each copy of T in F the edge bc corresponds to an edge uw with $w \in N_u$ or an edge vw with $w \in N_v$. In particular choosing $Y_{uv} = \{uw \colon w \in N_u\} \cup \{vw \colon w \in N_v\}$ yields the desired edge set, provided that $|N_u|$, $|N_v| \leq k$. In the following, we prove $|N_u| \leq k$. By symmetry the same bound holds for $|N_v|$.

For a contradiction, assume that $|N_u| \geq k+1$, which in particular implies that u is of degree at least $k+1$ in F'. We shall prove that there is a copy of T in F' under this assumption. For each $w \in N_u$, choose a star in F' with center vertex w and exactly s leaves not containing u, and let $\mathcal{S}$ denote the set of all chosen stars. For any two such stars S, $S' \in \mathcal{S}$ there are at least $k + 1 - 2(s+1) \geq s > s'$ neighbors of u in F' not contained in $V(S) \cup V(S')$ (see Figure 3.4 middle). Since F' does not

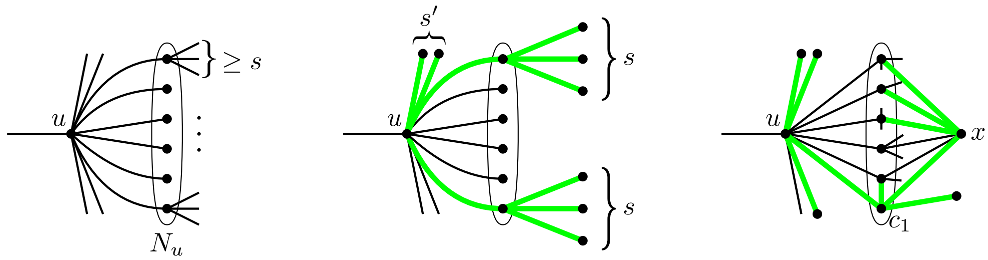

FIG. 3.4. *Left: The vertex u and the set N_u of at least $k+1$ neighbors of u of degree at least $s+1$ each. Middle: If two vertices in N_u have only u as a common neighbor, then there is copy of T (green/thick gray). Right: Otherwise, there is a vertex x that is a leaf in at least $2s+1$ copies of $K_{1,s}$ centered in N_u and there is a copy of T (here a 3-suitable caterpillar) as well (green/thick gray).*

contain a copy of T, the stars S and S' must intersect in some vertex, which could be the center of one of the stars but not of both.

Now, consider some fixed star $S \in \mathcal{S}$. By the pigeonhole principle, there is a vertex x in $V(S)$ that is contained in at least $|\mathcal{S} \setminus \{S\}|/|V(S)| = (|N_u|-1)/(s+1) \geq \frac{k}{s+1} = 2(s+1)$ of the stars in $\mathcal{S}$. It may happen that x is the center vertex of one such star, but in any case there is a family of $2s+1$ stars in $\mathcal{S}$ that have x as a leaf. Let $X = \{c_1, c_2, \ldots, c_{2s+1}\} \subseteq N_u$ denote the set of their centers. Then we find a copy of T in F' as follows: Let X_1 denote a set of s' neighbors of c_1 distinct from u and x in F', which exists since $s' < s$ and $c_1 \in X$ is of degree $s+1$ in F'. Let X_2 denote a set of s neighbors of x in X disjoint from $X_1 \cup \{c_1\}$, which exists since $|X| = 2s+1 \geq s+s'+2$. Finally, let X_3 denote a set of s neighbors of u in F' disjoint from $X_1 \cup X_2 \cup \{x, c_1\}$, which exists since the degree of u in F' is at least $k+1 = 2(s+1)^2+1 \geq 2s+s'+2$. Then the path uc_1x together with the vertices in X_1, X_2, and X_3 induces a copy of T in F', a contradiction (see Figure 3.4 right). This shows that $|N_u| \leq k$ and, by symmetry, $|N_v| \leq k$. Hence, $Y_{uv} = \{uw \colon w \in N_u\} \cup \{vw \colon w \in N_v\}$ is the desired edge set. ◻

4. Non-equivalence results for trees and cliques. In this section, we prove each part of Theorem 1.7 in turn. When constructing appropriate distinguishing graphs in our proofs, we will often combine several smaller graphs, which we call building blocks, by identifying some of their vertices or edges. We will assume that, except for the specified intersections, all of these building blocks are disjoint from one another.

4.1. Proof of Theorem 1.7(a). Recall that the diameter of a tree T, denoted $\mathrm{diam}(T)$, is the length of its longest path. Our construction gives $(T, K_t) \not\sim (T, H)$ for each tree T from the following slightly larger class $\mathcal{T}'$ consisting of all trees T that either have odd diameter, or have even diameter and additionally satisfy the following:

- The central vertex of T has at most one neighbor of degree at least three that is contained in a longest path in T.
- If T is of diameter four, the central vertex is of degree at least three.

See Figure 4.1 for an illustration.

Theorem 1.7(a) is clearly a direct consequence of Theorem 4.1 below.

THEOREM 4.1. *Let $t \geq 3$, let H be a connected graph, and let $T \in \mathcal{T}'$. Then*

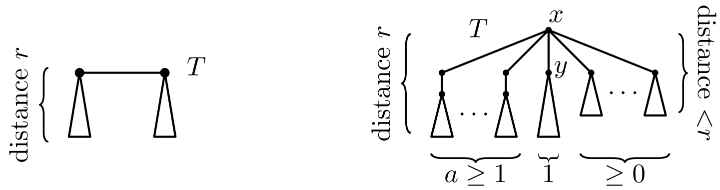

FIG. 4.1. *An odd diameter tree (left) and an even diameter tree (right) from the class $\mathcal{T}'$.*

$(T, K_t) \not\sim (T, H)$.

Proof. Consider a tree $T \in \mathcal{T}'$. If $\omega(H) \neq t$, then $(T, K_t) \not\sim (T, H)$ by Theorem 1.4. So we assume $K_t \subsetneq H$ for the remainder of the proof. We will construct a Ramsey graph for (T, K_t) that is not Ramsey for $(T, K_t \cdot K_2)$ and hence not Ramsey for (T, H). The construction differs slightly depending on the parity of the diameter of T. We begin by introducing a useful gadget graph.

Throughout the proof, we let $U_{k,i}$ denote the rooted tree in which every leaf is at distance i from the root and every vertex that is not a leaf has exactly k children. Here, the distance between two vertices x and y is the length of a shortest path that has x and y as its endpoints. Such a tree is sometimes called a perfect k-ary tree of depth i, but in the literature this term is also used with other meanings. Note that $U_{k,i}$ contains every tree of diameter at most $2i$ and maximum degree at most k.

Let d denote the maximum degree of T. Let Γ be a Ramsey graph for (T, K_{t-1}) that does not contain a copy of K_t, which exists by Theorem 1.4. Write $k = d|V(\Gamma)|$. For a positive integer i, let $\Lambda_i = \Lambda_i(T, \Gamma)$ denote the graph obtained from a copy of $U_{k,i}$ by adding edges so that, for each non-leaf vertex of $U_{k,i}$, its set of children induces d vertex-disjoint copies of Γ. We refer to the root of $U_{k,i}$ as the root of Λ_i. Let $\Phi_i = \Phi_i(\Lambda_i)$ be the red/blue-coloring that assigns red to all edges in $U_{k,i}$ and blue to all the other edges, see Figure 4.2 (left) for an illustration. Observe that, if $2i < \operatorname{diam}(T)$, then Φ_i is a (T, K_t)-free coloring of Λ_i. We have the following Ramsey property of Λ_i.

CLAIM 4.2. *Every red/blue-coloring of Λ_i yields a red copy of T, a blue copy of K_t, or a red copy of $U_{d,i}$ whose root is the root of Λ_i.*

Proof. To see why this is true, consider an arbitrary 2-coloring of Λ_i with no red copy of T. Then each copy of Γ contains a blue copy of K_{t-1}. If some non-leaf vertex in $U_{k,i}$ has only blue edges to one of the copies of Γ formed by its children, then there is a blue copy of K_t. Otherwise, every such vertex has a red edge to each of the d copies of Γ formed by its children, yielding a copy of $U_{d,i}$ as required. ◻

First consider the case where T is of diameter $2r + 1$ for some integer r. We construct a graph F as follows: Start with a copy K of K_t. For each vertex u of K, add a copy of Λ_r rooted at u so that the copies of Λ_r are pairwise disjoint. We claim that $F \to (T, K_t)$ and $F \not\to (T, K_t \cdot K_2)$.

To prove the first claim, we consider an arbitrary 2-coloring of F with no red copy of T. By Claim 4.2, some copy of Λ_r contains a blue copy of K_t or each vertex of K is the root of a red copy of $U_{d,r}$. If we find a blue copy of K_t, we are done, and hence we may assume that the latter happens for every vertex of K. If there is a red edge in K, then this edge and the red copies of $U_{d,r}$ rooted at its endpoints form a graph which contains a red copy of T. Otherwise, all edges of K are colored blue, yielding

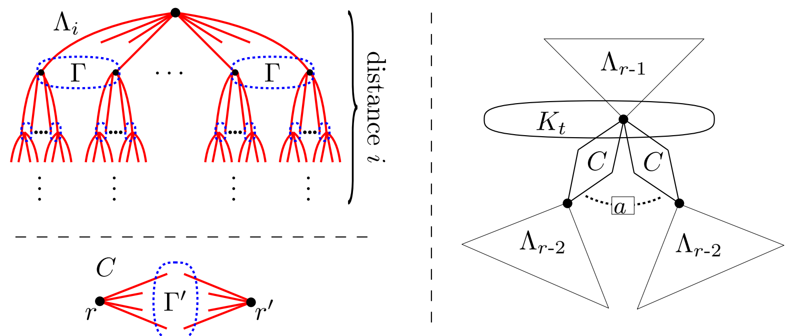

FIG. 4.2. *Left: Graphs Λ_i and C with the respective $(T, K_t \cdot K_2)$-free colorings. Right: A graph F with $F \to (T, K_t)$ and $F \not\to (T, K_t \cdot K_2)$ in case T is of even diameter.*

a blue copy of K_t. This shows $F \to (T, K_t)$.

To see that $F \not\to (T, K_t \cdot K_2)$, color all edges of K blue and give all copies of Λ_r the coloring Φ_r. Then K is the only blue copy of K_t and it cannot be extended to a copy of $K_t \cdot K_2$, as all edges leaving K are colored red and all the other blue edges form vertex-disjoint copies of Γ, which was chosen such that $K_t \nsubseteq \Gamma$. The red edges form vertex-disjoint trees of diameter $2r < \mathrm{diam}(T)$. Hence, there is no red copy of T and no blue copy of $K_t \cdot K_2$ and so $F \not\to (T, K_t \cdot K_2)$.

Now consider the case where T is of diameter $2r$ for some integer r. In this case the assumptions on $\mathcal{T}'$ imply that at most one neighbor of the central vertex is of degree at least three and is contained in a longest path. Further, if the diameter is exactly four, the central vertex of T is of degree at least three. Let x denote the central vertex of T, let y denote a neighbor of x in T that is of largest degree among all neighbors of x contained in a longest path in T, and let a denote the number of all other neighbors of x contained in a longest path in T (see Figure 4.1 (right) for an illustration). By the assumption on the structure of T, all neighbors of x counted by a are of degree exactly two in T. As in the previous case, we will use the graphs Λ_i as building blocks. We now define the second type of building block that we will use in the construction.

Let J denote a graph containing no copy of K_t such that, for any 2-coloring of the *vertices* of J, there is a vertex-monochromatic copy of K_{t-1}. Such a graph exists by [14]. Let Γ' be a Ramsey graph for (T, J) not containing a copy of K_t, which exists by Theorem 1.4, and let $k' = |V(\Gamma')|$. Let C denote the graph obtained from a copy of Γ' by adding two non-adjacent vertices r and r' and a complete bipartite graph between these two vertices and the vertices of the copy of Γ'. For convenience we call r the root of C. See Figure 4.2 (left) for an illustration.

CLAIM 4.3. *In every red/blue-coloring of C, there exists a red copy of T, a blue copy of K_t, or a red path rvr' for some $v \in V(\Gamma')$.*

Proof. To see why this is true, consider a red/blue-coloring of C with no red copy of T. Then the copy of Γ' in C contains a blue copy J' of J. In particular, each copy of K_{t-1} in J' is blue. Moreover, either each copy of K_{t-1} has a red edge going to each of r and r', or there is a blue copy of K_t. In the latter case, we are done, so assume the former. Consider a vertex coloring of J' obtained by coloring each vertex v in J' with the color of the edge rv. Since there cannot be a vertex-monochromatic blue copy of K_{t-1}, there is a vertex-monochromatic red copy of K_{t-1}. We know that there is a red edge between this copy of K_{t-1} and r', and hence we find a red path rvr', where $v \in V(J')$. □

We now construct a graph F' as follows: Start with a copy K' of K_t. For each vertex u of K', add a copies of C and a copy of Λ_{r-1} all rooted at u. Further, for each copy of C, add a copy of Λ_{r-2} rooted at the copy of r'. See Figure 4.2 (right) for an illustration. We claim that $F' \to (T, K_t)$ and $F' \not\to (T, K_t \cdot K_2)$.

To prove the first claim we consider an arbitrary 2-coloring of F' with no red copy of T. By Claim 4.2, either there is a blue copy of K_t in some copy of Λ_{r-1} or Λ_{r-2} or the root of each copy of Λ_{r-1} or Λ_{r-2} is the root of a red copy of $U_{d,r-1}$ or $U_{d,r-2}$, respectively. Assume the latter is true, since otherwise we are done. By Claim 4.3, either there exists a blue copy of K_t in some copy of C or each copy of C in F contains a red path of length two connecting the copies of r and r'. Again, we may assume that we are in the latter case. For each copy of C, the copy of r' is the root of a copy of Λ_{r-2}. Now, every vertex of K' is the root of a copies of C and a copy of Λ_{r-1}. If there is a red edge e in K, then there is a red copy of T formed by e and subtrees of the red trees rooted at its endpoints, with e playing the role of the edge xy in T. Otherwise all edges of K are colored blue, proving the claim.

To show that $F' \not\to (T, K_t \cdot K_2)$ we color the edges of F' as follows: All edges of K' are colored blue and all mentioned copies of Λ_i ($i \in \{r-1, r-2\}$) are colored according to the coloring Φ_i, as defined earlier. For each mentioned copy of C, all edges in the copy of Γ' are blue and all other edges red. Then K' is the only blue copy of K_t, as all other blue edges form vertex-disjoint copies of Γ or Γ' and these graphs do not contain copies of K_t. Moreover, K' has only red incident edges. So there is no blue copy of $K_t \cdot K_2$. Now consider the red subgraph of F', and recall that the central vertex of T has $a+1$ neighbors contained in paths of length $2r$. If $r > 2$, then each longest red path in F' has $2r$ edges and the middle vertex of each such path is in K'. In particular, the central vertex of each red tree of diameter $2r$ is in K'. But for each vertex $u \in V(K')$ there are at most a red paths of length $2r$ that meet at u and are otherwise pairwise vertex-disjoint, so there is no red copy of T. If $r = 2$, then for the same reason there is no red copy of T rooted in K'. In this case there are also red paths of length $2r = 4$ whose central vertex is in the neighborhood of K'. However, these vertices are of degree at most two in the red subgraph and, by assumption, the root of T is of degree at least three in this case. This shows that $F \not\to (T, K_t \cdot K_2)$. □

4.2. Proof of Theorem 1.7(b). We first introduce some definitions. A *cycle of length s* in a hypergraph $\mathcal{H}$ is a sequence $e_1, v_1, e_2, v_2 \dots, e_s, v_s$ of distinct hyperedges and vertices such that $v_i \in e_i \cap e_{i+1}$ for all $1 \le i \le s$ where $e_{s+1} = e_1$. The *girth* of a hypergraph $\mathcal{H}$ is the length of a shortest cycle in $\mathcal{H}$ (if no cycle exists, then we say that the girth of $\mathcal{H}$ is infinity). The *chromatic number* of a hypergraph $\mathcal{H}$ is the minimum number r for which there exists an r-coloring of the vertex set of $\mathcal{H}$ with no monochromatic edges. A hypergraph is *d-degenerate* if every subhypergraph contains a vertex of degree at most d, and we define its *degeneracy* to be the smallest d for which this property holds.

Proof of Theorem 1.7(b). Suppose that H contains a copy K of K_t, and H contains a cycle with vertices from both $V(K)$ and $V(H) \setminus V(K)$. Let $g \ge 3$ denote the length of a shortest such cycle in H and let $k = |E(T)|$.

We shall use a hypergraph of high girth and high minimum degree. The existence of such a hypergraph follows from a well-known result of Erdős and Hajnal [12]; we sketch the argument here for the sake of completeness. Erdős and Hajnal [12] showed that there exists a t-uniform hypergraph $\mathcal{H}'$ with girth at least $g+1$ and chromatic number at least $kt+1$. It is not difficult to show, using a greedy algorithm, that a d-degenerate hypergraph has chromatic number at most $d+1$. Hence, the degeneracy of $\mathcal{H}'$ must be at least kt, and thus $\mathcal{H}'$ must contain a t-uniform subhypergraph $\mathcal{H}$ with girth at least $g+1$ and minimum degree at least kt.

We construct a graph F with vertex set $V(\mathcal{H})$ by embedding a copy of K_t into each edge of $\mathcal{H}$. First observe that F does not contain a copy of H, since $\mathcal{H}$ has girth larger than g and hence each cycle of length at most g is fully contained in one of the copies of K_t, that is, in one of the hyperedges, and no two copies of K_t in F share an edge. In particular $F \not\to (T, H)$. Next we shall prove that $F \to (G, K_t)$. Consider a 2-coloring of F without blue copies of K_t. Then each copy of K_t in F (each hyperedge of $\mathcal{H}$) contains a red edge. Since $\mathcal{H}$ has minimum degree kt and is t-uniform, there are at least $v(\mathcal{H})k$ red edges, that is, the average red degree of F is at least $2k$. It follows from a standard greedy argument that the red subgraph of F contains a subgraph of minimum degree at least k. By greedily embedding the vertices of T in this subgraph, we find a red copy of T. Hence $F \to (T, K_t)$ and $(T, K_t) \not\sim (G, H)$. □

4.3. Proof of Theorem 1.7(c). In order to prove part (c) of Theorem 1.7, we will use a gadget graph known as a determiner. A graph D with a distinguished edge $\beta \in E(D)$ is called (G, H, β)-*determiner*, if $D \not\to (G, H)$, and in every (G, H)-free red/blue-coloring of D, the edge β is colored red. Moreover, we call such determiner *well-behaved* if it has an (G, H)-free coloring in which all edges incident to β are blue. Burr, Erdős, Faudree, Rousseau, and Schelp [7] showed that well-behaved determiners exist for any pair (T, K_t) when both the tree and the clique have at least three vertices. In fact, their construction satisfies some further properties which we will use in the proof of Theorem 1.7(c). We summarize those in the following proposition.

PROPOSITION 4.4 ([7, Proof of Theorem 8, Lemmas 9 & 10]). *Let $t \geq 3$ be an integer and T be a tree with at least three vertices. There exists a well-behaved (T, K_t, β)-determiner D. Moreover, the graph induced by the endpoints of β and the union of their neighborhoods is isomorphic to K_t.*

Proof of Theorem 1.7(c). We may assume that $G \not\subseteq T$, since otherwise G is a tree and we can switch the graphs G and T in the statement. Suppose that the pairs (T, K_t) and (G, K_t) are Ramsey equivalent. In order to reach a contradiction, we will construct a graph F which is Ramsey for (T, K_t) but not Ramsey for (G, K_t). To do so, first fix a (T, K_t, β)-determiner D, as given by Proposition 4.4. To create F, we start with a copy T_0 of T, and for each edge e of T_0 we take a copy D_e of D on a new set of vertices and identify e with the copy of β in D_e.

We first observe that F is a Ramsey graph for (T, K_t). Indeed, if we assume that F has a (T, K_t)-free coloring, then this induces a (T, K_t)-free coloring on each copy of D, so each copy of β needs to be red by the definition of a determiner. But then T_0 becomes a red copy of T, a contradiction.

It remains to prove that F is not Ramsey for (G, K_t), i.e., to find a (G, K_t)-free coloring of F. For this, fix any edge $e_0 \in E(T_0)$. We first observe that the graph $F - e_0$ is not Ramsey for (T, K_t) by considering the following coloring: give each copy of D a (T, K_t)-free coloring such that its copy of β is red (or not colored if $\beta = e_0$) and all edges incident to β in D are blue. The existence of such a coloring is guaranteed by the fact that D is well-behaved.

By our assumption that (T, K_t) and (G, K_t) are Ramsey equivalent, we conclude that $F - e_0$ is not a Ramsey graph for (G, K_t). Therefore, we can find a (G, K_t)-free coloring c of $F - e_0$. We now extend this coloring to F by assigning the color blue to e_0. If this does not create a blue copy of K_t, we have already found the required coloring. So we may assume that this extension leads to a blue copy K of K_t. Notice that every copy of K_t in F is fully contained in a copy of the determiner D. Then by Proposition 4.4 this blue copy of K_t is the graph induced by the endpoints of e_0 and the union of their neighborhood in D_{e_0}, i.e., it must be contained in the copy D_{e_0} of D and is unique (as the coloring c is (G, K_t)-free). We now use this information to recolor all other copies of $D - \beta$ in F using the coloring of $D_{e_0} - e_0$; we further color T_0 fully red. In this new coloring of F, there cannot be a blue copy of K_t as there were none in $D_{e_0} - e_0$. Moreover, there cannot be a red copy of G, since every copy of $D - \beta$ has a (G, K_t)-free coloring, every edge incident to T_0 is blue, and $G \not\subseteq T$. This is a contradiction to the assumption $F \to (G, K_t)$ and hence $(T, K_t) \not\sim (G, K_t)$.□

5. Concluding remarks and open problems. In this paper we identify a non-trivial infinite family of Ramsey equivalent pairs of connected graphs of the form $(T, K_t) \sim (T, K_t \cdot K_2)$, where T is a non-trivial star or a so-called suitable caterpillar. We also prove that $(T, K_t) \not\sim (T, K_t \cdot K_2)$ for a large class of other trees T including all trees of odd diameter. It remains open whether for the remaining trees the respective pairs are Ramsey equivalent or not. Our proof actually shows $(G, K_t) \sim (G, K_t \cdot K_2)$ for all so-called woven graphs G and sufficiently large t. This leads to the following two questions: Are there any woven graphs other than the trees mentioned in Theorem 1.6(b)? Are there non-woven graphs G and integers t with $(G, K_t) \sim (G, K_t \cdot K_2)$?

One of the questions that drove the study of Ramsey equivalence is: What graphs H are Ramsey equivalent to the clique K_t? This question was addressed in [3, 15, 28]. In particular, it follows from the results of Folkman [14] and Nešetřil and Rödl [21] and Fox, Grinshpun, Liebenau, Person, and Szabó [15] that there is no connected graph $H \neq K_t$ such that $H \sim K_t$. It is then natural to ask: what about an asymmetric pair of connected graphs?

Question 5.1. *Are there connected graphs G and H and an integer t such that, for $(G, H) \neq (K_t, K_t)$ it holds $(G, H) \sim (K_t, K_t)$?*

Some known results allow us to easily exclude many possible pairs (G, H). For example, the results of Folkman [14] and Nešetřil and Rödl [21], as stated in Theorem 1.4 above, show that, if $\max\{\omega(G), \omega(H)\} \neq t$, then $(G, H) \not\sim (K_t, K_t)$, while the work of Fox, Grinshpun, Liebenau, Person, and Szabó [15] shows that we cannot have $\omega(G) = \omega(H) = t$. Thus, we can restrict our attention to pairs (G, H) with $\omega(G) < t$ and $\omega(H) = t$. Combining several results concerning Ramsey properties of the random graph $G(n, p)$ [5, 17, 20, 25, 26], we can restrict (G, H) even further: namely, we can show that $m_2(G) = m_2(H) = m_2(K_t)$. Using the ideas developed by Savery in [27], we can also prove that the chromatic numbers of the graphs G and H must satisfy either $\chi(G) = t - 1$ and $\chi(H) = t + 1$, or $\chi(G) = t$ and $H = K_t$. In addition, the theory of determiners developed in [9] for 3-connected graphs allows us to conclude that G and H cannot both be 3-connected. It would be very interesting to provide a complete answer to Question 5.1.

Our study focuses on pairs of connected graphs. Disconnected graphs have also received some attention in the symmetric setting; the central question here asks which graphs are Ramsey equivalent to a complete graph [3, 15, 28]. Similar questions arise in the asymmetric setting, for instance for which graphs G and integers t we have $(G, K_t) \sim (G, K_t + K_{t-1})$, where $K_t + K_{t-1}$ is the disjoint union of K_t and K_{t-1} (this holds in case $G = K_t$ by [3]).

Acknowledgments. The authors thank an anonymous referee for many helpful comments.

REFERENCES

[1] M. Axenovich, J. Rollin, and T. Ueckerdt, *Conditions on Ramsey nonequivalence*, J. Graph Theory **86** (2017), no. 2, 159–192.
[2] H.-B. Belck, *Reguläre Faktoren von Graphen*, J. Reine Angew. Math. **188** (1950), 228–252.
[3] T. F. Bloom and A. Liebenau, *Ramsey equivalence of K_n and K_n+K_{n-1}*, Electron. J. Combin. **25** (2018), no. 3, P3.4.
[4] M. Borowiecki, I. Schiermeyer, and E. Sidorowicz, *Ramsey $(K_{1,2}, K_3)$-minimal graphs*, Electron. J. Combin. **12** (2005), R20.
[5] J. Böttcher, *Large-scale structures in random graphs*, Surveys in Combinatorics 2017, London Math. Soc. Lecture Note Ser., vol. 440, Cambridge Univ. Press, Cambridge, 2017, pp. 87–140.
[6] S. A. Burr, P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, *Ramsey-minimal graphs for star-forests*, Discrete Math. **33** (1981), no. 3, 227–237.
[7] ______, *Ramsey-minimal graphs for forests*, Discrete Math. **38** (1982), no. 1, 23–32.
[8] S. A. Burr, P. Erdős, and L. Lovász, *On graphs of Ramsey type*, Ars Combinatoria **1** (1976), no. 1, 167–190.
[9] S. A. Burr, J. Nešetřil, and V. Rödl, *On the use of senders in generalized Ramsey theory for graphs*, Discrete Math. **54** (1985), no. 1, 1–13.
[10] D. Clemens, A. Liebenau, and D. Reding, *On minimal Ramsey graphs and Ramsey equivalence in multiple colours*, Combin. Probab. Comput. **29** (2020), no. 4, 537–554.
[11] D. Conlon, J. Fox, and B. Sudakov, *Recent developments in graph Ramsey theory.*, Surveys in Combinatorics **424** (2015), 49–118.
[12] P. Erdős and A. Hajnal, *On chromatic number of graphs and set-systems*, Acta Math. Hungarica **17** (1966), no. 1, 61–99.
[13] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, *The size Ramsey number*, Period. Math. Hungar. **9** (1978), no. 1-2, 145–161.
[14] J. Folkman, *Graphs with monochromatic complete subgraphs in every edge coloring*, SIAM J. Appl. Math. **18** (1970), no. 1, 19–24.
[15] J. Fox, A. Grinshpun, A. Liebenau, Y. Person, and T. Szabó, *What is Ramsey-equivalent to a clique?*, J. Comb. Theory B **109** (2014), 120–133.
[16] A. Grinshpun, *Some problems in graph Ramsey theory*, Ph.D. thesis, Massachusetts Institute of Technology, 2015.
[17] R. Hancock, K. Staden, and A. Treglown, *Independent sets in hypergraphs and Ramsey properties of graphs and the integers*, SIAM J. Discret. Math. **33** (2019), no. 1, 153–188.
[18] P. Keevash, E. Long, and J. Skokan, *Cycle-complete ramsey numbers*, Int. Math. Res. Not. IMRN **2021** (2021), no. 1, 275–300.
[19] T. Łuczak, *On Ramsey minimal graphs*, Electron. J. Combin. **1** (1994), R4.
[20] F. Mousset, R. Nenadov, and W. Samotij, *Towards the Kohayakawa–Kreuter conjecture on asymmetric Ramsey properties*, Combin. Probab. Comput. **29** (2020), no. 6, 943–955.
[21] J. Nešetřil and V. Rödl, *The Ramsey property for graphs with forbidden complete subgraphs*, J. Comb. Theory B **20** (1976), no. 3, 243–249.
[22] T. Niessen and B. Randerath, *Regular factors of simple regular graphs, and factor spectra*, Discrete Math. **85** (1998), 89–103.
[23] J. Petersen, *Die Theorie der Regulären Graphen*, Acta. Math. **15** (1891), 193–220.
[24] F. P. Ramsey, *On a problem in formal logic*, Proc. Lond. Math. Soc. **30** (1930), 264–286.
[25] V. Rödl and A. Ruciński, *Lower bounds on probability thresholds for Ramsey properties*, Combinatorics, Paul Erdős is eighty **1** (1993), 317–346.
[26] ______, *Threshold functions for Ramsey properties*, J. Amer. Math. Soc. **8** (1995), no. 4, 917–942.
[27] M. Savery, *Chromatic number is Ramsey distinguishing*, J. Graph Theory **99** (2022), no. 1, 152–161.
[28] T. Szabó, P. Zumstein, and S. Zürcher, *On the minimum degree of minimal Ramsey graphs*, J. Graph Theory **64** (2010), no. 2, 150–164.
[29] W. T. Tutte, *The factors of graphs*, Canad. J. Math. **4** (1952), 314–328.