

The codegree Turán density of tight cycles minus one edge

Piga, Simón; Sales, Marcelo; Schülke, Bjarne

DOI:

[10.1017/S0963548323000196](https://doi.org/10.1017/S0963548323000196)

License:

Creative Commons: Attribution (CC BY)

Document Version

Publisher's PDF, also known as Version of record

Citation for published version (Harvard):

Piga, S, Sales, M & Schülke, B 2023, 'The codegree Turán density of tight cycles minus one edge', *Combinatorics, Probability and Computing*, pp. 1-4. <https://doi.org/10.1017/S0963548323000196>

[Link to publication on Research at Birmingham portal](#)

General rights

Unless a licence is specified above, all rights (including copyright and moral rights) in this document are retained by the authors and/or the copyright holders. The express permission of the copyright holder must be obtained for any use of this material other than for purposes permitted by law.

- Users may freely distribute the URL that is used to identify this publication.
- Users may download and/or print one copy of the publication from the University of Birmingham research portal for the purpose of private study or non-commercial research.
- User may use extracts from the document in line with the concept of 'fair dealing' under the Copyright, Designs and Patents Act 1988 (?)
- Users may not further distribute the material nor use it for the purposes of commercial gain.

Where a licence is displayed above, please note the terms and conditions of the licence govern your use of this document.

When citing, please reference the published version.

Take down policy

While the University of Birmingham exercises care and attention in making items available there are rare occasions when an item has been uploaded in error or has been deemed to be commercially or otherwise sensitive.

If you believe that this is the case for this document, please contact UBIRA@lists.bham.ac.uk providing details and we will remove access to the work immediately and investigate.

ARTICLE

The codegree Turán density of tight cycles minus one edge

Simón Piga¹ , Marcelo Sales²  and Bjarne Schülke³

¹School of Mathematics, University of Birmingham, Birmingham, UK, ²Mathematics Department, Emory University, Atlanta, GA, USA, and ³Mathematics Department, California Institute of Technology, Pasadena, CA, USA

Corresponding author: Simón Piga; Email: s.piga@bham.ac.uk

(Received 23 November 2022; revised 15 May 2023; accepted 17 May 2023)

Abstract

Given $\alpha > 0$ and an integer $\ell \geq 5$, we prove that every sufficiently large 3-uniform hypergraph H on n vertices in which every two vertices are contained in at least αn edges contains a copy of C_ℓ^- , a tight cycle on ℓ vertices minus one edge. This improves a previous result by Balogh, Clemen, and Lidický.

Keywords: codegree density; hypergraphs

2020 MSC Codes: Primary: 05D99. Secondary: 05C65

1. Introduction

A k -uniform hypergraph H consists of a vertex set $V(H)$ together with a set of edges $E(H) \subseteq V(H)^{(k)} = \{S \subseteq V(H) : |S| = k\}$. Throughout this note, if not stated otherwise, by *hypergraph* we always mean a 3-uniform hypergraph. Given a hypergraph F , the extremal number of F for n vertices, $\text{ex}(n, F)$, is the maximum number of edges an n -vertex hypergraph can have without containing a copy of F . Determining the value of $\text{ex}(n, F)$, or the Turán density $\pi(F) = \lim_{n \rightarrow \infty} \frac{\text{ex}(n, F)}{\binom{n}{3}}$, is one of the core problems in combinatorics. In particular, the problem of determining the Turán density of the complete 3-uniform hypergraph on four vertices, i.e., $\pi(K_4^{(3)})$, was asked by Turán in 1941 [13] and Erdős [4] offered 1000\$ for its resolution. Despite receiving a lot of attention (see for instance the survey by Keevash [8]) this problem, and even the seemingly simpler problem of determining $\pi(K_4^{(3)-})$, where $K_4^{(3)-}$ is the $K_4^{(3)}$ minus one edge, remain open.

Several variations of this type of problem have been considered, see for instance [1, 7, 12] and the references therein. The one that we are concerned with in this note asks how large the minimum codegree of an F -free hypergraph can be. Given a hypergraph H and $S \subseteq V$, we define the degree $d(S)$ of S (in H) as the number of edges containing S , i.e., $d(S) = |\{e \in E(H) : S \subseteq e\}|$. If $S = \{v\}$ or $S = \{u, v\}$ (and H is 3-uniform), we omit the parentheses and speak of $d(v)$ or $d(uv)$ as the degree of v or codegree of u and v , respectively. We further write $\delta(H) = \delta_1(H) = \min_{v \in V(H)} d(v)$ and $\delta_2(H) = \min_{uv \in V(H)^{(2)}} d(uv)$ for the minimum degree and the minimum codegree of H , respectively.

Given a hypergraph F and $n \in \mathbb{N}$, Mubayi and Zhao [11] introduced the *codegree Turán number* $\text{ex}_2(n, F)$ of n and F as the maximum d such that there is an F -free hypergraph H on n vertices

S. Piga, is supported by EPSRC grant EP/V002279/1. The second author is partially supported by NSF grant DMS 1764385. There are no additional data beyond that contained within the main manuscript.

© The Author(s), 2023. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (<https://creativecommons.org/licenses/by/4.0/>), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

with $\delta_2(H) \geq d$. Moreover, they defined the *codegree Turán density* of F as

$$\gamma(F) := \lim_{n \rightarrow \infty} \frac{ex_2(n, F)}{n}$$

and proved that this limit always exists. It is not hard to see that

$$\gamma(F) \leq \pi(F).$$

The codegree Turán density is known only for a few (non-trivial) hypergraphs (and blow-ups of these), see the table in [1]. The first result that determined $\gamma(F)$ exactly is due to Mubayi [9] who showed that $\gamma(\mathbb{F}) = 1/2$, where \mathbb{F} denotes the ‘Fano plane’. Later, using a computer assisted proof, Falgas–Ravry, Pikhurko, Vaughan, and Volec [6] proved that $\gamma(K_4^{(3)-}) = 1/4$. As far as we know, the only other hypergraph for which the codegree Turán density is known is $F_{3,2}$, a hypergraph with vertex set [5] and edges 123, 124, 125, and 345 [5]. The problem of determining the codegree Turán density of $K_4^{(3)}$ remains open, and Czygrinow and Nagle [2] conjectured that $\gamma(K_4^{(3)}) = 1/2$. For more results concerning $\pi(F)$, $\gamma(F)$, and other variations of the Turán density see [1].

Given an integer $\ell \geq 3$, a *tight cycle* C_ℓ is a hypergraph with vertex set $\{v_1, \dots, v_\ell\}$ and edge set $\{v_i v_{i+1} v_{i+2} : i \in \mathbb{Z}/\ell\mathbb{Z}\}$. Moreover, we define C_ℓ^- as C_ℓ minus one edge. In this note, we prove that the Turán codegree density of C_ℓ^- is zero for every $\ell \geq 5$.

Theorem 1.1. *Let $\ell \geq 5$ be an integer. Then $\gamma(C_\ell^-) = 0$.*

The previously known best upper bound was given by Balogh, Clemen, and Lidický [1] who used flag algebras to prove that $\gamma(C_\ell^-) \leq 0.136$.

2. Proof of Theorem 1.1

For singletons, pairs, and triples, we may omit the set parentheses and commas. For a hypergraph $H = (V, E)$ and $v \in V$, the *link* of v (in H) is the graph $L_v = (V \setminus v, \{e \setminus v : v \in e \in E\})$. For $x, y \in V$, the neighbourhood of x and y (in H) is the set $N(xy) = \{z \in V : xyz \in E\}$. For positive integers ℓ, k and a hypergraph F on k vertices, denote the ℓ -blow-up of F by $F(\ell)$. This is the k -partite hypergraph $F(\ell) = (V, E)$ with $V = V_1 \dot{\cup} \dots \dot{\cup} V_k$, $|V_i| = \ell$ for $1 \leq i \leq k$, and $E = \{v_{i_1} v_{i_2} v_{i_3} : v_{i_j} \in V_{j_j} \text{ and } i_1 i_2 i_3 \in E(F)\}$.

In their seminal paper, Mubayi and Zhao [11] proved the following supersaturation result for the codegree Turán density.

Proposition 2.1 (Mubayi and Zhao [11]). *For every hypergraph F and $\varepsilon > 0$, there are n_0 and $\delta > 0$ such that every hypergraph H on $n \geq n_0$ vertices with $\delta_2(H) \geq (\gamma(F) + \varepsilon)n$ contains at least $\delta n^{\nu(F)}$ copies of F . Consequently, for every positive integer ℓ , $\gamma(F) = \gamma(F(\ell))$.*

Proof of Theorem 1.1. We begin by noting that it is enough to show that $\gamma(C_5^-) = 0$. Indeed, we shall prove by induction that $\gamma(C_\ell^-) = 0$ for every $\ell \geq 5$. For $\ell = 6$, the result follows since C_6^- is a subgraph of $C_3(2)$. Hence, by Proposition 2.1, we have $\gamma(C_6^-) \leq \gamma(C_3(2)) = \gamma(C_3) = 0$. For $\ell = 7$, note that C_7^- is a subgraph of $C_5^-(2)$. To see that, let v_1, \dots, v_5 be the vertices of a C_5^- with edge set $\{v_i v_{i+1} v_{i+2} : i \neq 4\}$, where the indices are taken modulo 5. Now add one copy v'_2 of v_2 and one copy v'_3 of v_3 . Then $v_1 v_3 v_2 v_4 v'_3 v_5 v'_2$ is the cyclic ordering of a C_7^- with the missing edge being $v'_3 v_5 v'_2$. Therefore, if $\gamma(C_5^-) = 0$, then, by Proposition 2.1, we have $\gamma(C_7^-) = 0$. Finally, for $\ell \geq 8$, $\gamma(C_\ell^-) = 0$ follows by induction using the same argument and observing that C_ℓ^- is a subgraph of $C_{\ell-3}^-(2)$.

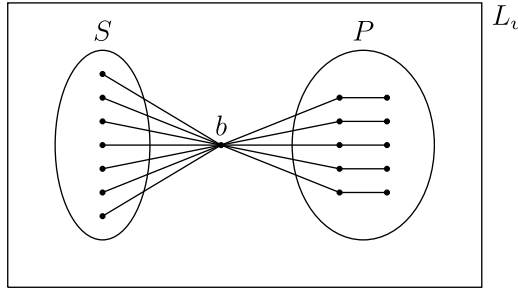


Figure 1. A nice picture (v, S, b, P) .

Given $\varepsilon \in (0, 1)$, consider a hypergraph $H = (V, E)$ on $n \geq (\frac{2}{\varepsilon})^{5/\varepsilon^2+2}$ vertices with $\delta_2(H) \geq \varepsilon n$. We claim that H contains a copy of a C_5^- .

Given $v, b \in V$, $S \subseteq V$, and $P \subseteq (V \setminus S)^2$, we say that (v, S, b, P) is a *nice picture* if it satisfies the following (see Figure 1):

- (i) $S \subseteq N_{L_v}(b)$, where $N_{L_v}(b)$ is the neighbourhood of b in the link L_v .
- (ii) For every vertex $u \in S$ and ordered pair $(x, y) \in P$, the sequence $ubvxy$ is a path of length 3 in L_v .

Note that if (v, S, b, P) is a nice picture and there exists $u \in S$ and $(x, y) \in P$ such that $uxy \in E$, then $ubvxy$ is a copy of C_5^- (with the missing edge being yub).

To find such a copy of C_5^- in H , we are going to construct a sequence of nested sets $S_t \subseteq S_{t-1} \subseteq \dots \subseteq S_0$, where $t = \lceil 5/\varepsilon^2 + 1 \rceil$, such that for $1 \leq i \leq t$ there are nice pictures (v_i, S_i, b_i, P_i) satisfying $v_i \in S_{i-1}$, $|S_i| \geq (\frac{\varepsilon}{2})^{i+1} n \geq 1$ and $|P_i| \geq \varepsilon^2 n^2 / 5$. Suppose that such a sequence exists. Then by the pigeonhole principle, there exist two indices $i, j \in [t]$ such that $P_i \cap P_j \neq \emptyset$ and $i < j$. Let (x, y) be an element of $P_i \cap P_j$. Hence, we obtain a nice picture (v_i, S_i, b_i, P_i) , $v_j \in S_i$ and $(x, y) \in P_i$ such that $v_jxy \in E$ (since xy is an edge in L_{v_j}). Consequently, $v_jb_iv_ixy$ is a copy of C_5^- in H .

It remains to prove that the sequence described above always exists. We construct it recursively. Let $S_0 \subseteq V$ be an arbitrary subset of size $\varepsilon n / 2$. Suppose we already found the sets S_i for $0 \leq i < k \leq t$, with the respective nice pictures (v_i, S_i, b_i, P_i) for $1 \leq i < k$. Now we want to construct (v_k, S_k, b_k, P_k) . Pick $v_k \in S_{k-1}$ arbitrarily. The minimum codegree of H implies that $\delta(L_{v_k}) \geq \varepsilon n$ and thus for every $u \in S_{k-1}$, we have that $d_{L_{v_k}}(u) \geq \varepsilon n$. Observe that

$$\sum_{b \in V \setminus v_k} |N_{L_{v_k}}(b) \cap S_{k-1}| = \sum_{u \in S_{k-1} \setminus v_k} d_{L_{v_k}}(u) \geq \varepsilon n (|S_{k-1}| - 1) \geq \left(\frac{\varepsilon}{2}\right)^{k+1} n^2$$

and therefore, by an averaging argument there is a vertex $b_k \in V \setminus v_k$ such that the subset $S_k := N_{L_{v_k}}(b_k) \cap S_{k-1} \subseteq S_{k-1}$ is of size at least $|S_k| \geq (\frac{\varepsilon}{2})^{k+1} n$. Let P_k be all the pairs $(x, y) \in (V \setminus S_k)^2$ such that for every vertex $v \in S_k$, the sequence v, b_k, x, y forms a path of length 3 in L_{v_k} . Since $|S_k| \leq \varepsilon n / 2$ and $\delta(L_{v_k}) \geq \varepsilon n$, it is easy to see that $|P_k| \geq (\varepsilon n / 2)(\varepsilon n / 2 - 1) \geq \varepsilon^2 n^2 / 5$. That is to say (v_k, S_k, b_k, P_k) is a nice picture satisfying the desired conditions. \square

3. Concluding remarks

A famous result by Erdős [3] asserts that a hypergraph F satisfies $\pi(F) = 0$ if F is tripartite (i.e., $V(F) = X_1 \dot{\cup} X_2 \dot{\cup} X_3$ and for every $e \in E(F)$ we have $|e \cap X_i| = 1$ for every $i \in [3]$). Note that if H is tripartite, then every subgraph of H is tripartite as well and there are tripartite hypergraphs H

with $|E(H)| = \frac{2}{9} \binom{|V(H)|}{3}$. Therefore, if F is not tripartite, then $\pi(F) \geq 2/9$. In other words, Erdős' result implies that there are no Turán densities in the interval $(0, 2/9)$. It would be interesting to understand the behaviour of the codegree Turán density in the range close to zero.

Question 3.1. Is it true that for every $\xi \in (0, 1]$, there exists a hypergraph F such that

$$0 < \gamma(F) \leq \xi ?$$

Mubayi and Zhao [11] answered this question affirmatively if we consider the codegree Turán density of a family of hypergraphs instead of a single hypergraph.

Since C_5^- is not tripartite, we have that $\pi(C_5^-) \geq 2/9$. The following construction attributed to Mubayi and Rödl (see e.g. [1]) provides a better lower bound. Let $H = (V, E)$ be a C_5^- -free hypergraph on n vertices. Define a hypergraph \tilde{H} on $3n$ vertices with $V(\tilde{H}) = V_1 \dot{\cup} V_2 \dot{\cup} V_3$ such that $\tilde{H}[V_i] = H$ for every $i \in [3]$ plus all edges of the form $e = \{v_1, v_2, v_3\}$ with $v_i \in V_i$. Then, it is easy to check that \tilde{H} is also C_5^- -free. We may recursively repeat this construction starting with H being a single edge and obtain an arbitrarily large C_5^- -free hypergraph with density $1/4 - o(1)$. In fact, those hypergraphs are C_ℓ^- -free for every ℓ not divisible by three. The following is a generalisation of a conjecture in [10].

Conjecture 3.2. If $\ell \geq 5$ is not divisible by three, then $\pi(C_\ell^-) = \frac{1}{4}$.

References

- [1] Balogh, J., Clemen, F. C. and Lidický, B. (2021) Hypergraph Turán Problems in ℓ_2 -Norm. arXiv preprint arXiv: 2108.10406.
- [2] Czygrinow, A. and Nagle, B. (2001) A note on codegree problems for hypergraphs. *Bull. Inst. Comb. Appl.* **32** 63–69.
- [3] Erdős, P. (1964) On extremal problems of graphs and generalized graphs. *Isr. J. Math.* **2** 183–190. DOI: 10.1007/BF02759942.
- [4] Erdős, P. (1977) Paul Turán, 1910–1976: his work in graph theory. *J. Graph Theory* **1**(2) 97–101. DOI: 10.1002/jgt.3190010204.
- [5] Falgas-Ravry, V., Marchant, E., Pikhurko, O. and Vaughan, E. R. (2015) The codegree threshold for 3-graphs with independent neighborhoods. *SIAM J. Discrete Math.* **29**(3) 1504–1539.
- [6] Falgas-Ravry, V., Pikhurko, O., Vaughan, E. and Volec, J. (2017) The codegree threshold of K_4^- . *Electron. Notes Discrete Math.* **61**, 407–413.
- [7] Glebov, R., Král', D. and Volec, J. (2016) A problem of Erdős and Sós on 3-graphs. *Isr. J. Math.* **211**(1) 349–366. DOI: 10.1007/s11856-015-1267-4.
- [8] Keevash, P. (2011) Hypergraph Turán problems. In *Surveys in Combinatorics 2011*, Vol. 392 of *London Mathematical Society Lecture Note series*, Cambridge University Press, pp. 83–139.
- [9] Mubayi, D. (2005) The co-degree density of the Fano plane. *J. Comb. Theory Ser. B* **95**(2) 333–337. DOI: 10.1016/j.jctb.2005.06.001.
- [10] Mubayi, D., Pikhurko, O. and Sudakov, B. (2011) *Hypergraph Turán Problem: Some Open Questions*. <https://homepages.https://homepages>.
- [11] Mubayi, D. and Zhao, Y. (2007) Co-degree density of hypergraphs. *J. Comb. Theory Ser. A* **114**(6) 1118–1132.
- [12] Reiher, C., Rödl, V. and Schacht, M. (2018) On a Turán problem in weakly quasirandom 3-uniform hypergraphs. *J. Eur. Math. Soc.* **20**(5) 1139–1159. DOI: 10.4171/JEMS/784.
- [13] Turán, P. (1941) Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok* **48** 436–452. (Hungarian, with German summary).