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Kurkofka, Jan; Melcher, Ruben; Pitz, Max

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RESEARCH ARTICLE

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A strengthening of Halin's grid theorem

Jan Kurkofka | **Ruben Melcher** | **Max Pitz**

Department of Mathematics, Universität Hamburg, Hamburg, Germany

Correspondence
Ruben Melcher, Universität Hamburg, Department of Mathematics, Bundesstrasse 55 (Geomatikum), 20146 Hamburg, Germany.
Email: ruben.melcher@uni-hamburg.de

Abstract

We show that for every infinite collection $\mathcal{R}$ of disjoint equivalent rays in a graph G there is a subdivision of the hexagonal half-grid in G such that all its vertical rays belong to $\mathcal{R}$. This result strengthens Halin's grid theorem by giving control over which specific set of rays is used, while its proof is significantly shorter.

MSC 2020
05C63 (primary), 05C40 (secondary)

1 | INTRODUCTION

An *end* of a graph G is an equivalence class of rays, where two rays of G are *equivalent* if there are infinitely many vertex-disjoint paths between them in G. The *degree* $\deg(\omega) \in \mathbb{N} \cup \{\infty\}$ of an end ω of G is the maximum size of a collection of pairwise disjoint rays in ω, see Halin [5]. Ends of infinite degree are also called *thick*. The *half-grid*, the graph on $\mathbb{N}^2$ in which two vertices (n, m) and (n', m') are adjacent if and only if $|n - n'| + |m - m'| = 1$, and its sibling the *hexagonal half-grid*, where one deletes every other rung from the half-grid as shown in Figure 1, are examples of graphs which have only one end, which is thick.

One of the cornerstones of infinite graph theory, *Halin's grid theorem* [5], says that grid-like graphs are the prototypes for ends of infinite degree. Recall that a *subdivision* of a graph G is any graph obtained from G by replacing some edges of G with new paths between their endvertices, so that none of these paths has an inner vertex in $V(G)$ or on another new path [3, § 1.7].

Halin's grid theorem. Every graph with an end of infinite degree contains a subdivision of the hexagonal half-grid whose rays belong to that end.

Halin's theorem is a precursor of the work by Robertson et al. on excluding infinite grid or clique minors [8] and has further influenced research in [1, 4, 6, 7]. It is curious, however, that

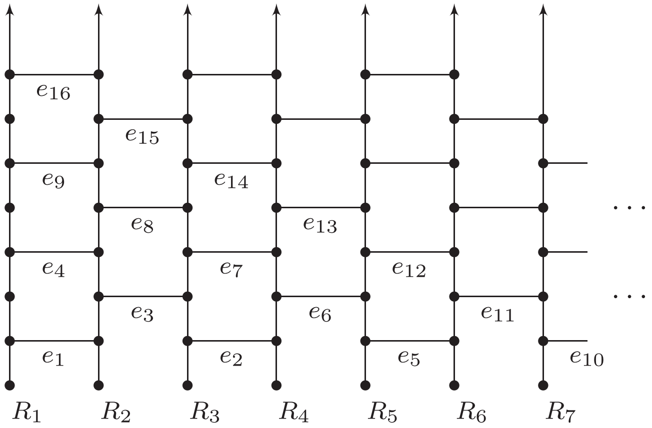

FIGURE 1 The hexagonal half-grid with vertical rays R_i

Halin's theorem does not mention any specific ray families, that is if one chooses a specific infinite collection $\mathcal{R}$ of disjoint rays witnessing that the end is thick, then neither the assertion of Halin's theorem nor its available proofs by Halin [5, Satz 4] and by Diestel [2, 3] make any assertion on how the resulting subdivided hexagonal half-grid relates to the collection of rays $\mathcal{R}$ one started with. Furthermore, in recent work of ours on an extension of Halin's grid theorem to higher cardinals [4], it became quite important to achieve more control of specific uncountable ray families, and the question arose whether this can be done also in the countable case. And indeed, the main result of this note is that this is in fact possible:

Theorem 1. *For every infinite collection $\mathcal{R}$ of disjoint equivalent rays in a graph G, there is a subdivision of the hexagonal half-grid in G such that all its vertical rays belong to $\mathcal{R}$.*

The known proofs of Halin's grid theorem are rather involved and require an elaborate recursive construction that runs close to five pages in Diestel's textbook. Our stronger result in Theorem 1 requires a different approach — which coincidentally provides a much shorter proof of Halin's original grid theorem.

2 | THE PROOF

Suppose we are handed a countably infinite collection $\mathcal{R}$ of disjoint equivalent rays in a graph G. The following routine argument shows that there is a ray S in G that meets each ray in $\mathcal{R}$ infinitely often: First choose an enumeration $(R_n)_{n\in\mathbb{N}}$ of $\mathcal{R}$ that lists each ray in $\mathcal{R}$ infinitely often. Recursively, build a sequence $(S_n)_{n\in\mathbb{N}}$ of longer and longer finite paths all starting in the same vertex and all extending each other such that each S_n intersects all R_i for all $i \leqslant n$ and has its last vertex v_n on R_n such that no vertex of R_n which comes after v_n belongs to S_n. Given S_n, since all rays in $\mathcal{R}$ are equivalent, there is an R_n–R_{n+1} path P_n disjoint from S_n whose endvertices w_n and v_{n+1} on R_n and R_{n+1} come later than all vertices of S_n on these rays. To obtain S_{n+1}, extend S_n along R_n from v_n to w_n, and then append P_n in order to reach v_{n+1}. Once the recursion is complete, $S := \bigcup_{n\in\mathbb{N}} S_n$ is a ray as desired.

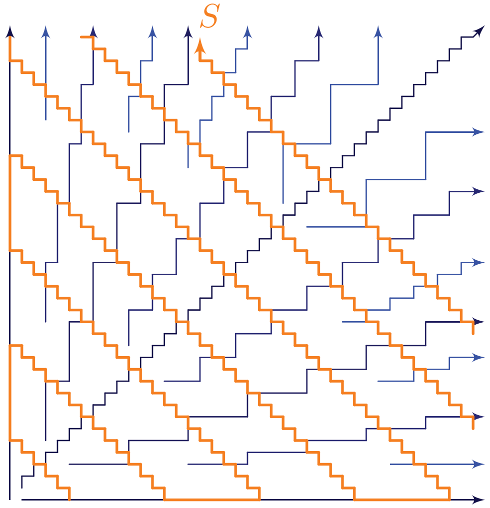

FIGURE 2 A configuration of rays in the half-grid with edge-less $M_\infty(\mathcal{R})$

Hence we may fix, for the remainder of this paper, a ray S in G that meets each ray in $\mathcal{R}$ infinitely often. Recall that a path P is *internally disjoint* from a vertex set X if P has no inner vertex in X, that is, only the endvertices of P may lie in X. An $\mathcal{R}$-segment (of S) is any maximal subpath of S which is internally disjoint from $V[\mathcal{R}] := \bigcup_{R\in\mathcal{R}} V(R)$. We say that an $\mathcal{R}$-segment is *between* two rays $R_1, R_2 \in \mathcal{R}$ if it has its endpoints on R_1 and R_2, respectively. Let $M(\mathcal{R})$ denote the auxiliary multigraph with vertex set $\mathcal{R}$ where the multiplicity of an edge R_1R_2 is equal to the number of $\mathcal{R}$-segments between R_1 and R_2. Finally, let $M_\infty(\mathcal{R})$ denote the spanning subgraph of $M(\mathcal{R})$ obtained by removing all edges of finite multiplicity.

Recall that a multigraph is *infinitely edge-connected* if it has at least two vertices and the deletion of finitely many edges does not disconnect it. Since S meets every ray in $\mathcal{R}$ infinitely often, it follows that $M(\mathcal{R})$ is infinitely edge-connected. As every infinite connected graph has a vertex of infinite degree or contains a ray [3, Proposition 8.2.1], we get the assertion of Theorem 1 immediately when $M_\infty(\mathcal{R})$ has a component with infinitely many vertices: In this case, $M_\infty(\mathcal{R})$ either contains a ray $R_1, R_2, \ldots$ or an infinite star with centre R and leaves $R_1, R_2, \ldots$. In the ray case, one recursively selects sufficiently late $\mathcal{R}$-segments to represent subdivided edges $e_1, e_2, e_3, \ldots$ of the hexagonal half-grid in the order indicated in Figure 1; in the star case, edges between R_i and R_j are represented by two sufficiently late $\mathcal{R}$-segments between R_i, R_j and R together with the subpath on R connecting the endpoints of those segments.

However, $M_\infty(\mathcal{R})$ might have no edges at all: Consider, for example, a collection of radial rays in the half-grid such that between any two rays there lies a third, see Figure 2. Still, by moving to an infinite subcollection $\mathcal{R}' \subseteq \mathcal{R}$ and considering auxiliary multigraphs $M(\mathcal{R}')$ and $M_\infty(\mathcal{R}')$ instead, the connectivity properties of $M_\infty(\mathcal{R}')$ might improve. Indeed, the auxiliary multigraphs for $\mathcal{R}'$ remain well defined as the same S still meets every ray in $\mathcal{R}'$ infinitely often. Note, however, that $\mathcal{R}$-segments of S may now be properly contained in $\mathcal{R}'$-segments of S. Our preceding discussion can be summarized as:

(1) the auxiliary multigraph $M(\mathcal{R}')$ is infinitely edge-connected for any infinite $\mathcal{R}' \subseteq \mathcal{R}$;

(2) if for some $\mathcal{R}' \subseteq \mathcal{R}$ the auxiliary multigraph $M_\infty(\mathcal{R}')$ has an infinite component, there is a subdivision of the hexagonal half-grid in G such that all its vertical rays belong to $\mathcal{R}'$.

Our next observation provides a sufficient condition for $M_\infty(\mathcal{R}')$ to have an infinite component. Recall that the degree of a vertex in a multigraph denotes the number of its neighbours.

(3) If $\mathcal{R}' \subseteq \mathcal{R}$ is infinite such that $M(\mathcal{R}')$ has only finitely many vertices of infinite degree, then $M_\infty(\mathcal{R}')$ has an infinite component.

Indeed, suppose for a contradiction that all components of $M_\infty(\mathcal{R}')$ are finite. Then there is also a finite component C of $M_\infty(\mathcal{R}')$ that contains none of the finitely many vertices that have infinite degree in $M(\mathcal{R}')$. Since $M(\mathcal{R}')$ is infinitely edge-connected by (1), there are infinitely many edges in $M(\mathcal{R}')$ from C to its complement. And since C consists of vertices of finite degree only, the neighbourhood of C in $M(\mathcal{R}')$ is finite. Thus, there is a vertex in C that sends infinitely many edges to some vertex outside of C, contradicting the choice of C. This establishes (3).

The idea of the proof of Theorem 1 is now as follows: If all vertices of $M(\mathcal{R})$ have finite degree, then $M_\infty(\mathcal{R})$ has an infinite component by (3) and we are done by (2). Otherwise, there is a ray $R_1 \in \mathcal{R}$ that has infinitely many neighbours $N(R_1)$ in $M(\mathcal{R})$, and we may restrict our collection of rays to $\mathcal{R}_1 := \{R_1\} \cup N(R_1)$. Next, if all but finitely many rays of $\mathcal{R}_1$ have finite degree in $M(\mathcal{R}_1)$, then $M_\infty(\mathcal{R}_1)$ has an infinite component by (3) and we are again done by (2). Thus, we may pick a second ray R_2 in $\mathcal{R}_1$ distinct from R_1 such that $N(R_2)$ is infinite in $M(\mathcal{R}_1)$, and restrict our collection of rays to $\mathcal{R}_2 := \{R_1, R_2\} \cup N(R_2)$. Repeating this step as often as possible gives rise to a sequence of rays $R_1, R_2, R_3, \ldots$. If this procedure ever stops because there are no more vertices of infinite degree to choose, then we are done by (3) and (2). Thus, the question becomes what to do when this procedure does not terminate.

Informally, the solution is to modify our construction so that besides the first n rays $R_1, \ldots, R_n$ we will also have chosen suitable paths $P_1, \ldots, P_{n-2}$ between them representing the subdivided edges $e_1, \ldots, e_{n-2}$ in the copy of the hexagonal half-grid from Figure 1.[†] Then, in the case where our procedure never stops, the chosen rays $R_1, R_2, \ldots$ become the vertical rays of a hexagonal half-grid where the subdivided paths corresponding to an edge e_i are given by the path P_i.

Formally, suppose that at step n we have chosen n distinct rays $R_1, \ldots, R_n$ from $\mathcal{R}$ and an infinite subcollection $\mathcal{R}_n \subseteq \mathcal{R}$ containing all chosen R_i such that in $M(\mathcal{R}_n)$ every R_i for $i = 1, \ldots, n$ is adjacent to all rays in $\mathcal{R}_n$. Further, suppose that we have chosen $n-2$ disjoint paths $P_1, \ldots, P_{n-2}$ internally disjoint from $V[\mathcal{R}_n]$, such that each P_i connects the same two rays from $\{R_1, \ldots, R_n\}$ as e_i in Figure 1, in a way such that whenever two paths P_i, P_j with $i < j$ have endvertices on the same ray R_k, then the endvertex of P_i comes before the endvertex of P_j on R_k.

Now if all but finitely many rays in $\mathcal{R}_n$ have finite degree in $M(\mathcal{R}_n)$, then we are done by (3) and (2). Hence, we may assume that there is a ray R_{n+1} in $\mathcal{R}_n \setminus \{R_1, \ldots, R_n\}$ that has infinitely many neighbours $N(R_{n+1})$ in $M(\mathcal{R}_n)$. Let $\mathcal{R}'_{n+1} := \{R_1, \ldots, R_{n+1}\} \cup N(R_{n+1})$, and note that in $M(\mathcal{R}'_{n+1})$, every R_i for $i = 1, \ldots, n+1$ is adjacent to all other rays in $\mathcal{R}'_{n+1}$. Now let i and j denote the indices of the rays in Figure 1 containing the endvertices of the edge e_{n-1}. Note that $i, j \leqslant n+1$. Since R_i and R_j are adjacent to all rays Q in $M(\mathcal{R}'_{n+1}) \backslash \{R_1, \ldots, R_{n+1}\}$, we also find such a common neighbour Q_{n-1} such that the corresponding $\mathcal{R}'_{n+1}$-segments of S between R_i, R_j and Q_{n-1} are disjoint from all earlier paths $P_1, \ldots, P_{n-2}$ and also have their endvertices on R_i, R_j later than the endvertices of any previous path $P_1, \ldots, P_{n-2}$. Then we may pick a new path P_{n-1} consisting of both these $\mathcal{R}'_{n+1}$-segments of S between R_i, R_j and Q_{n-1} together with a suitable subpath of Q_{n-1}. Finally, set $\mathcal{R}_{n+1} := \mathcal{R}'_{n+1} \setminus \{Q_{n-1}\}$. This completes the induction step, and the proof is complete. □

[†] The index shift just has the purpose that when choosing a path for e_2 we have already selected R_3 and R_4.

We remark that only in the case where our procedure stops and (2) yields a ray one can just build a grid, in all other cases one can build a clique of rays, that is, one finds an infinite $\mathcal{R}' \subseteq \mathcal{R}$ and a family of internally disjoint $\mathcal{R}'$-paths witnessing that any two rays in $\mathcal{R}'$ are equivalent.

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