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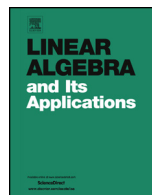
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## Bounding the row sum arithmetic mean by Perron roots of row-permuted matrices

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### ABSTRACT

$\mathbb{R}_+^{n \times n}$  denotes the set of  $n \times n$  non-negative matrices. For  $A \in \mathbb{R}_+^{n \times n}$  let  $\Omega(A)$  be the set of all matrices that can be formed by permuting the elements within each row of  $A$ . Formally:

$$\Omega(A) = \{B \in \mathbb{R}_+^{n \times n} : \forall i \exists \text{ a permutation } \phi_i \text{ s.t. } b_{i,j} = a_{i,\phi_i(j)} \forall j\}.$$

For  $B \in \Omega(A)$  let  $\rho(B)$  denote the spectral radius or largest non-negative eigenvalue of  $B$ . We show that the arithmetic mean of the row sums of  $A$  is bounded by the maximum and minimum spectral radius of the matrices in  $\Omega(A)$ . Formally, we show that

$$\min_{B \in \Omega(A)} \rho(B) \leq \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \leq \max_{B \in \Omega(A)} \rho(B).$$

For positive  $A$  we obtain necessary and sufficient conditions for these inequalities to become an equality. We also give criteria which an irreducible matrix  $C$  should satisfy so that  $\rho(C) = \min_{B \in \Omega(A)} \rho(B)$  or  $\rho(C) = \max_{B \in \Omega(A)} \rho(B)$ . These

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criteria are used to derive algorithms for finding such  $C$  when all the entries of  $A$  are positive.

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### 1. Introduction

In what follows,  $\mathbb{R}_+^n$  denotes the set of non-negative vectors with length  $n$  and  $\mathbb{R}_+^{n \times n}$  denotes the set of non-negative  $n \times n$  matrices. For  $x \in \mathbb{R}_+^n$  or, respectively,  $A \in \mathbb{R}_+^{n \times n}$  we write  $x > 0$  or, respectively,  $A > 0$ , if all entries of vector  $x$  or matrix  $A$  are positive. We will work with the following matrix set, which can be defined for any matrix  $A$ .

**Definition 1.1.** For  $A \in \mathbb{R}_+^{n \times n}$ , the matrix set  $\Omega(A)$  consists of the *row-permuted* matrices, whose entries in each row are a permutation of entries in the corresponding row of  $A$ . Formally:

$$\Omega(A) = \{B \in \mathbb{R}_+^{n \times n} : \forall i \exists \text{ a permutation } \phi_i \text{ s.t. } b_{i,j} = a_{i,\phi_i(j)} \forall j\}. \tag{1}$$

We will use the following standard notation for the Perron roots of matrices.

**Definition 1.2.** The Perron root (i.e. the largest non negative eigenvalue, or spectral radius) of a matrix  $B \in \Omega(A)$  will be denoted by  $\rho(B)$ .

For  $A \in \mathbb{R}_+^{n \times n}$  the following row sum inequality

$$\min_{i=1}^n \sum_{j=1}^n a_{i,j} \leq \rho(A) \leq \max_{i=1}^n \sum_{j=1}^n a_{i,j}$$

was first observed by Frobenius. The geometric means of the row sums as bounds for  $\rho(A)$  were explored by Al'pin [1] and Elsner and van Driessche [2], and further generalized by Engel et al. [5]. In this paper we are interested in establishing a different connection between Perron roots and row sums. Namely, we show that the arithmetic mean of the row sums satisfies

$$\min_{B \in \Omega(A)} \rho(B) \leq \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \leq \max_{B \in \Omega(A)} \rho(B). \tag{2}$$

For  $A > 0$  we obtain necessary and sufficient conditions for any of these inequalities to turn into equalities. For  $A \in \mathbb{R}_+^{n \times n}$  we also give necessary and sufficient criteria for an irreducible matrix  $C \in \Omega(A)$  to have  $\rho(C) = \min_{B \in \Omega(A)} \rho(B)$  or  $\rho(C) = \max_{B \in \Omega(A)} \rho(B)$ .

To obtain these results we make use, in particular, of the following well-known facts. These facts, which we are going to use throughout the paper, are closely related to the famous Collatz-Wielandt inequality and are summarized in the following proposition:

**Proposition 1.3** (e.g., [3], Theorem 1.11). For  $A \in \mathbb{R}_+^{n \times n}$  and constants  $\alpha > 0, \beta > 0$  and nonzero vector  $x \in \mathbb{R}_+^n$  we have:

- (i)  $\alpha x \leq Ax$  implies  $\alpha \leq \rho(A)$ ,
- (ii)  $Ax \leq \beta x$  with  $x > 0$  implies  $\rho(A) \leq \beta$ .

In addition, if  $A$  is irreducible then the following implications hold:

- (iii) if  $\alpha x \leq Ax$  and  $\exists i$  such that  $\alpha x_i < \sum_{j=1}^n a_{i,j} x_j$  then  $\rho(A) > \alpha$ ,
- (iv) if  $Ax \leq \beta x$  and  $\exists i$  such that  $\sum_{j=1}^n a_{i,j} x_j < \beta x_i$  then  $\rho(A) < \beta$ .

The next result, which we will use to derive the criteria for  $\rho(C) = \max_{B \in \Omega(A)} \rho(B)$  and  $\rho(C) = \min_{B \in \Omega(A)} \rho(B)$ , is known as the *rearrangement inequality*.

**Proposition 1.4** (e.g., [6], page 261). Let  $x, y \in \mathbb{R}_+^n$  be such that  $x_1 \leq x_2 \leq \dots \leq x_n$  and  $y_1 \leq y_2 \leq \dots \leq y_n$ , and let  $\phi: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  be an arbitrary permutation. Then the following inequalities hold:

$$\sum_{i=1}^n x_i y_{n+1-i} \leq \sum_{i=1}^n x_i y_{\phi(i)} \leq \sum_{i=1}^n x_i y_i.$$

## 2. Preliminary lemmas

The following lemma, related to the rearrangement inequality, establishes that the maximum Perron root is achieved on a matrix with all positive entries for which the correlation between the order of the components of its Perron eigenvector and each of its row vectors is maximized. The minimum Perron root is achieved when the correlation between the order of the components of its Perron eigenvector and each of its row vectors is minimized.

**Lemma 2.1.** Let  $A \in \mathbb{R}_+^{n \times n}$  be irreducible. Then the following implications hold:

- (i) if  $\rho(A) = \max_{B \in \Omega(A)} \rho(B)$  and  $x$  is a Perron eigenvector of  $A$  then for  $1 \leq i, j, k \leq n$ :

$$x_k < x_j \text{ implies } a_{i,k} \leq a_{i,j},$$

- (ii) if  $\rho(A) = \min_{B \in \Omega(A)} \rho(B)$  and  $x$  is a Perron eigenvector of  $A$  then for  $1 \leq i, j, k \leq n$ :

$$x_k < x_j \text{ implies } a_{i,k} \geq a_{i,j}.$$

**Proof.** (i): Assume that  $A \in \mathbb{R}_+^{n \times n}$  is irreducible and  $\rho(A) = \max_{B \in \Omega(A)} \rho(B)$ . Then  $\exists x > 0$  such that  $Ax = \rho(A)x$ . By contradiction, assume that there exist  $i, j, k$  such

that  $x_j > x_k$  but  $a_{i,j} < a_{i,k}$ . Let  $B$  be the matrix formed by swapping the two entries  $a_{i,j}$  and  $a_{i,k}$  so that  $b_{i,j} = a_{i,k}$  and  $b_{i,k} = a_{i,j}$ , with all other entries of  $B$  equal to the entries of  $A$ . Then  $B$  is in  $\Omega(A)$ . We have  $\rho(A)x_i = \sum_{j=1}^n a_{i,j}x_j < \sum_{j=1}^n b_{i,j}x_j$  and  $\sum_{j=1}^n a_{s,j}x_j = \sum_{j=1}^n b_{s,j}x_j$  for  $s \neq i$ . Thus by Proposition 1.3 part (iii),  $\rho(B) > \rho(A)$ . Since this contradicts that  $\rho(A) = \max_{B \in \Omega(A)} \rho(B)$  it follows that for  $1 \leq i, j, k \leq n$   $x_k < x_j$  implies  $a_{i,k} \leq a_{i,j}$  and (i) is established.

(ii): The proof is similar to the previous part, with the difference that here we assume that  $\rho(A) = \min_{B \in \Omega(A)} \rho(B)$ . Upon assuming by contradiction that there exist  $i, j, k$  such that  $x_j > x_k$  but  $a_{i,j} > a_{i,k}$  we define matrix  $B$  by swapping the entries  $a_{i,j}$  and  $a_{i,k}$  so that  $b_{i,j} = a_{i,k}$  and  $b_{i,k} = a_{i,j}$ , with all other entries of  $B$  equal to the entries of  $A$ . Observing that  $\rho(A)x_i = \sum_{j=1}^n a_{i,j}x_j > \sum_{j=1}^n b_{i,j}x_j$  and  $\sum_{j=1}^n a_{s,j}x_j = \sum_{j=1}^n b_{s,j}x_j$  for  $s \neq i$ , we use Proposition 1.3 part (iv) to obtain  $\rho(B) < \rho(A)$ , a contradiction establishing part (ii).  $\square$

Proof of the next lemma follows the reasoning used in the proof of Tchebychef’s inequality [6] page 43.

**Lemma 2.2.** *Let  $A \in \mathbb{R}_+^{n \times n}$  have a Perron eigenvector  $x \in \mathbb{R}_+^n$  satisfying  $\sum_{i=1}^n x_i = 1$ . Then the following properties hold:*

(i) *if  $\forall i, j, k$   $x_k < x_j$  implies  $a_{i,k} \leq a_{i,j}$  then*

$$\forall i \frac{\sum_{j=1}^n a_{i,j}}{n} \leq \sum_{j=1}^n a_{i,j}x_j = \rho(A)x_i,$$

(ii) *if  $\forall i, j, k$   $x_k < x_j$  implies  $a_{i,k} \geq a_{i,j}$  then*

$$\forall i \frac{\sum_{j=1}^n a_{i,j}}{n} \geq \sum_{j=1}^n a_{i,j}x_j = \rho(A)x_i,$$

(iii) *if  $\forall i, j, k$   $x_k < x_j$  implies  $a_{i,k} \leq a_{i,j}$  or  $\forall i, j, k$   $x_k < x_j$  implies  $a_{i,k} \geq a_{i,j}$ , then the following are equivalent:*

(a)  $\frac{\sum_{j=1}^n a_{i,j}}{n} = \sum_{j=1}^n a_{i,j}x_j = \rho(A)x_i$  for all  $i$ ;

(b) either  $x_i = \frac{1}{n}$  for all  $i$ , or for each  $i$  there is  $c_i$  such that  $c_i = a_{i,j}$  for all  $j$ .

**Proof.** (i): The property that  $\forall 1 \leq i, j, k \leq n$   $x_k < x_j$  implies  $a_{i,k} \leq a_{i,j}$  is equivalent to  $(a_{i,j} - a_{i,k})(x_j - x_k) \geq 0$ . From this we obtain

$$\begin{aligned} \forall i \ 2nx_i\rho(A) &= \sum_{j=1}^n \sum_{k=1}^n (a_{i,j}x_j + a_{i,k}x_k) \geq \sum_{j=1}^n \sum_{k=1}^n (a_{i,j}x_k + a_{i,k}x_j) \\ &\geq 2 \left( \sum_{j=1}^n a_{i,j} \right) \left( \sum_{j=1}^n x_j \right) = 2 \sum_{j=1}^n a_{i,j}. \end{aligned}$$

This implies

$$\forall i \ \frac{\sum_{i=1}^n a_{i,j}}{n} \leq \sum_{i=1}^n a_{i,j}x_j = \rho(A)x_i$$

establishing part (i).

(ii): The proof of this part is similar to the proof of part (i). Here we first observe that the property that  $\forall 1 \leq i, j, k \leq n \ x_k < x_j$  implies  $a_{i,k} \geq a_{i,j}$  is equivalent to  $(a_{i,j} - a_{i,k})(x_j - x_k) \leq 0$ . Using this inequality in the same way as in the proof of part (i) the opposite inequality is used, we obtain

$$\forall i \ \frac{\sum_{i=1}^n a_{i,j}}{n} \geq \sum_{i=1}^n a_{i,j}x_j = \rho(A),$$

establishing part (ii).

(iii): To establish (3) (a) implies (b) assume  $\frac{\sum_{i=1}^n a_{i,j}}{n} = \sum_{i=1}^n a_{i,j}x_j = \rho(A)x_i$  and that either  $\forall 1 \leq i, j, k \leq n \ x_k < x_j$  implies  $a_{i,k} \leq a_{i,j}$  or  $\forall 1 \leq i, j, k \leq n \ x_k < x_j$  implies  $a_{i,k} \geq a_{i,j}$ .

In the first case for any  $i, j, k$  we have that  $(a_{i,j} - a_{i,k})(x_j - x_k) \geq 0$  and in the second case we have that  $(a_{i,j} - a_{i,k})(x_j - x_k) \leq 0$ . In the first case, if there exists  $i$  such that  $(a_{i,j} - a_{i,k})(x_j - x_k) > 0$  for some  $j$  and  $k$  then  $\frac{\sum_{i=1}^n a_{i,j}}{n} < \sum_{i=1}^n a_{i,j}x_j = \rho(A)x_i$ . Similarly in the second case if there exists  $i$  such that  $(a_{i,j} - a_{i,k})(x_j - x_k) < 0$ , then  $\frac{\sum_{i=1}^n a_{i,j}}{n} > \sum_{i=1}^n a_{i,j}x_j = \rho(A)x_i$ . Since none of these strict inequalities holds, we have

$$\forall i, j, k \ (a_{i,j} - a_{i,k})(x_j - x_k) = 0. \tag{3}$$

For any  $i = 1, \dots, n$  let  $t(i)$  and  $d(i)$  be defined (non-uniquely) by

$$a_{i,t(i)} = \min_j a_{i,j}, \quad a_{i,d(i)} = \max_j a_{i,j} \tag{4}$$

and suppose that  $x_i = x_k$  does not hold for all  $i \neq k$ . Our aim is to show that then the coefficients in every row of  $A$  are equal to each other. Since either  $\forall i, j, k \ x_k < x_j$  implies  $a_{i,k} \leq a_{i,j}$  or  $\forall i, j, k \ x_k < x_j$  implies  $a_{i,k} \geq a_{i,j}$ , we can let  $t(i)$  and  $d(i)$  be defined in such a way that not only equalities (4) hold but also in the first case  $x_{t(i)} = \min_j x_j$  and  $x_{d(i)} = \max_j x_j$  and in the second case  $x_{t(i)} = \max_j x_j$  and  $x_{d(i)} = \min_j x_j$ . In both cases (3) entails that  $(a_{i,t(i)} - a_{i,d(i)})(x_{t(i)} - x_{d(i)}) = 0$  and hence  $a_{i,t(i)} = a_{i,d(i)}$ . By (4)

we obtain that all entries in the  $i$ th row of  $A$  are equal to each other, establishing the implication (a) $\Rightarrow$ (b).

To prove that (b) implies (a), first observe that obviously if  $x_j = \frac{1}{n}$  then  $\frac{\sum_{i=1}^n a_{i,j}}{n} = \sum_{i=1}^n a_{i,j}x_j = \rho(A)x_i$ . If instead for each  $i$  there is  $c_i$  such that  $a_{i,j} = c_i$  for all  $j$ , then the unique Perron eigenvector  $x$  with  $\sum_{j=1}^n x_j = 1$  has coordinates  $x_i = c_i / \sum_{j=1}^n c_j$  for all  $i$  and the Perron root is  $\rho(A) = \sum_{j=1}^n c_j$ . Indeed, we have

$$\sum_{j=1}^n a_{i,j}x_j = c_i \sum_{i=1}^n x_j = c_i = \sum_{j=1}^n c_j \cdot \frac{c_i}{\sum_{j=1}^n c_j} = \rho(A)x_i.$$

In this case  $\frac{\sum_{i=1}^n a_{i,j}}{n} = c_i = \sum_{j=1}^n a_{i,j}x_j$ , establishing (a).  $\square$

### 3. Main results

We begin this section by establishing the inequality between the arithmetic mean of the rows and the largest and smallest Perron roots of matrices in  $\Omega(A)$ .

**Theorem 3.1.** *For any  $A \in \mathbb{R}^{n \times n}_+$*

$$\min_{B \in \Omega(A)} \rho(B) \leq \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \leq \max_{B \in \Omega(A)} \rho(B). \tag{5}$$

**Proof.** We first assume that  $A > 0$  and establish  $\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \leq \max_{B \in \Omega(A)} \rho(B)$  for such  $A$ . Select  $C \in \Omega(A)$  such that  $\rho(C) = \max_{B \in \Omega(A)} \rho(B)$ . Let  $x > 0$  be a Perron eigenvector of  $C$  such that  $\sum_{i=1}^n x_i = 1$ . By Lemma 2.1 part (i) we have that for  $1 \leq i, j, k \leq n$   $x_k < x_j$  implies  $c_{i,k} \leq c_{i,j}$ . Then by Lemma 2.2 part (i) we have  $\forall i \frac{1}{nx_i} \sum_{j=1}^n c_{i,j} \leq \sum_{j=1}^n c_{i,j} \frac{x_j}{x_i} = \max_{B \in \Omega(A)} \rho(B)$  and since  $\sum_{j=1}^n c_{i,j} = \sum_{j=1}^n a_{i,j}$  for all  $i$ , we obtain

$$\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \leq \sum_{i=1}^n \max_{B \in \Omega(A)} \rho(B)x_i = \max_{B \in \Omega(A)} \rho(B).$$

Still assuming  $A > 0$ , we can establish  $\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \geq \min_{B \in \Omega(A)} \rho(B)$  in a similar way. For this we select  $C \in \Omega(A)$  such that  $\rho(C) = \min_{B \in \Omega(A)} \rho(B)$  and let  $x > 0$  be a Perron eigenvector of  $C$  such that  $\sum_{i=1}^n x_i = 1$ . Combining Lemma 2.1 part (ii) with Lemma 2.2 part (ii) we obtain  $\forall i \frac{1}{nx_i} \sum_{j=1}^n c_{i,j} \geq \sum_{j=1}^n c_{i,j} \frac{x_j}{x_i}$  and hence

$$\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \geq \sum_{i=1}^n \min_{B \in \Omega(A)} \rho(B)x_i = \min_{B \in \Omega(A)} \rho(B).$$

Now for arbitrary  $A \in \mathbb{R}^{n \times n}_+$  and  $\epsilon > 0$  we define  $A^\epsilon = (a_{i,j}^\epsilon) = (a_{i,j} + \epsilon)$ . Then since  $0 < A^\epsilon$  we have that  $\min_{B \in \Omega(A^\epsilon)} \rho(B) \leq \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n (a_{i,j} + \epsilon) \leq \max_{B \in \Omega(A^\epsilon)} \rho(B)$ .

Thus by continuity of the Perron root and letting  $\epsilon$  go to zero we obtain the desired inequality for  $A$ .  $\square$

We now establish the conditions when any of the inequalities in Theorem 3.1 becomes an equality.

**Theorem 3.2.** For  $0 < A \in \mathbb{R}_+^{n \times n}$  the following are equivalent:

- (i)  $\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,j} = \max_{B \in \Omega(A)} \rho(B)$ .
- (ii) Either the flat vector  $x = (x_i)$  where  $\forall i \ x_i = 1$  is a Perron eigenvector of  $A$  or there exists a non singular diagonal matrix  $\exists D \geq 0$  such that  $DA$  is a flat matrix (i.e.  $\forall i, j \ d_i a_{i,j} = 1$ ).
- (iii)  $\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,j} = \min_{B \in \Omega(A)} \rho(B)$ .

**Proof.** We first establish (i) $\Rightarrow$ (ii). By (i),  $\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,j} = \max_{B \in \Omega(A)} \rho(B)$ . Since the set  $\Omega(A)$  is finite, there exist  $C \in \Omega(A)$  and  $y \in \mathbb{R}_+^{n \times n}$  with  $\sum_{i=1}^n y_i = 1$  such that  $Cy = (\max_{B \in \Omega(A)} \rho(B))y$ . Thus

$$\sum_{i=1}^n \sum_{j=1}^n c_{i,j}y_j = \left( \max_{B \in \Omega(A)} \rho(B) \right) \left( \sum_{i=1}^n y_i \right) = \max_{B \in \Omega(A)} \rho(B) = \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \frac{1}{n}.$$

By Lemma 2.1 part (i)  $\forall i, j, k: 1 \leq i, j, k \leq n$  we have that  $y_k < y_j$  implies  $c_{i,k} \leq c_{i,j}$ . Then by Lemma 2.2 part (i)

$$\forall i \ \frac{1}{n} \sum_{j=1}^n c_{i,j} \leq \sum_{j=1}^n c_{i,j}y_j = \max_{B \in \Omega(A)} \rho(B)y_i.$$

Since  $\forall i \ \frac{1}{n} \sum_{j=1}^n a_{i,j} = \frac{1}{n} \sum_{j=1}^n c_{i,j}$  we can rewrite this as

$$\forall i \ \frac{1}{n} \sum_{j=1}^n a_{i,j} \leq \max_{B \in \Omega(A)} \rho(B)y_i.$$

As by (i) we have

$$\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,j} = \max_{B \in \Omega(A)} \rho(B) = \max_{B \in \Omega(A)} \rho(B) \sum_{i=1}^n y_i,$$

if there exists  $i$  such that  $\frac{1}{n} \sum_{j=1}^n a_{i,j} < \max_{B \in \Omega(A)} \rho(B)y_i$  then there would have to exist  $k$  such that  $\frac{1}{n} \sum_{j=1}^n a_{k,j} > \max_{B \in \Omega(A)} \rho(B)y_k$ , which is a contradiction, hence

$$\forall i \ \max_{B \in \Omega(A)} \rho(B)y_i = \frac{1}{n} \sum_{j=1}^n a_{i,j} = \sum_{j=1}^n c_{i,j}y_j = \frac{1}{n} \sum_{j=1}^n c_{i,j}.$$



Applying Lemma 2.2 part (iii), we obtain that either  $\forall i y_i = 1/n$  or  $\forall i, j, k c_{i,j} = c_{i,k}$ . If  $\forall i y_i = 1/n$  then  $\forall i \frac{1}{n} \sum_{j=1}^n a_{i,j} = \max_{B \in \Omega(A)} \rho(B) \frac{1}{n}$ , from which it follows that flat vector  $x = (x_i)$  where  $\forall i x_i = 1$  is a Perron eigenvector of  $A$ . If  $\forall i, j, k c_{i,j} = c_{i,k}$  then  $\forall i, j, k a_{i,j} = c_{i,j} = c_{i,k} = a_{i,k}$ . Let  $D$  be the diagonal matrix where  $\forall i d_{i,i} = \frac{1}{a_{i,i}}$  and the rest of the entries of  $D$  are 0. Thus  $DA$  is the flat matrix such that  $\forall i, j d_{i,i} a_{i,j} = 1$ .

We now show (ii) $\Rightarrow$ (i), (iii). Assume first that the flat vector  $x = (x_i)$  where  $\forall i x_i = 1$  is a Perron eigenvector of  $A$ . This is equivalent to all row sums of  $A$  being equal to each other. If this property holds for  $A$  then it also holds for all  $B \in \Omega(A)$ , so the flat vector is a Perron eigenvector of any such  $B$  with the same Perron root (equal to any of the row sums). Thus we have both (i) and (iii), i.e.,

$$\max_{B \in \Omega(A)} \rho(B) = \min_{B \in \Omega(A)} \rho(B) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,j}. \tag{6}$$

Now assume that there exists a non-singular diagonal matrix  $D \geq 0$  such that  $DA$  is a flat matrix. In this case the entries in each row of  $A$  are equal to each other, implying that  $\Omega(A) = \{A\}$ . As the left hand side and the right hand side of (5) are equal to each other, we obtain (6).

Finally, the proof of (iii) $\Rightarrow$ (ii) is similar to the proof of (i) $\Rightarrow$ (ii) and will be described more briefly. By (ii),  $\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n a_{i,j} = \min_{B \in \Omega(A)} \rho(B)$ . Since the set  $\Omega(A)$  is finite, there exist  $C \in \Omega(A)$  and  $y \in \mathbb{R}_+^{n \times n}$  with  $\sum_{i=1}^n y_i = 1$  such that  $Cy = (\min_{B \in \Omega(A)} \rho(B))y$ . Thus

$$\sum_{i=1}^n \sum_{j=1}^n c_{i,j} y_j = \left( \min_{B \in \Omega(A)} \rho(B) \right) \left( \sum_{i=1}^n y_i \right) = \min_{B \in \Omega(A)} \rho(B) = \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \frac{1}{n}.$$

Next, combining Lemma 2.1 part (ii) and Lemma 2.2 part (ii) and using that for each  $i$  the sum of the  $i$ th row of  $A$  equals the sum of the  $i$ th row of  $C$ , we obtain

$$\forall i \frac{1}{n} \sum_{j=1}^n a_{i,j} \geq \min_{B \in \Omega(A)} \rho(B) y_i.$$

Using condition (iii), however, we see that the strict inequality cannot hold for any  $i$  and therefore we have

$$\forall i \min_{B \in \Omega(A)} \rho(B) y_i = \frac{1}{n} \sum_{j=1}^n a_{i,j} = \sum_{j=1}^n c_{i,j} y_j = \frac{1}{n} \sum_{j=1}^n c_{i,j}.$$

Condition (ii) then follows by applying Lemma 2.2 part (iii) (see the end of the proof of (i) $\Rightarrow$ (ii) written above.)  $\square$

The following result applies the rearrangement inequality (Proposition 1.4) to yield a sufficient condition for establishing when  $\rho(A) = \max_{B \in \Omega(A)} \rho(B)$  and  $\rho(A) = \min_{B \in \Omega(A)} \rho(B)$ .

**Theorem 3.3.** *Let  $A \in \mathbb{R}_+^{n \times n}$  and  $0 \leq x \in \mathbb{R}_+^n$  be a Perron eigenvector of  $A$ . Then*

$$(\forall i, j, k \ 1 \leq i, j, k \leq n : x_k < x_j \Rightarrow a_{i,k} \leq a_{i,j}) \implies \rho(A) = \max_{B \in \Omega(A)} \rho(B) \quad (7)$$

$$(\forall i, j, k \ 1 \leq i, j, k \leq n : x_k < x_j \Rightarrow a_{i,k} \geq a_{i,j}) \implies \rho(A) = \min_{B \in \Omega(A)} \rho(B) \quad (8)$$

**Proof.** Consider the condition on the left hand side of (7). Observe that we can assume without loss of generality that  $a_{i,k} \leq a_{i,j} \Leftrightarrow a_{l,k} \leq a_{l,j}$  for any two rows  $i$  and  $l$  of  $A$ . Indeed, if  $x_k < x_j$  then this is the case (by the condition), and if  $x_k = x_j$  then the entries  $a_{i,k}$  and  $a_{i,j}$  or  $a_{l,k}$  and  $a_{l,j}$  can be swapped without changing  $Ax$ , so that the modified matrix belongs to  $\Omega(A)$  and has the same Perron eigenvector  $x$  and the same Perron root  $\rho(A)$ . Then we can also assume without loss of generality that simultaneously  $x_1 \leq x_2 \leq \dots \leq x_n$  and  $a_{i,1} \leq a_{i,2} \leq \dots \leq a_{i,n}$  for all  $i$ . If we consider any matrix  $B \in \Omega(A)$ , then the rearrangement inequality implies that  $Bx \leq Ax = \rho(A)x$  and hence  $\rho(B) \leq \rho(A)$ .

Similarly, to prove the sufficiency of the condition on the right hand side of (10), we can assume without loss of generality that  $a_{i,k} \geq a_{i,j} \Leftrightarrow a_{l,k} \geq a_{l,j}$  for any two rows  $i$  and  $l$  of  $A$ . Indeed, if  $x_k < x_j$  then this is the case (by the condition), and if  $x_k = x_j$  then the corresponding non-aligning entries in any row can be swapped to obtain the alignment. Then we can also assume without loss of generality that simultaneously  $x_1 \leq x_2 \leq \dots \leq x_n$  and  $a_{i,1} \geq a_{i,2} \geq \dots \geq a_{i,n}$  for all  $i$ . If we consider any matrix  $B \in \Omega(A)$ , then the rearrangement inequality implies that  $Bx \geq Ax = \rho(A)x$  and hence  $\rho(B) \geq \rho(A)$ .  $\square$

The following result applies Lemma 2.1 to show that for irreducible matrices conditions (7) and (8) of Theorem 3.3 are necessary and sufficient for  $\rho(A) = \max_{B \in \Omega(A)} \rho(B)$  or  $\rho(A) = \min_{B \in \Omega(A)} \rho(B)$ .

**Theorem 3.4.** *Let  $A \in \mathbb{R}_+^{n \times n}$  be irreducible and  $0 < x \in \mathbb{R}_+^n$  be a Perron eigenvector of  $A$ . Then*

$$\rho(A) = \max_{B \in \Omega(A)} \rho(B) \iff (\forall i, j, k \ 1 \leq i, j, k \leq n : x_k < x_j \Rightarrow a_{i,k} \leq a_{i,j}) \quad (9)$$

$$\rho(A) = \min_{B \in \Omega(A)} \rho(B) \iff (\forall i, j, k \ 1 \leq i, j, k \leq n : x_k < x_j \Rightarrow a_{i,k} \geq a_{i,j}) \quad (10)$$

**Proof.** By Lemma 2.1, the conditions on the right hand sides of (9) and (10) are necessary. The fact that they are sufficient follows immediately from Theorem 3.3.  $\square$

#### 4. Solving $\max_{B \in \Omega(A)} \rho(B)$ and $\min_{B \in \Omega(A)} \rho(B)$

Below we give two simple iterative procedures for solving  $\max_{B \in \Omega(A)} \rho(B)$  and  $\min_{B \in \Omega(A)} \rho(B)$ . Note that the computation of the minimum and maximum spectral radius over sets more general than  $\Omega(A)$  was investigated by Protasov [7] where similar iterative procedures were suggested.

Before presenting the iterative procedures we first establish the following lemmas.

**Lemma 4.1.** *Let  $A \in \mathbb{R}_+^{n \times n}$  and  $P$  be a permutation matrix. Then  $\rho(PA) = \rho(AP)$ .*

**Proof.** It is easy to see that any eigenvalue of  $PA$  is an eigenvalue of  $AP$  and the other way around:

$$\begin{aligned} PAx = \alpha x &\Rightarrow AP(P^{-1}x) = \alpha(P^{-1}x), \\ APy = \beta y &\Rightarrow PA(Py) = \beta(Py). \quad \square \end{aligned}$$

**Lemma 4.2.** *For  $A \in \mathbb{R}_+^{n \times n}$  and all permutation matrices  $P$*

$$\max_{B \in \Omega(A)} \rho(B) = \max_{B \in \Omega(PA)} \rho(B)$$

and

$$\min_{B \in \Omega(A)} \rho(B) = \min_{B \in \Omega(PA)} \rho(B).$$

**Proof.** Take arbitrary  $B \in \Omega(PA)$ . Then  $B = PC$ , where  $C \in \Omega(A)$ , and by Lemma 4.1  $\rho(B) = \rho(CP)$ , where  $CP \in \Omega(A)$ . This observation implies that

$$\max_{B \in \Omega(A)} \rho(B) \geq \max_{B \in \Omega(PA)} \rho(B), \quad \min_{B \in \Omega(A)} \rho(B) \leq \min_{B \in \Omega(PA)} \rho(B).$$

The reverse inequalities follow from a similar argument where we start with  $B = \Omega(A)$  and represent  $B = P^{-1}C$  with  $C \in \Omega(PA)$ .  $\square$

**Definition 4.3.** A  $n \times n$  matrix  $A$  is said to be fully indecomposable if  $PAQ$  is irreducible for all permutation matrices  $P$  and  $Q$ .

We now argue that Algorithm 1 (see below) is valid. Observe that if in step 6 vector  $x$  is not in ascending order and hence  $P$  is not the identity matrix, then  $C(Px) \geq Cx = \rho(C)x$  with at least one strict inequality, since all rows of  $C$  as well as  $Px$  are aligned together in ascending order, but this is not true about all rows of  $C$  and vector  $x$ . Then we obtain  $(PC)Px \geq \rho(C)Px$  with at least one strict inequality, and by Proposition 1.3 part (iii)  $\rho(C) < \rho(PC)$ . If  $P$  is the identity matrix then  $\forall i, j, k \ 1 \leq i, j, k \leq n : x_k < x_j \Rightarrow c_{i,k} \leq c_{i,j}$  and by Theorem 3.4  $\rho(C) = \max_{B \in \Omega(C)} \rho(B)$ . By Lemma 4.2 it

**Algorithm 1** Solving  $\max_{B \in \Omega(A)} \rho(B)$ .

**Input:**  $A \in \mathbb{R}_+^{n \times n}$  with  $A$  fully indecomposable.

- 1: Define matrix  $C_0 \in \Omega(A)$  by placing the entries in each row of  $A$  in ascending order.
- 2: Find a permutation matrix  $Q \in \mathbb{R}_+^{n \times n}$  such that the Euclidean norms of the rows of  $QC$  are in ascending order.
- 3:  $P \in \mathbb{R}_+^{n \times n}$  is the zero matrix,  $C := QC_0$ .
- 4: **while**  $P$  is not the identity matrix **do**
- 5:     Find a Perron eigenvector  $x \in \mathbb{R}_+^n$  of  $C$
- 6:     **if** the entries of  $x$  are not in ascending order **then**
- 7:         Find a permutation matrix  $P \in \mathbb{R}_+^{n \times n}$  so that entries of  $Px$  are in ascending order.
- 8:     **else**
- 9:         Set  $P$  to be the identity matrix.
- 10:    **end if**
- 11:     $C := PC, Q := PQ$ .
- 12: **end while**

**Output:**  $C_0Q, \rho(C_0Q) = \max_{B \in \Omega(A)} \rho(B)$

**Algorithm 2** Solving  $\min_{B \in \Omega(A)} \rho(B)$ .

**Input:**  $A \in \mathbb{R}_+^{n \times n}$  with  $A$  fully indecomposable.

- 1: Define matrix  $C_0 \in \Omega(A)$  by placing the entries in each row of  $A$  in descending order.
- 2: Find a permutation matrix  $Q \in \mathbb{R}_+^{n \times n}$  such that the Euclidean norms of the row sums of  $QC$  are in descending order.
- 3:  $P \in \mathbb{R}_+^{n \times n}$  is the zero matrix,  $C := QC_0$ .
- 4: **while**  $P$  is not the identity matrix **do**
- 5:     Find a Perron eigenvector  $x \in \mathbb{R}_+^n$  of  $C$
- 6:     **if** the entries of  $x$  are not in descending order **then**
- 7:         Find a permutation matrix  $P \in \mathbb{R}_+^{n \times n}$  so that entries of  $Px$  are in descending order.
- 8:     **else**
- 9:         Set  $P$  to be the identity matrix.
- 10:    **end if**
- 11:     $C := PC, Q := PQ$ .
- 12: **end while**

**Output:**  $C_0Q, \rho(C_0Q) = \min_{B \in \Omega(A)} \rho(B)$

follows that  $\rho(C) = \max_{B \in \Omega(A)} \rho(B)$ . The algorithm terminates in a finite number of iterations since  $\rho(C)$  is strictly increasing so matrices  $C$  do not repeat, and since the number of permutations is finite. Lemma 4.1 also implies that for the final matrix  $C$  we have  $\rho(C) = \rho(C_0Q)$ , implying that  $C_0Q$  solves the problem of maximizing  $\rho(B)$  over  $\Omega(A)$  (while belonging to  $\Omega(A)$ ).

Algorithm 2 is valid for the reasons similar to those explained above for Algorithm 1. We now demonstrate the work of Algorithm 1 on the following small example.

**Example 4.4.** Consider matrix

$$A = \begin{pmatrix} 2 & 5 & 2 & 2 & 5 \\ 6 & 6 & 2 & 3 & 1 \\ 7 & 3 & 5 & 5 & 3 \\ 3 & 3 & 4 & 6 & 8 \\ 2 & 4 & 2 & 5 & 5 \end{pmatrix}$$

First we align all rows of this matrix in ascending order thus obtaining  $C_0$ . The Euclidian norms of the row sums if  $C$  are 11, 12, 18, 27 and 31. Thus initially  $Q = I$  and  $C = QC_0 = C_0$  with its Perron vector  $x$ :

$$C = QC_0 = \begin{pmatrix} 2 & 2 & 2 & 5 & 5 \\ 1 & 2 & 3 & 6 & 6 \\ 3 & 3 & 5 & 5 & 7 \\ 3 & 3 & 4 & 6 & 8 \\ 2 & 2 & 4 & 5 & 5 \end{pmatrix}, \quad x \approx \begin{pmatrix} 0.3561 \\ 0.4098 \\ 0.5091 \\ 0.5301 \\ 0.4063 \end{pmatrix}$$

The components of  $x$  are not ascending and we have:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The next while loop proceeds, since  $P \neq I$ . We compute the next matrix  $C$  and its Perron eigenvector  $x$ :

$$C := PQC_0 = \begin{pmatrix} 2 & 2 & 2 & 5 & 5 \\ 2 & 2 & 4 & 5 & 5 \\ 1 & 2 & 3 & 6 & 6 \\ 3 & 3 & 5 & 5 & 7 \\ 3 & 3 & 4 & 6 & 8 \end{pmatrix}, \quad x \approx \begin{pmatrix} 0.3595 \\ 0.3987 \\ 0.4116 \\ 0.5055 \\ 0.5355 \end{pmatrix}$$

Here,  $x$  is in the ascending order. The algorithm ends and returns

$$C_0PQ = \begin{pmatrix} 2 & 2 & 5 & 5 & 2 \\ 1 & 3 & 6 & 6 & 2 \\ 3 & 5 & 5 & 7 & 3 \\ 3 & 4 & 6 & 8 & 3 \\ 2 & 4 & 5 & 5 & 2 \end{pmatrix}, \quad \rho(C_0PQ) \approx 20.9863.$$

**Remark 4.5.** We conducted a number of numerical experiments, in which we increased the matrix dimension from 5 to 200. For each dimension we generated 50 random instances of  $A$  and counted the number of while loops that Algorithms 1 and 2 require before convergence. For the whole dimension range, the average number of while loops stayed with the maximum number of loops not exceeding 3. Finding a reasonable upper bound on the number of loops before convergence is an open problem. Note that Cvetković and Protasov [4] establish that a similar algorithm has local quadratic convergence (see [4], page 19).

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Data availability

No data was used for the research described in the article.

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