Upper density of monochromatic paths in edge-coloured infinite complete graphs and bipartite graphs

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UPPER DENSITY OF MONOCHROMATIC PATHS IN EDGE-COLOURED INFINITE COMPLETE GRAPHS AND BIPARTITE GRAPHS

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Abstract. The upper density of an infinite graph G with $V(G) \subseteq \mathbb{N}$ is defined as $\overline{d}(G) = \limsup_{n\to\infty} |V(G) \cap \{1, \dots, n\}|/n$. Let $K_{\mathbb{N}}$ be the infinite complete graph with vertex set $\mathbb{N}$. Cortsen, DeBiasio, Lamaison and Lang showed that in every 2-edge-colouring of $K_{\mathbb{N}}$, there exists a monochromatic path with upper density at least $(12 + \sqrt{8})/17$, which is best possible. In this paper, we extend this result to k-edge-colouring of $K_{\mathbb{N}}$ for $k \geq 3$. We conjecture that every k-edge-coloured $K_{\mathbb{N}}$ contains a monochromatic path with upper density at least $1/(k-1)$, which is best possible (when $k-1$ is a prime power). We prove that this is true when $k = 3$ and asymptotically when $k = 4$. Furthermore, we show that this problem can be deduced from its bipartite variant, which is of independent interest.

1. Introduction

Throughout the paper, a k-edge-colouring of a graph uses colours $1, 2, \dots, k$. Given a k-edge-coloured K_n, how long is the longest monochromatic path? This question is equivalent to asking for the k-colour Ramsey number of paths P_n, denoted by $R_k(P_n)$. When $k = 2$, Gerencsér and Gyárfás [8] show that $R_2(P_n) = \lfloor 3n/2 \rfloor - 1$. When $k = 3$, Gyárfás, Ruszinkó, Sárközy and Szemerédi [9] show that $R_3(P_n) = n + 2\lceil n/2 \rceil - 2$. For $k \geq 4$, we know that $(k - 1 + o(1))n \leq R_k(P_n) \leq (k - 1/2 + o(1))n$ by Sun, Yang, Xu and Li [16] and Knierim and Su [10], respectively. This implies that every k-edge-coloured K_n contains a monochromatic path of density between $2/(2k-1)$ and $1/(k-1)$. We ask the analogous question for infinite complete graphs.

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Let $K_{\mathbb{N}}$ be the infinite complete graph with vertex set $\mathbb{N}$, where $\mathbb{N}$ is the set of strictly positive integers (i.e. without zero). Given a set $A \subseteq \mathbb{N}$, the *upper density of A* is defined as

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [n]|}{n},$$

where $[n] = \{1, \ldots, n\}$. Similarly, if G is a graph with $V(G) \subseteq \mathbb{N}$, then the *upper density of G* is defined as $\overline{d}(G) = \overline{d}(V(G))$. Hence, the main focus of this paper is determine the largest upper density of a monochromatic path guaranteed in any k-edge-colouring of $K_{\mathbb{N}}$.

A result of Rado [14] states that any k-edge-coloured $K_{\mathbb{N}}$ can be partitioned into at most k monochromatic paths. This implies that one of these paths must have upper density at least $1/k$. When $k = 2$, Erdős and Galvin [6] proved that any 2-edge-coloured $K_{\mathbb{N}}$ contains a monochromatic path with upper density between $2/3$ and $8/9$. The lower bound was further improved in [5, 13]. Finally, Corsten, DeBiasio, Lamaison and Lang [4] proved that any 2-edge-coloured $K_{\mathbb{N}}$ contains a monochromatic path with upper density at least $(12+\sqrt{8})/17$, which is best possible. See [1, 3, 11] for other upper densities of monochromatic subgraphs in 2-edge-coloured $K_{\mathbb{N}}$.

In this paper, we consider k-edge-colourings of $K_{\mathbb{N}}$ for $k \geq 3$; we conjecture that the picture is quite different from the case $k = 2$.

Conjecture 1.1. *Let $k \geq 3$. In any k-edge-colouring of $K_{\mathbb{N}}$, there exists a monochromatic path P with $\overline{d}(P) \geq 1/(k-1)$.*

We prove that this conjecture holds when $k = 3$ and asymptotically when $k = 4$.

Theorem 1.2. *In any 3-edge-colouring of $K_{\mathbb{N}}$, there exists a monochromatic path P with $\overline{d}(P) \geq 1/2$. In any 4-edge-colouring of $K_{\mathbb{N}}$, $\sup\{\overline{d}(P) : P \text{ is a monochromatic path}\} \geq 1/3$.*

The following result shows that if Conjecture 1.1 holds, then it is sharp when $k-1$ is a prime power. In particular, Theorem 1.2 is best possible.

Proposition 1.3 ([5, Corollary 3.5])**.** *Let $k \geq 3$ and let q be a prime power with $q \leq k-1$. Then there exists a k-edge-colouring of $K_{\mathbb{N}}$ in which every monochromatic path P satisfies $\overline{d}(P) \leq 1/q$.*

1.1. **Monochromatic paths in complete bipartite infinite graphs.** We now look at the bipartite variant of the above problem, which turns out to be closely related. Let $K_{\mathbb{N},\mathbb{N}}$ denote the set of all complete bipartite graphs on $\mathbb{N}$ where both vertex classes are infinite. We will write $K_{V,W} \in K_{\mathbb{N},\mathbb{N}}$ to denote that $K_{V,W}$ is a graph in $K_{\mathbb{N},\mathbb{N}}$ such that V and W are both infinite, disjoint and together partition $\mathbb{N}$.

We investigate the largest guaranteed upper density of monochromatic paths in a k-edge-coloured $K_{V,W} \in K_{\mathbb{N},\mathbb{N}}$. A result of D. Soukup [15, Theorem 2.4.1] states that any k-edge-coloured $K_{V,W}$ can be partitioned into at most $2k-1$ monochromatic paths.

So one of them will have upper density at least $1/(2k-1)$. By considering a k-edge-colouring of $K_{V,W}$ derived from a proper k-edge-colouring of $K_{k,k}$ (see Section 3 for details), it is not difficult to see that $1/k$ is an upper bound on the upper density of a monochromatic path in $K_{V,W}$.

Proposition 1.4. *Let $K_{V,W} \in K_{\mathbb{N},\mathbb{N}}$. For all $k \in \mathbb{N}$, there exists a k-edge-colouring of $K_{V,W}$ in which every monochromatic path P satisfies $\overline{d}(P) \leqslant 1/k$.*

We conjecture that this bound is in fact tight.

Conjecture 1.5. *In any k-edge-colouring of $K_{V,W} \in K_{\mathbb{N},\mathbb{N}}$, there exists a monochromatic path P with $\overline{d}(P) \geq 1/k$.*

The conjecture is trivial for $k = 1$. In this paper, we show that the conjecture is true for $k = 2$ and a weaker result when $k \geq 3$, which improves the lower bound of $1/(2k-1)$ coming from Soukup's result [15].

Theorem 1.6. *Let $K_{V,W} \in K_{\mathbb{N},\mathbb{N}}$. In any 2-edge-colouring of $K_{V,W}$, there exists a monochromatic path P with $\overline{d}(P) \geq 1/2$. In any k-edge-colouring of $K_{V,W}$ with $k \geq 3$, $\sup\{\overline{d}(P) : P \text{ is a monochromatic path}\} \geq 1/(2k-3)$.*

The case $k = 2$ for both Theorem 1.6 and Conjecture 1.5 are also implied by a result of Corsten, DeBiasio and McKenney [3, Theorem 1.15]. They also conjectured a stronger version of Conjecture 1.5.

We show that one can deduce Conjecture 1.1 from Conjecture 1.5 using the following result.

Theorem 1.7. *Let $k \geq 3$. Let ϕ_k be such that, for all $K_{V,W} \in K_{\mathbb{N},\mathbb{N}}$ and all k-edge-colourings of $K_{V,W}$, there exists a monochromatic path P with $\overline{d}(P) \geq \phi_k$. Then, in any k-edge-colouring of $K_{\mathbb{N}}$, there exists a monochromatic path P with $\overline{d}(P) \geq \phi_{k-1}$.*

Therefore, Theorem 1.2 is a corollary of Theorems 1.6 and 1.7.

A natural open problem is to show that Conjecture 1.5 for all $k \geq 3$. Note that when $k = 3$, Theorem 1.7 shows that the supremum of the upper densities of monochromatic paths is at least $1/3$. It would be good to replace the supremum with the maximum.

1.2. **Notations and layout.** Given a graph G, a vertex $v \in V(G)$, and a colouring of the edges of G, we write $N_i(v)$ for the set of vertices x in G such that edge vx has colour i. A path of colour i refers to a monochromatic path of colour i.

The layout of this paper is as follows. In Section 2, we employ some ideas from graph regularity to prove Theorem 1.7. In Section 3, we investigate k-edge-coloured $K_{\mathbb{N},\mathbb{N}}$. In particular, we prove Proposition 1.4 and Theorem 1.6.

2. Proof of Theorem 1.7

We need a Ramsey-type result on matchings. Cockayne and Lorimer [2] showed that in any k-edge-colouring of K_n, there exists a monochromatic matching of size at least

$\left\lfloor \frac{n+k-1}{k+1} \right\rfloor$. We need the analogous result by Omidi, Raesi and Rahimi [12], which replaces K_n with graphs with large minimum degree.

Theorem 2.1 (Omidi, Raesi and Rahimi [12, Corollary 1.4])**.** *Let G be a graph on n vertices with $\delta(G) \geq \frac{kn}{k+1}$. Then, in any k-edge-colouring of G, there exists a monochromatic matching of size at least $\left\lfloor \frac{n+k-1}{k+1} \right\rfloor$.*

By a standard application of the regularity lemma, we can deduce that any large k-edge-coloured K_n contains a constant number of monochromatic paths covering about $2/(k+1)$ fraction of the vertices. (Namely, we first apply the Szemerédi's regularity lemma to K_n, then obtain a large monochromatic matching in the reduced graph using Theorem 2.1, and finally, convert the edges of the monochromatic matchings into long monochromatic paths by the blow-up lemma. For example, see the proof of Lemma 3 in [7].)

Corollary 2.2. *For all $k \geq 3$ and all $\varepsilon > 0$, there exists $t_0 = t_0(k,\varepsilon) > 0$ and $n_0 = n_0(k,\varepsilon) \in \mathbb{N}$ such that the following holds. In any k-edge-colouring of K_n with $n \geq n_0$, there exist vertex-disjoint paths $P_1, \dots, P_t$ such that $t \leq t_0$, $\bigcup_{i\in[t]} P_i$ is monochromatic and $|\bigcup_{i\in[t]} P_i| \geq \frac{(2-\varepsilon)n}{k+1}$.*

We now prove Theorem 1.7.

Proof of Theorem 1.7. Fix a k-edge-colouring of $K_{\mathbb{N}}$. Suppose to the contrary that there is no monochromatic path P with $\overline{d}(P) \geq \phi_{k-1}$. We say that a vertex $v \in V(K_{\mathbb{N}})$ is *of colour i* if $N_i(v)$ is infinite. Note that a vertex can have more than one colour.

Claim 2.3. *For every finite set $S \subseteq \mathbb{N}$, every $i \in [k]$ and every pair of vertices $x, y \in \mathbb{N}\backslash S$ of colour i, there exists a path of colour i from x to y that avoids S.*

Proof of Claim. Suppose to the contrary that there exists a finite set $S \subseteq \mathbb{N}$, $i \in [k]$ and $x, y \in \mathbb{N} \setminus S$ of colour i such that there is no path from x to y of colour i which avoids S. Let X be the set of vertices that can be reached from x by a path of colour i that avoids S, and let $Y = \mathbb{N} \setminus (X \cup S)$. Since $N_i(x) \setminus S \subseteq X$ and $N_i(y) \setminus S \subseteq Y$, each of X and Y is infinite. Moreover, there is no edge of colour i between X and Y. Thus the infinite complete bipartite graph $K[X,Y]$ (with vertex classes X and Y) is $(k-1)$-coloured. So there exists a monochromatic path P in $K[X,Y] \subseteq K_{\mathbb{N}}$ with

$$\overline{d}(P) = \limsup_{n\to\infty} \frac{|V(P) \cap [n]|}{n} = \limsup_{n\to\infty} \frac{|(V(P) \cap [n]) \setminus S|}{|[n] \setminus S|} \geq \phi_{k-1},$$

a contradiction. ■

For each $i \in [k]$, let U_i be the set of vertices which is of colour i only. Suppose that there exists $i \in [k]$ such that U_i is finite, so all but finitely many vertices is of some colour in $[k] \setminus \{i\}$. So there exists $j \in [k] \setminus \{i\}$ and a set $A \subseteq \mathbb{N}$ such that each vertex in A is of colour j and A has upper density at least $1/(k-1)$. By Claim 2.3, we obtain a monochromatic path P of colour j containing A. Thus P has upper density at least $1/(k-1) \geq \phi_{k-1}$ by Proposition 1.4, a contradiction. Therefore, U_i is infinite for all $i \in [k]$.

Let $\varepsilon_j = 2^{-j}$ for $j \in \mathbb{N} \cup \{0\}$. We will now construct monochromatic paths as follows.

Claim 2.4. *For $j \in \mathbb{N} \cup \{0\}$, there exist an integer ℓ_j and paths $P_1^j, \dots, P_k^j$ such that*

(i) *each P_i^j is monochromatic of colour i with vertices in $[\ell_j]$ and both endvertices in U_i;*

(ii) *each P_i^j is an extension of P_i^{j-1} and, moreover, P_i^{j-1} is the subgraph of P_i^j induced by the vertex set $[\ell_{j-1}]$;*

(iii) *there exists $i_j \in [k]$ such that $\frac{|V(P_{i_j}^j) \cap [\ell_j]|}{\ell_j} \geq \frac{2(1-\varepsilon_j)}{k+1}$;*

(iv) *$\ell_j \geq \varepsilon_j^{-1} = 2^j$.*

Proof of Claim. Let $j = 0$. Since each U_i is infinite, let ℓ_0 be the smallest ℓ such that $U_i \cap [\ell] \neq \emptyset$ for all $i \in [\ell]$. For each $i \in [k]$, set P_i^0 be a vertex in $U_i \cap [\ell_0]$. Thus the claim holds for $j = 0$. Consider $j \in \mathbb{N}$. Suppose we have already constructed $P_1^{j-1}, \dots, P_k^{j-1}$ and we construct $P_1^j, \dots, P_k^j$ as follows.

Let $t_0 = t_0(k, \varepsilon_j)$ and $n_0 = n_0(k, \varepsilon_j)$ be given by Corollary 2.2. Let $\ell'_j > \ell_{j-1}$ be the smallest integer such that $|U_i \cap [\ell_{j-1}+1, \ell'_j]| \geq t_0$ for all $i \in [k]$. Let $m > \ell'_j$ be the smallest integer such that, for all $i \in [k]$, all edges between $U_i \cap [\ell'_j]$ and $[m, \infty)$ have colour i.

Let $n' = \max\{n_0, 2m/\varepsilon_j\}$ and $\ell_j = m + n'$. Let $I' = [m+1, \ell_j]$. Consider the complete subgraph $K_{I'}$ of $K_{\mathbb{N}}$ induced by I'. By Corollary 2.2, $K_{I'}$ contains vertex-disjoint paths $P_1, \dots, P_t$ such that $t \leq t_0$, $\bigcup_{j \in [t]} P_j$ is monochromatic of colour i_0 say, and

$$\left| \bigcup_{j \in [t]} P_j \right| \geq \frac{(2-\varepsilon_j)n'}{k+1} \geq \frac{2(1-\varepsilon_j)\ell_j}{k+1}.$$

Recall that $|U_{i_0} \cap [\ell_{j-1}+1, \ell'_j]| \geq t_0 \geq t$ and all edges between $U_{i_0} \cap [\ell'_j]$ and $V(\bigcup_{j \in [t]} P_j) \subseteq [m+1, \infty)$ have colour i_0. Together with (i) and using vertices in $U_{i_0} \cap [\ell_{j-1}+1, \ell'_j]$, we join $P_{i_0}^{j-1}, P_1, \dots, P_t$ into a monochromatic path $P_{i_0}^j$ of colour i_0 with endvertices in $U_{i_0} \cap [\ell'_j]$. We are done by setting $P_i^j = P_i^{j-1}$ for $i \in [k] \setminus \{i_0\}$. ∎

Note that there exists a colour i such that $i_j = i$ for infinitely many $j \in \mathbb{N}$. Then the monochromatic path $P' = \bigcup_{j \in \mathbb{N}} P_i^j$ satisfies $\overline{d}(P') \geq 2/(k+1) \geq \phi_{k-1}$, where the last inequality holds by Proposition 1.4. This is a contradiction. □

3. Complete bipartite infinite graphs

We now prove Proposition 1.4, that is, bounding the upper density of monochromatic paths in k-edge-coloured $K_{V,W} \in K_{\mathbb{N},\mathbb{N}}$ from above.

Proof of Proposition 1.4. Let c be a proper k-edge-colouring of $K_{k,k}$ with vertex classes $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$. Let $\phi_X : V \to X$ and $\phi_Y : W \to Y$ be such that $\overline{d}(\phi_X^{-1}(x_i) \cup \phi_Y^{-1}(y_j)) = 1/k$ for all $i, j \in [k]$. (For instance, if $V = \{v_1, v_2, \dots\}$ and

$W = \{w_1, w_2, \ldots\}$ with $v_i < v_{i+1}$ and $w_j < w_{j+1}$, then set $\phi_X(v_i) = x_{i'}$ and $\phi_Y(w_j) = y_{j'}$ such that $i \equiv i' \pmod{k}$ and $j \equiv j' \pmod{k}$.) We now edge-colour $K_{V,W}$ such that the edge vw with $v \in V$ and $w \in W$ has colour $c(\phi_X(v)\phi_Y(w))$. Since each colour class of c is a perfect matching, any monochromatic path in $K_{V,W}$ lies in $\phi_X^{-1}(x_i) \cup \phi_Y^{-1}(y_j)$ for some $i, j \in [k]$. Hence the result follows. □

In order to prove Theorem 1.6, we use the notion of an ultrafilter. Given an infinite set X, a family $\mathcal{U}$ of subsets of X is an *ultrafilter* if $\mathcal{U}$ is closed under finite intersections and supersets, the empty set is not in $\mathcal{U}$, and for every set $Y \subseteq X$, we have that either $Y \in \mathcal{U}$ or $X \setminus Y \in \mathcal{U}$. Thus if $\mathcal{U}$ is an ultrafilter on X, and $\{X_1, \ldots, X_n\}$ is a finite partition of X, then exactly one X_i is in $\mathcal{U}$. Finally, an ultrafilter $\mathcal{U}$ is *nonprincipal* if no set in $\mathcal{U}$ is finite. By Zorn's Lemma, for any infinite set X, there exists a nonprincipal ultrafilter on X.

Let $K_{V,W} \in K_{\mathbb{N},\mathbb{N}}$. Let $A \subseteq V$ and $B \subseteq W$ be infinite sets. A pair $(\mathcal{V}_A, \mathcal{W}_B)$ is an *ultrafilter-pair* of (A, B), if $\mathcal{V}_A$ and $\mathcal{W}_B$ are nonprincipal ultrafilters on A and B, respectively. Given an ultrafilter-pair $\mathcal{S} = (\mathcal{V}_A, \mathcal{W}_B)$ of (A, B), define the k-vertex-colouring $c_{\mathcal{S}}$ of $V \cup W$ such that $c_{\mathcal{S}}(v) = i$ if $N_i(v) \cap (A \cup B) \in \mathcal{V}_A \cup \mathcal{W}_B$ for $v \in V \cup W$. Note that every vertex gets exactly one colour. Moreover, for each $i \in [k]$, let

$$V_i(\mathcal{S}) = \{v \in V : c_{\mathcal{S}}(v) = i\} \text{ and } W_i(\mathcal{S}) = \{w \in W : c_{\mathcal{S}}(w) = i\}.$$

When it is clear which ultrafilter-pair we are referring to, we will often omit the $\mathcal{S}$ and write V_i and W_i instead. We make use of the following lemma.

Lemma 3.1. *Let $K_{V,W} \in K_{\mathbb{N},\mathbb{N}}$ be k-edge-coloured and $i_0 \in [k]$. Let $\mathcal{S} = (\mathcal{V}', \mathcal{W}')$ be an ultrafilter-pair on (V', W') with $V' \subseteq V$ and $W' \subseteq W$. Then there exists a monochromatic path of colour i_0 containing V_{i_0}.*

Moreover, let $U^*_{i_0}$ be the set of vertices $v \in V_{i_0} \cup W_{i_0}$ such that $N_{i_0}(v) \cap (V_{i_0} \cup W_{i_0})$ is infinite. If $U^*_{i_0}$ itself is infinite, then there exists a monochromatic path P of colour i_0 containing $V_{i_0} \cup W_{i_0}$.

Proof. Without loss of generality, we may assume that $i_0 = 1$. Let $V_1 = \{a_1, a_2, \ldots\}$ with $a_j < a_{j+1}$. For each $j \in \mathbb{N}$, note that $N_1(a_j) \cap W', N_1(a_{j+1}) \cap W' \in \mathcal{W}'$ and recall that $\mathcal{W}'$ is closed under finite intersections, so $N_1(a_j) \cap N_1(a_{j+1}) \cap W' \in \mathcal{W}'$ is infinite. Hence, we can find distinct vertices $w_1, w_2, \dots \in W'$ such that $w_j \in N_1(a_j) \cap N_1(a_{j+1})$. Then $P = v_1 w_1 v_2 w_2 \ldots$ is a monochromatic path of colour 1 containing V_1.

We now prove the moreover statement. Without loss of generality, we may assume $V_1^* = V_1 \cap U_1^*$ is infinite. Let $V_1 \cup W_1 = \{a_1, a_2, \ldots\}$ with $a_j < a_{j+1}$. Let $A_j = \{a_1, \ldots, a_j\}$. We will construct monochromatic path P_j of colour 1 containing A_j with endpoints in V_1^*. Set P_0 be a single vertex in V_1^*. Suppose that we have constructed P_{j-1} and construct P_j as follows. If $a_j \in V(P_{j-1})$, then set $P_j = P_{j-1}$.

Suppose that $a_j \notin V(P_{j-1})$ and v_{j-1} be an endpoint of P_{j-1}. If $a_j \in V_1$, then pick $v_j \in V_1^* \setminus (V(P_{j-1}) \cup \{a_j\})$. Note that $v_{j-1}, v_j, a_j \in V_1$, so $N_1(v_{j-1}) \cap W'$, $N_1(v_j) \cap W'$

and $N_1(a_j) \cap W'$ are members of $\mathcal{W}'$. Hence, $N_1(v_{j-1}) \cap N_1(v_j) \cap N_1(a_j) \cap W' \in \mathcal{W}'$ is an infinite set. Pick distinct vertices $w_{j-1}, w_j \in (N_1(v_{j-1}) \cap N_1(v_j) \cap N_1(a_j) \cap W') \setminus V(P_{j-1})$. Note that $P_j = P_{j-1}v_{j-1}w_{j-1}a_jw_jv_j$ is a path of colour 1 as desired.

If $a_j \in W_1$, then pick distinct vertices $v_j \in V_1^* \setminus V(P_{j-1})$, $w_{j-1} \in (N_1(v_{j-1}) \cap W_1) \setminus V(P_{j-1})$ and $w_j \in (N_1(v_j) \cap W_1) \setminus V(P_{j-1})$. Similarly, pick distinct vertices $v'_{j-1}, v'_j \in (N_1(w_{j-1}) \cap N_1(w_j) \cap N_1(a_j) \cap V') \setminus V(P_{j-1})$. Note that $P_j = P_{j-1}v_{j-1}w_{j-1}v'_{j-1}a_jv'_jw_jv_j$ is a path of colour 1 as desired. We are done by setting $P = \bigcup_{i \in \mathbb{N}} P_i$. □

First we prove Theorem 1.6 when $k = 2$.

Proof of Theorem 1.6 when $k = 2$. Fix a 2-edge-colouring of $K_{V,W}$, and let $\mathcal{S} = (\mathcal{V}, \mathcal{W})$ be an ultrafilter-pair on (V, W). Note that $\overline{d}(V_1 \cup W_1) + \overline{d}(V_2 \cup W_2) \geq 1$. Thus, by relabelling colours if necessary, we may assume that

$$\overline{d}(V_1 \cup W_1) \geq 1/2.$$

We may assume that $\overline{d}(V_1), \overline{d}(W_1) > 0$ (or else, $\overline{d}(V_1) \geq 1/2$ or $\overline{d}(W_1) \geq 1/2$ and we are done by Lemma 3.1). Hence V_1 and W_1 are infinite.

Let U_1^* be the set of vertices $v \in V_1 \cup W_1$ such that $N_1(v) \cap (V_1 \cup W_1)$ is infinite. If U_1^* is infinite, then Lemma 3.1 implies that there is a path of colour 1 containing $V_1 \cup W_1$, as required. Thus we may assume that U_1^* is finite.

Thus $V_1 \setminus U_1^*$ and $W_1 \setminus U_1^*$ are infinite. Futhermore, every $v \in V_1 \setminus U_1^*$ (and $w \in W_1 \setminus U_1^*$) sends finitely many edges of colour 1 to W_1 (and to V_1, respectively). It is easy to construct a monochromatic path P of colour 2 with vertex set $V(P) = (V_1 \cup W_1) \setminus U_1^*$ (see the proof of the moreover statement of Lemma 3.1). Note that $\overline{d}(P) = \overline{d}((V_1 \cup W_1) \setminus U_1^*) = \overline{d}(V_1 \cup W_1) \geq 1/2$ as required. □

Before proving Theorem 1.6 when $k \geq 3$, we would need to define the lower density of a set. Given a set $A \subseteq \mathbb{N}$, the *lower density of A* is defined as

$$\underline{d}(A) = \liminf_{n \to \infty} \frac{|A \cap [n]|}{n}.$$

For sets $U, W \subseteq \mathbb{N}$ with $U \cap W$ finite (i.e. almost disjoint sets of $U, W \subseteq \mathbb{N}$), the following standard inequality holds:

$$\underline{d}(U) + \underline{d}(W) \leq \underline{d}(U \cup W) \leq \underline{d}(U) + \overline{d}(W) \leq \overline{d}(U \cup W) \leq \overline{d}(U) + \overline{d}(W). \tag{3.1}$$

Proof of Theorem 1.6 when $k \geq 3$. Let $\alpha = 1/(2k-3)$. Suppose to the contrary that there exists $\varepsilon > 0$ and a k-edge-coloured $K_{V,W}$ such that every monochromatic path P in $K_{V,W}$ has $\overline{d}(P) \leqslant \alpha - 2\varepsilon$.

Let β be the supremum of $\max_{i \in [k]}\{\overline{d}(V_i(\mathcal{S}) \cup W_i(\mathcal{S}))\}$ taken over all ultrafilter-pairs $\mathcal{S}$ of (V', W') with infinite sets $V' \subseteq V$ and $W' \subseteq W$. Clearly,

$$\beta \geq 1/k. \tag{3.2}$$

Let $\mathcal{S}^0 = (\mathcal{V}^0, \mathcal{W}^0)$ be an ultrafilter-pair on (V^0, W^0) with infinite sets $V^0 \subseteq V$ and $W^0 \subseteq W$ such that $\max_{i \in [k]}\{\overline{d}(V_i(\mathcal{S}^0) \cup W_i(\mathcal{S}^0))\} \geq \beta - \varepsilon$. For each $i \in [k]$, let $V_i^0 =$

$V_i(\mathcal{S}^0)$ and $W_i^0 = W_i(\mathcal{S}^0)$. By relabelling if necessary, we may assume that

$$\overline{d}(V_1^0 \cup W_1^0) = \max_{i\in[k]}\{\overline{d}(V_i^0 \cup W_i^0)\} \geq \beta - \varepsilon. \tag{3.3}$$

By Lemma 3.1 (with $\mathcal{S} = \mathcal{S}^0$ and $i_0 = 1$), there exists a path of colour 1 containing V_1^0, so $\overline{d}(V_1^0) \leq \alpha - 2\varepsilon$. Hence

$$\overline{d}(W_1^0) \overset{(3.1)}{\geq} \overline{d}(V_1^0 \cup W_1^0) - \overline{d}(V_1^0) \overset{(3.3)}{\geq} \beta - \alpha + \varepsilon \overset{(3.2)}{>} 0$$

and so W_1^0 is infinite. Similarly, $\overline{d}(W_1^0) \leq \alpha - 2\varepsilon$ and V_1^0 is infinite. Moreover,

$$\beta \overset{(3.3)}{\leq} \overline{d}(V_1^0 \cup W_1^0) + \varepsilon \overset{(3.1)}{\leq} \overline{d}(V_1^0) + \overline{d}(W_1^0) + \varepsilon \leq 2\alpha - 3\varepsilon < 2\alpha. \tag{3.4}$$

Let $\mathcal{S}' = (\mathcal{V}', \mathcal{W}')$ be an ultrafilter-pair on (V_1^0, W_1^0). Let $V_i' = V_i(\mathcal{S}')$ and $W_i' = W_i(\mathcal{S}')$ for all $i \in [k]$.

Claim 3.2. *$(V_1' \cup W_1') \cap (V_1^0 \cup W_1^0)$ is finite.*

Proof of Claim. Suppose to the contrary that $(V_1' \cup W_1') \cap (V_1^0 \cup W_1^0)$ is infinite. Without loss of generality $V_1^0 \cap V_1'$ is infinite. For all $v \in V_1^0 \cap V_1' \subseteq V_1'$, we have that $N_1(v) \cap W_1^0 \in \mathcal{W}'$ is infinite. Lemma 3.1 (with $\mathcal{S} = \mathcal{S}$, $i_0 = 1$ and $U_1^* \supseteq V_1^0 \cap V_1'$) implies that there exists a path of colour 1 containing $V_1^0 \cup W_1^0$ with upper density at least

$$\overline{d}(V_1^0 \cup W_1^0) \geq \beta - \varepsilon \overset{(3.2)}{\geq} 1/k - \varepsilon \geq \alpha - \varepsilon,$$

a contradiction. ■

Consider the ultrafilter-pair $\mathcal{S}^* = (\mathcal{V}', \mathcal{W}^0)$, so $V_1(\mathcal{S}^*) = V_1^0$ and $W_1(\mathcal{S}^*) = W_1'$. Moreover, for all $w \in W_1'$, $N_1(w) \cap V_1^0 \in \mathcal{V}'$ is infinite. If W_1' is finite, then $\overline{d}(V_1^0 \cup W_1') = \overline{d}(V_1^0) \leq \alpha - 2\varepsilon$. If W_1' is infinite, then Lemma 3.1 (with $\mathcal{S} = \mathcal{S}^*$, $i_0 = 1$ and $U_1^* \supseteq W_1'$) implies that there exists a path of colour 1 containing $V_1^0 \cup W_1'$. In both cases, we have $\overline{d}(V_1^0 \cup W_1') \leq \alpha - 2\varepsilon$. Similarly by considering the ultrafilter-pair $(\mathcal{V}^0, \mathcal{W}')$, we deduce that $\overline{d}(W_1^0 \cup V_1') \leq \alpha - 2\varepsilon$. Together with Claim 3.2 and (3.1), we deduce that

$$\begin{aligned}\underline{d}(V_1' \cup W_1') + \beta - \varepsilon &\overset{(3.3)}{\leq} \underline{d}(V_1' \cup W_1') + \overline{d}(V_1^0 \cup W_1^0)\\ &\leq \overline{d}(V_1' \cup W_1' \cup V_1^0 \cup W_1^0)\\ &\leq \overline{d}(V_1^0 \cup W_1') + \overline{d}(W_1^0 \cup V_1') \leq 2(\alpha - 2\varepsilon),\end{aligned}$$

which implies that

$$\underline{d}(V_1' \cup W_1') \leq 2\alpha - \beta. \tag{3.5}$$

Since $\mathcal{V}'$ is an ultrafilter on V_1^0, there exists some $i_0 \in [k]$ such that $V_1^0 \cap V_{i_0}' \in \mathcal{V}'$. Note that $V_1^0 \cap V_{i_0}'$ is infinite, so $i_0 \neq 1$ by Claim 3.2. Therefore, without loss of generality, we may assume that $i_0 = 2$. Recall that $V_1^0 \cap V_2' \in \mathcal{V}'$. For all $w \in W_2'$, note that $N_2(w) \cap V_1^0 \in \mathcal{V}'$ implying that $N_2(w) \cap V_1^0 \cap V_2' = (N_2(w) \cap V_1^0) \cap (V_1^0 \cap V_2') \in \mathcal{V}'$ is infinite. If W_2' is infinite, then Lemma 3.1 (with $\mathcal{S} = \mathcal{S}'$, $i_0 = 2$ and $U_2^* \supseteq W_2'$) implies that there exists a path of colour 2 containing $V_2' \cup W_2'$. If W_2' is finite, then we have

$\overline{d}(V_2' \cup W_2') = \overline{d}(V_2')$ and Lemma 3.1 implies that there is a path of colour 2 containing V_2'. In both cases, we deduce that

$$\overline{d}(V_2' \cup W_2') \leq \alpha - 2\varepsilon < \alpha. \tag{3.6}$$

Recall the definition of β that $\overline{d}(V_i' \cup W_i') \leq \beta$ for all $i \in [k]$. Putting these all together, we get that

$$\begin{aligned} 1 = \underline{d}(V \cup W) &\leq \underline{d}(V_1' \cup W_1') + \overline{d}\left(\bigcup_{2 \leq i \leq k} (V_i' \cup W_i')\right) \leq \underline{d}(V_1' \cup W_1') + \sum_{2 \leq i \leq k} \overline{d}(V_i' \cup W_i') \\ &\overset{(3.5),\,(3.6)}{<} (2\alpha - \beta) + \alpha + \sum_{3 \leq i \leq k} \overline{d}(V_i' \cup W_i') \leq 3\alpha - \beta + (k-2)\beta = 3\alpha + (k-3)\beta \\ &\overset{(3.4)}{<} (2k-3)\alpha = 1, \end{aligned}$$

a contradiction. □

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