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Partition functions and fibering operators on the Coulomb branch of 5d SCFTs

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Abstract: We study 5d $\mathcal{N}=1$ supersymmetric field theories on closed five-manifolds $\mathcal{M}_5$ which are principal circle bundles over simply-connected Kähler four-manifolds, $\mathcal{M}_4$, equipped with the Donaldson-Witten twist. We propose a new approach to compute the supersymmetric partition function on $\mathcal{M}_5$ through the insertion of a fibering operator, which introduces a non-trivial fibration over $\mathcal{M}_4$, in the 4d topologically twisted field theory. We determine the so-called Coulomb branch partition function on any such $\mathcal{M}_5$, which is conjectured to be the holomorphic 'integrand' of the full partition function. We precisely match the low-energy effective field theory approach to explicit one-loop computations, and we discuss the effect of non-perturbative 5d BPS particles in this context. When $\mathcal{M}_4$ is toric, we also reconstruct our Coulomb branch partition function by appropriately gluing Nekrasov partition functions. As a by-product of our analysis, we provide strong new evidence for the validity of the Lockhart-Vafa formula for the five-sphere partition function.

Keywords: Nonperturbative Effects, Supersymmetric Effective Theories, Supersymmetric Gauge Theory

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Contents

1 Introduction

The study of supersymmetric quantum field theories on non-trivial Riemannian manifolds often opens an important window into their strongly coupled dynamics [1–4]. In this work, we consider five-dimensional superconformal field theories (5d SCFTs) [5] on five-manifolds $\mathcal{M}_5$ that can be constructed as circle fibrations over a Kähler four-manifold:

$$S^1 \longrightarrow \mathcal{M}_5 \xrightarrow{\pi} \mathcal{M}_4 \ . \tag{1.1}$$

For simplicity, we will only consider principal circle bundles[1] over $\mathcal{M}_4$. A particularly important example of this construction is the five-sphere (with its round metric), viewed as a circle bundle over the complex projective space:

$$S^1 \longrightarrow S^5 \xrightarrow{\pi} \mathbb{P}^2 \ . \tag{1.2}$$

The partition function of a 5d SCFT on the five-sphere,

$$F_{S^5} = \log |\mathbf{Z}_{S^5}| \, , \tag{1.3}$$

can be defined for any 5d conformal field theory, irrespective of supersymmetry. It is conjecturally a good measure of the number of 'degrees of freedom' of the strongly-coupled fixed point [6]. Despite some heroic computations, e.g. as in [7], to compute this object exactly and efficiently in 5d SCFTs remains a challenge.

[1]By which we mean 'principal $U(1)$ bundles'. The case of more general fibrations is left for future work.

1.1 The Donaldson-Witten twist approach

This paper initiates a new approach to computing the $\mathcal{M}_5$ supersymmetric partition function, $\mathbf{Z}_{\mathcal{M}_5}$, following a line of ideas which was successfully applied to 3d $\mathcal{N}=2$ theories on Seifert manifolds [8] — see also [9–13]. As our starting point, we first consider the 5d theory on a product manifold

$$\mathcal{M}_5 = \mathcal{M}_4 \times S^1 . \tag{1.4}$$

The presence of the S^1 factor allows us to consider the 4d $\mathcal{N}=2$ Kaluza-Klein (KK) theory that one obtains by compactifying the 5d SCFT on a circle [14]. The low-energy physics of that KK theory on its Coulomb branch (CB) is governed by a certain Seiberg-Witten geometry [14–17], like for any 4d $\mathcal{N}=2$ supersymmetric field theory.[2] The simplest way to put this KK theory on $\mathcal{M}_4$ while preserving supersymmetry is to consider the ordinary topological twist, also known as the Donaldson-Witten (DW) twist [1]. We choose $\mathcal{M}_4$ to be Kähler so that we can preserve two supercharges [2] which anticommute to a translation along the S^1. Supersymmetric operators must then be extended along the circle.

The topologically twisted partition function $\mathbf{Z}_{\mathcal{M}_4\times S^1}$, also called the twisted index [23], is expected to capture (generalised) K-theoretic Donaldson invariants of $\mathcal{M}_4$ [24]. More precisely, those invariants should arise from the insertions of some operators $\mathcal{O}$ wrapping the circle. Let us denote by

$$\langle \mathcal{O} \rangle^{\rm DW}_{\mathcal{M}_4\times S^1} , \tag{1.5}$$

any observable in the four-dimensional topological quantum field theory (TQFT) obtained by the DW twist. The direct path integral computation of such observables is a famous and famously challenging problem [25–28] which, however, is not the focus of this paper (see [29–39] for a lot of more recent progress, and especially [40] for the case of 4d $\mathcal{N}=2$ KK theories). Instead, let us assume that we know how to compute (1.5) for any $\mathcal{O}$. Then, we claim that one can compute the $\mathcal{M}_5$ partition function as a particular observable in the DW theory, namely as the expectation value of a so-called *fibering operator*, denoted by $\mathscr{F}$. We write this as:

$$\mathbf{Z}_{\mathcal{M}_5} = \langle \mathscr{F}_{\mathfrak{p}} \rangle^{\rm DW}_{\mathcal{M}_4\times S^1} , \tag{1.6}$$

where $\mathfrak{p}$ denotes the first Chern class of the principal circle bundle (1.1). In this approach, we should be able to compute seemingly 'non-topological' 5d observables, like the S^5 partition function, in the four-dimensional DW theory.[3] (Similarly, in [8], the S^3 partition function was understood as an observable in the A-model, a 2d TQFT.)

Thanks to topological invariance, one can use the low-energy Seiberg-Witten description of the KK theory to compute the partition function on $\mathcal{M}_4 \times S^1$ and hence, upon insertion of the fibering operator, on $\mathcal{M}_5$. It is useful to consider the theory at any given

[2] The CB physics of 4d $\mathcal{N}=2$ KK theories has been revisited in recent works [18–22].

[3] To prove this expectation *a priori*, one should show explicitly that one can continuously deform the 'DW-twist' supersymmetric background used in this paper to the 'round S^5' supersymmetric background with vanishing $SU(2)_R$ background gauge field, and furthermore prove that this deformation in $\mathcal{Q}$-exact. This would amount to an 8-supercharge equivalent to the 4-supercharge analysis of [41], which goes well beyond the scope of this paper. Instead, we simply observe that our computations match previous computations performed on different supersymmetric backgrounds.

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point on the Coulomb branch, with $\boldsymbol{a}$ denoting the scalars in the low-energy abelian vector multiplets.[4] One can then ask what is the 'partition function' on $\mathcal{M}_5$ at a fixed value of $\boldsymbol{a}$, and with some fixed background fluxes $\mathfrak{m}$ for the abelian gauge fields turned on. By a slight abuse of terminology, this 'off-shell' quantity will be called the *CB partition function*, denoted by:

$$Z_{\mathcal{M}_5}(\boldsymbol{a})_{\mathfrak{m}} \ . \tag{1.7}$$

On any geometry with the topology of $\mathbb{R}^4 \times S^1$, this would correspond to the DW-twist of the Seiberg-Witten theory, which is fully determined by the Seiberg-Witten prepotential, $\mathcal{F}(\boldsymbol{a})$. On a closed space, we must integrate over the full moduli space, and the partition function on $\mathcal{M}_5$ will be obtained after integrating out the dynamical low-energy vector multiplets (we usually retain a dependence on background vector multiplets, which keep track of the flavour symmetry). Based on a number of previous results and conjectures in the literature (see e.g. [23, 35, 42]), we expect an explicit formula of the form:

$$\mathbf{Z}_{\mathcal{M}_5} = \sum_{\mathfrak{m}} \oint_{\mathcal{C}} d\boldsymbol{a}\, Z_{\mathcal{M}_5}(\boldsymbol{a})_{\mathfrak{m}} \,, \tag{1.8}$$

where the precise form of the sum over fluxes and of the integration contour have to be determined. We hope to address this in future work. In this paper, we shall be more modest and focus on computing the integrand, namely the CB partition function $Z_{\mathcal{M}_5}$. As we will see, this already entails a number of subtleties and leads to new and interesting results.

The CB partition function (1.7) is a holomorphic function of $\boldsymbol{a}$. The schematic formula (1.8) is inspired by the 'holomorphic approach' to the DW twist [43], and by a conjecture of Nekrasov for toric four-manifolds [42]. By contrast, the Moore-Witten u-plane integral approach deals with a non-holomorphic integrand, which renders the vector-multiplet integration better defined. While it would be important to reconcile the two approaches, in this paper we shall be wilfully naïve and ignore all $\mathcal{Q}$-exact terms in the effective action. We will not discuss the contributions from Seiberg-Witten invariants, either, even though they are expected to appear prominently in the general story [25].

On any Kähler four-manifold, we have the relations:

$$\chi_h \equiv 1 - h^{0,1} + h^{0,2} = \frac{\chi + \sigma}{4} \,, \qquad\qquad b_2^+ = 1 + 2h^{0,2} \,, \tag{1.9}$$

where χ and σ are the Euler characteristic $\chi = \sum_{i=0}^{4}(-1)^i b_i$ and the signature $\sigma = b_2^+ - b_2^-$ of $\mathcal{M}_4$, respectively, and $h^{p,q}$ are its Hodge numbers. In this work, we will further assume that the Kähler manifold $\mathcal{M}_4$ is simply connected (in particular, $h^{0,1} = 0$), as is often done in the study of Donaldson invariants.[5] This is to avoid some additional zero-mode contributing to the CB partition function, and it also simplifies the computation of certain one-loop determinants. (The case with $\pi_1(\mathcal{M}_4) \neq 0$ is nonetheless rather interesting, and it is left for future work.)

[4]We will also use $\boldsymbol{a}$ to denote mass terms, which arise as background vector multiplets. These will be discussed more explicitly in the main text.

[5]From the point of view of 4-manifold invariants, this assumption is necessary to have any hope of classifying smooth manifolds. For us, it is just a technical restriction. See also [44] for a discussion of the u-plane approach when $\pi_1(\mathcal{M}_4) \neq 0$.

1.2 The S^5 partition function

The computation of the S^5 partition function of 5d SCFTs has been approached in various ways in the literature [45–51], mostly from the perspective of the 5d $\mathcal{N}=1$ supersymmetric gauge theories that may arise as infrared phases — see e.g. the reviews [52, 53]. Conjecturally, the full partition function takes the form:

$$\mathbf{Z}_{S^5} = \int d\sigma \prod_{l=1}^{3} Z_{\mathbb{C}^2\times S^1}(i\sigma^{(l)}, \epsilon_1^{(l)}, \epsilon_2^{(l)})\,, \tag{1.10}$$

where the integral is over the 5d CB parameters, and the integrand is a product of three K-theoretic Nekrasov partition functions (including the classical and perturbative contributions), with the Ω-deformation parameters ϵ_1, ϵ_2 which become supersymmetric squashing parameters on S^5. It was further conjectured by Lockhart and Vafa [49] that the S^5 partition function can also be written as:

$$\mathbf{Z}_{S^5} = \int d\sigma \prod_{\alpha\ \mathrm{BPS}} Z^{(\alpha)}_{S^5,\mathrm{BPS}}(\sigma)\,, \tag{1.11}$$

schematically, where the integrand is a product over all 5d BPS states — that is, the electrically-charged particles on the 5d Coulomb branch. The present paper corroborates the Lockhart-Vafa formula (1.11). We also revisit and clarify the factorisation formula (1.10), and we generalise it to other five-manifolds.

Using the DW-twist approach outlined above, we find the following CB partition function on S^5:

$$Z_{S^5}(\boldsymbol{a}) = \boldsymbol{A}(\boldsymbol{a})^3\, \boldsymbol{B}(\boldsymbol{a})\, \mathscr{F}(\boldsymbol{a})^{\frac{1}{2}}\ . \tag{1.12}$$

with the fibering operator:

$$\mathscr{F}(\boldsymbol{a}) = \exp\left(-4\pi i\left(\mathcal{F}(\boldsymbol{a}) - \boldsymbol{a}\frac{\partial \mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}} + \frac{1}{2}\boldsymbol{a}^2\frac{\partial^2\mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^2}\right)\right)\ . \tag{1.13}$$

Here, $\boldsymbol{A}(\boldsymbol{a})$ and $\boldsymbol{B}(\boldsymbol{a})$ are the standard gravitational couplings in the DW theory [54], and $\mathcal{F}(\boldsymbol{a})$ is the full effective prepotential of the 4d $\mathcal{N}=2$ KK theory. The formula (1.13) is our general result for the fibering operator. Note that we could set the gauge fluxes to zero ($\mathfrak{m}=0$) in (1.12) because all fluxes on $\mathbb{P}^2$ trivialise when lifted to S^5. In writing (1.12) we also assumed, for simplicity, that the KK theory can be consistently coupled to $\mathcal{M}_4=\mathbb{P}^2$, which is only true if a certain condition on the BPS spectrum is satisfied (roughly speaking, there should be no hypermultiplets). We will explain how to lift this artificial assumption momentarily.

We are then naturally led to the following conjecture for the round S^5 partition function of any 5d SCFT:

$$\mathbf{Z}_{S^5} = \int_{i\mathbb{R}} d\boldsymbol{a}\, Z_{S^5}(\boldsymbol{a})\,, \tag{1.14}$$

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where the integration contour is over the imaginary axis for the CB variables $\boldsymbol{a}$.[6] This matches many previous computations whenever a comparison is possible, including the $\epsilon_1, \epsilon_2 \to 0$ limit of (1.10) with $\boldsymbol{a} = i\sigma$. In our approach, the Lockhart-Vafa factorisation (1.11) into BPS states is completely equivalent to the following expansion of the effective prepotential of the 4d $\mathcal{N} = 2$ KK theory:

$$\mathcal{F}(\boldsymbol{a}) = \mathcal{F}_{\rm cl}(\boldsymbol{a}) - \frac{1}{(2\pi i)^3} \sum_{\alpha \text{ BPS}} d^{(\alpha)} \operatorname{Li}_3(Q^{(\alpha)}) \,, \tag{1.15}$$

with $Q = e^{2\pi i \boldsymbol{a}}$, schematically. An analogous expansion must also hold for $\log \boldsymbol{A}$ and $\log \boldsymbol{B}$. Whenever the 5d SCFT can be engineered at a Calabi-Yau threefold singularity in M-theory, the sum in (1.15) is a sum over wrapped M2-branes near the resolved singularity, with Q the exponentiated Kähler parameters, and the numerical constants $d^{(\alpha)}$ are then essentially the (refined) Gopakumar-Vafa invariants of the resolved threefold [55–58]. The term $\mathcal{F}_{\rm cl}(\boldsymbol{a})$ in (1.15) denotes the 'classical' terms, which are supergravity contributions in M-theory; for our purposes in this paper, we will essentially ignore these terms.

1.3 Three ways to obtain the CB partition functions

We can compute the CB partition function on $\mathcal{M}_5$ using three complementary methods, which all yield the same answer:

(1) **One-loop determinants.** In this approach, we first study ordinary 5d $\mathcal{N} = 1$ gauge theories on our supersymmetric background $\mathcal{M}_5$. This is in keeping with standard supersymmetric localisation computations [52], and it would seem to only capture the 'perturbative' part of the full 5d SCFT partition function. However, if we assume the validity of the Lockhart-Vafa factorisation:

$$Z_{\mathcal{M}_5}(\boldsymbol{a})_{\mathfrak{m}} = \prod_{\alpha \text{ BPS}} Z^{(\alpha)}_{\mathcal{M}_5, \text{BPS}}(\boldsymbol{a})_{\mathfrak{m}} \,, \tag{1.16}$$

on any $\mathcal{M}_5$, then each factor $Z^{(\alpha)}_{\mathcal{M}_5,\text{BPS}}$ can be computed by a simple generalisation of the one-loop determinant computation for a free hypermultiplet. We pay particular attention to regularising the one-loop results in a way that fully respects five-dimensional gauge invariance (and hence, in general, breaks parity), following [59].

(2) **Low-energy effective action.** In this approach, which forms the core of the paper, we study the low-energy effective couplings on the Coulomb branch. This is the TQFT logic summarised in section 1.1 above. The fibering operator arises as a background flux insertion for the $U(1)_{\rm KK}$ symmetry (that is, the conserved momentum along the circle). This general method does not require us to assume the Lockhart-Vafa factorisation, or to assume anything about the 5d SCFT, and it works for any (simply connected) Kähler manifold $\mathcal{M}_4$. (A somewhat similar approach was taken in [60–63] for geometries that are more closely related to the 3d setup of [8].)

[6] In general, we expect that the contour should be slightly deformed so that the integral converges. This is the analogue of the 'σ-contour' discussed in [8]. There will also be a representation of $\mathbf{Z}_{S^5}$ as in (1.8) that involves a sum over $\mathbb{P}^2$ fluxes. This and the relations between different formulas for the full partition function will be discussed elsewhere.

(3) **Nekrasov partition function gluing.** Our third approach is only valid for $\mathcal{M}_5$ a circle bundle over $\mathcal{M}_4$ a toric manifold. Then, we can construct $\mathcal{M}_5$ as a toric gluing of $\chi(\mathcal{M}_4)$ distinct patches $\mathbb{C}^2\times S^1$, with the CB partition function obtained by gluing Nekrasov partition functions $Z_{\mathbb{C}^2\times S^1}$ — this approach has been discussed extensively in the literature, see e.g. [48–50, 64–66].[7] We revisit the case of the five-sphere, and we propose a generalisation of (1.10) to any such $\mathcal{M}_5$, which reads:

$$\mathbf{Z}_{\mathcal{M}_5} = \sum_{\mathfrak{n}_l} \oint d\boldsymbol{a} \prod_{l=1}^{\chi(\mathcal{M}_4)} Z_{\mathbb{C}^2\times S^1}\left(\frac{\boldsymbol{a}+\tau_1^{(l)}\mathfrak{n}_l+\tau_2^{(l)}\mathfrak{n}_{l+1}}{\gamma^{(l)}}, \frac{\tau_1^{(l)}}{\gamma^{(l)}}, \frac{\tau_2^{(l)}}{\gamma^{(l)}}\right) . \tag{1.17}$$

The quantities in the arguments will be explained in section 6. (In particular, here $\tau_i = \beta\epsilon_i$ denotes the Ω-deformation parameters, with β the radius of the S^1 fiber.) In the non-equivariant limit, $\tau_1, \tau_2 \to 0$, we recover the CB partition function for the DW twist.

An important subtlety, which we will address in detail, arises because the DW twist is actually not well-defined for every 5d SCFT. This is true whenever $\mathcal{M}_4$ is not spin and the twisted theory contains fields that transform as spinors. For instance, this is the case for the simplest theory, a free hypermultiplet on a non-spin manifold like $\mathbb{P}^2$. In general one needs to consider (background) spin^c connections instead of ordinary $U(1)$ connections [25, 36, 72]. In physics language, this simply means that we should consider abelian gauge fields A with some half-integer-quantised fluxes:

$$\frac{1}{2\pi}\int_{\mathrm{S}} F = \mathfrak{m} + \varepsilon\mathbf{k} , \qquad \varepsilon \in \frac{1}{2}\mathbb{Z} . \tag{1.18}$$

with $F = dA$. Here $\mathbf{k}$ denotes the first Chern class of the canonical line bundle $\mathcal{K}$ over $\mathcal{M}_4$, and the 'ordinary' fluxes $\mathfrak{m}$ are integer-quantised. Recall that a Kähler manifold $\mathcal{M}_4$ is spin if and only if the 'square root' $\mathcal{K}^{\frac{1}{2}}$ exists. The partition function will ultimately depend on the choice of ε in (1.18). We call the corresponding supersymmetric background the *extended DW twist*.[8] The allowed values of the extended DW-twist parameter ε depend on the theory. Moreover, requiring that the 5d theory can be defined on any $\mathcal{M}_4$ (with a consistent choice of ε) actually imposes the existence of an interesting 'spin/charge relation' [73, 74], which is a non-trivial constraint on the 5d BPS spectrum.[9]

For $\mathcal{M}_5$ a principal circle bundle of first Chern class $\mathfrak{p}$ over $\mathcal{M}_4$, the full CB partition function takes the form:

$$Z_{\mathcal{M}_5}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}} = \boldsymbol{A}(\boldsymbol{a})^{\chi}\boldsymbol{B}(\boldsymbol{a})^{\sigma}\boldsymbol{G}(\boldsymbol{a};\varepsilon)^{2\chi+3\sigma}\boldsymbol{\Pi}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{m}+2\varepsilon\mathbf{k},\mathfrak{m})}\mathscr{K}(\boldsymbol{a})^{(\mathfrak{p},\mathfrak{m}+\varepsilon\mathbf{k})}\,\mathscr{F}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{p},\mathfrak{p})} . \tag{1.19}$$

Here, χ and σ are the Euler characteristic and signature of $\mathcal{M}_4$, respectively. The first three couplings, $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{G}$, are effective gravitational couplings (including $\boldsymbol{G}$, the coupling to

[7] The gluing approach is more general and can also be applied beyond the topological twist by gluing 'topological' and 'anti-topological' Nekrasov partition functions, as first discussed by Pestun for S^4 [4] and later generalised in various directions [67–71].

[8] Slightly more general choices of spin^c connections are possible (see e.g. [36]), but we will restrict ourselves to the quantisation condition (1.18) for simplicity.

[9] It would be interesting to know whether any 5d SCFT necessarily satisfies this spin/charge relation. We will only show that it holds explicitly in some rank-1 theories.

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the spin^c connections), the fourth coupling, $\mathbf{\Pi}$, is dubbed the 'flux operator', which inserts some abelian (background) flux on $\mathcal{M}_4$, and

$$\widehat{\mathscr{F}}_{\mathfrak{p}}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}} \equiv \mathscr{K}(\boldsymbol{a})^{(\mathfrak{p},\mathfrak{m}+\varepsilon\mathbf{k})}\, \mathscr{F}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{p},\mathfrak{p})}\,, \tag{1.20}$$

gives us the fibering operator, which is a flux operator for the $U(1)_{\rm KK}$ symmetry. The relation (1.12) is a simple instance of this master formula, for S^5 with $\mathfrak{m}=0$ and $\varepsilon=0$.

Given the very general result (1.19), the next step will be to compute the full partition function by performing the integration over the Coulomb branch of the 4d $\mathcal{N}=2$ KK theory, with the hope of deriving (1.8) as well as our conjecture (1.14). For theories of rank one, such computations would build upon recent analyses of the Coulomb branch geometry [19, 21, 75]. This is left for future work.

This paper is organised as follows. In section 2, we discuss the topological twist on $\mathcal{M}_4$, setting up our conventions. In section 3, we define and study a class of supersymmetric backgrounds for 5d $\mathcal{N}=1$ theories on $\mathcal{M}_5$. In section 4, we compute the relevant one-loop determinants on $\mathcal{M}_5$. In section 5, we derive the master formula (1.19) using the low-energy effective action in curved space. Finally, in section 6, we study the gluing of Nekrasov partition functions for $\mathcal{M}_4$ a toric surface. Our geometry and supersymmetry conventions, as well as some useful additional results, are collected in various appendices.

2 Topological twist on a Kähler surface

In this section, we review the topological twist of four-dimensional $\mathcal{N}=2$ supersymmetric gauge theories [1, 2]. We focus on Kähler four-manifolds (also known as Kähler surfaces), on which we can preserve two supercharges, and we discuss the 'twisted variables' that are most useful in that context. By a slight abuse of terminology, we call this the Donaldson-Witten (DW) twist. We will particularly insist on the 'extended DW twist' of the hypermultiplet.

2.1 Kähler surfaces and the $\mathcal{N}=2$ topological twist

Consider a Kähler four-manifold, $\mathcal{M}_4$, viewed as a hermitian manifold $(\mathcal{M}_4, J, g)$ whose complex structure is covariantly constant, $\nabla_\mu J^\nu{}_\rho = 0$. Let us use some local complex coordinates $(z^i) = (z^1, z^2)$, in terms of which the Kähler metric g reads:

$$ds^2 = 2g_{i\bar{j}}dz^i d\bar{z}^{\bar{j}}\,, \qquad g_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^{\bar{j}}}\,, \tag{2.1}$$

where K is the Kähler potential. Our geometric conventions are spelled out in appendix A.1.

We are interested in 4d $\mathcal{N}=2$ quantum field theories in Euclidean space-time with an exact $SU(2)_R$ R-symmetry. Let us first recall the standard definition of the topological twist, as originally given by Witten [1]. We start with a theory with a global symmetry that includes the Euclidean rotation $\mathrm{Spin}(4) \cong SU(2)_l \times SU(2)_r$ and the R-symmetry, and we relabel the spins of fields according to the new 'twisted spin',

$$SU(2)_l \times SU(2)_D\,, \qquad SU(2)_D \equiv \mathrm{diag}(SU(2)_r \times SU(2)_R)\,. \tag{2.2}$$

Alternatively, the DW twist can be understood as a supergravity background (i.e. a rigid limit of some 4d $\mathcal{N}=2$ supergravity), consisting of a metric (2.1) and of a background gauge field $\mathbf{A}^{(R)}$ for the R-symmetry, preserving some fraction of the flat-space supersymmetry (see e.g. [76–78]). One preserves a right-chiral supersymmetry:

$$\delta_{\widetilde{\xi}} \equiv \widetilde{\xi}^I_{\dot{\alpha}} \widetilde{Q}^{\dot{\alpha}}_I \,, \tag{2.3}$$

on any background $(\mathcal{M}_4, g, \mathbf{A}^{(R)})$ that admits a covariantly-constant spinor $\widetilde{\xi}_I$:

$$D_\mu \widetilde{\xi}_I \equiv \left(\nabla_\mu \delta_I{}^J - i(\mathbf{A}^{(R)}_\mu)_I{}^J\right) \widetilde{\xi}_J = 0 \,. \tag{2.4}$$

Here and in the following, $I=1,2$ are $SU(2)_R$ indices.[10] Such a background exists on any Riemannian four-manifold: one obtains a solution to (2.4) by identifying the $SU(2)_R$ connection with the spin connection [1]. In our conventions, we have:

$$(\mathbf{A}^{(R)}_\mu)_I{}^J = \frac{i}{2}\omega_{\mu ab}(\widetilde{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}}\, \delta^J{}_{\dot{\alpha}}\, \delta_I{}^{\dot{\beta}} \,, \qquad (\widetilde{\xi}^{\dot{\alpha}}_I) = (\delta^{\dot{\alpha}}{}_I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,. \tag{2.5}$$

More invariantly, the Killing spinor is a section of a complex vector bundle:

$$\widetilde{\xi} \in \Gamma[S_+ \otimes E_R] \,, \tag{2.6}$$

where E_R is a rank-2 $SU(2)_R$ vector bundle. The topological twist (2.5) consists in choosing $E_R \cong S_+$, in which case $S_+ \otimes E_R$ decomposes as a direct sum

$$S_+ \otimes E_R \cong \mathcal{O} \oplus \Omega^+ \,, \tag{2.7}$$

where Ω^+ is the rank-3 vector bundle of self-dual 2-forms.[11] Then our Killing spinor $\widetilde{\xi}$ is simply the constant section of the trivial line bundle $\mathcal{O}$. It is also important to note that the topological twist is defined on any four-manifold, irrespective of whether it is a spin manifold, because the bundle (2.7) is well-defined even when S_+ is not.

We are interested in Kähler surfaces, in which case we actually preserve two distinct supersymmetries:

$$\delta_1 \equiv \widetilde{\xi}^I_{(1)\dot{\alpha}} \widetilde{Q}^{\dot{\alpha}}_I \,, \qquad \delta_2 \equiv \widetilde{\xi}^I_{(2)\dot{\alpha}} \widetilde{Q}^{\dot{\alpha}}_I \,. \tag{2.8}$$

The Levi-Civita connection on a Kähler manifold has reduced holonomy $U(2) \cong SU(2)_l \times U(1)_r \subset SU(2)_l \times SU(2)_r$, and we then only need to 'twist' $U(1)_r$ by turning on a non-trivial gauge field for the R-symmetry subgroup $U(1)_R \subset SU(2)_R$. In the complex frame basis (e^1, e^2), the spin connection reads:

$$\frac{1}{2}\omega_{\mu ab}\widetilde{\sigma}^{ab} = -(\omega_{\mu 1\bar{1}} + \omega_{\mu 2\bar{2}})\tau^3 \,, \qquad \tau^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,. \tag{2.9}$$

By choosing the background $SU(2)_R$ gauge field:

$$(\mathbf{A}^{(R)}_\mu)_I{}^J = \sum_{\mathbf{a}=1}^{3} A^{\mathbf{a}}_\mu (\tau^{\mathbf{a}})_I{}^J = A^{\mathbf{3}}_\mu (\tau^3)_I{}^J \,, \qquad A^{\mathbf{3}}_\mu dx^\mu = -i\left(\omega_{\mu 1\bar{1}} + \omega_{\mu 2\bar{2}}\right) \,, \tag{2.10}$$

[10] We usually keep the $SU(2)_R$ indices explicit, while suppressing the spinor indices $\alpha, \dot{\alpha}$.

[11] At the level of Spin(4) representations, we have $(0, \frac{1}{2}) \otimes (0, \frac{1}{2}) = (0,0) \oplus (0,1)$.

with $\tau^{\mathbf{a}}$ the Pauli matrices, we preserve the two Killing spinors:

$$(\widetilde{\xi}^{\dot\alpha}_{(1)I}) = (\delta^{\dot\alpha\dot 1}\delta_{I1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad (\widetilde{\xi}^{\dot\alpha}_{(2)I}) = (\delta^{\dot\alpha\dot 2}\delta_{I2}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.11}$$

Note that the solution $\widetilde{\xi}_I$ in (2.5) is the sum of these two Killing spinors,

$$\widetilde{\xi} = \widetilde{\xi}_{(1)} + \widetilde{\xi}_{(2)} . \tag{2.12}$$

Correspondingly, we preserve the flat-space supercharges $\widetilde{Q}^2_{\dot 2}$ and $\widetilde{Q}^1_{\dot 1}$ on any Kähler manifold, while on a generic four-manifold we only preserve their sum, $\widetilde{Q}^1_{\dot 1} + \widetilde{Q}^2_{\dot 2}$.

Let us describe this Donaldson-Witten twist more covariantly. On any Kähler surface $\mathcal{M}_4$, the spin bundle $\mathbf{S} \equiv S_- \oplus S_+$ formally decomposes as:

$$S_- \cong \mathcal{K}^{\frac{1}{2}} \otimes \Omega^{0,1}, \qquad S_+ \cong \mathcal{K}^{\frac{1}{2}} \oplus \mathcal{K}^{-\frac{1}{2}}, \tag{2.13}$$

with $\mathcal{K}$ the canonical line bundle. Here, $\mathcal{M}_4$ is spin if and only if the 'square-root' $\mathcal{K}^{\frac{1}{2}}$ actually exists. Recall that the second Stiefel-Withney class of a complex surface $\mathcal{M}_4$ is related to its first Chern class, namely

$$w_2(\mathcal{M}_4) \cong c_1(\mathcal{K}) \mod 2 . \tag{2.14}$$

Let us choose an $SU(2)_R$ vector bundle of the form

$$E_R = L_R^{-1} \oplus L_R , \tag{2.15}$$

for L_R some $U(1)_R$ line bundle. The Killing spinors (2.11) are really sections

$$\widetilde{\xi}_{(1)} \in \Gamma[S_+ \otimes L_R], \qquad \widetilde{\xi}_{(2)} \in \Gamma[S_+ \otimes L_R^{-1}], \tag{2.16}$$

and the DW twist amounts to the formal identification

$$L_R \cong \mathcal{K}^{-\frac{1}{2}} . \tag{2.17}$$

In general, $\mathcal{M}_4$ is not spin and therefore $\mathcal{K}^{\frac{1}{2}}$ does not exist, but the bundles $S_+ \otimes L_R^{\pm 1}$ are nonetheless well-defined spin^c bundles. We will further comment on this point in section 2.3 below, where we discuss the topological twist of the hypermultiplet.

Note also that the Killing spinors (2.16) are precisely the ones that give rise to a single curved-space supercharge for $\mathcal{N} = 1$ supersymmetric field theories on $\mathcal{M}_4$ a Kähler manifold [2, 79]. The two distinct $\mathcal{N} = 1$ subalgebras correspond to $\widetilde{Q}^{I=1}$ and $\widetilde{Q}^{I=2}$ for the spinors $\widetilde{\xi}_{(1)}$ and $\widetilde{\xi}_{(2)}$, respectively.

Spinor bilinears and Kähler structure. Given the Killing spinors introduced so far, one can construct well-defined two-forms on $\mathcal{M}_4$. First of all, given any solution $\widetilde{\xi}$ to the Killing spinor equation (2.4), we can define the $SU(2)_R$-neutral anti-self-dual two-form:

$$\mathcal{J}_{\mu\nu}[\widetilde{\xi}] \equiv -2i \frac{\widetilde{\xi}^{\dagger I} \widetilde{\sigma}_{\mu\nu} \widetilde{\xi}_I}{|\widetilde{\xi}|^2}, \qquad |\widetilde{\xi}|^2 \equiv \widetilde{\xi}^{\dagger I} \widetilde{\xi}_I , \tag{2.18}$$

where the sum over repeated indices is understood. For the Killing spinor (2.5) on a general four-manifold, the bilinear (2.18) identically vanishes. On the other hand, from the Killing spinors (2.11), we obtain:

$$J_{\mu\nu} \equiv \mathcal{J}_{\mu\nu}[\widetilde{\xi}_{(1)}] = -\mathcal{J}_{\mu\nu}[\widetilde{\xi}_{(2)}]\,, \tag{2.19}$$

which satisfies:

$$J^{\mu}{}_{\nu} J^{\nu}{}_{\rho} = -\delta^{\mu}{}_{\rho}\,, \qquad \nabla_\mu J_{\nu\rho} = 0\ . \tag{2.20}$$

Thus, (2.19) gives us the complex structure (and the associated Kähler form) of the hermitian Kähler manifold $\mathcal{M}_4$. In this way, one can show that there are two linearly independent solutions to (2.4) if and only $\mathcal{M}_4$ is Kähler [79]. Given the two Killing spinors (2.16), we may also write down the bilinears:

$$\begin{aligned} p^{2,0}_{(1)} &\equiv \widetilde{\xi}_{(1)}\sigma_{\mu\nu}\widetilde{\xi}_{(1)}\, dx^\mu \wedge dx^\nu \quad &\in \Gamma[\mathcal{K}\otimes L_R^2]\,, \\ p^{0,2}_{(2)} &\equiv \widetilde{\xi}_{(2)}\sigma_{\mu\nu}\widetilde{\xi}_{(2)}\, dx^\mu \wedge dx^\nu \quad &\in \Gamma[\mathcal{K}^{-1}\otimes L_R^{-2}]\ . \end{aligned} \tag{2.21}$$

In the frame basis, we have:

$$p^{2,0}_{(1)} = -e^1\wedge e^2\,, \qquad p^{0,2}_{(2)} = -e^{\bar 1}\wedge e^{\bar 2}\,, \tag{2.22}$$

for the solutions (2.11). These are nowhere-vanishing sections of line bundles, therefore the corresponding line bundles are trivial. This is another way to see that (2.17) must hold, or more precisely $L_R^2 \cong \mathcal{K}^{-1}$ if $\mathcal{M}_4$ is not spin. This topological twist is therefore also an '$\mathcal{N}=1$ holomorphic twist' (see e.g. [79, 80]) for either of the two $\mathcal{N}=1$ subalgebras mentioned above.

2.2 The vector multiplet on $\mathcal{M}_4$

Let us consider the $\mathcal{N}=2$ vector multiplet $\mathcal{V}$, in the adjoint representation of some Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, on $\mathcal{M}_4$ a Kähler manifold. It consists of a gauge field A_μ, two sets of gauginos λ^I, and a triplet of auxiliary scalar fields $D_{IJ} = D_{JI}$:

$$\mathcal{V} = \Big(A_\mu, \phi, \widetilde{\phi}, \lambda^I, \widetilde{\lambda}_I, D_{IJ}\Big)\ . \tag{2.23}$$

Its off-shell supersymmetry transformations on $\mathbb{R}^4$ are reviewed in appendix A.1.3. The gauge connection $A = A_\mu dx^\mu$ is well defined on any four-manifold. After the topological twist, the gauginos are also well-defined on $\mathcal{M}_4$. The left-chiral gauginos $\lambda = (\lambda^I)$ are sections

$$\lambda \in \Gamma[S_-\otimes E_R] \cong \Gamma[\Omega^{0,1}\oplus(\mathcal{K}\otimes\Omega^{0,1})] \cong \Gamma[\Omega^{0,1}\oplus\Omega^{1,0}]\ . \tag{2.24}$$

We use the Hodge star operator to map $(2,1)$-forms (the sections of $\mathcal{K}\otimes\Omega^{1,0}$) to $(1,0)$-forms, according to $\omega^{1,0} = \star\omega^{2,1}$. Similarly, the right-chiral gauginos $\widetilde{\lambda} = (\widetilde{\lambda}_I)$ are sections

$$\widetilde{\lambda} \in \Gamma[S_+\otimes \bar{E}_R] \cong \Gamma[\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{K}\oplus\mathcal{K}^{-1}]\ . \tag{2.25}$$

These (p,q)-forms can be constructed explicitly from the ordinary (flat-space) spinors, by contracting the gauginos with the Killing spinors (2.11) to form $SU(2)_R$-neutral tensors.

For the left-chiral gauginos, the holomorphic and anti-holomorphic 1-forms in (2.24) are given by:

$$\Lambda^{1,0} \equiv \widetilde{\xi}_{(1)I}\widetilde{\sigma}_\mu\lambda^I\, dx^\mu\,, \qquad \Lambda^{0,1} \equiv \widetilde{\xi}_{(2)I}\widetilde{\sigma}_\mu\lambda^I\, dx^\mu\;. \tag{2.26}$$

Similarly, starting from the right-chiral gauginos $\widetilde{\lambda} = (\widetilde{\lambda}_I)$, we define the two scalars

$$\widetilde{\Lambda}^{0,0}_{(1)} \equiv \widetilde{\xi}^I_{(1)}\widetilde{\lambda}_I\,, \qquad \widetilde{\Lambda}^{0,0}_{(2)} \equiv \widetilde{\xi}^I_{(2)}\widetilde{\lambda}_I\,, \tag{2.27}$$

and the holomorphic and anti-holomorphic two-forms:

$$\begin{aligned}
\widetilde{\Lambda}^{2,0} &\equiv \left(\widetilde{\xi}^I_{(1)}\widetilde{\sigma}_{\mu\nu}\widetilde{\lambda}_I + \frac{i}{2}\widetilde{\xi}^I_{(1)}\widetilde{\lambda}_I\, J_{\mu\nu}\right) dx^\mu\wedge dx^\nu = \frac{1}{2}\widetilde{\Lambda}^{2,0}_{\mu\nu}dx^\mu\wedge dx^\nu\,,\\
\widetilde{\Lambda}^{0,2} &\equiv \left(\widetilde{\xi}^I_{(2)}\widetilde{\sigma}_{\mu\nu}\widetilde{\lambda}_I - \frac{i}{2}\widetilde{\xi}^I_{(2)}\widetilde{\lambda}_I\, J_{\mu\nu}\right) dx^\mu\wedge dx^\nu = \frac{1}{2}\widetilde{\Lambda}^{0,2}_{\mu\nu}dx^\mu\wedge dx^\nu\,,
\end{aligned} \tag{2.28}$$

in agreement with (2.25). In our choice of local frame basis, we have:

$$\begin{aligned}
\Lambda^{1,0} &= -\lambda_1^1\, e^1 + \lambda_2^1\, e^2\,, & \Lambda^{0,1} &= \lambda_2^2\, e^{\bar{1}} + \lambda_1^2\, e^{\bar{2}}\,,\\
\widetilde{\Lambda}^{0,0}_{(1)} &= \widetilde{\lambda}^{\dot{2}}_2\,, & \widetilde{\Lambda}^{0,0}_{(2)} &= \widetilde{\lambda}^{\dot{1}}_1\,,\\
\widetilde{\Lambda}^{2,0} &= -\widetilde{\lambda}^{\dot{1}}_2\, e^1\wedge e^2\,, & \widetilde{\Lambda}^{0,2} &= \widetilde{\lambda}^{\dot{2}}_1\, e^{\bar{1}}\wedge e^{\bar{2}}\;.
\end{aligned} \tag{2.29}$$

Let us also define the $SU(2)_R$-neutral auxiliary fields:

$$\begin{aligned}
\mathcal{D}^{2,0} &\equiv -iD_{IJ}\,\xi^I_{(1)}\widetilde{\sigma}_{\mu\nu}\widetilde{\xi}^J_{(1)}\, dx^\mu\wedge dx^\nu & &= -iD_{22}\,p^{2,0}_{(1)}\,,\\
\mathcal{D}^{0,2} &\equiv iD_{IJ}\,\xi^I_{(2)}\widetilde{\sigma}_{\mu\nu}\widetilde{\xi}^J_{(2)}\, dx^\mu\wedge dx^\nu & &= -iD_{11}\,p^{0,2}_{(2)}\,,\\
\mathcal{D}^{0,0} &\equiv \frac{1}{2}J^{\mu\nu}\left(-iD_{IJ}\,\xi^I_{(1)}\widetilde{\sigma}_{\mu\nu}\widetilde{\xi}^J_{(2)} + F_{\mu\nu}\right) & &= D_{12} + \widehat{F}\,,
\end{aligned} \tag{2.30}$$

with

$$\widehat{F} \equiv \frac{1}{2}J^{\mu\nu}F_{\mu\nu} = 2i(F_{1\bar{1}} + F_{2\bar{2}})\;. \tag{2.31}$$

The flat-space supersymmetry transformations of the vector multiplet are written explicitly in appendix A.1.3 — see equation (A.27). Using the twisted variables, one obtains the following curved-space supersymmetry transformations under the two supercharges (2.8):

$$\begin{aligned}
&\delta_1\phi = 0\,, & &\delta_2\phi = 0\,,\\
&\delta_1\widetilde{\phi} = \sqrt{2}\widetilde{\Lambda}^{0,0}_{(1)}\,, & &\delta_2\widetilde{\phi} = \sqrt{2}\widetilde{\Lambda}^{0,0}_{(2)}\,,\\
&\delta_1 A = -i\Lambda^{1,0}\,, & &\delta_2 A = -i\Lambda^{0,1}\,,\\
&\delta_1\Lambda^{1,0} = 0\,, & &\delta_2\Lambda^{1,0} = 2i\sqrt{2}\partial_A\phi\,,\\
&\delta_1\Lambda^{0,1} = 2i\sqrt{2}\bar{\partial}_A\phi\,, & &\delta_2\Lambda^{0,1} = 0\,,\\
&\delta_1\widetilde{\Lambda}^{0,0}_{(1)} = 0\,, & &\delta_2\widetilde{\Lambda}^{0,0}_{(1)} = i\mathcal{D}^{0,0} - i[\widetilde{\phi},\phi]\,,\\
&\delta_1\widetilde{\Lambda}^{0,0}_{(2)} = -i\mathcal{D}^{0,0} - i[\widetilde{\phi},\phi]\,, & &\delta_2\widetilde{\Lambda}^{0,0}_{(2)} = 0\,,\\
&\delta_1\widetilde{\Lambda}^{2,0} = \mathcal{D}^{2,0}\,, & &\delta_2\widetilde{\Lambda}^{2,0} = 4F^{2,0}\,,\\
&\delta_1\widetilde{\Lambda}^{0,2} = 4F^{0,2}\,, & &\delta_2\widetilde{\Lambda}^{0,2} = \mathcal{D}^{0,2}\,,\\
&\delta_1\mathcal{D}^{0,0} = \sqrt{2}[\phi,\widetilde{\Lambda}^{0,0}_{(1)}]\,, & &\delta_2\mathcal{D}^{0,0} = -\sqrt{2}[\phi,\widetilde{\Lambda}^{0,0}_{(2)}]\,,\\
&\delta_1\mathcal{D}^{2,0} = 0\,, & &\delta_2\mathcal{D}^{2,0} = 4i\partial_A\Lambda^{1,0} + 2i\sqrt{2}[\phi,\widetilde{\Lambda}^{2,0}]\,,\\
&\delta_1\mathcal{D}^{0,2} = 4i\bar{\partial}_A\Lambda^{0,1} + 2i\sqrt{2}[\phi,\widetilde{\Lambda}^{0,2}]\,, & &\delta_2\mathcal{D}^{0,2} = 0\;.
\end{aligned} \tag{2.32}$$

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Here, ∂_A and $\bar{\partial}_A$ denote the Dolbeault operators twisted by the gauge field $A = A_\mu dx^\mu$:

$$d_A = d - iA = \partial_A + \bar{\partial}_A \ . \tag{2.33}$$

Moreover, $F^{2,0}$ and $F^{0,2}$ denote the $(2,0)$ and $(0,2)$ projection of the field strength $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$. Note that we have:

$$\partial_A^2 = -iF^{2,0}\wedge\,, \qquad \bar{\partial}_A^2 = -iF^{0,2}\wedge\,, \qquad \{\partial_A, \bar{\partial}_A\} = -iF^{1,1}\wedge \ . \tag{2.34}$$

It is straightforward to check that the transformations (2.32) realize the supersymmetry algebra:

$$\delta_1^2 = 0\,, \qquad \delta_2^2 = 0\,, \qquad \{\delta_1, \delta_2\} = 2\sqrt{2}\delta_{g(\phi)}\,, \tag{2.35}$$

where $\delta_{g(\phi)}$ is a gauge transformation with parameter ϕ. In particular, we have $\delta_{g(\phi)}A = d_A\phi = d\phi + i[\varphi, A]$ for the gauge field, and $\delta_{g(\phi)}\varphi = i[\phi,\varphi]$ for any field φ transforming in the adjoint representation of $\mathfrak{g}$.

Supersymmetric Lagrangian. The 4d $\mathcal{N}=2$ SYM Lagrangian (given by eq. (A.23) in flat-space) can be written compactly as

$$\mathscr{L}_{\rm SYM} = \frac{1}{g^2}\delta_1\delta_2\,{\rm tr}\left(\frac{1}{8}\star\left(\widetilde{\Lambda}^{2,0}\wedge\widetilde{\Lambda}^{0,2}\right) - i\frac{\sqrt{2}}{4}\widetilde{\phi}\left(\mathcal{D}^{0,0} - 2\widehat{F}\right)\right) - \frac{1}{2g^2}\star{\rm tr}\left(F\wedge F\right) \ . \tag{2.36}$$

Note that it is "mostly" $\mathcal{Q}$-exact. The bosonic terms read:

$$\begin{aligned}\mathscr{L}_{\rm SYM}\big|_{\rm bos} = \frac{1}{g^2}\,{\rm tr}\Bigg(&\star\left(2F^{2,0}\wedge F^{0,2} - \frac{1}{2}F\wedge F\right) - \widetilde{\phi}D_\mu D^\mu\phi \\ &- \frac{1}{8}\star\left(\mathcal{D}^{2,0}\wedge\mathcal{D}^{0,2}\right) - \frac{1}{2}\mathcal{D}^{0,0}\left(\mathcal{D}^{0,0} - 2\widehat{F}\right)\Bigg) \ .\end{aligned} \tag{2.37}$$

On supersymmetric configurations, the $\mathcal{Q}$-exact terms in (2.36) evaluate to zero. Defining the 'instanton number' — more precisely, (minus) the second Chern class of any holomorphic vector bundle associated to a principal G bundle — as

$$k = -\frac{1}{16\pi^2}\int_{\mathcal{M}_4}{\rm tr}\left(F\wedge F\right)\,, \tag{2.38}$$

and adding the topological coupling

$$S_{\rm top} = i\frac{\theta}{16\pi^2}\int{\rm tr}\left(F\wedge F\right) = -i\theta k\,, \tag{2.39}$$

to the Lagrangian, any supersymmetric vector-multiplet configuration is weighted by a factor:

$$e^{-S_{\rm SYM} - S_{\rm top}} = e^{2\pi i\tau k}\,, \tag{2.40}$$

where we defined the holomorphic gauge coupling as

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \ . \tag{2.41}$$

In particular, the classical saddles are Yang-Mills instantons[12] and they contribute in this way.

[12] More precisely, *anti-instantons*, satisfying the anti-self-duality condition $F = -\star F$. We can call them 'instantons' since self-dual instantons do not play a role in Donaldson-Witten theory.

2.3 The hypermultiplet on $\mathcal{M}_4$

Consider a hypermultiplet $\mathcal{H}$ charged under a Lie group G. When considered as part of a larger gauge theory, G will include both the gauge group, with its dynamical gauge fields, and the flavor symmetry group, with its background gauge fields. On-shell, this 4d $\mathcal{N}=2$ multiplet consists of two complex scalars, q^I, forming a doublet of $SU(2)_R$, and of two $SU(2)_R$-neutral Dirac fermions:

$$\mathcal{H} = \left(q^I\,,\, \widetilde{q}_I\,,\, \eta\,,\, \widetilde{\eta}\,,\, \chi\,,\, \widetilde{\chi}\right)\,. \tag{2.42}$$

In addition, we will need to introduce some auxiliary fields in order to realize the two supersymmetries δ_1 and δ_2 off-shell.[13] The fields q^I, η and $\widetilde{\chi}$ transform in some representation $\mathfrak{R}$ of the gauge algebra $\mathfrak{g}$, and the fields $\widetilde{q}_I$, χ and $\widetilde{\eta}$ transform in the conjugate representation $\bar{\mathfrak{R}}$. After the topological twist, the scalars q^I, $\widetilde{q}_I$ become right-chiral spinors, which are therefore not well-defined unless $\mathcal{M}_4$ is a spin manifold [81]. For charged hypermultiplets, this issue can be remedied by introducing a spin^c structure [25, 82, 83]. Such a structure exists on any oriented closed four-manifold, but it is important to emphasise that this is an additional choice that we make when considering hypermultiplets. We thus call this an 'extended DW twist'.

Without too much loss of generality, let us consider $\mathcal{H}$ charged under some gauge group $\widetilde{G} = U(1)\times G$, where the $U(1)$ gauge field is really a spin^c connection. It is associated with a line bundle $\mathcal{L}_0$ such that $\mathcal{L}_0^{\frac{1}{2}} \otimes S_+$ is well-defined. On a Kähler manifold, with the spin bundle formally given by (2.13), we will choose:

$$\mathcal{L}_0 \cong \mathcal{K}^{-1}\,. \tag{2.43}$$

We insist on the fact that this is a somewhat arbitrary choice, however natural it appears on a Kähler manifold. For our purposes, it will be important to consider the more general case:

$$\mathcal{L}_0 \cong \mathcal{K}^{2\varepsilon}\,, \tag{2.44}$$

with $\varepsilon \in \frac{1}{2}\mathbb{Z}$ a free parameter. Roughly speaking, the extended topological twist is simply a choice of ε for each hypermultiplet in a theory; this must be done in a consistent way, as we will discuss further in later sections.

The ordinary DW twist of the hypermultiplet scalars gives us the right-chiral spinors

$$\mathbf{q} \equiv \widetilde{\xi}_I q^I\,, \qquad\qquad \widetilde{\mathbf{q}} \equiv \epsilon^{IJ}\widetilde{\xi}_I \widetilde{q}_J\,, \tag{2.45}$$

where we are using the DW Killing spinor (2.12). On an arbitrary Kähler manifold, the extended topological twist exists when these spinors are further valued in $\mathcal{L}_0^{\frac{1}{2}}$, namely:

$$\mathbf{q} \in \Gamma(S_+ \otimes \mathcal{K}^{\varepsilon})\,, \qquad\qquad \widetilde{\mathbf{q}} \in \Gamma(S_+ \otimes \mathcal{K}^{-\varepsilon})\,. \tag{2.46}$$

Note that $\mathbf{q}$ being valued in $\mathcal{K}^{\varepsilon}$ means that the charged conjugate scalar $\widetilde{\mathbf{q}}$ is valued in $\mathcal{K}^{-\varepsilon}$. In the rest of this section, we will set $\varepsilon = -\frac{1}{2}$. Reinstating a general ε will simply

[13] It is well-known that one cannot realize the full flat-space $\mathcal{N}=2$ supersymmetry off-shell with a finite number of auxiliary fields, but there is no problem with realizing these two particular supercharges off-shell.

correspond to having the twisted hypermultiplet, as described below, also valued in a line bundle $\mathcal{K}^{\varepsilon+\frac{1}{2}}$. Thus, setting $\varepsilon = -\frac{1}{2}$, the scalars become (p,q)-forms:

$$\mathbf{q} \in \Gamma(\mathcal{O} \oplus \mathcal{K}^{-1})\,, \qquad\qquad \widetilde{\mathbf{q}} \in \Gamma(\mathcal{K} \oplus \mathcal{O})\,. \tag{2.47}$$

All fields are also valued in the appropriate vector bundles $E_{\mathfrak{R}}$ or $E_{\bar{\mathfrak{R}}}$ determined by the representation $\mathfrak{R}$ of G — we omitted this from the notation to avoid clutter. To perform the explicit change of basis between flat-space and twisted variables, it is convenient to introduce the spinors

$$(\epsilon_+^{\dot\alpha}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \Gamma\left(\mathcal{K}^{\frac{1}{2}}\right), \qquad\qquad (\epsilon_-^{\dot\alpha}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \Gamma\left(\mathcal{K}^{-\frac{1}{2}}\right), \tag{2.48}$$

which are, formally, sections of $\mathcal{K}^{\pm\frac{1}{2}}$, as indicated. For any right-chiral spinor $\widetilde\psi^{\dot\alpha}$, let us define the contractions

$$\begin{aligned}
\mathbf{C}^{0,2}(\widetilde\psi) &= \left(\widetilde\epsilon_-\widetilde\sigma_{\mu\nu}\widetilde\psi - i\,\widetilde\epsilon_-\widetilde\psi\, J_{\mu\nu}\right) dx^\mu \wedge dx^\nu &&= \widetilde\psi^{\dot 2} e^{\bar 1} \wedge e^{\bar 2}\,,\\
\mathbf{C}^{0,2}(\widetilde\psi) &= -\left(\widetilde\epsilon_+\widetilde\sigma_{\mu\nu}\widetilde\psi + i\,\widetilde\epsilon_+\widetilde\psi\, J_{\mu\nu}\right) dx^\mu \wedge dx^\nu &&= \widetilde\psi^{\dot 1} e^{1} \wedge e^{2}\,.
\end{aligned} \tag{2.49}$$

The 'twisted scalars' for this extended topological twist are then

$$\begin{aligned}
Q^{0,0} &= \epsilon_-\mathbf{q} = q^1\,, & \widetilde Q^{0,0} &= -\epsilon_+\mathbf{q} = \widetilde q_1\,,\\
Q^{0,2} &= \mathbf{C}^{0,2}(\widetilde{\mathbf{q}}) = q^2\, e^{\bar 1} \wedge e^{\bar 2}\,, & \widetilde Q^{2,0} &= \mathbf{C}^{2,0}(\widetilde{\mathbf{q}}) = \widetilde q_2\, e^1 \wedge e^2\,.
\end{aligned} \tag{2.50}$$

We also have the two Dirac spinors:

$$\Psi = \begin{pmatrix} \eta_\alpha \\ \widetilde\chi^{\dot\alpha} \end{pmatrix} \in \Gamma[S \otimes \mathcal{K}^{-\frac{1}{2}}]\,, \qquad\qquad \widetilde\Psi = \begin{pmatrix} \chi_\alpha \\ \widetilde\eta^{\dot\alpha} \end{pmatrix} \in \Gamma[S \otimes \mathcal{K}^{\frac{1}{2}}]\,, \tag{2.51}$$

which are sections of spin^c bundles as explained above. They can be conveniently decomposed into (p,q)-forms:

$$\Psi^{0,\bullet} = (\eta^{0,1}, \widetilde\chi^{0,0}, \widetilde\chi^{0,2}) \in \Omega^{0,\bullet}\,, \qquad\qquad \widetilde\Psi^{\bullet,0} = (\chi^{1,0}, \widetilde\eta^{0,0}, \widetilde\eta^{2,0}) \in \Omega^{\bullet,0}\,. \tag{2.52}$$

For instance, the spinor χ is a section of $S_- \otimes \mathcal{K}^{\frac{1}{2}} \cong \Omega^{0,1} \otimes \mathcal{K} \cong \Omega^{2,1} \cong \Omega^{1,0}$, where we find it convenient to use $\chi^{1,0} \equiv \star\chi^{2,1}$. The explicit change of variables is given by

$$\begin{aligned}
\eta^{0,1} &= \widetilde\epsilon_-\widetilde\sigma_\mu\eta\, dx^\mu\,, & \chi^{1,0} &= \widetilde\epsilon_+\widetilde\sigma_\mu\chi\, dx^\mu\,,\\
\widetilde\chi^{0,0} &= \widetilde\epsilon_-\widetilde\chi\,, & \widetilde\eta^{0,0} &= \widetilde\epsilon_+\widetilde\eta\,,\\
\widetilde\chi^{0,2} &= \mathbf{C}^{0,2}(\widetilde\chi)\,, & \widetilde\eta^{0,0} &= \mathbf{C}^{2,0}(\widetilde\eta)\,.
\end{aligned} \tag{2.53}$$

We also need to introduce the auxiliary one-forms $h^{0,1}$ and $\widetilde h^{1,0}$ in order to close the curved-space supersymmetry algebra off-shell. In fact, under the two supersymmetries δ_1, δ_2, the hypermultiplet splits into two off-shell multiplets (coupled to the vector multiplet):

$$\mathcal{H} \cong (Q^{0,0}, Q^{0,2}, \Psi^{0,\bullet}, h^{0,1}) \oplus (\widetilde Q^{0,0}, \widetilde Q^{2,0}, \widetilde\Psi^{\bullet,0}, \widetilde h^{1,0})\,, \tag{2.54}$$

which consist of purely anti-holomorphic and holomorphic forms, respectively. The supersymmetry transformations read:

$$\begin{aligned}
&\delta_1 Q^{0,0} = 0\,, && \delta_2 Q^{0,0} = \sqrt{2}\widetilde{\chi}^{0,0}\,,\\
&\delta_1 Q^{0,2} = \sqrt{2}\widetilde{\chi}^{0,2}\,, && \delta_2 Q^{0,2} = 0\,,\\
&\delta_1 \eta^{0,1} = 2i\sqrt{2}\bar{\partial}_A Q^{0,0} + h^{0,1}\,, && \delta_2 \eta^{0,1} = i\sqrt{2}\star\left(\partial_A Q^{0,2}\right)\,,\\
&\delta_1 \widetilde{\chi}^{0,0} = 2i\phi Q^{0,0}\,, && \delta_2 \widetilde{\chi}^{0,0} = 0\,,\\
&\delta_1 \widetilde{\chi}^{0,2} = 0\,, && \delta_2 \widetilde{\chi}^{0,2} = 2i\phi Q^{0,2}\,,\\
&\delta_1 h^{0,1} = 0\,, && \delta_2 h^{0,1} = X^{0,1}\,,
\end{aligned} \tag{2.55}$$

and:

$$\begin{aligned}
&\delta_1 \widetilde{Q}^{0,0} = -\sqrt{2}\widetilde{\eta}^{0,0}\,, && \delta_2 \widetilde{Q}^{0,0} = 0\,,\\
&\delta_1 \widetilde{Q}^{2,0} = 0\,, && \delta_2 \widetilde{Q}^{2,0} = \sqrt{2}\widetilde{\eta}^{2,0}\,,\\
&\delta_1 \chi^{1,0} = i\sqrt{2}\star\left(\bar{\partial}_A \widetilde{Q}^{2,0}\right)\,, && \delta_2 \chi^{1,0} = 2i\sqrt{2}\partial_A \widetilde{Q}^{0,0} + \widetilde{h}^{1,0}\,,\\
&\delta_1 \widetilde{\eta}^{0,0} = 0\,, && \delta_2 \widetilde{\eta}^{0,0} = 2i\widetilde{Q}^{0,0}\phi\,,\\
&\delta_1 \widetilde{\eta}^{2,0} = -2i\widetilde{Q}^{2,0}\phi\,, && \delta_2 \widetilde{\eta}^{2,0} = 0\,,\\
&\delta_1 \widetilde{h}^{1,0} = \widetilde{X}^{1,0}\,, && \delta_2 \widetilde{h}^{1,0} = 0\,,
\end{aligned} \tag{2.56}$$

with:

$$\begin{aligned}
X^{0,1} \equiv\; & -4i\bar{\partial}_A \widetilde{\chi}^{0,0} - 2i\star(\partial_A \widetilde{\chi}^{0,2}) + 2i\sqrt{2}\Lambda^{0,1}Q^{0,0} + i\sqrt{2}\star(\Lambda^{1,0}\wedge Q^{0,2})\\
& + 2i\sqrt{2}\phi\eta^{0,1}\,,\\
\widetilde{X}^{1,0} \equiv\; & 4i\partial_A \widetilde{\eta}^{0,0} - 2i\star(\bar{\partial}_A \widetilde{\eta}^{2,0}) - 2i\sqrt{2}\widetilde{Q}^{0,0}\Lambda^{1,0} - i\sqrt{2}\star(\widetilde{Q}^{2,0}\wedge\Lambda^{0,1})\\
& - 2i\sqrt{2}\chi^{1,0}\phi\,.
\end{aligned} \tag{2.57}$$

In the above expressions, the covariant derivatives act as

$$D_\mu f = \partial_\mu f - iA_\mu f\,, \qquad D_\mu \widetilde{f} = \partial_\mu \widetilde{f} + i\,\widetilde{f}A_\mu\,, \tag{2.58}$$

on any fields $f = (Q, \widetilde{\chi}, \eta, h)$ and $\widetilde{f} = (\widetilde{Q}, \widetilde{\eta}, \chi, \widetilde{h})$ in the gauge representation $\mathfrak{R}$ and $\bar{\mathfrak{R}}$, respectively. The equations of motion for the fermions are $X^{0,1} = 0$ and $\widetilde{X}^{1,0} = 0$. Note that we have:

$$\begin{aligned}
&\delta_1 X^{0,1} = 2i\sqrt{2}\phi h^{0,1}\,, && \delta_2 X^{0,1} = 0\,,\\
&\delta_1 \widetilde{X}^{1,0} = 0\,, && \delta_2 \widetilde{X}^{1,0} = -2i\sqrt{2}\,\widetilde{h}^{1,0}\phi\,.
\end{aligned} \tag{2.59}$$

These transformations reproduce the supersymmetry algebra (2.35). In particular, we have

$$\{\delta_1, \delta_2\}\, f = 2i\sqrt{2}\,\phi\, f\,, \qquad \{\delta_1, \delta_2\}\,\widetilde{f} = -2i\sqrt{2}\,\widetilde{f}\,\phi\,. \tag{2.60}$$

The hypermultiplet Lagrangian on $\mathcal{M}_4$ can be obtained by starting from the flat-space Lagrangian (A.30), writing it in twisted variables, and adding in the auxiliary fields to ensure off-shell supersymmetry. The important fact is that it is $\mathcal{Q}_{DW}$-exact. We find

$$\begin{aligned}\mathscr{L}_{\mathcal{H}} = \frac{1}{4}(\delta_1+\delta_2)\star\Big(&\widetilde{h}^{1,0}\wedge\star\,\eta^{0,1} - 2i\sqrt{2}\,\chi^{1,0}\wedge\star\,\bar{\partial}_A Q^{0,0} + i\sqrt{2}\,\chi^{1,0}\wedge\partial_A Q^{0,2}\\ &-\frac{i}{4}\,\widetilde{\eta}^{0,0}\,\widetilde{\phi}\,Q^{0,0}\,\mathrm{dvol} + i\,\widetilde{\eta}^{2,0}\wedge\widetilde{\phi}\,Q^{0,2} - i\,\widetilde{Q}^{2,0}\wedge\widetilde{\Lambda}^{0,2}\,Q^{0,0}\\ &+\frac{i}{4}\,\widetilde{Q}^{0,0}\,\widetilde{\Lambda}^{0,0}_{(1)}\,Q^{0,0}\,\mathrm{dvol} + i\,\widetilde{Q}^{2,0}\wedge\widetilde{\Lambda}^{0,0}_{(2)}\,Q^{0,2} + i\,\widetilde{Q}^{0,0}\,\widetilde{\Lambda}^{2,0}\wedge Q^{0,2}\Big)\,,\end{aligned} \tag{2.61}$$

with $d\mathrm{vol} = \frac{1}{4}e^1\wedge e^2\wedge e^{\bar{1}}\wedge e^{\bar{2}} = \star 1$. We will discuss the corresponding kinetic terms in detail in section 4.1.

3 Five-dimensional DW twist on circle bundles

In this section, we uplift the topological twist of the previous section to a supersymmetric background for 5d $\mathcal{N}=1$ supersymmetric field theories on any five-manifold $\mathcal{M}_5$ which is a principal circle bundle over a Kähler surface,

$$S^1 \longrightarrow \mathcal{M}_5 \xrightarrow{\pi} \mathcal{M}_4\ . \tag{3.1}$$

Supersymmetric backgrounds of similar geometries were discussed by many authors — see e.g. [45, 46, 84–88]. Our approach here is limited to a supersymmetric background that reduces to the (extended) topological twist on $\mathcal{M}_4$.

3.1 Circle-bundle geometries and the 5d Killing spinor equation

Let the five-manifold $\mathcal{M}_5$ be a principal circle bundle over a Kähler four-manifold $\mathcal{M}_4$. This fibration is fully determined by the first Chern class:

$$c_1(\mathcal{L}_{\mathrm{KK}}) = \frac{1}{2\pi}\hat{\mathrm{F}} \in H^2(\mathcal{M}_4,\mathbb{Z})\ , \tag{3.2}$$

where the 'defining line bundle' $\mathcal{L}_{\mathrm{KK}}$ is the complex line bundle associated to the S^1 bundle. We define the Chern numbers $\mathfrak{p}_k$ by:

$$c_1(\mathcal{L}_{\mathrm{KK}}) = \sum_k \mathfrak{p}_k[S_k]\,, \tag{3.3}$$

with the 2-cycles $\mathrm{S}_k \subset \mathcal{M}_4$ forming a basis of $H_2(\mathcal{M}_4,\mathbb{Z})$, and $[S_k]\in H^2(\mathcal{M}_4,\mathbb{Z})$ their Poincaré duals. We have:

$$\mathbf{I}_{kl} = \int_{\mathrm{S}_k}[\mathrm{S}_l] = \mathrm{S}_k\cdot\mathrm{S}_l\,, \tag{3.4}$$

the intersection numbers on $\mathcal{M}_4$. Given a Kähler metric (2.1) on $\mathcal{M}_4$ with local coordinates (z^1,z^2), we choose the five-dimensional metric

$$ds^2(\mathcal{M}_5) = ds^2(\mathcal{M}_4)+\eta^2\,,\qquad \eta\equiv\beta(d\psi+\mathrm{C})\,, \tag{3.5}$$

with the fiber coordinate ψ subject to the identification $\psi \sim \psi + 2\pi$, and the connection C on $\mathcal{M}_4$ such that:

$$d\mathrm{C} = \hat{\mathrm{F}} = 2\pi\, c_1(\mathcal{L}_{\mathrm{KK}}) \ . \tag{3.6}$$

In (3.5) we also introduced β, the radius of the circle fiber. It has been shown in [88] that theories with $\mathcal{N} = 1$ supersymmetry can be defined on five-manifolds that admit such a metric. The existence of curved-space supersymmetry is related to the existence of a transversely holomorphic foliation (THF) structure defined by the one-form η, similarly to the three-dimensional geometries studied in [8, 89].

By assumption, since we have a fibration structure, the metric (3.5) admits a Killing vector K with dual one-form given by η, $K^M \equiv \eta^M$, namely:

$$K = \frac{1}{\beta}\partial_\psi \ . \tag{3.7}$$

Note that we have $d\eta = \beta\hat{\mathrm{F}} = 2\pi\beta \sum_k \mathfrak{p}_i[\mathrm{S}_k]$ and $\nabla_M \eta_N + \nabla_N \eta_M = 0$, which both follow from the relation:

$$\nabla_M \eta_N = \frac{\beta}{2}\hat{\mathrm{F}}_{MN} \ . \tag{3.8}$$

Here ∇_M is the 5d Levi-Civita connection. We would like to construct a supersymmetric background on $\mathcal{M}_5$ which is the uplift of the four-dimensional DW twist. In particular, such a background should admit two five-dimensional Killing spinors $\zeta^I_{(i)}$, for $i = 1, 2$, related to the four-dimensional Killing spinors (2.11) by

$$\zeta_{(i)I} = \begin{pmatrix} 0 \\ \widetilde{\xi}_{(i)I} \end{pmatrix} , \tag{3.9}$$

with $\widetilde{\xi}^{\dot{\alpha}}_{(1)I} = \delta^{\dot{\alpha}\dot{1}}\delta_{I1}$ and $\widetilde{\xi}^{\dot{\alpha}}_{(2)I} = \delta^{\dot{\alpha}\dot{2}}\delta_{I2}$. Once we posit the Killing spinors (3.9), we must reconstruct the 5d Killing spinor equations that they satisfy. The trivial uplift of the 4d Killing spinor equation,

$$\left(\nabla_M \delta_I{}^J - i(A^R_M)_I{}^J\right)\zeta_J = 0\,, \tag{3.10}$$

only holds for the trivial circle fibration. This is related to the fact that the connection ∇_M does not preserve the decomposition of tensors into vertical and horizontal components with respect to the fibration, since

$$\nabla_M \eta_N \neq 0\,, \tag{3.11}$$

unless $\mathfrak{p}_i = 0$. To correct this, we can simply introduce a new connection that preserves the fibration structure. The price to pay is that such a connection will have non-zero torsion.

3.1.1 Transversely holomorphic foliation and adapted connection

The five-dimensional manifolds that we are considering here are fibrations with a Kähler base — in particular, they are transversely holomorphic foliations (THF) with an adapted metric. A one-dimensional foliation structure on the five-manifold $\mathcal{M}_5$ is generated by a nowhere-vanishing vector field $K^M = g^{MN}\eta_N$. The foliation is transversely holomorphic if there exists a tangent bundle endomorphism Φ — i.e. a two-tensor $\Phi^M{}_N$ — whose restriction to the kernel of the one-form η gives an integrable complex structure,

$$\Phi\Big|_{\ker(\eta)} = J \ . \tag{3.12}$$

In particular, we have the relation

$$\Phi^M{}_N \Phi^N{}_P = -\delta^M{}_P + K^M \eta_P \ . \tag{3.13}$$

As shown in [88], the existence of one supercharge on a five-manifold[14] implies the existence of a THF. In the present case, we have two supercharges and the foliation is actually a fibration. We further restrict ourselves to the case when the holomorphic base manifold is Kähler, as required by the DW twist with two supercharges.

Focussing then on the class of fibered five-manifolds introduced above, with the adapted metric (3.5), it is convenient to introduce a modified connection $\hat{\nabla}$ that preserves the THF and fibration structure,

$$\hat{\nabla}_M g_{NP} = 0 \,, \qquad \hat{\nabla}_M \eta_N = 0 \,, \qquad \hat{\nabla}_M \Phi^N{}_P = 0 \ . \tag{3.14}$$

This connection can be expressed in terms of the Levi-Civita connection as

$$\hat{\Gamma}^P_{MN} = \Gamma^P_{MN} + K^P_{MN} \,, \tag{3.15}$$

where K is the cotorsion tensor. In terms of the circle-bundle curvature $\hat{\mathtt{F}}_{MN}$, this becomes:

$$K_{PMN} = \frac{\beta}{2} \left(\eta_P \hat{\mathtt{F}}_{MN} - \eta_N \hat{\mathtt{F}}_{MP} - \eta_M \hat{\mathtt{F}}_{NP} \right) \ . \tag{3.16}$$

The torsion tensor of the modified connection reads

$$T^P{}_{MN} = K^P{}_{MN} - K^P{}_{NM} = \beta \eta^P \hat{\mathtt{F}}_{MN} \ . \tag{3.17}$$

From here on, we will denote by $\hat{D}_M$ the covariant derivative with respect to the modified connection (which is also $SU(2)_R$- and gauge-covariant, as the case may be). For instance, for a scalar field ϕ we have

$$\left[\hat{D}_M, \hat{D}_N\right] \phi = -T^P{}_{MN} \hat{D}_P \phi \ . \tag{3.18}$$

3.1.2 Killing spinor equation and spinor bilinears

Given the adapted connection $\hat{D}_M$ on $\mathcal{M}_5$, we choose the Killing spinor equation

$$\hat{D}_M \zeta_I \equiv \left(\hat{\nabla}_M \delta^I{}_J - i (\mathbf{A}^{(R)}_M)^I{}_J \right) \zeta_J = 0 \,, \tag{3.19}$$

with $\mathbf{A}^{(R)}_M$ the $SU(2)_R$ background gauge field. One can easily check that the 5d Killing spinors (3.9) are indeed solutions to (3.19), once we take $\mathbf{A}^{(R)}_M$ to be the pull-back of the corresponding DW-twist connection on $\mathcal{M}_4$. In fact, we only need to turn on a $U(1)_R \subset$

[14] Together with a few other assumptions about the type of theories considered.

$SU(2)_R$ background, as in the four-dimensional case. As a result, we can introduce the spinors:

$$\zeta_{(1)} = \zeta_{(1)\, I=1}\,, \qquad \zeta_{(2)} = \zeta_{(2)\, I=2}\,, \tag{3.20}$$

for which we have:

$$\left(\hat{\nabla}_M - iA_M^{(R)}\right)\zeta_{(1)} = 0\,, \qquad \left(\hat{\nabla}_M + iA_M^{(R)}\right)\zeta_{(2)} = 0\,, \tag{3.21}$$

with the $U(1)_R$ gauge field

$$A^{(R)} = -i\left(\omega_{\mu 1\bar{1}} + \omega_{\mu 2\bar{2}}\right) dx^\mu\big|_{\mathcal{M}_4}\,. \tag{3.22}$$

THF from spinors. Given two distinct nowhere-vanishing solutions to (3.19), we can reconstruct THF tensors. We define the one-form

$$\eta_M = -\frac{1}{|\zeta_{(1)}|^2}\zeta^{\dagger\, I}_{(1)}\gamma_M\zeta_{(1)\, I} = -\frac{1}{|\zeta_{(2)}|^2}\zeta^{\dagger\, I}_{(2)}\gamma_M\zeta_{(2)\, I}\,, \tag{3.23}$$

with the Hermitian conjugate defined as for four-dimensional spinors [79, 90]. Additionally, similarly to the three-dimensional analysis of [89], the quantity defined as:

$$K^M = \zeta_{(1)I}\gamma^M\zeta^I_{(2)} = -\zeta_{(1)}\gamma^M\zeta_{(2)}\,, \tag{3.24}$$

is a non-vanishing Killing vector, whose orbits define a foliation of $\mathcal{M}_5$. Note that $K^M = \eta^M$ for our choice of metric. For future reference, let us also define the scalar

$$\kappa \equiv \zeta^I_{(1)}\zeta_{(2)\, I}\,. \tag{3.25}$$

Note that, when plugging in (3.9), we have $K^M = \delta^{M5}$ and $\kappa = 1$. Finally, we define a two-tensor

$$\Phi^{MN} = \frac{i\zeta^\dagger_{(1)}\gamma^{MN}\zeta_{(1)}}{|\zeta_{(1)}|^2}\,, \tag{3.26}$$

with $\gamma^{MN} \equiv \frac{1}{2}[\gamma^M, \gamma^N]$, which satisfies (3.12) and (3.13). The Killing spinor equation (3.19) also implies (3.14).

The 5d DW twist. The five-dimensional uplift of the DW twist on $\mathcal{M}_4$ can be formulated a little bit more covariantly. To do this, it is useful to consider the two-forms

$$\begin{aligned} \mathcal{P}_{(1)} &\equiv \zeta_{(1)}\Sigma_{MN}\zeta_{(1)}\, dx^M \wedge dx^N = ie^1 \wedge e^2\,, \\ \mathcal{P}_{(2)} &\equiv \zeta_{(2)}\Sigma_{MN}\zeta_{(2)}\, dx^M \wedge dx^N = ie^{\bar{1}} \wedge e^{\bar{2}}\,, \end{aligned} \tag{3.27}$$

where we use the complex frame (A.49). Let us define the canonical line bundle $\mathcal{K}_{\mathcal{M}_5}$ on $\mathcal{M}_5$ as the pull-back of the canonical line bundle $\mathcal{K}_{\mathcal{M}_4}$ on the Kähler manifold $\mathcal{M}_4$, using the fibration structure $\pi : \mathcal{M}_5 \to \mathcal{M}_4$, namely:

$$\mathcal{K}_{\mathcal{M}_5} = \pi^*\mathcal{K}_{\mathcal{M}_4}\,. \tag{3.28}$$

Since the spinors $\zeta_{(1)}$ and $\zeta_{(2)}$ have $U(1)_R$ charges $+1$ and -1, respectively, the two forms (3.27) are nowhere-vanishing sections

$$\mathcal{P}_{(1)} \in \Gamma[\mathcal{K} \otimes L_R^2]\,, \qquad\qquad \mathcal{P}_{(2)} \in \Gamma[\bar{\mathcal{K}} \otimes L_R^{-2}]\,, \tag{3.29}$$

with L_R the $U(1)_R$ bundle on $\mathcal{M}_5$. We, therefore, have the 5d uplift of the DW twist,

$$L_R \cong \mathcal{K}^{-\frac{1}{2}}\,, \tag{3.30}$$

literally as in (2.17) but now written in terms of line bundles on $\mathcal{M}_5$. As before, $\mathcal{K}^{\frac{1}{2}}$ will not be well-defined unless $\mathcal{M}_4$ is spin, but the Killing spinors are well-defined sections of appropriate spin^c bundles nonetheless.

3.1.3 (p, q)-forms and twisted Dolbeault operators on $\mathcal{M}_5$

The 5d uplift of the DW twist remains independent of the choice of Kähler metric on $\mathcal{M}_4$. To make this property manifest, we express all fields in terms of differential forms, exactly like in 4d. On $\mathcal{M}_5$, differential forms can be further decomposed into horizontal and vertical forms (i.e. along the base $\mathcal{M}_4$ and the circle fiber, respectively). This can be done explicitly by using the projectors:

$$\begin{aligned}
\Pi^M{}_N &= \frac{1}{2}\left(\delta^M{}_N - i\Phi^M{}_N - K^M\eta_N\right)\,,\\
\bar{\Pi}^M{}_N &= \frac{1}{2}\left(\delta^M{}_N + i\Phi^M{}_N - K^M\eta_N\right)\,,\\
\Theta^M{}_N &= K^M\eta_N\;.
\end{aligned} \tag{3.31}$$

Any k-form on $\mathcal{M}_5$ decomposes into $(p, q|\ell)$-forms, with $p+q+\ell = k$. Here, ℓ denotes the form degree along the fiber. By abuse of notation, a five-dimensional $(p, q|0)$-form is called (p, q)-form, denoted by $\omega^{p,q}$. Any $(p, q|1)$-form can be written as

$$\omega^{(p,q|1)} = \omega^{p,q} \wedge \eta\;. \tag{3.32}$$

For instance, for any one form $\omega = \omega_M dx^M$, we have

$$\omega = \omega^{1,0} + \omega^{0,1} + \omega_5\eta = \omega^{1,0}_i dz^i + \omega^{0,1}_{\bar{i}} d\bar{z}^{\bar{i}} + \omega_5\eta\,, \tag{3.33}$$

where:

$$\omega_M\Pi^M{}_N = \omega^{1,0}_N\,, \qquad \omega_M\bar{\Pi}^M{}_N = \omega^{0,1}_N\,, \qquad \omega_5 \equiv K^M\omega_M\;. \tag{3.34}$$

In particular, the vertical component is defined by contracting with K. For future reference, we also note that any 2-form F decomposes as:

$$F = F^{2,0} + F^{0,2} + F^{1,1} + F^{1,0}\wedge\eta + F^{0,1}\wedge\eta\;. \tag{3.35}$$

In particular, $(2, 0)$-forms are sections of the 5d canonical line bundle (3.28).

Dolbeault operators on $\mathcal{M}_5$. The differential $d : \Omega^k \to \Omega^{k+1}$ on $\mathcal{M}_5$ decomposes as

$$d = \partial + \bar{\partial} + \widehat{\partial}_5 \,, \tag{3.36}$$

where ∂ and $\bar{\partial}$ denote the twisted Dolbeault operators:

$$\partial : \Omega^{(p,q|\ell)} \to \Omega^{(p+1,q|\ell)} \,, \qquad \bar{\partial} : \Omega^{(p,q|\ell)} \to \Omega^{(p,q+1|\ell)} \,, \tag{3.37}$$

and $\widehat{\partial}_5 : \Omega^{(p,q|\ell)} \to \Omega^{(p,q|\ell+1)}$ is given by:[15]

$$\widehat{\partial}_5 \equiv \eta \wedge \partial_5 \,, \qquad \partial_5 \equiv \mathcal{L}_K = K^M \partial_M \ . \tag{3.38}$$

In terms of the local coordinates $(x^M) = (z^i, \bar{z}^{\bar{i}}, \psi)$, the twisted Dolbeault operators are given explicitly by:

$$\partial = dz^i \wedge (\partial_i - \mathrm{C}_i \partial_\psi) \,, \qquad \bar{\partial} = d\bar{z}^{\bar{i}} \wedge (\partial_{\bar{i}} - \mathrm{C}_{\bar{i}} \partial_\psi) \,, \tag{3.39}$$

where C_i and $\mathrm{C}_{\bar{i}}$ and the holomorphic and anti-holomorphic component of the connection C introduced in (3.5). Whenever the fibration is non-trivial, the twisted Dolbeault operators are not nilpotent. Instead, they satisfy the relations:

$$\partial^2 = -\beta \hat{\mathrm{F}}^{2,0} \wedge \partial_5 \,, \qquad \bar{\partial}^2 = -\beta \hat{\mathrm{F}}^{0,2} \wedge \partial_5 \,, \qquad \{\partial, \bar{\partial}\} = -\beta \hat{\mathrm{F}}^{1,1} \wedge \partial_5 \ . \tag{3.40}$$

Of course, they reduce to the ordinary Dolbeault operators on $\mathcal{M}_4$ upon dimensional reduction along the fiber. We also have that

$$\{\partial + \bar{\partial}, \widehat{\partial}_5\} = \beta \hat{\mathrm{F}} \wedge \partial_5 \,, \qquad \widehat{\partial}_5^2 = 0 \,, \tag{3.41}$$

where $\beta \hat{\mathrm{F}} = d\eta$. Note that $\hat{\mathrm{F}}$ is a horizontal 2-form on $\mathcal{M}_5$.

3.1.4 Background fluxes on $\mathcal{M}_5$

Let us consider supersymmetry-preserving background fluxes for gauge fields on $\mathcal{M}_5$. Equivalently, we consider line bundles $L_{\mathcal{M}_5}$ with first Chern class

$$c_1(L_{\mathcal{M}_5}) \in H^2(\mathcal{M}_5, \mathbb{Z}) \ . \tag{3.42}$$

The supersymmetry-preserving line bundles are pull-back of holomorphic line bundles on $\mathcal{M}_4$:

$$L_{\mathcal{M}_5} = \pi^* L_{\mathcal{M}_4} \ . \tag{3.43}$$

Given our assumption that $\mathcal{M}_4$ is simply connected, the Gysin sequence implies the following simple relation between the second cohomologies of $\mathcal{M}_4$ and $\mathcal{M}_5$:

$$H^2(\mathcal{M}_5, \mathbb{Z}) = \mathrm{coker}\Big(c_1(\mathcal{L}_{\mathrm{KK}}) \,:\, H^0(\mathcal{M}_4, \mathbb{Z}) \to H^2(\mathcal{M}_4, \mathbb{Z})\Big) \ . \tag{3.44}$$

[15] Note that the operator ∂_5 does not change the form degree. We denote this way the Lie derivative along K, which is equal to $K^M \partial_M$ on forms because $\iota_K \omega = 0$ for any horizontal form, and moreover $\mathcal{L}_K \eta = 0$ because $K^M = \eta^M$ is a Killing vector.

Let us introduce the notation $\mathfrak{m}$ for the abelian flux on $\mathcal{M}_4$:

$$c_1(L_{\mathcal{M}_4}) = \sum_k \mathfrak{m}_k [S_k]\,, \tag{3.45}$$

as in (3.3).[16] The relation (3.44) means that we can write any five-dimensional flux, denoted by $\mathfrak{m}_{5\mathrm{d}} \in H^2(\mathcal{M}_5, \mathbb{Z})$, as:

$$\mathfrak{m}_{5\mathrm{d}} = \mathfrak{m} \mod \mathfrak{p}\ . \tag{3.46}$$

One important example is the lens space $S^5/\mathbb{Z}_p$ obtained as a degree-p fibration over $\mathbb{P}^2$ (hence $\mathfrak{p} = p$), in which case we have $\mathfrak{m}_{5\mathrm{d}} \in \mathbb{Z}_p$, with $p = 1$ corresponding to the five-sphere.

For future reference, let us introduce the intersection pairing. Given two line bundles L and L' on $\mathcal{M}_4$ with fluxes $\mathfrak{m}$ and $\mathfrak{m}'$, respectively, we define:

$$(\mathfrak{m}, \mathfrak{m}') = \int_{\mathcal{M}_4} c_1(L) \wedge c_1(L') = \sum_{k,l} \mathfrak{m}_k \mathbf{I}_{kl} \mathfrak{m}_l\,, \tag{3.47}$$

with $\mathbf{I}_{kl}$ defined in (3.4).

3.2 The 5d $\mathcal{N} = 1$ vector multiplet on $\mathcal{M}_5$

Let us now consider the simplest supersymmetry multiplets on $\mathcal{M}_5$. (See appendix A.2 for a review of the standard flat-space results.) The 5d vector multiplet contains a gauge field A_M, a real scalar σ, an $SU(2)_R$ doublet of gauginos, Λ_I, transforming as a Majorana-Weyl spinor, and an $SU(2)_R$ triplet of auxiliary scalars D_{IJ}, with the flat-space supersymmetry transformations reviewed in appendix A.2.3. On our curved-space background $\mathcal{M}_5$, the supersymmetry transformations read:

$$\begin{aligned}
\delta A_M &= i\zeta_I \gamma_M \Lambda^I\,,\\
\delta\sigma &= -\zeta_I \Lambda^I\,,\\
\delta\Lambda_I &= -i\Sigma^{MN}\zeta_I\,\left(F_{MN} - i\beta\sigma\hat{\mathtt{F}}_{MN}\right) + i\gamma^M \zeta_I \hat{D}_M \sigma - iD_{IJ}\zeta^J\,,\\
\delta D_{IJ} &= \zeta_I \gamma^M \hat{D}_M \Lambda_J + \zeta_J \gamma^M \hat{D}_M \Lambda_I + \zeta_I[\sigma, \Lambda_J] + \zeta_J[\sigma, \Lambda_I]\ .
\end{aligned} \tag{3.48}$$

Note that the difference from the flat-space algebra arises due to the expression for the field strength, which, when written in terms of the new covariant derivative, reads:

$$F_{MN} = \hat{\nabla}_M A_N - \hat{\nabla}_N A_M - i[A_M, A_N] + \beta\eta^P \hat{\mathtt{F}}_{MN} A_P\ . \tag{3.49}$$

The curved-space supersymmetry algebra on $\mathcal{M}_5$ reads:

$$\delta_1^2 = 0\,, \qquad \delta_2^2 = 0\,, \qquad \{\delta_1, \delta_2\} = -2i\mathcal{L}_K^{(A)} + 2\kappa\delta_{g(\sigma)}\,, \tag{3.50}$$

where $\mathcal{L}_K^{(A)}$ is the gauge-covariant Lie derivative along K^M, κ is defined in (3.25), and $\delta_{g(\sigma)}$ denotes a gauge transformation with parameter σ. The supersymmetry algebra (3.50) reproduces (2.35) upon dimensional reduction along the fiber direction. It is equivalent to the flat-space algebra (A.61) with the derivatives D_M replaced with $\hat{D}_M$.

[16] Assuming that $\mathcal{M}_4$ is simply connected, all 2-cycles in $\mathcal{M}_5$ are inherited from 2-cycles in $\mathcal{M}_4$. More generally, the same would remain true of supersymmetry-preserving fluxes.

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Upon topological twisting, the various fields become differential forms on $\mathcal{M}_5$, exactly like in 4d. We decompose them into $(p,q|\ell)$-forms, following the notation introduced in section 3.1.3. In particular, the decomposition (3.33) holds for the 5d connection $A \equiv A_M dx^M$, and the field-strength 2-form (3.49) decomposes as in (3.35). We then write the supersymmetry variations in terms of the twisted Dolbeault operator (3.37), which need to be further twisted by the gauge fields:

$$\partial_A \equiv \partial - iA^{1,0}\,, \qquad \bar{\partial}_A \equiv \bar{\partial} - iA^{0,1}\,, \qquad \partial_{5,A} \equiv \partial_5 - iA_5\,, \tag{3.51}$$

to preserve gauge covariance. Note that they satisfy:

$$\begin{aligned} \partial_A^2 &= -\,iF^{2,0} \wedge -\beta \hat{\mathrm{F}}^{2,0} \wedge \partial_5\,,\\ \bar{\partial}_A^2 &= -\,iF^{0,2} \wedge -\beta \hat{\mathrm{F}}^{0,2} \wedge \partial_5\,,\\ \{\partial_A, \bar{\partial}_A\} &= -\,iF^{1,1} \wedge -\beta \hat{\mathrm{F}}^{1,1} \wedge \partial_5\,, \end{aligned} \tag{3.52}$$

generalising (2.34). We then have the vector-multiplet supersymmetry variations:

$$\begin{aligned} \delta_1\sigma &= \widetilde{\Lambda}^{0,0}_{(1)}\,, & \delta_2\sigma &= \widetilde{\Lambda}^{0,0}_{(2)}\,,\\ \delta_1 A &= -i\Lambda^{1,0} + i\widetilde{\Lambda}^{0,0}_{(1)}\eta\,, & \delta_2 A &= -i\Lambda^{0,1} + i\widetilde{\Lambda}^{0,0}_{(2)}\eta\,,\\ \delta_1\Lambda^{1,0} &= 0\,, & \delta_2\Lambda^{1,0} &= 2i\partial_A\sigma - 2F^{1,0}\,,\\ \delta_1\Lambda^{0,1} &= 2i\bar{\partial}_A\sigma - 2F^{0,1}\,, & \delta_2\Lambda^{0,1} &= 0\,,\\ \delta_1\widetilde{\Lambda}^{0,0}_{(1)} &= 0\,, & \delta_2\widetilde{\Lambda}^{0,0}_{(1)} &= i\widehat{\mathcal{D}}^{0,0} - i\partial_{5,A}\sigma\,,\\ \delta_1\widetilde{\Lambda}^{0,0}_{(2)} &= -i\widehat{\mathcal{D}}^{0,0} - i\partial_{5,A}\sigma\,, & \delta_2\widetilde{\Lambda}^{0,0}_{(2)} &= 0\,,\\ \delta_1\widetilde{\Lambda}^{2,0} &= \mathcal{D}^{2,0}\,, & \delta_2\widetilde{\Lambda}^{2,0} &= 4(F^{2,0} - i\beta\sigma\hat{\mathrm{F}}^{2,0})\,,\\ \delta_1\widetilde{\Lambda}^{0,2} &= 4(F^{0,2} - i\beta\sigma\hat{\mathrm{F}}^{0,2})\,, & \delta_2\widetilde{\Lambda}^{0,2} &= \mathcal{D}^{0,2}\,,\\ \delta_1\widehat{\mathcal{D}}^{0,0} &= [\sigma, \widetilde{\Lambda}^{0,0}_{(1)}] - \partial_{5,A}\widetilde{\Lambda}^{0,0}_{(1)}\,, & \delta_2\widehat{\mathcal{D}}^{0,0} &= -[\sigma, \widetilde{\Lambda}^{0,0}_{(2)}] + \partial_{5,A}\widetilde{\Lambda}^{0,0}_{(2)}\,, \end{aligned} \tag{3.53}$$

and:

$$\begin{aligned} \delta_1\mathcal{D}^{2,0} &= 0, & \delta_2\mathcal{D}^{2,0} &= 4i\partial_A\Lambda^{1,0} + 2i[\sigma, \widetilde{\Lambda}^{2,0}] - 2i\partial_{5,A}\widetilde{\Lambda}^{2,0},\\ \delta_1\mathcal{D}^{0,2} &= 4i\bar{\partial}_A\Lambda^{0,1} + 2i[\sigma, \widetilde{\Lambda}^{0,2}] - 2i\partial_{5,A}\widetilde{\Lambda}^{0,2}, & \delta_2\mathcal{D}^{0,2} &= 0. \end{aligned} \tag{3.54}$$

The scalar appearing in $\widehat{\mathcal{D}}^{0,0}$ in (3.53) is related to the 4d scalar $\mathcal{D}^{0,0}$ defined in (2.30) by:

$$\widehat{\mathcal{D}}^{0,0} \equiv \mathcal{D}^{0,0} + 2\beta\sigma\hat{\mathrm{F}}^{0,0}\,, \tag{3.55}$$

where $\hat{\mathrm{F}}^{0,0}$ is defined as $\hat{\mathrm{F}}^{0,0} \equiv \frac{1}{4}\Phi^{MN}\hat{\mathrm{F}}_{MN}$. To check the closure of (3.53)–(3.54), it is useful to first compute the supersymmetry variations of the field strength:

$$\begin{aligned} \delta_1 F^{2,0} &= -i\partial_A\Lambda^{1,0} + i\beta\hat{\mathrm{F}}^{2,0}\widetilde{\Lambda}^{0,0}_{(1)}\,, & \delta_2 F^{2,0} &= i\beta\hat{\mathrm{F}}^{2,0}\widetilde{\Lambda}^{0,0}_{(2)}\,,\\ \delta_1 F^{0,2} &= i\beta\hat{\mathrm{F}}^{0,2}\widetilde{\Lambda}^{0,0}_{(1)}\,, & \delta_2 F^{0,2} &= -i\bar{\partial}_A\Lambda^{0,1} + i\beta\hat{\mathrm{F}}^{0,2}\widetilde{\Lambda}^{0,0}_{(2)}\,,\\ \delta_1 F^{1,0} &= i\partial_A\Lambda^{0,0}_{(1)} + i\partial_{5,A}\Lambda^{1,0}\,, & \delta_2 F^{1,0} &= i\partial_A\Lambda^{0,0}_{(2)}\,,\\ \delta_1 F^{0,1} &= i\bar{\partial}_A\Lambda^{0,0}_{(1)}\,, & \delta_2 F^{0,1} &= i\bar{\partial}_A\Lambda^{0,0}_{(2)} + i\partial_{5,A}\Lambda^{0,1}\,. \end{aligned} \tag{3.56}$$

It is then straightforward to check that $\delta_1^2 = 0 = \delta_2^2$, while $\{\delta_1, \delta_2\} f = 2i[\sigma, f] - 2i\partial_{5,A} f$ for any of the fields f in the vector multiplet.[17] The twisted vector multiplet therefore realises the supersymmetry algebra (3.50), namely:

$$\delta_1^2 = 0\,, \qquad \delta_2^2 = 0\,, \qquad \{\delta_1, \delta_2\} = -2i\mathcal{L}_K + 2\delta_{g(\sigma + i\,\iota_K A)}\,, \tag{3.57}$$

where $\mathcal{L}_K$ is the usual Lie derivative along K, $\iota_K A = K^M A_M = A_5$ is the contraction with the vector field K, while $\delta_{g(\epsilon)}$ is the gauge transformation with parameter ϵ introduced in (2.35). Finally, one can check that the 5d $\mathcal{N} = 1$ SYM Lagrangian on $\mathcal{M}_5$ is 'almost' $\mathcal{Q}$-exact, similarly to the 4d Lagrangian (2.36).

3.3 The 5d $\mathcal{N} = 1$ hypermultiplet on $\mathcal{M}_5$

The 5d $\mathcal{N} = 1$ hypermultiplet consists of an $SU(2)_R$ doublet of complex scalar fields q^I and of a Dirac spinor $\Psi, \widetilde{\Psi}$. (See appendix A.2 for our 5d conventions.) As in the case of the 4d $\mathcal{N} = 2$ hypermultiplet, we can realise the two supercharges of the DW twist off-shell by introducing some appropriate auxiliary fields. This is explained in appendix A.2.3 in the case of flat-space supersymmetry. On our curved space background, one simply replaces the derivatives D_M in (A.73)–(A.74) with the torsionfull adapted connection $\hat{D}_M$.

The hypermultiplet can be recast in twisted variables, exactly as in 4d. The field content is formally the same as in (2.54), with the (p, q)-forms being now interpreted as forms on $\mathcal{M}_5$, following the discussion of section 3.1.3. The supersymmetry variations read:

$$\begin{aligned}
\delta_1 Q^{0,0} &= 0\,, & \delta_2 Q^{0,0} &= \sqrt{2}\,\widetilde{\chi}^{0,0}\,,\\
\delta_1 Q^{0,2} &= \sqrt{2}\,\widetilde{\chi}^{0,2}\,, & \delta_2 Q^{0,2} &= 0\,,\\
\delta_1 \eta^{0,1} &= 2i\sqrt{2}\,\bar{\partial}_A Q^{0,0} + h^{0,1}\,, & \delta_2 \eta^{0,1} &= i\sqrt{2} \star \left(\partial_A Q^{0,2}\right)\,,\\
\delta_1 \widetilde{\chi}^{0,0} &= i\sqrt{2}(\sigma - \partial_{5,A}) Q^{0,0}\,, & \delta_2 \widetilde{\chi}^{0,0} &= 0\,,\\
\delta_1 \widetilde{\chi}^{0,2} &= 0\,, & \delta_2 \widetilde{\chi}^{0,2} &= i\sqrt{2}(\sigma - \partial_{5,A}) Q^{0,2}\,,\\
\delta_1 h^{0,1} &= 0\,, & \delta_2 h^{0,1} &= X^{0,1}\,,
\end{aligned} \tag{3.58}$$

and:

$$\begin{aligned}
\delta_1 \widetilde{Q}^{0,0} &= -\sqrt{2}\,\widetilde{\eta}^{0,0}\,, & \delta_2 \widetilde{Q}^{0,0} &= 0\,,\\
\delta_1 \widetilde{Q}^{2,0} &= 0\,, & \delta_2 \widetilde{Q}^{2,0} &= \sqrt{2}\,\widetilde{\eta}^{2,0}\,,\\
\delta_1 \chi^{1,0} &= i\sqrt{2} \star \left(\bar{\partial}_A \widetilde{Q}^{2,0}\right)\,, & \delta_2 \chi^{1,0} &= 2i\sqrt{2}\,\partial_A \widetilde{Q}^{0,0} + \widetilde{h}^{1,0}\,,\\
\delta_1 \widetilde{\eta}^{0,0} &= 0\,, & \delta_2 \widetilde{\eta}^{0,0} &= i\sqrt{2}\left(\widetilde{Q}^{0,0}\sigma + \partial_{5,A} \widetilde{Q}^{0,0}\right)\,,\\
\delta_1 \widetilde{\eta}^{2,0} &= -i\sqrt{2}\left(\widetilde{Q}^{2,0}\sigma + \partial_{5,A} \widetilde{Q}^{2,0}\right)\,, & \delta_2 \widetilde{\eta}^{2,0} &= 0\,,\\
\delta_1 \widetilde{h}^{1,0} &= \widetilde{X}^{1,0}\,, & \delta_2 \widetilde{h}^{1,0} &= 0\,,
\end{aligned} \tag{3.59}$$

[17] This also holds for the field-strength F upon using the Bianchi identity.

where we defined:

$$\begin{aligned}X^{0,1} &\equiv -4i\,\bar\partial_A\widetilde\chi^{0,0} - 2i\star(\partial_A\widetilde\chi^{0,2}) + 2i\sqrt{2}\,\Lambda^{0,1}Q^{0,0} + i\sqrt{2}\star(\Lambda^{1,0}\wedge Q^{0,2})\\&\quad + 2i(\sigma-\partial_{5,A})\eta^{0,1}\,,\\ \widetilde X^{1,0} &\equiv 4i\,\partial_A\widetilde\eta^{0,0} - 2i\star(\bar\partial_A\widetilde\eta^{2,0}) - 2i\sqrt{2}\,\widetilde Q^{0,0}\Lambda^{1,0} - i\sqrt{2}\star(\widetilde Q^{2,0}\wedge\Lambda^{0,1})\\&\quad - 2i\left(\chi^{1,0}\sigma + \partial_{5,A}\chi^{1,0}\right)\,.\end{aligned}\tag{3.60}$$

This naturally generalises (2.55)–(2.56). Note that the four-dimensional scalar is replaced by a differential operator:

$$\phi \to \frac{1}{\sqrt{2}}(\sigma \mp \partial_{5,A})\,,\tag{3.61}$$

when acting on a field in the representation $\mathfrak{R}$ or $\bar{\mathfrak{R}}$, respectively. Let us also point out that the Hodge star operator used above is in fact the Hodge dual on $\mathcal{M}_4$, obtained from the 5d Hodge dual by the contraction $\star \equiv \iota_K \star_5$. One can easily check that the supersymmetry algebra (3.57) is satisfied. The kinetic Lagrangian is again $\mathcal{Q}$-exact. The five-dimensional uplift of (2.61) reads:

$$\begin{aligned}\mathscr{L}_H = \frac{1}{4}\star(\delta_1+\delta_2)\bigg(&\widetilde h^{1,0}\wedge\star\eta^{0,1} - 2i\sqrt{2}\,\chi^{1,0}\wedge\star\bar\partial_A Q^{0,0} + i\sqrt{2}\,\chi^{1,0}\wedge\partial_A Q^{0,2}\\&-\frac{i\sqrt{2}}{8}\,\widetilde\eta^{0,0}(\sigma+\partial_{5,A})\,Q^{0,0}\,\mathrm{dvol} + \frac{i\sqrt{2}}{2}\,\widetilde\eta^{2,0}\wedge(\sigma+\partial_{5,A})\,Q^{0,2} - i\,\widetilde Q^{2,0}\wedge\widetilde\Lambda^{0,2}\,Q^{0,0}\\&+\frac{i}{4}\,\widetilde Q^{0,0}\,\widetilde\Lambda^{0,0}_{(1)}\,Q^{0,0}\,\mathrm{dvol} + i\,\widetilde Q^{2,0}\wedge\widetilde\Lambda^{0,0}_{(2)}\,Q^{0,2} + i\,\widetilde Q^{0,0}\,\widetilde\Lambda^{2,0}\wedge Q^{0,2}\bigg)\,.\end{aligned}\tag{3.62}$$

4 One-loop determinants on $\mathcal{M}_5$

In this section, we compute one-loop determinants on $\mathcal{M}_5$. We start by considering the contribution of a free hypermultiplet in 4d, before obtaining the 5d result by summing over the KK modes. We then generalise this result to derive the one-loop contribution of any 5d BPS particle running along the circle fiber.

4.1 Free hypermultiplet on $\mathcal{M}_4$

Consider the 4d $\mathcal{N}=2$ hypermultiplet on $\mathcal{M}_4$ with the extended topological twist discussed in section 2.3. We couple the hypermultiplet to a background vector multiplet, which should preserve our two supercharges on $\mathcal{M}_4$. Hence, we impose:

$$\mathcal{D}^{2,0}=0\,,\qquad \mathcal{D}^{0,2}=0,\qquad F^{0,2}=0\,,\qquad F^{2,0}=0\,,\tag{4.1}$$

and:

$$\partial_A\phi=0\,,\qquad \bar\partial_A\phi=0\,,\qquad \mathcal{D}^{0,0}=0\,,\qquad [\widetilde\phi,\phi]=0\,.\tag{4.2}$$

These conditions are obtained by imposing that the gaugino variations in (2.32) vanish. Note that, in particular, the background gauge field must be the connection of a holomorphic vector bundle. Given the hypermultiplet Lagrangian (2.61), let us extract the quadratic fermionic and bosonic fluctuations around this supersymmetric background. They can be expressed in the following compact form:

$$\mathscr{L}_{\rm bos} = \star\left(\widetilde{Q}\wedge\star\Delta_{\rm bos}Q\right)\,, \qquad \mathscr{L}_{\rm fer} = \star\left(\widetilde{\Psi}\wedge\star\Delta_{\rm fer}\Psi\right)\,. \tag{4.3}$$

Here the bosonic fields are written as $\widetilde{Q} = (\widetilde{Q}^{0,0}\,, \widetilde{Q}^{2,0})$ and $Q = (Q^{0,0}\,, Q^{0,2})^T$, while the fermionic fields are given by $\widetilde{\Psi} = (\chi^{1,0}\,, \widetilde{\eta}^{\,0,0}\,, \widetilde{\eta}^{\,2,0})$ and $\Psi = (\eta^{0,1}\,, \widetilde{\chi}^{\,0,0}\,, \widetilde{\chi}^{\,0,2})^T$. The bosonic kinetic operator reads:

$$\Delta_{\rm bos} = \begin{pmatrix} -2\star\partial_A\star\bar{\partial}_A + 2\phi\widetilde{\phi} & 0 \\ 0 & -\frac{1}{2}\star\bar{\partial}_A\star\partial_A + \frac{1}{2}\phi\widetilde{\phi}\end{pmatrix}\,, \tag{4.4}$$

and the fermionic operator takes the form

$$\Delta_{\rm fer} = \begin{pmatrix} -\frac{i\sqrt{2}}{2}\,\phi & i\,\bar{\partial}_A & \frac{i}{2}\star\partial_A \\ -\,i\star\partial_A\star & i\sqrt{2}\,\widetilde{\phi} & 0 \\ -\frac{i}{2}\star\bar{\partial}_A & 0 & -\,\frac{i\sqrt{2}}{4}\,\widetilde{\phi}\end{pmatrix}\,. \tag{4.5}$$

(Note that $\star^2 = \pm 1$, depending on whether the form is of even or odd degree.) The corresponding Gaussian integration will give the exact answer for the hypermultiplet in this supersymmetric background, by a standard scaling argument, because the kinetic Lagrangian (2.61) is $\mathcal{Q}$-exact. The partition function of the 4d hypermultiplet on $\mathcal{M}_4$ is then given by:

$$Z^{\mathcal{H}}_{\mathcal{M}_4} = \frac{\det\Delta_{\rm fer}}{\det\Delta_{\rm bos}}\,. \tag{4.6}$$

Supersymmetry leads to huge cancellations between fermionic and bosonic fluctuations, so that (4.6) can be further simplified. With the DW twist on $\mathcal{M}_4$, the boson-fermion degeneracy is almost perfect — we discuss the pairing more explicitly in appendix B.1. Here, we follow a standard argument (see e.g. [4, 91]) which gives us the final answer almost immediately. First, we note that the bosonic and fermionic kinetic operators are related by:

$$\Delta_{\rm fer}\begin{pmatrix} 1 & -2i\bar{\partial}_A & i\star\partial_A \\ 0 & -i\sqrt{2}\phi & 0 \\ 0 & 0 & i\sqrt{2}\phi\end{pmatrix} = \begin{pmatrix} -\frac{i\sqrt{2}}{2}\phi & 0 & 0 \\ -i\star\partial_A\star & \multicolumn{2}{c}{\multirow{2}{*}{$\Delta_{\rm bos}$}} \\ \frac{i}{2}\star\bar{\partial}_A & & \end{pmatrix}\,, \tag{4.7}$$

which is a consequence of supersymmetry. It will be convenient to introduce the quantity

$$\mathbb{L} = -i\sqrt{2}\phi\,, \tag{4.8}$$

viewed as an operator on our bosonic and fermionic $\mathfrak{R}$-valued forms. Taking the determinant of both sides of (4.7), we obtain:

$$Z^{\mathcal{H}}_{\mathcal{M}_4} = \frac{\det\Delta_{\rm fer}}{\det\Delta_{\rm bos}} = \frac{\det(\mathbb{L}^{0,1})}{\det(\mathbb{L}^{0,0})\det(\mathbb{L}^{0,2})}\,, \tag{4.9}$$

with the superscript indicating the type of (p,q)-forms that $\mathbb{L} : \Omega^{p,q} \to \Omega^{p,q}$ acts upon. Now, recall that the Dolbeault operators $\bar{\partial} : \Omega^{p,q-1} \to \Omega^{p,q}$ and $\partial : \Omega^{p-1,q} \to \Omega^{p,q}$ have adjoints $\bar{\partial}^* : \Omega^{p,q} \to \Omega^{p,q-1}$ and $\partial^* : \Omega^{p,q} \to \Omega^{p-1,q}$, respectively, defined as:

$$\bar{\partial}^* = -\star\partial\star\,, \qquad \partial^* = -\star\bar{\partial}\star\,, \tag{4.10}$$

and similarly for the gauge-covariant generalisation. Then, one can check that:

$$\begin{aligned} \ker(\bar{\partial}_A) &= \ker(\star\partial_A \star \bar{\partial}_A)\,, \quad \ker(\bar{\partial}_A^*) = \ker(\bar{\partial}_A \star \partial_A \star)\,, \\ \ker(\partial_A) &= \ker(\star\bar{\partial}_A \star \partial_A)\,, \quad \ker(\partial_A^*) = \ker(\partial_A \star \bar{\partial}_A \star)\ . \end{aligned} \tag{4.11}$$

As a result, the non-zero eigenvalues of $\bar{\partial}_A^*\bar{\partial}_A$ and those of $\bar{\partial}_A\bar{\partial}_A^*$ are in one-to-one correspondence, and similarly for the operators in the second line. Note that these operators do not change the degree of the differential form they act upon. Moreover, they clearly commute with the operators $\mathbb{L}$ introduced above and, as a result, the eigenvalues of $\mathbb{L}$ that lie outside these kernels will cancel in the one-loop determinant. Thus, we have:

$$Z^{\mathcal{H}}_{\mathcal{M}_4} = \frac{\det_{\ker(\bar{\partial}_A^*)\oplus\ker(\bar{\partial}_A)}(\mathbb{L}^{0,1})}{\det_{\ker(\bar{\partial}_A)}(\mathbb{L}^{0,0})\det_{\ker(\bar{\partial}_A^*)}(\mathbb{L}^{0,2})}\ . \tag{4.12}$$

That is, we are restricting attention to the zero modes:

$$\bar{\partial}_A Q^{0,0} = 0\,, \qquad \bar{\partial}_A^*(\star Q^{0,2}) = 0\,, \qquad \bar{\partial}_A^*\eta^{0,1} = 0\,, \qquad \bar{\partial}_A\eta^{0,1} = 0\,, \tag{4.13}$$

using the fact that $\star : \Omega^{0,2} \to \Omega^{0,2}$. Up to an irrelevant numerical factor, we find that:

$$Z^{\mathcal{H}}_{\mathcal{M}_4} = \left(-i\sqrt{2}\phi\right)^{-\mathcal{I}}\,, \tag{4.14}$$

with $\mathcal{I}$ the net number of zero-modes (4.13) contributing. It is given by:

$$\begin{aligned} \mathcal{I} = {} & \dim\ker(\bar{\partial}_A : E^{0,0} \to E^{0,1}) + \dim\ker(\bar{\partial}_A^* : E^{0,2} \to E^{0,1}) \\ & - \dim\ker(\bar{\partial}_A^* : E^{0,1} \to E^{0,0}) - \dim\ker(\bar{\partial}_A : E^{0,1} \to E^{0,2})\,, \end{aligned} \tag{4.15}$$

where we denoted by $E^{0,q} \equiv \Omega^{0,q}\otimes E$ the space of $(0,q)$-forms valued in the gauge bundle E with connection A. Let $\mathrm{ind}(\bar{\partial}_A)$ denote the index of the Dolbeault complex twisted by E:

$$0 \longrightarrow \Omega^{0,0}\otimes E \xrightarrow{\bar{\partial}_A} \Omega^{0,1}\otimes E \xrightarrow{\bar{\partial}_A} \Omega^{0,2}\otimes E \longrightarrow 0\ . \tag{4.16}$$

Formally, one finds:

$$\mathcal{I} = \mathrm{ind}(\bar{\partial}_A) - \dim(\Omega^{0,1}\otimes E)\ . \tag{4.17}$$

By the assumption that $\mathcal{M}_4$ is simply connected, however, we have $\dim(\Omega^{0,1}\otimes E) = 0$ and thus $\mathcal{I} = \mathrm{ind}(\bar{\partial}_A)$. In summary, the one-loop determinant of the hypermultiplet on a simply-connected Kähler manifold reads:

$$Z^{\mathcal{H}}_{\mathcal{M}_4} = \left(-i\sqrt{2}\phi\right)^{-\mathrm{ind}(\bar{\partial}_A)}\ . \tag{4.18}$$

Some useful facts about the twisted Dolbeault complex index are reviewed in appendix B.3.

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4.2 Five-dimensional uplift of the hypermultiplet

Let us now compute the partition function of a charged 5d $\mathcal{N}=1$ hypermultiplet on $\mathcal{M}_5$. The simplest way to do this is to expand the 5d fields into 4d modes, by KK reduction along the S^1 fiber, and use the 4d result (4.18) for the 4d modes of fixed KK charge. Due to the non-trivial fibration structure on $\mathcal{M}_5$, the supersymmetric background for the 5d vector multiplet is slightly more complicated than the 4d background (4.1)–(4.2). We have the conditions:

$$\mathcal{D}^{2,0}=0\,,\qquad \mathcal{D}^{0,2}=0\,,\qquad F^{2,0}=i\beta\sigma\hat{\mathrm{F}}^{2,0}\,,\qquad F^{0,2}=i\beta\sigma\hat{\mathrm{F}}^{0,2}\,,\tag{4.19}$$

together with:

$$i\partial_A\sigma=F^{1,0}\,,\qquad i\bar{\partial}_A\sigma=F^{0,1}\,,\qquad \widehat{\mathcal{D}}^{0,0}=\partial_{5,A}\sigma=0\,,\tag{4.20}$$

in terms of the 5d twisted Dolbeault operators. On the four-dimensional supersymmetric background (4.1), the condition (4.1) implied $\partial_A^2=0$ and $\bar{\partial}_A^2=0$, which simplified the computation. This is no longer the case on $\mathcal{M}_5$ if the fibration is non-trivial. The kinetic operators for the bosonic and fermionic fluctuations around the 5d background read:

$$\begin{aligned}\Delta_{\rm bos}&=\begin{pmatrix}-2\star\partial_A\star\bar{\partial}_A+(\sigma\sigma-\partial_{5,A}\partial_{5,A}) & *\partial_A\partial_A\\ -\star\bar{\partial}_A\bar{\partial}_A & -\frac{1}{2}\star\bar{\partial}_A\star\partial_A+\frac{1}{4}(\sigma\sigma-\partial_{5,A}\partial_{5,A})\end{pmatrix},\\ \Delta_{\rm fer}&=\begin{pmatrix}-\frac{i}{2}(\sigma-\partial_{5,A}) & i\,\bar{\partial}_A & \frac{i}{2}\star\partial_A\\ -i\star\partial_A\star & i\,(\sigma+\partial_{5,A}) & 0\\ -\frac{i}{2}\star\bar{\partial}_A & 0 & -\frac{i}{4}(\sigma+\partial_{5,A})\end{pmatrix}.\end{aligned}\tag{4.21}$$

They are related as follows:

$$\Delta_{\rm fer}\begin{pmatrix}1 & -2i\bar{\partial}_A & i\star\partial_A\\ 0 & -i\,(\sigma-\partial_{5,A}) & 0\\ 0 & 0 & i\,(\sigma-\partial_{5,A})\end{pmatrix}=\begin{pmatrix}-\frac{i}{2}(\sigma-\partial_{5,A}) & 0 \quad 0\\ -i\star\partial_A\star & \\ \frac{i}{2}\star\bar{\partial}_A & \Delta_{\rm bos}\end{pmatrix}.\tag{4.22}$$

As a result, the one-loop determinant reduces to:

$$Z^{\mathcal{H}}_{\mathcal{M}_5}=\frac{\det(\Delta_{\rm fer})}{\det(\Delta_{\rm bos})}=\frac{\det(\mathbb{L}^{(0,1)})}{\det(\mathbb{L}^{(0,0)})\det(\mathbb{L}^{(0,2)})}\,,\qquad \mathbb{L}=i\,(\sigma-\partial_{5,A})\,,\tag{4.23}$$

giving us the five-dimensional uplift of (4.9). Using a similar argument to the one given above, the only modes that contribute to (4.23) are the zero modes of the twisted Dolbeault operator on $\mathcal{M}_5$. To evaluate this explicitly, we expand the 5d fields φ into 4d KK modes, as a Fourier decomposition along the S^1 fiber:

$$\varphi(z,\bar{z},\psi)=\sum_{n\in\mathbb{Z}}\varphi_{(n)}(z,\bar{z},\psi)\,,\qquad \varphi_{(n)}(z,\bar{z},\psi)\equiv e^{-in\psi}\varphi_n(z,\bar{z})\,.\tag{4.24}$$

We then have:

$$\mathbb{L}\varphi_n=\lambda_n\varphi_n\,,\qquad \lambda_n=i\beta(\sigma+iA_5)+n\,.\tag{4.25}$$

From the 4d point of view, each KK mode $\varphi^{p,q}$ of a given (p,q)-degree is a section of a bundle

$$\Omega^{p,q} \otimes V_n \,, \qquad\qquad V_n = E_{\mathfrak{R}} \otimes (\mathcal{L}_{\mathrm{KK}})^n \,, \tag{4.26}$$

where $E_{\mathfrak{R}}$ is the gauge bundle and $\mathcal{L}_{\mathrm{KK}}$ is the defining line bundle introduced in section 3.1. The 5d hypermultiplet partition function then takes the form:

$$Z^{\mathcal{H}}_{\mathcal{M}_5} = \prod_{n\in\mathbb{Z}} \lambda_n^{-\mathrm{ind}(\bar\partial_{V_n})} \,, \tag{4.27}$$

where $\mathrm{ind}(\bar\partial_{V_n})$ is the index of the 4d Dolbeault complex twisted by the vector bundle V_n. This infinite product needs to be properly regularised, as we discuss next.

4.3 Regularisation: summing up the KK tower

For our purposes, we will only consider abelian gauge bundles, by choosing a maximal torus of the (background or dynamical) gauge group. Then, without loss of generality, we can consider the hypermultiplet coupled to a single $U(1)$ gauge field with background flux $\mathfrak{m}$, as discussed in section 3.1.4. The complex scalar in the effective 4d $\mathcal{N}=2$ vector multiplet is denoted by

$$a \equiv i\beta(\sigma + iA_5) \,, \tag{4.28}$$

with the identification $a \sim a+1$ under a $U(1)$ large gauge transformation. The 5d hypermultiplet partition function is given formally by the infinite product:

$$Z^{\mathcal{H}}_{\mathcal{M}_5} = \prod_{n\in\mathbb{Z}} \left(\frac{1}{a+n}\right)^{\mathrm{ind}(\bar\partial_{V_{n,\varepsilon}})} \,, \tag{4.29}$$

in terms of the index of the $V_{n,\varepsilon}$-twisted Dolbeault complex. Here, we take $V_{n,\varepsilon}$ to be the line bundle:

$$V_{n,\varepsilon} \cong \mathcal{K}^{\varepsilon+\frac{1}{2}} \otimes L \otimes (\mathcal{L}_{\mathrm{KK}})^n \,, \tag{4.30}$$

where the L connection is the background $U(1)$ gauge field with flux $\mathfrak{m}$, and ε indexes our choice of extended twist for the hypermultiplet, as discussed around (2.44). The canonical choice on a generic Kähler base $\mathcal{M}_4$ is $\varepsilon = -\frac{1}{2}$, while if $\mathcal{M}_4$ is spin it is also natural to choose $\varepsilon = 0$. Note that:

$$c_1(V_{n,\varepsilon}) = \sum_l \left(\left(\varepsilon + \frac{1}{2}\right) \mathbf{k}_l + \mathfrak{m}_l + n\,\mathfrak{p}_l\right) [\mathrm{S}_l] \,, \tag{4.31}$$

where $\mathbf{k}$ denotes the first Chern class of the canonical line bundle on $\mathcal{M}_4$,

$$c_1(\mathcal{K}) = \sum_l \mathbf{k}_l [S_l] \,, \tag{4.32}$$

and $\mathfrak{p}$ was defined in (3.3). For simplicity of notation, we can absorb the $(\varepsilon+\frac{1}{2})\mathbf{k}$ term into $\mathfrak{m}$, effectively setting $\varepsilon = -\frac{1}{2}$ in what follows. Using the index theorem (see appendix B.3), we find:

$$\mathrm{ind}(\bar\partial_{V_n}) = \int_{\mathcal{M}_4} \mathrm{Td}(T\mathcal{M}_4) \wedge \mathrm{ch}(V_n) = \chi_h + \frac{1}{2}(\mathfrak{m} + n\,\mathfrak{p} - \mathbf{k}, \mathfrak{m} + n\,\mathfrak{p}) \,, \tag{4.33}$$

with $\chi_h = \frac{\chi+\sigma}{4}$ the holomorphic Euler characteristic, and with the intersection pairing $(-,-)$ on $\mathcal{M}_4$ as defined in (3.47).

Regularisation of the result. Given (4.33), the infinite product to be regularised takes the explicit form:

$$Z^{\mathcal{H}}_{\mathcal{M}_5}(a)_{\mathfrak{m}} = \prod_{n\in\mathbb{Z}} \left(\frac{1}{a+n}\right)^{\chi_h+\frac{1}{2}(\mathfrak{m}+n\,\mathfrak{p}-\mathbf{k},\mathfrak{m}+n\,\mathfrak{p})} . \tag{4.34}$$

The notation $Z^{\mathcal{H}}_{\mathcal{M}_5}(a)_{\mathfrak{m}}$ makes the dependence on a and $\mathfrak{m}$ manifest. It is convenient to factor (4.34) as follows:

$$Z^{\mathcal{H}}_{\mathcal{M}_5}(a)_{\mathfrak{m}} = \mathbf{\Pi}^{\mathcal{H}}(a)^{\chi_h+\frac{1}{2}(\mathfrak{m}-\mathbf{k},\mathfrak{m})}\, \mathscr{K}^{\mathcal{H}}(a)^{(\mathfrak{m}-\frac{1}{2}\mathbf{k},\mathfrak{p})}\, \mathscr{F}^{\mathcal{H}}(a)^{\frac{1}{2}(\mathfrak{p},\mathfrak{p})} . \tag{4.35}$$

Here, we formally defined the following functions in terms of divergent products:

$$\mathbf{\Pi}^{\mathcal{H}}(a) \equiv \prod_{n\in\mathbb{Z}} \frac{1}{a+n} , \qquad \mathscr{K}^{\mathcal{H}}(a) \equiv \prod_{n\in\mathbb{Z}} \left(\frac{1}{a+n}\right)^{n} , \qquad \mathscr{F}^{\mathcal{H}}(a) \equiv \prod_{n\in\mathbb{Z}} \left(\frac{1}{a+n}\right)^{n^2} . \tag{4.36}$$

These formal products give us information on the analytic structure of the corresponding meromorphic functions, with poles or zeros at $a \in \mathbb{Z}$. Namely, $\mathbf{\Pi}^{\mathcal{H}}$ has poles of order 1 at any integer $a \in \mathbb{Z}$, $\mathscr{K}^{\mathcal{H}}$ has poles of order n at $a = n$ for every negative integer n (and zeros at the positive integers), and $\mathscr{F}^{\mathcal{H}}$ has poles of order n^2 at $a = n$ for any integer n. Following the discussion in [8, 12, 59], we choose the gauge-invariant regularisation, also known as the '$U(1)_{-\frac{1}{2}}$ quantisation'. We then find:

$$\begin{aligned}
\mathbf{\Pi}^{\mathcal{H}}(a) &= \frac{1}{1-e^{2\pi i a}} ,\\
\mathscr{K}^{\mathcal{H}}(a) &= \exp\left(\frac{1}{2\pi i}\mathrm{Li}_2(e^{2\pi i a}) + a\log(1-e^{2\pi i a})\right) ,\\
\mathscr{F}^{\mathcal{H}}(a) &= \exp\left(-\frac{1}{2\pi^2}\mathrm{Li}_3(e^{2\pi i a}) - \frac{a}{\pi i}\mathrm{Li}_2(e^{2\pi i a}) - a^2\log(1-e^{2\pi i a})\right) .
\end{aligned} \tag{4.37}$$

Despite the appearance of polylogarithms, these functions are meromorphic in a, with the poles mentioned above. They also have simple transformation properties under large gauge transformations, $a \sim a+1$, with $\mathbf{\Pi}^{\mathcal{H}}(a+1) = \mathbf{\Pi}^{\mathcal{H}}(a)$ and

$$\mathscr{K}^{\mathcal{H}}(a+1) = \mathbf{\Pi}^{\mathcal{H}}(a)^{-1}\mathscr{K}^{\mathcal{H}}(a) , \qquad \mathscr{F}^{\mathcal{H}}(a+1) = \mathbf{\Pi}^{\mathcal{H}}(a)\mathscr{K}^{\mathcal{H}}(a)^{-2}\mathscr{F}^{\mathcal{H}}(a) . \tag{4.38}$$

Using these relations, we can check that the partition function is gauge invariant. Whenever the circle is non-trivially fibered over $\mathcal{M}_4$, a large gauge transformation amounts to the simultaneous shift $(a, \mathfrak{m}) \to (a+1, \mathfrak{m}+\mathfrak{p})$. More invariantly, this corresponds to tensoring the $U(1)$ line bundle with the defining line bundle, $L \to L \otimes \mathcal{L}_{\mathrm{KK}}$. We indeed find that:

$$Z^{\mathcal{H}}_{\mathcal{M}_5}(a+1)_{\mathfrak{m}+\mathfrak{p}} = Z^{\mathcal{H}}_{\mathcal{M}_5}(a)_{\mathfrak{m}} , \tag{4.39}$$

as expected.

Example: trivial fibrations. Let us first consider the case $\mathcal{M}_5 = \mathcal{M}_4 \times S^1$. Since $\mathfrak{p} = 0$, the partition function takes the simple form:

$$Z^{\mathcal{H}}_{\mathcal{M}_4\times S^1}(a)_{\mathfrak{m}} = \mathbf{\Pi}^{\mathcal{H}}(a)^{\chi_h+\frac{1}{2}(\mathfrak{m}-\mathbf{k},\mathfrak{m})} = \left(\frac{1}{1-e^{2\pi i a}}\right)^{\chi_h+\frac{1}{2}(\mathfrak{m}-\mathbf{k},\mathfrak{m})} , \tag{4.40}$$

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for $\varepsilon = -\frac{1}{2}$. This agrees with previous results [23], up to some differences in conventions.[18] For a more general choice of extended DW twist, we find:

$$Z^{\mathcal{H}}_{\mathcal{M}_4\times S^1}(a;\varepsilon)_{\mathfrak{m}} = \left(\frac{1}{1-e^{2\pi i a}}\right)^{-\frac{\sigma}{8}+\frac{\varepsilon^2}{2}(2\chi+3\sigma)+\frac{1}{2}(\mathfrak{m}+2\varepsilon\mathbf{k},\mathfrak{m})}\,, \tag{4.41}$$

where we used the relation $(\mathbf{k},\mathbf{k}) = 2\chi+3\sigma$. (See appendix B.3.)

Example: the five-sphere S^5. Let us now consider the example of S^5, fibered over $\mathbb{P}^2$, setting $\varepsilon = -\frac{1}{2}$ for simplicity. We have $\chi = 3$, $\sigma = 1$ and $\mathbf{k} = -3$, and thus (4.40) gives us:

$$Z^{\mathcal{H}}_{\mathbb{P}^2\times S^1}(a)_{\mathfrak{m}} = \left(\frac{1}{1-e^{2\pi i a}}\right)^{1+\frac{1}{2}\mathfrak{m}(\mathfrak{m}+3)}\,, \tag{4.42}$$

for any flux $\mathfrak{m}\in\mathbb{Z}$. The S^5 is obtained by a fibration with $\mathfrak{p}=1$, so that:

$$Z^{\mathcal{H}}_{\mathcal{M}_5}(a)_{\mathfrak{m}} = Z^{\mathcal{H}}_{\mathbb{P}^2\times S^1}(a)_{\mathfrak{m}}\, \mathscr{K}^{\mathcal{H}}(a)^{\mathfrak{m}-\frac{3}{2}}\, \mathscr{F}^{\mathcal{H}}(a)^{\frac{1}{2}}\,, \tag{4.43}$$

and we can set $\mathfrak{m}=0$ by a large gauge transformation (4.39), reflecting the fact that $H^2(S^5)$=0. Hence we find:

$$Z^{\mathcal{H}}_{\mathcal{M}_5}(a) = \exp\left(-\frac{1}{4\pi^2}\mathrm{Li}_3(e^{2\pi i a}) - \frac{2a-3}{4\pi i}\mathrm{Li}_2(e^{2\pi i a}) - \frac{a^2-3a+2}{2}\log(1-e^{2\pi i a})\right)\,. \tag{4.44}$$

This is in good agreement with previous results [46].

Example: the five-manifold T^{p_1,p_2}. As another example, consider the fibration over $\mathcal{M}_4 = \mathbb{F}_0 = \mathbb{P}^1\times\mathbb{P}^1$, with $\mathfrak{p} = (p_1,p_2)$, which is sometimes called T^{p_1,p_2}.[19] (We take the two $\mathbb{P}^1$ factors as our basis curves, $\mathbb{F}_0 \cong \mathrm{S}_1\times\mathrm{S}_2$.) Then we have $\chi = 4$, $\sigma = 0$ and $\mathbf{k} = (-2,-2)$, hence:

$$Z^{\mathcal{H}}_{\mathbb{P}^2\times S^1}(a)_{\mathfrak{m}} = \left(\frac{1}{1-e^{2\pi i a}}\right)^{4\varepsilon^2-2\varepsilon\mathfrak{m}_1\mathfrak{m}_2+\mathfrak{m}_1\mathfrak{m}_2}\,, \tag{4.45}$$

for any flux $\mathfrak{m} = (\mathfrak{m}_1,\mathfrak{m}_2)$, and keeping an arbitrary ε. (Since $\mathbb{F}_0$ is spin, we can choose it as we like, including choosing the DW twist value $\varepsilon = 0$.) We then have:

$$Z^{\mathcal{H}}_{T^{p_1,p_2}}(a)_{\mathfrak{m}} = Z^{\mathcal{H}}_{\mathbb{P}^2\times S^1}(a)_{\mathfrak{m}}\, \mathscr{K}^{\mathcal{H}}(a)^{(p_1\mathfrak{m}_2+p_2\mathfrak{m}_1)-2\varepsilon(\mathfrak{m}_1+\mathfrak{m}_2)}\, \mathscr{F}^{\mathcal{H}}(a)^{p_1p_2}\,. \tag{4.46}$$

For $T^{1,1}$ with $\varepsilon = 0$, for instance, this gives:

$$\begin{aligned} Z^{\mathcal{H}}_{T^{1,1}}(a)_{\mathfrak{m}} \;=\; \exp\Bigg(&-\frac{1}{2\pi^2}\mathrm{Li}_3(e^{2\pi i a}) - \frac{2a-\mathfrak{m}_1-\mathfrak{m}_2}{4\pi i}\mathrm{Li}_2(e^{2\pi i a}) \\ &- (a^2-(\mathfrak{m}_1+\mathfrak{m}_2)a-\mathfrak{m}_1\mathfrak{m}_2)\log(1-e^{2\pi i a})\Bigg)\,. \end{aligned} \tag{4.47}$$

Note that we have the gauge equivalence $(a,\mathfrak{m}_1,\mathfrak{m}_2)\sim(a+1,\mathfrak{m}_1+1,\mathfrak{m}_2+1)$. Using (3.44), one can check that $H^2(T^{1,1},\mathbb{Z})\cong\mathbb{Z}$.

[18] The most important difference is in the choice of regularisation. As emphasised in [59], our choice is singled out by requiring gauge invariance under large gauge transformations.

[19] In particular, for $p_1 = p_2 = 1$, $T^{1,1}$ famously admits a Sasaki-Einstein metric [92].

4.4 Higher-spin particles on $\mathcal{M}_5$

By a small generalisation of the above computation, one can also capture the contribution of higher-spin states. Such electrically-charged states generally appear on the real Coulomb branch of 5d SCFTs. For instance, when we have an infrared non-abelian gauge theory phase, the W-bosons give spin-one states. More generally, 5d BPS particles of arbitrary spin can contribute. Following the approach of [49, 55, 56], we expect that, in the topologically-twisted theory, they contribute to the partition function on the Coulomb branch as KK towers of 4d off-shell hypermultiplets of $SU(2)_l \times SU(2)_r$ spin (j_l, j_r). In the 5d interpretation, (j_l, j_r) is the representation under the little group of the massive particle.

Let us first recall some elementary properties of the half-BPS massive representations of the 5d $\mathcal{N} = 1$ supersymmetry algebra. The BPS states saturate the BPS mass bound with $M = Z_{\rm 5d}$, where we take the fifth direction to be time, with $P_M = (0, 0, 0, 0, -M)$. Such states are annihilated by the 'right-chiral' supercharges (in the 4d notation), and the supersymmetry algebra (A.51), after Wick rotation, is realised as:

$$\{Q^I_\alpha, Q^J_\beta\} = -4iM\epsilon^{IJ}\epsilon_{\alpha\beta}\,, \qquad\qquad \widetilde{Q}^{\dot\alpha}_I = 0\,. \tag{4.48}$$

Picking the supercharges $Q^{I=1}_\alpha$ and $Q^{I=2}_\alpha$ (of R-charge $R = \pm 1$, under $U(1)_R \subset SU(2)_R$, respectively) as the creation and annihilation operators, we obtain the supermultiplet:

$$\left(j_l, j_r; \frac{1}{2}\right)^{(-1)^{2j_l+2j_r}} \oplus \left(j_l + \frac{1}{2}, j_r; 0\right)^{-(-1)^{2j_l+2j_r}} \oplus \left(j_l - \frac{1}{2}, j_r; 0\right)^{-(-1)^{2j_l+2j_r}}\,, \tag{4.49}$$

for any spin (j_l, j_r) for the 'ground state' — this is for $j_l > 0$, while for $j_l = 0$ there is no third summand in (4.49). Here $(j_l, j_r, s)^\pm$ denotes a 5d massive state of spin (j_l, j_r) and of $SU(2)_R$ 'isospin' s, with the superscript $\pm$ corresponding to bosons and fermions, respectively. The statistics is determined by the spin-statistics theorem.

Now, consider the standard DW twist of the multiplet (4.49). We obtain states of twisted $SU(2)_l \times SU(2)_D$ spins:

$$\left(j_l, j_r \pm \frac{1}{2}\right)^{(-1)^{2j_l+2j_r}} \oplus \left(j_l \pm \frac{1}{2}, j_r\right)^{-(-1)^{2j_l+2j_r}}\,, \tag{4.50}$$

which are most conveniently written as:

$$\left[\left(0, \frac{1}{2}\right)^{(-1)^{2j_l+2j_r}} \oplus \left(\frac{1}{2}, 0\right)^{-(-1)^{2j_l+2j_r}}\right] \otimes (j_l, j_r)\,. \tag{4.51}$$

The states in the bracket give us a standard massive hypermultiplet in the twisted theory (up to a choice of statistics), and we simply need to tensor by the general spin (j_l, j_r).

For $(j_l, j_r) = (0, 0)$, we recover the standard hypermultiplet. As a first non-trivial example, it is interesting to consider the massive vector multiplet after the DW twist. Such massive vectors appear as W-bosons through the Higgs mechanism on the CB, for instance. In this case, we can compute their contribution explicitly, as a one-loop computation, by considering the gauge-fixed SYM Lagrangian. This is discussed in some detail in appendix B.2. After the topological twist, we have the multiplet:

$$\left[\left(0,\frac{1}{2}\right)^{-}\oplus\left(\frac{1}{2},0\right)^{+}\right]\otimes\left(0,\frac{1}{2}\right)=(0,1)^{-}\oplus(0,0)^{-}\oplus\left(\frac{1}{2},\frac{1}{2}\right)^{+}\,,\tag{4.52}$$

corresponding to the on-shell gauginos and the massive vector, respectively. Thus the massive vector multiplet corresponds to $(j_l,j_r)=(0,\frac{1}{2})$.

Extended topological twist at higher spin. Consider a massive particle of spin (j_l,j_r) charged under some abelian gauge symmetry $\prod_K U(1)_K$ with charges $q_K\in\mathbb{Z}$, after the standard DW twist. The corresponding KK tower of fields on $\mathcal{M}_4$ are valued in the bundles:

$$[S_-\oplus S_+]\otimes\mathcal{K}^{\varepsilon}\otimes\bigotimes_K(L_K)^{q_K}\otimes S^{2j_l}(S_-)\otimes S^{2j_r}(S_+)\otimes(\mathcal{L}_{\rm KK})^n\,,\tag{4.53}$$

where L_K are $U(1)_K$ bundles (to be discussed further in section 5.1 below), and $S^k(E)$ denotes the symmetrised product of k copies of the bundle E. When the Kähler manifold $\mathcal{M}_4$ is not spin, the extended twist parameter ε cannot be zero unless $2j_l+2j_r$ is odd. In general, we need to choose ε so that:

$$\varepsilon+j_l+j_r+\frac{1}{2}\in\mathbb{Z}\,,\tag{4.54}$$

which ensures that the bundle (4.53) is well-defined. This generalises the discussion of section 2.3. In a given 5d theory, there might be any number of massive particles of various spins that will contribute in this way, and the ε parameters for each cannot be chosen independently. We will come back to this important point in section 5.3.1 below.

Partition function at spin (j_l,j_r). We can now generalise the previous results for the partition function of a hypermultiplet with spin (j_l,j_r). It is determined in terms of the Dolbeault complex twisted by the KK tower of 'higher-spin' bundles:

$$V_{n,\varepsilon;(j_l,j_r)}=\mathcal{K}^{\frac{1}{2}}\otimes\bigotimes_K\left(\mathcal{K}^{\varepsilon^K}\otimes L_K\right)^{q_K}\otimes S^{2j_l}(S_-)\otimes S^{2j_r}(S_+)\otimes(\mathcal{L}_{\rm KK})^n\,.\tag{4.55}$$

The computation of that index is discussed in appendix B.3. Using the notation $\mathfrak{m}\equiv q_K\mathfrak{m}^K$ and $\varepsilon=q_K\varepsilon^K$, one finds:

$$\begin{aligned}\mathrm{ind}(\bar\partial_{V_{n,\varepsilon;(j_l,j_r)}})=(2j_l+1)(2j_r+1)\Bigg[&-\frac{\sigma}{8}+\frac{1}{2}\varepsilon^2(2\chi+3\sigma)-\frac{2}{3}j_l(j_l+1)\chi\\&+\frac{j_l(j_l+1)+j_r(j_r+1)}{6}(2\chi+3\sigma)+\frac{1}{2}(\mathfrak{m}+n\mathfrak{p}+2\varepsilon\mathbf{k},\mathfrak{m}+n\mathfrak{p})\Bigg]\,.\end{aligned}\tag{4.56}$$

Then, using the building blocks (4.37) and the notation $a=q_Ka^K$, we obtain:

$$\begin{aligned}Z^{(j_l,j_r)}_{\mathcal{M}_5}(a)_{\mathfrak{m}}&=\mathbf{\Pi}^{\mathcal{H}}(a)^{c_{\mathcal{A}}\chi+c_{\mathcal{B}}\sigma+c_0\left[\frac{1}{2}\varepsilon^2(2\chi+3\sigma)+\frac{1}{2}(\mathfrak{m}+2\varepsilon\mathbf{k},\mathfrak{m})\right]}\\&\quad\times\mathscr{K}^{\mathcal{H}}(a)^{c_0(\mathfrak{m}+\varepsilon\mathbf{k},\mathfrak{p})}\,\mathscr{F}^{\mathcal{H}}(a)^{\frac{1}{2}c_0(\mathfrak{p},\mathfrak{p})}\,,\end{aligned}\tag{4.57}$$

in terms of the following spin-dependent numbers:

$$
\begin{aligned}
c_{\mathcal{A}} &= (-1)^{2j_l+2j_r}(2j_l+1)(2j_r+1)\frac{j_r(j_r+1)-j_l(j_l+1)}{3}\,,\\
c_{\mathcal{B}} &= (-1)^{2j_l+2j_r}(2j_l+1)(2j_r+1)\left(-\frac{1}{8}+\frac{j_l(j_l+1)+j_r(j_r+1)}{2}\right)\,,\\
c_0 &= (-1)^{2j_l+2j_r}(2j_l+1)(2j_r+1)\,,
\end{aligned}
\tag{4.58}
$$

which are independent of the geometry.

5 Flux and fibering operators on the Coulomb branch

In this section, we consider the low-energy effective action of a 5d $\mathcal{N}=1$ field theory compactified on a circle. The 5d theories we have in mind are 5d SCFTs, but the following infrared approach is independent of the exact UV completion. We consider the effective 4d $\mathcal{N}=2$ KK theory compactified on $\mathcal{M}_4$, at arbitrary fixed values of the extended Coulomb branch vector multiplets. As explained in the introduction, this is a crucial step towards a systematic computation of the U-plane integral.

5.1 KK theory on $\mathcal{M}_4\times S^1$: flux operators and gravitational couplings

Consider any 4d $\mathcal{N}=2$ theory on $\mathcal{M}_4$. For definiteness, let us assume it is a KK theory, so that we have a scale β^{-1} set by the inverse radius of the circle. We wish to study the Coulomb branch of this theory, where the low-energy degrees of freedom are r 4d $\mathcal{N}=2$ abelian vector multiplets — r is the 'rank' of the 5d theory, by definition. We denote by a^i the scalars in the $U(1)^r$ vector multiplets, which are related to the 5d $\mathcal{N}=1$ abelian vector multiplets as:

$$
a^i = i\beta\left(\sigma^i + iA_5^i\right)\,,\qquad\qquad i=1,\cdots,r\,. \tag{5.1}
$$

Note that a^i is dimensionless, in our conventions. Furthermore, large-gauge transformations along the fifth direction give us the periodicity $a^i\sim a^i+1$, $\forall i$. We also consider background vector multiplets for some maximal torus of the flavour symmetry group, $U(1)^{r_F}\subset G_F$, where r_F denotes the rank of the flavour group. The corresponding background scalars are simply complex masses, denoted by:

$$
\mu^\alpha = i\beta\left(m^\alpha + iA^\alpha_{F,5}\right)\,,\qquad\qquad \alpha=1,\cdots,r_F\,, \tag{5.2}
$$

with the identification $\mu^\alpha\sim\mu^\alpha+1$. The total space of values for (a^i,μ^α) is called the extended Coulomb branch, of dimension $r+r_F$. It is convenient to introduce the notation:[20]

$$
(\boldsymbol{a}^I) = (a^i,\mu^\alpha)\,,\qquad\qquad I=(i,\alpha)\,, \tag{5.3}
$$

[20] *Beware the indices:* in this section and the next, the indices $I,J,\cdots$ run over the gauge and flavor maximal torus, while $i,j,\cdots$ are gauge indices and α,β are flavor indices. This is distinct from the conventions in other sections, for instance z^i denoted holomorphic coordinates on $\mathcal{M}_4$, and α,β are also 4d left-chiral spinor indices; no confusion is likely there. Note also that I,J were previsouly used as $SU(2)_R$ indices, but we are now dealing with DW-twisted fields which are $SU(2)_R$-neutral, therefore this notation switch should cause no confusion.

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which treats dynamical and background vector multiplets democratically. We will furthermore assume that the vector multiplets are the only massless degrees of freedom at generic points on the (extended) CB.[21]

The low-energy 4d $\mathcal{N} = 2$ effective field theory in flat space is then governed by the effective prepotential, denoted by $\mathcal{F}(a, \mu)$. We define $\mathcal{F}(\boldsymbol{a})$ for the KK theory to be dimensionless (it is related to the usual 4d prepotential, $\mathcal{F}_{\text{4d}}$, by $\mathcal{F} = \beta^2 \mathcal{F}_{\text{4d}}$). The flat-space Lagrangian can be coupled to the DW-twist background on $\mathcal{M}_4$. Its key property is that it is 'almost' $\mathcal{Q}$-exact, similarly to (2.36). Discarding the $\mathcal{Q}$-exact pieces, we are left with the following topological action, which is well-defined on any $\mathcal{M}_4$ [43]:

$$\begin{aligned} S_{\text{flat}} = \frac{i}{4\pi} \int_{\mathcal{M}_4} \bigg(& F^I \wedge F^J \frac{\partial^2 \mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J} - \frac{i}{2} F^I \wedge \Lambda^J \wedge \Lambda^K \frac{\partial^3 \mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J \partial \boldsymbol{a}^K} \\ & - \frac{1}{48} \Lambda^I \wedge \Lambda^J \wedge \Lambda^K \wedge \Lambda^L \frac{\partial^4 \mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J \partial \boldsymbol{a}^K \partial \boldsymbol{a}^L} \bigg) , \end{aligned} \tag{5.4}$$

where the sum over repeated indices is understood. Here $F = dA$ for an abelian gauge field, and we also introduced the one-form $\Lambda = \Lambda^{1,0} + \Lambda^{0,1}$, in the notation of (2.32). Formally, (5.4) can be viewed as the fourth descendant, $\int_{\mathcal{M}_4} \mathcal{O}^{(4)}$, with respect to the DW supercharge $\delta = \delta_1 + \delta_2$, of the 0-form:

$$\mathcal{O}^{(0)} = -\frac{2i}{\pi} \mathcal{F}(\boldsymbol{a}) , \tag{5.5}$$

where we used the descent relations $\delta \mathcal{O}^{(n)} = d\mathcal{O}^{(n-1)}$ with the supersymmetry variations:

$$\delta a = 0 , \qquad \delta \Lambda = 2 da , \qquad \delta F = -i d\Lambda , \tag{5.6}$$

for an abelian vector multiplet, with $a = i\sqrt{2}\phi$. The fermionic terms in (5.4) only depend on the one-form Λ. Correspondingly, they will only affect the low-energy physics on $\mathcal{M}_4$ if the Λ fields have zero-modes, which is to say if $H^1(\mathcal{M}_4, \mathbb{R})$ is non-trivial. Since we have assumed that $H^1(\mathcal{M}_4, \mathbb{R}) = 0$ — i.e. $b_1 = 0$ — in this paper, we can ignore the effect of these fermionic couplings in the following. We hope to address the more general case in future work.

Let us now consider any background gauge field configuration for the $U(1)_I$ symmetries, assuming it preserves our two supercharges. We denote the corresponding fluxes on $\mathcal{M}_4$ by

$$c_1(F^I) = \frac{1}{2\pi} F^I = \sum_k \mathfrak{m}_k^I [\mathrm{S}_k] . \tag{5.7}$$

Recall that we denote the intersection pairing on $H_2(\mathcal{M}_4, \mathbb{Z})$ by $(-,-)$, so that we have:

$$(\mathfrak{m}^I, \mathfrak{m}^J) = \frac{1}{4\pi^2} \int_{\mathcal{M}_4} F^I \wedge F^J = \sum_{k,l} \mathfrak{m}_k^I \mathbf{I}_{kl} \mathfrak{m}_l^J , \tag{5.8}$$

[21] More generally, there could be additional massless hypermultiplets, giving us a so-called enhanced CB. We will not consider this possibility in this paper.

with $\mathbf{I}_{kl}$ as in (3.4). At any generic point on the Coulomb branch, taking $\boldsymbol{a}^I$ to be constant, the action (5.4) evaluates to:

$$S_{\text{flux}} \equiv S_{\text{flat}}\Big|_{\text{CB}} = \pi i (\mathfrak{m}^I, \mathfrak{m}^J) \frac{\partial^2 \mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J} \,. \tag{5.9}$$

In addition to (5.4), the infrared theory compactified on $\mathcal{M}_4$ is governed by well-studied gravitational couplings [26, 28, 43, 54, 93]. Up to $\mathcal{Q}$-exact terms and away from Seiberg-Witten singularities, the topologically-twisted Coulomb-branch theory takes the simple form:

$$S_{\text{TFT}} = S_{\text{flat}} + S_{\text{grav}} \,. \tag{5.10}$$

The second term in (5.10) consists of couplings to the background metric:

$$\begin{aligned} S_{\text{grav}} &= \frac{i}{64\pi} \int d^4x \sqrt{g}\, \epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} R_{\mu\nu\alpha\beta} R_{\rho\sigma\gamma\delta}\, \mathcal{A}(\boldsymbol{a}) \\ &\quad + \frac{i}{48\pi} \int d^4x \sqrt{g}\, \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\alpha}{}^{\beta} R_{\rho\sigma\beta}{}^{\alpha}\, \mathcal{B}(\boldsymbol{a}) \,. \end{aligned} \tag{5.11}$$

At constant values of the extended CB parameters, this evaluates to:

$$S_{\text{grav}} = 2\pi i \Big(\chi\, \mathcal{A}(\boldsymbol{a}) + \sigma\, \mathcal{B}(\boldsymbol{a}) \Big) \,, \tag{5.12}$$

where χ and σ are the topological Euler characteristic and the signature of $\mathcal{M}_4$, respectively. This gives the famous contribution:

$$e^{-S_{\text{grav}}} = \boldsymbol{A}(\boldsymbol{a})^{\chi} \boldsymbol{B}(\boldsymbol{a})^{\sigma} \,, \qquad \boldsymbol{A}(\boldsymbol{a}) \equiv e^{-2\pi i \mathcal{A}(\boldsymbol{a})} \,, \qquad \boldsymbol{B}(\boldsymbol{a}) \equiv e^{-2\pi i \mathcal{B}(\boldsymbol{a})} \,. \tag{5.13}$$

The prepotential $\mathcal{F}$ and the gravitational couplings $\mathcal{A}$ and $\mathcal{B}$ can be determined from the Seiberg-Witten geometry of the 5d theory on a circle, in principle, or else from an explicit instanton counting computation on the Ω-background — for the rank-one 5d SCFTs with E_n symmetry, this was discussed at length in [19].

On general grounds, the prepotential $\mathcal{F}$ suffers from branch-cut ambiguities:

$$\mathcal{F}(\boldsymbol{a}) \sim \mathcal{F}(\boldsymbol{a}) + \frac{n_2}{2} \boldsymbol{a}^2 + n_1 \boldsymbol{a} + \frac{n_0}{2} \,, \qquad n_0, n_1, n_2 \in \mathbb{Z} \,. \tag{5.14}$$

Such shifts are incurred, in particular, when performing large gauge transformations along the 5d circle. It follows that the exponentiated action $\exp(-S_{\text{flux}})$ is singled-valued if and only if the intersection pairing is even, so that $(\mathfrak{m}, \mathfrak{m}) \in 2\mathbb{Z}$ for any integer-quantized flux $\mathfrak{m}$, which is true if $\mathcal{M}_4$ is spin. More generally, we need to modify the quantization condition on our fluxes, so that A^I describe spin^c connections rather than $U(1)$ gauge fields. For $U(1)_I$ bundles, we would have $\mathfrak{m}_k^I \in \mathbb{Z}$, while more generally we may choose:

$$\frac{1}{2\pi} F^I = \sum_k \left(\varepsilon^I \mathbf{k}_k + \mathfrak{m}_k^I \right) [\mathrm{S}_k] \,. \tag{5.15}$$

Here, $\mathbf{k}$ was defined in (4.32), $\varepsilon^I \in \frac{1}{2}\mathbb{Z}$, and $\mathfrak{m}_k^I \in \mathbb{Z}$. The parameters ε^I must be carefully chosen depending on the theory so that it be well-defined on $\mathcal{M}_4$, as we will discuss in more detail in section 5.3.1 below. They are the infrared analogue of the extended DW-twist parameter ε introduced in section 2.3 for the hypermultiplet. The spin^c connections A^I can be formally viewed as connections on the ill-defined line bundles

$$\mathcal{L}_I = \mathcal{K}^{\varepsilon_I} \otimes L_I\,, \tag{5.16}$$

where L_I is a $U(1)$ line bundle with first Chern class $\mathfrak{m}^I$. The necessity of introducing spin^c connections arises from the fact that our 4d $\mathcal{N}=2$ KK theories generally contain spinors even after the standard DW twist — in the infrared description on the Coulomb branch, these arise as massive BPS particles coupled to the low-energy (background and dynamical) photons, which can have arbitrary (twisted) spin. We will give the precise condition on ε^I in section 5.3 below. For now, we claim that the ε^I's can always be chosen so that the low-energy theory is well-defined; in particular, choosing these parameters correctly will render $e^{-S_{\rm TFT}(\boldsymbol{a})}$ fully gauge-invariant, single-valued and locally holomorphic in $\boldsymbol{a}$.[22]

Let us now define the 'flux operators':

$$\boldsymbol{\Pi}_{I,J}(\boldsymbol{a}) = \exp\left(-2\pi i \frac{\partial^2 \mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}_I \partial \boldsymbol{a}_J}\right)\,, \tag{5.17}$$

which are meromorphic functions on the ECB parameters $\boldsymbol{a}_I$. Such insertions can be understood as local operators in the twisted infrared theory. Alternatively, we consider the insertion of (5.4) for specific fluxes, which can be viewed as the top-dimensional topological descendant of $\mathcal{F}(\boldsymbol{a})$, viewed itself as a local operator (at least formally). Using the topological invariance, we can localise $F^I \wedge F^J$ to have support at a point on $\mathcal{M}_4$, giving rise to the local insertion:[23]

$$e^{-S_{\rm flux}} = \prod_{I,J} \boldsymbol{\Pi}_{I,J}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{m}^I+\varepsilon^I \mathbf{k}, \mathfrak{m}^J+\varepsilon^J\mathbf{k})}\,. \tag{5.18}$$

Using the fact that $(\mathbf{k},\mathbf{k}) = 2\chi + 3\sigma$, it is convenient to factorise these contributions as:

$$e^{-S_{\rm flux}} = Z^{\rm flux}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}}\, \boldsymbol{G}(\boldsymbol{a};\varepsilon)^{2\chi+3\sigma}\,, \tag{5.19}$$

where we defined:

$$Z^{\rm flux}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}} \equiv \prod_{I,J} \boldsymbol{\Pi}_{I,J}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{m}^I+\varepsilon^I\mathbf{k},\mathfrak{m}^J)} = \exp\left(-\pi i \sum_{I,J} (\mathfrak{m}^I + 2\varepsilon^I \mathbf{k}, \mathfrak{m}^J) \frac{\partial^2\mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J}\right)\,, \tag{5.20}$$

and:

$$\boldsymbol{G}(\boldsymbol{a};\varepsilon) \equiv e^{-2\pi i \mathcal{G}(\boldsymbol{a};\varepsilon)}\,, \qquad \mathcal{G}(\boldsymbol{a};\varepsilon) \equiv \frac{1}{2}\sum_{I,J} \varepsilon^I \varepsilon^J \frac{\partial^2 \mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J}\,. \tag{5.21}$$

The full exponentiated topological field theory action (5.10) evaluated on the CB then gives us the 'CB partition function' on $\mathcal{M}_4$ with gauge and flavor fluxes $\mathfrak{m}$:

$$Z_{\mathcal{M}_4\times S^1}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}} = Z^{\rm geom}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon)\, Z^{\rm flux}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}}\,. \tag{5.22}$$

[22] Here, holomorphy is a formal consequence of supersymmetry since anti-holomorphic terms are $\mathcal{Q}$-exact.

[23] What we call the 'flux operator' has been denoted 'the C coupling' in recent works [36, 39]. The flux operator insertion can be also be interpreted as a contact term localised at the intersection of the 2-cycles carrying the flux [40].

This object is really the *holomorphic integrand* that will enter the U-plane integral of the 4d $\mathcal{N}=2$ KK theories, as discussed in the introduction. Here, we conjecture that the two factors in (5.22) are separately well-defined on any Kähler manifold (this is clearly true when $\mathcal{M}_4$ is spin, but not so obvious in the non-spin case). Consider first the "flux operator" contribution (5.20). The ε^I parameters should be such that (5.20) is single-valued. A sufficient set of conditions would be

$$\begin{aligned} \frac{1}{2}(\mathfrak{m}^I,\mathfrak{m}^J)+\varepsilon^I(\mathbf{k},\mathfrak{m}^J)\in\mathbb{Z} \quad &\text{if } I=J\,,\\ \varepsilon^I(\mathbf{k},\mathfrak{m}^J)+\varepsilon^J(\mathbf{k},\mathfrak{m}^I)\in\mathbb{Z} \quad &\text{if } I\neq J\,, \end{aligned} \tag{5.23}$$

for any $\mathfrak{m}_k^I\in\mathbb{Z}$, but this is much too strong in general. Instead, the correct condition on the ε^I's will depend on the 5d BPS spectrum of the field theory (see section 5.3.1.

The "geometrical" factor in (5.22) has contributions from the ordinary gravitational couplings (5.13) and from (5.21), which is dictated by our choice of (background) spin^c connections. It is given by:

$$Z^{\rm geom}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon)=\boldsymbol{A}(\boldsymbol{a})^{\chi}\boldsymbol{B}(\boldsymbol{a})^{\sigma}\boldsymbol{G}(\boldsymbol{a};\varepsilon)^{2\chi+3\sigma}\ . \tag{5.24}$$

The $\boldsymbol{A}$ and $\boldsymbol{B}$ couplings are given in terms of the low-energy Seiberg-Witten geometry as [26, 43, 54]:

$$\boldsymbol{A}=\alpha\left(\det_{ij}\frac{dU_i}{da^j}\right)^{\frac{1}{2}}\ ,\qquad\qquad \boldsymbol{B}=\beta\left(\Delta^{\rm phys}\right)^{\frac{1}{8}}\ , \tag{5.25}$$

with α,β some numerical constants. Here, $U_i(\boldsymbol{a})$ are the gauge-invariant U-parameters, which parametrise the Coulomb branch of the 4d $\mathcal{N}=2$ KK theory, and $\Delta^{\rm phys}$ is the so-called physical discriminant [93] of the Seiberg-Witten fibration (see [19] for a recent discussion). The gravitational couplings can also be extracted from the Nekrasov partition function (see section 6), as discussed in [19, 94, 95]. Our conjecture is then that the branch cuts ambiguities in $\boldsymbol{A}^{\chi}\boldsymbol{B}^{\sigma}$, that would generally arise from the expressions (5.25), are precisely cancelled by the third factor $\boldsymbol{G}^{2\chi+3\sigma}$ in (5.24).[24]

The free hypermultiplet. Let us consider the 5d hypermultiplet on $\mathcal{M}_4\times S^1$ coupled to a single $U(1)$ vector multiplet with charge 1, whose partition function we computed in the previous section. In the present CB approach, we simply need to know the effective prepotential and gravitational couplings for the free hypermultiplet. They are given by:

$$\mathcal{F}=-\frac{1}{(2\pi i)^3}{\rm Li}_3(Q)\,,\qquad \mathcal{A}=0\,,\qquad \mathcal{B}=-\frac{1}{16\pi i}\log(1-Q)\,, \tag{5.26}$$

with $Q\equiv e^{2\pi i a}$. These gravitational couplings will be further discussed from the perspective of the Ω-background in the next section. We then have:

$$\boldsymbol{A}=1\,,\qquad\qquad \boldsymbol{B}=(1-Q)^{\frac{1}{8}}\ . \tag{5.27}$$

[24] If the Kähler manifold $\mathcal{M}_4$ is spin, we have $\chi\in 4\mathbb{Z}$ and $\sigma\in 16\mathbb{Z}$. Then the $\boldsymbol{G}$ factor is well-defined by itself, and it can be reabsorbed into the flux operator.

The non-trivial physical discriminant $\Delta^{\rm phys} = (1-Q)$ encodes the singularity on the (extended) Coulomb branch at $Q=1$, where the hypermultiplet becomes massless. Taking the extended topological twist with $\varepsilon = -\frac{1}{2}+\delta$, and some background flux $\mathfrak{m}$, we also have

$$\boldsymbol{G} = (1-Q)^{-\frac{\varepsilon^2}{2}} = (1-Q)^{-\frac{1}{8}}(1-Q)^{-\frac{1}{2}\delta(\delta-1)}\,, \qquad Z^{\rm flux}_{\mathcal{M}_4} = (1-Q)^{-\frac{1}{2}(\mathfrak{m}+2\varepsilon\mathbf{k},\mathfrak{m})}\,. \tag{5.28}$$

Then, the formula (5.22) gives us:

$$Z^{\mathcal{H}}_{\mathcal{M}_4\times S^1}(a;\varepsilon)_{\mathfrak{m}} = \left(\frac{1}{1-Q}\right)^{\chi_h+\frac{1}{2}(\mathfrak{m}+\delta\mathbf{k}-\mathbf{k},\mathfrak{m}+\delta\mathbf{k})}\,, \tag{5.29}$$

in perfect agreement with (4.41).

5.2 KK theory on $\mathcal{M}_5$: the fibering operator

We now consider the non-trivial fibration $S^1 \to \mathcal{M}_5 \to \mathcal{M}_4$. From the 4d point of view, all 5d fields decompose in KK towers and there is always a distinguished $U(1)_{\rm KK}$ symmetry in 4d corresponding to the momentum along the fifth direction. A non-trivial fibration of the circle amounts to introducing background fluxes for the KK symmetry on $\mathcal{M}_4$:

$$\int_{{\rm S}_k} c_1(\mathcal{L}_{\rm KK}) = \frac{1}{2\pi}\int_{{\rm S}_k}\hat{\rm F} = \sum_l \mathbf{I}_{kl}\mathfrak{p}_l\,. \tag{5.30}$$

On the CB of the infrared topologically-twisted 4d $\mathcal{N}=2$ KK theory, the non-trivial fibration of the fifth direction over $\mathcal{M}_4$ is then encoded in a 'flux operator' for $U(1)_{\rm KK}$, which we call the *fibering operator*. The expression for the latter is easily determined by dimensional analysis. Reinstating dimensions, the mass parameter for $U(1)_{\rm KK}$ is really $\mu_{\rm KK} = 1/\beta$, so that $\mathcal{F}_{\rm 4d} = \mu_{\rm KK}^2\mathcal{F}$ and one finds:

$$\frac{\partial^2\mathcal{F}_{\rm 4d}}{\partial\mu_{\rm KK}^2} = 2\left(1 - \boldsymbol{a}^I\frac{\partial}{\partial\boldsymbol{a}^I} + \frac{1}{2}\boldsymbol{a}^I\boldsymbol{a}^J\frac{\partial^2}{\partial\boldsymbol{a}^I\partial\boldsymbol{a}^J}\right)\mathcal{F}(\boldsymbol{a})\,, \tag{5.31}$$

and:

$$\frac{\partial^2\mathcal{F}_{\rm 4d}}{\partial\mu_{\rm KK}\partial(\mu_{\rm KK}\boldsymbol{a}^I)} = \left(1 - \boldsymbol{a}^J\frac{\partial}{\partial\boldsymbol{a}^J}\right)\frac{\partial\mathcal{F}}{\partial\boldsymbol{a}^I}\,. \tag{5.32}$$

For a principal circle bundle with first Chern numbers $\mathfrak{p}_k$, we then write down the fibering operator:

$$\widehat{\mathscr{F}}_{\mathfrak{p}}(\boldsymbol{a};\varepsilon) \equiv \mathscr{F}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{p},\mathfrak{p})}\prod_I \mathscr{K}_I(\boldsymbol{a})^{(\mathfrak{p},\mathfrak{m}^I+\varepsilon^I\mathbf{k})}\,, \tag{5.33}$$

where we defined:

$$\mathscr{F}(\boldsymbol{a}) \equiv \exp\left(-4\pi i\left(1 - \boldsymbol{a}^I\frac{\partial}{\partial\boldsymbol{a}^I} + \frac{1}{2}\boldsymbol{a}^I\boldsymbol{a}^J\frac{\partial^2}{\partial\boldsymbol{a}^I\partial\boldsymbol{a}^J}\right)\mathcal{F}(\boldsymbol{a})\right)\,, \tag{5.34}$$

and:

$$\mathscr{K}_I(\boldsymbol{a}) \equiv \exp\left(-2\pi i\left(1 - \boldsymbol{a}^J\frac{\partial}{\partial\boldsymbol{a}^J}\right)\frac{\partial\mathcal{F}}{\partial\boldsymbol{a}^I}\right)\,. \tag{5.35}$$

The functions (5.34) and (5.35) are entirely determined by the exact effective prepotential of the 4D $\mathcal{N}=2$ KK theory, and they are unaffected by the ambiguities (5.14). Moreover, while $\mathscr{F}(\boldsymbol{a})^{\frac{1}{2}}$ and $\mathscr{K}_I(\boldsymbol{a})^{\frac{1}{2}}$ suffer from branch-cut ambiguities in general, the product (5.33) is expected to be unambiguous. This is exactly like in the case of the flavor flux operators discussed above. For spin manifolds, the intersection form is even and the factors in (5.33) are individually well-defined, while on a non-spin $\mathcal{M}_4$ we again conjecture that the fibering operator (5.33) remains well-defined once the parameters ε^I are correctly chosen.

The $\mathcal{M}_5$ partition function and gauge invariance. Putting all the contributions together, we arrive at the full $\mathcal{M}_5$ partition function at fixed values of the (gauge and flavor) $U(1)_I$ vector multiplets. We have:

$$Z_{\mathcal{M}_5}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}} = Z^{\text{geom}}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon)\, Z^{\text{flux}}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}}\, \widehat{\mathscr{F}}_{\mathfrak{p}}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}}\,, \tag{5.36}$$

with:

$$\begin{aligned} Z^{\text{geom}}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon) &= \boldsymbol{A}(\boldsymbol{a})^{\chi}\boldsymbol{B}(\boldsymbol{a})^{\sigma}\boldsymbol{G}(\boldsymbol{a};\varepsilon)^{2\chi+3\sigma}\,,\\ Z^{\text{flux}}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon) &= \boldsymbol{\Pi}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{m}+2\varepsilon\mathbf{k},\mathfrak{m})}\,,\\ \widehat{\mathscr{F}}_{\mathfrak{p}}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}} &= \mathscr{K}(\boldsymbol{a})^{(\mathfrak{p},\mathfrak{m}+\varepsilon\mathbf{k})}\,\mathscr{F}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{p},\mathfrak{p})}\,. \end{aligned} \tag{5.37}$$

Here we suppressed the I, J indices.[25] Importantly, the partition function (5.36) is fully gauge invariant. Consider the large gauge transformations along $U(1)_I$:

$$\boldsymbol{a}_J \to \boldsymbol{a}_J + \delta_{IJ}\,, \qquad \mathfrak{m}_J \to \mathfrak{m}_J + \delta_{IJ}\mathfrak{p}\,, \tag{5.38}$$

which we denote by the shorthand $(\boldsymbol{a},\mathfrak{m}) \to (\boldsymbol{a}+\delta_I, \mathfrak{m}+\delta_I\mathfrak{p})$. Gauge invariance implies that:

$$Z_{\mathcal{M}_5}(\boldsymbol{a}+\delta_I)_{\mathfrak{m}+\delta_I\mathfrak{p}} = Z_{\mathcal{M}_5}(\boldsymbol{a})_{\mathfrak{m}}\,. \tag{5.39}$$

This is indeed the case. To check this, note that $Z^{\text{geom}}_{\mathcal{M}_4}(\boldsymbol{a})$ is invariant by itself, and that we have the following large gauge transformations of the building blocks:

$$\begin{aligned} \boldsymbol{\Pi}_{J,K}(\boldsymbol{a}+\delta_I) &= \boldsymbol{\Pi}_{J,K}(\boldsymbol{a})\,,\\ \mathscr{K}_J(\boldsymbol{a}+\delta_I) &= \boldsymbol{\Pi}_{I,J}(\boldsymbol{a})^{-1}\,\mathscr{K}_J(\boldsymbol{a})\,,\\ \mathscr{F}(\boldsymbol{a}+\delta_I) &= \boldsymbol{\Pi}_{I,I}(\boldsymbol{a})\,\mathscr{K}_I(\boldsymbol{a})^{-2}\,\mathscr{F}(\boldsymbol{a})\,. \end{aligned} \tag{5.40}$$

Matching the one-loop computation. Consider the free hypermultiplet coupled to a $U(1)$ vector multiplet. By an application of the general formulas (5.34)–(5.35), using the hypermultiplet prepotential (5.26), we find:

$$\mathscr{K} = \mathscr{K}^{\mathcal{H}}(a)\,, \qquad \mathscr{F} = \mathscr{F}^{\mathcal{H}}(a)\,, \tag{5.41}$$

in terms of the meromorphic functions introduced in (4.37), so that:

$$\begin{aligned} \widehat{\mathscr{F}}_{\mathfrak{p}}(a;\varepsilon)_{\mathfrak{m}} = \mathscr{F}^{\mathcal{H}}_{\mathfrak{p}}(a)_{\mathfrak{m}} \equiv \exp\Bigg(&-\frac{(\mathfrak{p},\mathfrak{p})}{4\pi^2}\mathrm{Li}_3(e^{2\pi i a}) - \frac{(\mathfrak{p},\mathfrak{p})a-(\mathfrak{p},\mathfrak{m}+\varepsilon\mathbf{k})}{2\pi i}\mathrm{Li}_2(e^{2\pi i a})\\ &-\frac{a((\mathfrak{p},\mathfrak{p})a-2(\mathfrak{p},\mathfrak{m}+\varepsilon\mathbf{k}))}{2}\log(1-e^{2\pi i a})\Bigg)\,. \end{aligned} \tag{5.42}$$

By multiplying with (5.29), we obtain the full partition function of a free hypermultiplet on $\mathcal{M}_5$. This matches precisely with the direct one-loop computation of section 4.3.

[25] We will also often omit the ε from the notation, from now on, to avoid clutter.

5.3 Higher-spin state contributions

The prepotential of many five-dimensional superconformal field theories compactified on S^1 admits an expansion in terms of 5d BPS states:[26]

$$\mathcal{F} = -\frac{1}{(2\pi i)^3} \sum_{\beta} \sum_{j_l, j_r} c_0^{(j_l, j_r)} N^{\beta}_{j_l, j_r} \, \mathrm{Li}_3(Q^\beta) \; . \tag{5.43}$$

Here, in keeping with common notation, we denote by $\boldsymbol{\beta}_I \equiv q_I$ the charges under the $U(1)^{r+r_F}$ symmetry on the extended Coulomb branch, with $Q^\beta \equiv \prod_I Q_I^{\beta_I}$ and $Q_I \equiv e^{2\pi i \boldsymbol{a}^I}$, and with the universal coefficients $c_0^{j_l, j_r}$ as in (4.58), namely

$$c_0^{(j_l, j_r)} = (-1)^{2j_l + 2j_r} (2j_l + 1)(2j_r + 1) \; . \tag{5.44}$$

In the context of geometrical engineering of 5d SCFTs in M-theory on a toric threefold, the theory-dependent non-negative integers $N^{\beta}_{j_l, j_r}$ in (5.43) are the refined Gopakumar-Vafa invariants [57, 58], as we will review momentarily. The expansion (5.45) can be written simply as:

$$\mathcal{F} = -\frac{1}{(2\pi i)^3} \sum_{\beta} d_\beta \, \mathrm{Li}_3(Q^\beta) \, , \tag{5.45}$$

with

$$d_\beta \equiv \sum_{j_l, j_r} (-1)^{2j_l + 2j_r} (2j_l + 1)(2j_r + 1) N^{\beta}_{j_l, j_r} \, , \tag{5.46}$$

the effective number of 5d BPS states of charge $\boldsymbol{\beta}$. Given the expression (5.45) for the prepotential, we can directly compute the CB fibering operator in terms of the hypermultiplet result (5.42), at least formally, as a product over the charge lattice:

$$\mathscr{F}_{\mathfrak{p}}(\boldsymbol{a})_{\mathfrak{m}} = \prod_{\beta} \left[\mathscr{F}^{\mathcal{H}}_{\mathfrak{p}}(\boldsymbol{\beta}(\boldsymbol{a}))_{\boldsymbol{\beta}(\mathfrak{m})} \right]^{d_\beta} \; . \tag{5.47}$$

Thus, the higher-spin contributions are the same as for d_β hypermultiplets, in perfect agreement with the second line of (4.57).

One can similarly expand the flux operators. To obtain the full CB partition function, we should also consider the contribution of higher-spin particles to the gravitational couplings $\mathcal{A}$ and $\mathcal{B}$. In section 6.1.4 below, we will show that:

$$\begin{aligned} \mathcal{A} &= \frac{1}{2\pi i} \sum_{\beta} \sum_{j_l, j_r} c_{\mathcal{A}}^{(j_l, j_r)} N^{\beta}_{j_l, j_r} \, \log(1 - Q^\beta) \, , \\ \mathcal{B} &= \frac{1}{2\pi i} \sum_{\beta} \sum_{j_l, j_r} c_{\mathcal{B}}^{(j_l, j_r)} N^{\beta}_{j_l, j_r} \, \log(1 - Q^\beta) \, , \end{aligned} \tag{5.48}$$

when expanding in terms of the refined GV invariants, with the coefficients $c_{\mathcal{A},\mathcal{B}}^{(j_l, j_r)}$ given in (4.58). One then easily checks that the CB partition function on $\mathcal{M}_5$ can be written entirely in terms of the refined GV invariants of the 5d theory, as:

$$Z_{\mathcal{M}_5}(\boldsymbol{a})_{\mathfrak{m}} = \prod_{\beta} \prod_{j_l, j_r} \left[Z^{(j_l, j_r)}_{\mathcal{M}_5}(\boldsymbol{\beta}(\boldsymbol{a}))_{\boldsymbol{\beta}(\mathfrak{m})} \right]^{N^{\beta}_{j_l, j_r}} \; , \tag{5.49}$$

[26] Here we ignore some possible ‘classical’ terms, which would contribute additional factors to the CB partition function.

using the explicit expression (4.57). The expression (5.49) is the partition function that we would obtain by combining the localisation results of section 4 with the assumption that the full partition function can be obtained as a product over the 5d BPS states. What we have just shown is that this factorisation is consistent with the low-energy approach of the present section. In fact, the factorisation (5.49) is simply equivalent to the expansions (5.43) and (5.48) of the low-energy effective couplings.

5.3.1 Spin/charge constraints on the 5d BPS spectrum

To conclude this section, let us mention an important constraint that arises when trying to put a general 5d SCFT on our supersymmetric $\mathcal{M}_5$, for a generic choice of our geometric background. Namely, every BPS particle of spin (j_l, j_r) and charge $\boldsymbol{\beta}$, at any point on the 5d Coulomb branch, should be coupled consistently to the base manifold $\mathcal{M}_4$, *at the same time*. Given the CB (gauge and flavor) symmetry $\prod_I U(1)_I$, we need to choose the ε parameters ε^I, which define the spin^c connections as in (5.15), in such a way that

$$\frac{1}{2} + j_l + j_r + \boldsymbol{\beta}(\varepsilon) \in \mathbb{Z}\,, \qquad \forall\, j_l, j_r, \boldsymbol{\beta} \text{ with } N^{\boldsymbol{\beta}}_{j_l, j_r} \neq 0\,. \tag{5.50}$$

Here, $\boldsymbol{\beta}(\varepsilon) \equiv q_I \varepsilon^I$ is the ε parameter of this particular BPS particle. For any fixed j_l, j_r, q_I, this condition is equivalent to the requirement that the vector bundle (4.55) be well-defined on any Kähler manifold $\mathcal{M}_4$. (Of course, if $\mathcal{M}_4$ is spin, then this condition is not necessary.)

Note that, once we fix ε^I, the condition (5.50) only holds if the spin and electric charges of the BPS states are appropriately correlated (mod 2). The theories for which this holds obey a "spin/charge" relation, which is somewhat reminiscent of the 3d spin/charge relation discussed in [73] for strongly-coupled electrons; this spin/charge relation for 4d $\mathcal{N} = 2$ theories was also discussed in [74].

6 Fibering operators from gluing Nekrasov partition functions

In this section, we give a complementary perspective on the CB partition function (5.36), including the fibering operator, by building up $\mathcal{M}_5$ as a toric gluing of $\mathbb{C}^2 \times S^1$ patches, in the case when the base $\mathcal{M}_4$ is a toric four-manifold. We can then obtain the CB partition function $Z_{\mathcal{M}_5}$ as an appropriate gluing of 5d Nekrasov partition functions, generalising well-known results for the five-sphere [48–50, 85].

6.1 Nekrasov partition functions and refined topological strings

Partition functions of 4d $\mathcal{N} = 2$ field theories on toric four-manifolds can be computed in terms of the partition functions on toric patches $\mathbb{C}^2$ [42], and similarly for the 5d uplift. On each patch, one considers the so-called Nekrasov partition function on $\mathbb{C}^2 \times S^1$ with the Ω-background, which is obtained by the identification

$$(z_1, z_2, x_5) \sim (e^{2\pi i \tau_1} z_1, e^{2\pi i \tau_2} z_2, x_5 + \beta)\,, \tag{6.1}$$

where (z_1, z_2, x_5) are the $\mathbb{C}^2 \times S^1$ coordinates, and we also introduced the dimensionless Ω-deformation parameters:

$$\tau_1 = \beta \epsilon_1\,, \qquad\qquad \tau_2 = \beta \epsilon_2\,, \tag{6.2}$$

not to be confused with the gauge couplings. The Ω-background is a $U(1)^2$-equivariant deformation of the topological twist which effectively compactifies the non-compact $\mathbb{C}^2$, with a finite 'volume' $1/(\tau_1\tau_2)$. Using topological invariance, one can equivalently consider a background geometry $D^2_{\tau_1} \times D^2_{\tau_2} \times S^1$, where $D^2_{\tau_{1,2}}$ are elongated cigars fibered over S^1 according to (6.1). Formally, we can assign the following Euler characteristic and signature to the Ω-deformed $\mathbb{C}^2$ geometry [96]:

$$\chi(\mathbb{C}^2) = \tau_1\tau_2\,, \qquad \sigma(\mathbb{C}^2) = \frac{\tau_1^2+\tau_2^2}{3}\,. \tag{6.3}$$

Similarly, the first Chern class of the canonical line bundle over $\mathbb{C}^2$ is formally given by:

$$c_1(\mathcal{K}_{\mathbb{C}^2}) = \tau_1 + \tau_2\,. \tag{6.4}$$

Note that we have $c_1(\mathcal{K})^2 = 2\chi + 3\sigma = (\tau_1+\tau_2)^2$. The partition function of a 5d $\mathcal{N}=1$ theory on $\mathbb{C}^2 \times S^1$ is known as the (K-theoretic) Nekrasov partition function [24, 97], and it will be denoted by:

$$Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a}, \tau_1, \tau_2)\,. \tag{6.5}$$

Here, the CB parameters $\boldsymbol{a}^I$ arise as Dirichlet boundary conditions for the $U(1)_I$ vector multiplets at infinity. Whenever we have a four-dimensional gauge-theory interpretation, the Nekrasov partition function admits an expansion in some instanton counting parameter $\mathfrak{q} = e^{2\pi i \tau_{\rm UV}}$, according to:

$$Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a}, \tau_1, \tau_2) = Z^{\rm cl}_{\mathbb{C}^2\times S^1}(\boldsymbol{a}, \tau_1, \tau_2) Z^{\rm pert}_{\mathbb{C}^2\times S^1}(\boldsymbol{a}, \tau_1, \tau_2)\left(1 + \sum_k \mathfrak{q}^k Z_k^{\rm Nek}(\boldsymbol{a}, \tau_1, \tau_2)\right)\,. \tag{6.6}$$

See e.g. [96, 98, 99] for reviews of instanton counting, and [100–102] for some more recent advances. When considering the Donaldson-Witten twist, we are interested in the non-equivariant limit $\tau_{1,2} \to 0$. In that limit, the partition function diverges in a way which precisely encodes the low-energy couplings $\mathcal{F}$, $\mathcal{A}$ and $\mathcal{B}$ of the CB theory, namely [42, 103]:

$$\log Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a}, \tau_1, \tau_2) \approx -\frac{2\pi i}{\tau_1\tau_2}\left(\mathcal{F}(\boldsymbol{a}) + (\tau_1+\tau_2)H(\boldsymbol{a}) + \tau_1\tau_2\mathcal{A}(\boldsymbol{a}) + \frac{\tau_1^2+\tau_2^2}{3}\mathcal{B}(\boldsymbol{a})\right)\,. \tag{6.7}$$

The term $H(\boldsymbol{a})$ in (6.7) is allowed by dimensional analysis, but it does not represent an additional effective coupling. In fact, for the $U(1)^2$-equivariant DW twist, we must have $H(\boldsymbol{a}) = 0$ because there are no supergravity background fields that could contribute to this coupling (see e.g. [104]). More generally, H is fully determined in terms of $\mathcal{F}$ by the choice of background $U(1)$ gauge fields, as we will see momentarily.

6.1.1 Nekrasov partition functions for the extended topological twist

When patching together Nekrasov partition functions into compact four- or five-manifolds, we will have to be careful about whether the base $\mathcal{M}_4$ is spin or not. In general, we should consider the possibility of an extended DW twist on $\mathbb{C}^2 \times S^1$, with parameters ε^I. We propose that this corresponds to twisting the background gauge fields at infinity according to:

$$\boldsymbol{a}^I \to \boldsymbol{a}^I + \varepsilon^I(\tau_1+\tau_2)\,, \tag{6.8}$$

in agreement with the identification (6.4). Namely, the Nekrasov partition function for the extended DW twist is simply given by:

$$Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a},\tau_1,\tau_2;\varepsilon) = Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a}+\varepsilon(\tau_1+\tau_2),\tau_1,\tau_2)\ . \tag{6.9}$$

Hence, the non-equivariant limit of the partition function reads:

$$\begin{aligned}&\log Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a},\tau_1,\tau_2;\varepsilon)\approx\\ &-\frac{2\pi i}{\tau_1\tau_2}\left(\mathcal{F}(\boldsymbol{a})+(\tau_1+\tau_2)H(\boldsymbol{a};\varepsilon)+\tau_1\tau_2\mathcal{A}(\boldsymbol{a})+\frac{\tau_1^2+\tau_2^2}{3}\mathcal{B}(\boldsymbol{a})+(\tau_1+\tau_2)^2\mathcal{G}(\boldsymbol{a};\varepsilon)\right)\,,\end{aligned}\tag{6.10}$$

with:

$$H(\boldsymbol{a};\varepsilon)=\varepsilon^I\frac{\partial\mathcal{F}}{\partial \boldsymbol{a}^I}\,,\qquad\qquad \mathcal{G}(\boldsymbol{a};\varepsilon)=\frac{1}{2}\varepsilon^I\varepsilon^J\frac{\partial^2\mathcal{F}(\boldsymbol{a})}{\partial\boldsymbol{a}^I\partial\boldsymbol{a}^J}\ . \tag{6.11}$$

This parameterisation of the non-equivariant limit naturally parallels the discussion of section 5.1 for the CB effective couplings, with $\mathcal{G}$ being exactly as in (5.21).

6.1.2 Gluing transformations in the non-equivariant limit

We wish to glue together Nekrasov partition functions from different patches to obtain the CB partition function of a compact five-manifold $\mathcal{M}_5$. The most general gluing rules between two patches, for our purposes, are:[27]

$$\tau_i\to\widetilde{\tau}_i\equiv\frac{\widehat{\tau}_i}{\gamma}\,,\qquad\qquad \boldsymbol{a}\to\widetilde{\boldsymbol{a}}\equiv\frac{\boldsymbol{a}+\widehat{\mathfrak{n}}}{\gamma}\,, \tag{6.12}$$

where we defined:

$$\widehat{\tau}_i\equiv\alpha_i\tau_1+\beta_i\tau_2\,,\qquad \gamma\equiv\gamma_1\tau_1+\gamma_2\tau_2+1\,, \tag{6.13}$$

for some integers α_i, β_i, γ_i (with $i=1,2$), and:

$$\widehat{\mathfrak{n}}\equiv\widehat{\mathfrak{n}}_1\widehat{\tau}_1+\widehat{\mathfrak{n}}_2\widehat{\tau}_2\ . \tag{6.14}$$

The parameters $\widehat{\mathfrak{n}}_i$ allow us to introduce background fluxes. To recover the DW twist on $\mathcal{M}_5$, we need to consider the non-equivariant limit of the Nekrasov partition function in the variables (6.12). In the limit $\tau_i\to 0$ and using the ansatz (6.7), one finds:

$$\begin{aligned}&\log Z_{\mathbb{C}^2\times S^1}(\widetilde{\boldsymbol{a}},\widetilde{\tau}_1,\widetilde{\tau}_2)\approx\\ &-\frac{2\pi i}{\widehat{\tau}_1\widehat{\tau}_2}\bigg(\mathcal{F}+(\widehat{\tau}_1+\widehat{\tau}_2)H+2(\gamma-1)\left(\mathcal{F}-\boldsymbol{a}^I\partial_I\mathcal{F}\right)+\widehat{\tau}_1\widehat{\tau}_2\mathcal{A}(\boldsymbol{a})+\frac{\widehat{\tau}_1^2+\widehat{\tau}_2^2}{3}\mathcal{B}(\boldsymbol{a})\\ &\quad+(\gamma-1)^2\left(\mathcal{F}-\boldsymbol{a}^I\partial_I\mathcal{F}+\frac{1}{2}\boldsymbol{a}^I\boldsymbol{a}^J\partial_I\partial_J\mathcal{F}\right)+(\gamma-1)(\widehat{\tau}_1+\widehat{\tau}_2)\left(H-\boldsymbol{a}^I\partial_I H\right)\\ &\quad+\widehat{\mathfrak{n}}^I\left(\gamma\partial_I\mathcal{F}+(\widehat{\tau}_1+\widehat{\tau}_2)\partial_I H-(\gamma-1)\boldsymbol{a}^J\partial_I\partial_J\mathcal{F}\right)+\frac{1}{2}\widehat{\mathfrak{n}}^I\widehat{\mathfrak{n}}^J\partial_I\partial_J\mathcal{F}\bigg)\,,\end{aligned}\tag{6.15}$$

[27] More general gluings could be considered (similarly to the 3d computations in [12]), but this would go beyond the class of principal circle bundles that we consider in this paper.

where we used the notation $\partial_I = \frac{\partial}{\partial \boldsymbol{a}^I}$. When considering the extended topological twist as in (6.9), the general non-equivariant limit is obtained from (6.15) through the substitution:

$$H \to \varepsilon^I \partial_I \mathcal{F}\,, \qquad \mathcal{A} \to \mathcal{A} + 2\mathcal{G}\,, \qquad \mathcal{B} \to \mathcal{B} + 3\mathcal{G}\,, \tag{6.16}$$

with $\mathcal{G}$ defined in (6.11).

6.1.3 Refined topological string partition function and refined GV invariants

We are particularly interested in the 5d SCFTs that can be engineered at canonical singularities in M-theory [105, 106]; see e.g. [59, 107–110] for recent studies. Then, the Coulomb-branch low-energy effective theory on the Ω-background is obtained by considering the low-energy limit of M-theory on:

$$\mathbb{C}^2 \times S^1 \times \widetilde{\mathbf{X}}\,, \tag{6.17}$$

with the Ω-background turned on along $\mathbb{C}^2$. Here, $\widetilde{\mathbf{X}}$ denotes the crepant resolution of a threefold canonical singularity $\mathbf{X}$. Let us further choose $\mathbf{X}$ and its resolution to be toric. Then, the Nekrasov partition function of the five-dimensional theory can be computed using the refined topological vertex formalism [58, 111].

In the geometric-engineering picture, the various Coulomb-branch parameters of the 5d theory are now Kähler parameters of the crepant resolution $\widetilde{\mathbf{X}}$. In keeping with standard notation, we use the fugacities q, $t = p^{-1}$ and Q, defined as:

$$q \equiv e^{2\pi i \tau_1}\,, \qquad p = t^{-1} \equiv e^{2\pi i \tau_2}\,, \qquad Q^{\boldsymbol{\beta}} \equiv e^{2\pi i \int_{\boldsymbol{\beta}} (B + iJ)} = e^{2\pi i \boldsymbol{\beta}(\boldsymbol{a})}\,, \tag{6.18}$$

where $\boldsymbol{\beta} \cong [\mathcal{C}] \in H_2(\widetilde{\mathbf{X}}, \mathbb{Z})$ denote the homology class of any effective curve in $\widetilde{\mathbf{X}}$, and $B + iJ$ is the complexified Kähler form in Type IIA string theory.

The Nekrasov partition function of the 5d theory is expected to be equivalent to the refined topological string partition function for the threefold $\widetilde{\mathbf{X}}$ [57, 58]. In the M-theory approach, the 5d BPS states arise as M2-branes wrapped over curves. One can then write the Nekrasov partition function as a product over these BPS states, of electric charge $\boldsymbol{\beta}$ and spin (j_l, j_r) [49, 58]:

$$Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a}, \tau_1, \tau_2) = \prod_{\boldsymbol{\beta}} \prod_{j_l, j_r = 0}^{\infty} \left[\mathbf{Z}^{j_l, j_r}_{\mathbb{C}^2\times S^1}(Q^{\boldsymbol{\beta}}, q, p)\right]^{N^{\boldsymbol{\beta}}_{j_l, j_r}}\,, \tag{6.19}$$

where the non-negative integers $N^{\boldsymbol{\beta}}_{j_l, j_r}$ are the refined Gopakumar-Vafa invariants. Here, the higher-spin particles contribute as:

$$\mathbf{Z}^{j_l, j_r}_{\mathbb{C}^2\times S^1}(Q, q, p) \equiv \prod_{m_l = -j_l}^{j_l} \prod_{m_r = -j_r}^{j_r} \left(Q q^{\frac{1}{2} + m_r + m_l} p^{\frac{1}{2} + m_r - m_l}; q, p\right)_{\infty}^{(-1)^{1 + 2j_l + 2j_r}}\,, \tag{6.20}$$

which is written in terms of the double-Pochhammer symbol:

$$(x; q, p)_{\infty} \equiv \prod_{j,k=0}^{\infty} \left(1 - x q^j p^k\right)\,. \tag{6.21}$$

Note that the definition (6.21) is only valid for $\mathrm{Im}(\tau_i) > 0$. This can be analytically continued to $|q| \neq 1$, $|p| \neq 1$ [112], which gives us the formal identities:

$$(x; q^{-1}, p)_\infty = (xq; q, p)_\infty^{-1} \,, \qquad (x; q, p^{-1})_\infty = (xp; q, p)_\infty^{-1} \,. \tag{6.22}$$

The expression (6.19) gives us the Nekrasov partition function for the ordinary Ω-deformed DW twist, and we can also obtain the extended DW twist expression by the substitution $Q \to Q(qp)^\varepsilon$. For completeness, let us also mention that the unrefined topological string limit corresponds to setting $t = p^{-1} = q$, giving us:

$$Z_{\mathrm{top}}(Q, q) = \prod_\beta \prod_{j_l=0}^{\infty} \widetilde{\mathbf{Z}}_{\mathrm{top}}^{j_l}(Q^\beta, q)^{N_{j_l}^\beta} \,, \tag{6.23}$$

with:

$$\widetilde{\mathbf{Z}}_{\mathrm{top}}^{j_l}(Q, q) = \prod_{m_l=-j_l}^{j_l} \prod_{k=1}^{\infty} \left[\left(1 - Qq^{k+2m_l}\right)^k\right]^{(-1)^{2j_l}} \,, \tag{6.24}$$

in terms of the unrefined GV invariants $N_{j_l}^\beta \equiv \sum_{j_r} (-1)^{2j_r}(2j_r + 1) N_{j_l, j_r}^\beta$. The (refined) GV invariants have to be computed explicitly, for any given toric threefold $\widetilde{\mathbf{X}}$, for instance using the (refined) topological vertex formalism [58].

6.1.4 Quantum trilogarithm and GV expansion in the non-equivariant limit

It is interesting to consider the non-equivariant limit of the expression (6.19). We find it useful to introduce the 'quantum trilogarithm' defined as:

$$\mathrm{Li}_3(x; q, p) \equiv -\log(x; q, p)_\infty = \sum_{n=1}^{\infty} \frac{x^n}{n} \frac{1}{(1-q^n)(1-p^n)} \,. \tag{6.25}$$

In the small-τ_i limit, it admits an asymptotic expansion:

$$\mathrm{Li}_3(x; q, p) = \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m}}{n!\, m!} B_n B_m (2\pi i \tau_1)^{n-1} (2\pi i \tau_2)^{m-1} \mathrm{Li}_{3-n-m}(x) \,, \tag{6.26}$$

where B_n are the Bernoulli numbers.[28] We then have:

$$\mathrm{Li}_3(x; q, p) \approx \frac{1}{(2\pi i)^2 \tau_1 \tau_2} \mathrm{Li}_3(x) - \frac{1}{4\pi i} \frac{\tau_1 + \tau_2}{\tau_1 \tau_2} \mathrm{Li}_2(x) - \frac{1}{12}\left(3 + \frac{\tau_1}{\tau_2} + \frac{\tau_2}{\tau_1}\right) \log(1-x) \,. \tag{6.27}$$

For a massive hypermultiplet with the extended DW twist, we have

$$\log Z^{\mathcal{H}}_{\mathbb{C}^2 \times S^1}(\boldsymbol{a}, \tau_1, \tau_2; \varepsilon) = \mathrm{Li}_3(Q(qp)^{\frac{1}{2}+\varepsilon}; q, p) \,. \tag{6.28}$$

Setting $\varepsilon = 0$ for simplicity, the $\tau_i \to 0$ limit reads:

$$\log Z^{\mathcal{H}}_{\mathbb{C}^2 \times S^1}(\boldsymbol{a}, \tau_1, \tau_2) \approx -\frac{2\pi i}{\tau_1 \tau_2} \left(-\frac{1}{(2\pi i)^3} \mathrm{Li}_3(Q) - \left(\frac{\tau_1^2 + \tau_2^2}{3}\right) \frac{1}{16\pi i} \log(1-Q)\right) \,, \tag{6.29}$$

[28] With the convention that $B_0 = 1$ and $B_1 = \frac{1}{2}$.

from which we can read off (5.26). More generally, for the equivariant DW twist ($\varepsilon = 0$), we have the refined GV expansion:

$$\log Z_{\mathbb{C}^2\times S^1} = \sum_{\beta}\sum_{j_l,j_r} N^{\beta}_{j_l,j_r} \log \mathbf{Z}^{j_l,j_r}_{\mathbb{C}^2\times S^1}(Q^\beta, q, p)\,, \tag{6.30}$$

with:

$$\log \mathbf{Z}^{j_l,j_r}_{\mathbb{C}^2\times S^1}(Q,q,p) = (-1)^{2j_l+2j_r} \sum_{m_l=-j_l}^{j_l}\sum_{m_r=-j_r}^{j_r} \mathrm{Li}_3(Q q^{\frac{1}{2}+m_r+m_l} p^{\frac{1}{2}+m_r-m_l}; q, p)\,. \tag{6.31}$$

By taking the small-τ_i limit and comparing to (6.7), one can extract the contribution of a spin-(j_l, j_r) particle (of unit electric charge, $\boldsymbol{\beta} = 1$) to the low-energy effective couplings. By a straightforward computation, one finds:

$$\begin{aligned} &\mathcal{F}^{j_l,j_r} = -\frac{c_0^{(j_l,j_r)}}{(2\pi i)^3}\mathrm{Li}_3(Q)\,,\\ &\mathcal{A}^{j_l,j_r} = \frac{1}{2\pi i}c_{\mathcal{A}}^{(j_l,j_r)}\log(1-Q)\,, \qquad \mathcal{B}^{j_l,j_r} = \frac{1}{2\pi i}c_{\mathcal{B}}^{(j_l,j_r)}\log(1-Q)\,, \end{aligned} \tag{6.32}$$

with the coefficients:

$$\begin{aligned} c_0^{(j_l,j_r)} &= (-1)^{2j_l+2j_r}(2j_l+1)(2j_r+1)\,,\\ c_{\mathcal{A}}^{(j_l,j_r)} &= (-1)^{2j_l+2j_r}(2j_l+1)(2j_r+1)\frac{j_r(j_r+1)-j_l(j_l+1)}{3}\,,\\ c_{\mathcal{B}}^{(j_l,j_r)} &= (-1)^{2j_l+2j_r}(2j_l+1)(2j_r+1)\left(-\frac{1}{8}+\frac{j_l(j_l+1)+j_r(j_r+1)}{2}\right)\,, \end{aligned} \tag{6.33}$$

exactly as anticipated in section 5.3, and in perfect agreement with the index computation of section 4.4. Note also that, for the extended topological twist, one has the additional terms H and $\mathcal{G}$ in (6.10), namely:

$$H^{j_l,j_r} = -\frac{c_0^{(j_l,j_r)}}{(2\pi i)^2}\varepsilon\,\mathrm{Li}_2(Q)\,, \qquad \mathcal{G}^{j_l,j_r} = \frac{c_0^{(j_l,j_r)}}{4\pi i}\varepsilon^2\log(1-Q)\,, \tag{6.34}$$

according to (6.11).

6.2 Gluing rules for circle fibrations

Let us now consider the explicit gluing of Nekrasov partition functions to obtain the circle-fibered five-manifold:

$$S^1 \to \mathcal{M}_5 \to \mathcal{M}_4\,, \tag{6.35}$$

where $\mathcal{M}_4$ is a toric four-manifold. For definiteness, we will mostly focus on the case when $\mathcal{M}_4$ is one of the five toric Fano surfaces, $\mathbb{P}^2$, $\mathbb{F}_0 \cong \mathbb{P}^1\times\mathbb{P}^1$, or dP_n (the blow-up of $\mathbb{P}^2$ at n points) with $n \leq 3$, whose Euler characteristic and signature are:

	$\mathbb{P}^2$	$\mathbb{F}_0$	dP_1	dP_2	dP_3
χ	3	4	4	5	6
σ	1	0	0	-1	-2

(6.36)

Let us first review the case of a trivial fibration, $\mathcal{M}_5 = \mathcal{M}_4 \times S^1$, before considering the case of a non-trivial fibration.

6.2.1 The $\mathcal{M}_4 \times S^1$ partition function

It was conjectured in [42] that the Ω-deformed Coulomb-branch partition function on a toric manifold $\mathcal{M}_4$ can be obtained by gluing Nekrasov partition functions for each fixed point of the toric action. The full partition function is then obtained, in principle, by a particular contour integral over the CB parameters, together with a sum over fluxes, both of which one should determine by a more careful analysis. This approach was further developed in [29, 30, 113], and generalised to 5d theories on $\mathcal{M}_4 \times S^1$ in [23]. The full partition function then reads:

$$\mathbf{Z}_{\mathcal{M}_4\times S^1} = \sum_{\mathfrak{n}_l} \oint d\boldsymbol{a} \prod_{l=1}^{\chi(\mathcal{M}_4)} Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a} + \tau_1^{(l)}\mathfrak{n}_l + \tau_2^{(l)}\mathfrak{n}_{l+1}, \tau_1^{(l)}, \tau_2^{(l)})\,, \tag{6.37}$$

with $\mathfrak{n}_l$ being fluxes associated with the toric divisors $D_l \subset \mathcal{M}_4$, corresponding to a line bundle:

$$L = \mathcal{O}(-\sum_l \mathfrak{n}_l D_l)\,, \tag{6.38}$$

over $\mathcal{M}_4$. Note that there are $\chi(\mathcal{M}_4)$ toric divisors, with 2 linear relations amongst them. The previously defined $U(1)_I$ background fluxes $\mathfrak{m}^I$ are then given by:

$$\sum_{k=1}^{\chi-2} \mathfrak{m}_k^I [\mathrm{S}_k] = -\sum_{l=1}^{\chi} \mathfrak{n}_l^I [D_l]\ . \tag{6.39}$$

The equivariant parameters $\tau_i^{(l)}$ are linear combinations of $\tau_{1,2}$, which we shall comment on momentarily. The non-equivariant limit of the integrand of (6.37) can be obtained by a direct computation using (6.15) (with $\gamma = 1$), wherein all divergent pieces cancel out between patches, leaving us with a finite quantity. One finds:

$$\begin{aligned}\log Z_{\mathcal{M}_4\times S^1}(\boldsymbol{a}) \approx -2\pi i\Bigg(&\chi\mathcal{A}(\boldsymbol{a}) + \sigma\mathcal{B}(\boldsymbol{a}) + \Big(\sum D_l\Big)\cdot\Big(\sum \mathfrak{n}_l^I D_l\Big)\frac{\partial H(\boldsymbol{a})}{\partial \boldsymbol{a}^I} + \\ &+\frac{1}{2}\Big(\sum \mathfrak{n}_l^I D_l\Big)\cdot\Big(\sum \mathfrak{n}_l^J D_l\Big)\frac{\partial^2 \mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J}\Bigg)\,.\end{aligned} \tag{6.40}$$

For the DW twist, we have $H = 0$ and therefore:

$$Z_{\mathcal{M}_4\times S^1}(\boldsymbol{a}) = \boldsymbol{A}(\boldsymbol{a})^\chi \boldsymbol{B}(\boldsymbol{a})^\sigma \boldsymbol{\Pi}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{m},\mathfrak{m})}\,, \tag{6.41}$$

in terms of the quantities defined in section 5.1. This reproduces and generalises the results of [23, 29, 35]. More generally, as explained at length in previous sections, we should consider the extended DW twist with $\varepsilon \neq 0$, in which case we should substitute (6.16) into (6.40). Then, using the fact that $\mathcal{K}_{\mathcal{M}_4} \cong \mathcal{O}(-\sum_l D_l)$, we find:

$$Z_{\mathcal{M}_4\times S^1}(\boldsymbol{a};\varepsilon) = \boldsymbol{A}(\boldsymbol{a})^\chi \boldsymbol{B}(\boldsymbol{a})^\sigma \boldsymbol{G}(\boldsymbol{a};\varepsilon)^{2\chi+3\sigma} \boldsymbol{\Pi}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{m}+2\varepsilon\mathbf{k},\mathfrak{m})}\,, \tag{6.42}$$

which exactly reproduces the formula (5.22). Next, let us explain how the Nekrasov partition functions have been glued together.

Equivariant parameters. The patch-dependent equivariant parameters $\tau^{(l)}$ in (6.37) are determined by the toric data, as follows (see e.g. [23] for a more detailed discussion). A compact toric surface $\mathcal{M}_4$ is described by a set of vectors $\vec{n}_l \in \mathbb{Z}^2$, with $l = 1, \ldots, d$, which we order such that $\vec{n}_l$ and $\vec{n}_{l+1}$ are adjacent (with $\vec{n}_{d+1} \equiv \vec{n}_1$). Each such vector is associated to a non-compact divisor D_l.

Each pair of vectors $(\vec{n}_l, \vec{n}_{l+1})$ defines a two-dimensional cone σ_l, to which we can associate an affine variety V_{σ_l}. The construction is based on the dual cone $\hat{\sigma}_l$ generated by the primitive integer vectors $\vec{m}_l$ and $\vec{m}_{l+1}$, which are orthogonal to $\vec{n}_{l+1}$ and $\vec{n}_l$, respectively, and point inwards inside σ_l. The set of holomorphic functions on V_{σ_l} is given by monomials $z_1^{\mu_1} z_2^{\mu_2}$, for all $\vec{\mu} \in \hat{\sigma}_l$. Then, since $V_{\sigma_l} \cong \mathbb{C}^2$ by assumption that $\mathcal{M}_4$ be smooth, the local coordinates on V_{σ_l} can be chosen as:

$$\rho_1^{(l)} = z_1^{m_{l,1}} z_2^{m_{l,2}} \,, \qquad \rho_2^{(l)} = z_1^{m_{l+1,1}} z_2^{m_{l+1,2}} \,. \tag{6.43}$$

The toric variety $\mathcal{M}_4$ is obtained by gluing together the affine varieties V_{σ_l}, by identifying dense open subsets associated with the common vectors spanning the neighbouring cones σ_l.

Due to the Ω-background, the $\mathcal{M}_4 \times S^1$ partition function only receives contribution from the $\chi(\mathcal{M}_4)$ fixed points of the toric action. There is then a single contribution from each chart V_{σ_l} of $\mathcal{M}_4$, as written explicitly in (6.37). Thus, the equivariant parameters will 'transform' under the $(\mathbb{C}^*)^2$ action similarly to (6.43), leading to:

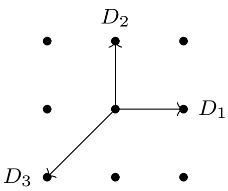
$$\tau_1^{(l)} = \vec{\tau} \cdot \vec{m}_l \,, \qquad \tau_2^{(l)} = \vec{\tau} \cdot \vec{m}_{l+1} \,. \tag{6.44}$$

Furthermore, the (background) gauge fluxes $\mathfrak{n}$, which appear as in (6.37), are similarly local contributions from each patch: thus, the Coulomb branch VEVs change according to:

$$\boldsymbol{a}^{(l)} = \boldsymbol{a} + \tau_1^{(l)} \mathfrak{n}_l + \tau_2^{(l)} \mathfrak{n}_{l+1} \,. \tag{6.45}$$

At this stage, it is natural to wonder how this procedure can be modified to account for a non-trivial $U(1)_{\mathrm{KK}}$ flux, leading to the non-trivial fibration $\mathcal{M}_5$. Before exploring this, let us briefly consider a couple of examples of toric gluings for $\mathcal{M}_4 \times S^1$.

The $\mathcal{M}_4 = \mathbb{P}^2$ case. The simplest example of a toric Kähler four manifold is that of $\mathbb{P}^2$, for which the toric fan and intersection numbers are:



	D_1	D_2	D_3	$\mathcal{K}$
S	1	1	1	-3

(6.46)

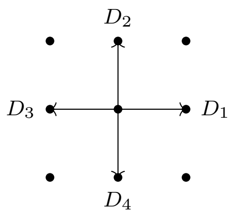
The toric divisors satisfy the linear relations $\mathrm{S} \cong D_1 \cong D_2 \cong D_3$, with $S \cong H$ the hyperplane class. Therefore, given the above triple intersection numbers, we have in our previous notation:

$$\mathfrak{m} = -(\mathfrak{n}_1 + \mathfrak{n}_2 + \mathfrak{n}_3) \,. \tag{6.47}$$

Note also that the canonical divisor is given by $\mathcal{K} \cong -\sum D_l \cong -3D_1$, and thus the Chern number is $\mathbf{k} = -3$. The equivariant parameters (6.44) are given by [23]:

$$\tau_1^{(l)} = (\tau_1, \, -\tau_1 + \tau_2, \, -\tau_2) \,, \qquad \tau_2^{(l)} = (\tau_2, \, -\tau_1, \, \tau_1 - \tau_2) \,. \tag{6.48}$$

The $\mathcal{M}_4 = \mathbb{F}_0$ case. Consider now the case of $\mathbb{F}_0 \cong S^1 \times S^1$, with the following toric data:



	D_1	D_2	D_3	D_4	$\mathcal{K}$
S_1	0	1	0	1	-2
S_2	1	0	1	0	-2

(6.49)

The toric divisors satisfy the linear relations $\mathrm{S}_1 \cong D_1 \cong D_3$ and $\mathrm{S}_2 \cong D_2 \cong D_4$ and, thus, there are two distinct compact curves S_1 and S_2, corresponding to the two $\mathbb{P}^1$ factors in $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. In this basis, the intersection form reads:

$$Q_{\mathbb{F}_0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{6.50}$$

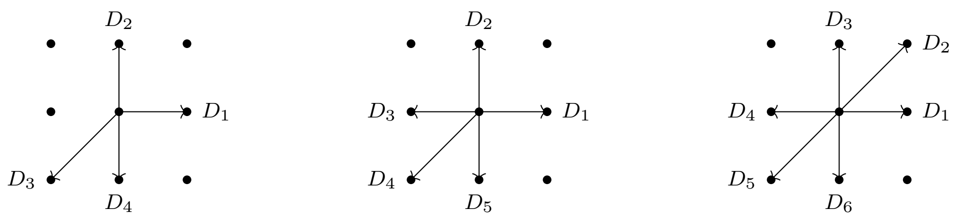
Furthermore, the canonical divisor is $\mathcal{K} = -2D_1 - 2D_2$, leading to the Chern numbers $\mathbf{k} = (-2, -2)$. Given the above triple intersection numbers, we also have the fluxes:

$$\mathfrak{m} = (\mathfrak{m}_1, \mathfrak{m}_2) = (-\mathfrak{n}_1 - \mathfrak{n}_3,\ -\mathfrak{n}_2 - \mathfrak{n}_4) . \tag{6.51}$$

Finally, the equivariant parameters are given by [23]:

$$\tau_1^{(l)} = (\tau_1,\ \tau_2,\ -\tau_1,\ -\tau_2) , \qquad \tau_2^{(l)} = (\tau_2,\ -\tau_1,\ -\tau_2,\ \tau_1) . \tag{6.52}$$

The dP_n cases. For completeness, let us also give the equivariant parameters for the remaining toric del Pezzo surfaces. The toric fans for dP_n with $n = 1, 2, 3$ read:



(6.53)

respectively, and the equivariant parameters are:

$$\begin{aligned}
dP_1: &\quad \tau_1^{(l)} = (\tau_1, -\tau_1 + \tau_2, -\tau_1, -\tau_2), &\quad \tau_2^{(l)} &= (\tau_2, -\tau_1, \tau_1 - \tau_2, \tau_1), \\
dP_2: &\quad \tau_1^{(l)} = (\tau_1, \tau_2, -\tau_1 + \tau_2, -\tau_1, -\tau_2), &\quad \tau_2^{(l)} &= (\tau_2, -\tau_1, -\tau_2, \tau_1 - \tau_2, \tau_1), \\
dP_3: &\quad \tau_1^{(l)} = (\tau_1 - \tau_2, \tau_1, \tau_2, -\tau_1 + \tau_2, -\tau_1, -\tau_2), &\quad \tau_2^{(l)} &= (\tau_2, -\tau_1 + \tau_2, -\tau_1, -\tau_2, \tau_1 - \tau_2, \tau_1).
\end{aligned}$$

6.2.2 The S^5 and $L(\mathfrak{p}; 1)$ partition functions

Let us now turn to the simplest and most important instance of a circle fibration $\mathcal{M}_5 \to \mathcal{M}_4$, which is the five-sphere viewed as a circle fibration over the complex projective plane:

$$S^1 \longrightarrow S^5 \longrightarrow \mathbb{P}^2 . \tag{6.54}$$

The gluing approach was first considered in [49], but it is worthwhile to discuss the argument in some detail. The metric of the round five-sphere can be written as $ds^2 = \sum dz_i d\bar{z}_i$, in terms of the coordinates $(z_i) \in \mathbb{C}^3$ subject to the constraint $\sum |z_i| = 1$. Alternatively, we can parametrise the five-sphere using the angles $\theta, \phi \in (0, \pi/2)$ and $\chi_i \in (0, 2\pi)$, as:

$$z_1 = e^{i\chi_1} \sin\theta \cos\phi\,, \qquad z_2 = e^{i\chi_2} \sin\theta \sin\phi\,, \qquad z_3 = e^{i\chi_3} \cos\theta\;. \tag{6.55}$$

In these coordinates, the S^5 metric reads:

$$ds^2 = d\theta^2 + \sin^2\theta\; d\phi^2 + \sin^2\theta \cos^2\phi\; d\chi_1^2 + \sin^2\theta \sin^2\phi\; d\chi_2^2 + \cos^2\theta\; d\chi_3\;. \tag{6.56}$$

To apply our formalism, we should write this metric in the general form (3.5), namely as $ds^2(S^5) = ds^2(\mathbb{P}^2) + (d\psi + \mathrm{C})^2$, where $ds^2(\mathbb{P}^2)$ is the Fubini-Study metric on the base $\mathbb{P}^2$. An important requirement is that the connection C should be well defined on each coordinate patch on the base. Let us then consider the Fubini-Study metric on each patch and subtract it from the S^5 metric, in order to find the connection C on that patch. The z_i coordinates of the five-sphere descend to coordinates of the projective space $\mathbb{P}^2$. As such, let us denote by $V_i \cong \mathbb{C}^2$ the patch with coordinates $w_j = z_j/z_i$, for $j \neq i$ and $z_i \neq 0$, and define the corresponding azimuthal coordinates:

$$\begin{aligned}
&\mathrm{Patch} V_1: \quad \rho_1^{(1)} = \chi_2 - \chi_1\,, \quad \rho_2^{(1)} = \chi_3 - \chi_1\,,\\
&\mathrm{Patch} V_2: \quad \rho_1^{(2)} = \chi_3 - \chi_2\,, \quad \rho_2^{(2)} = \chi_1 - \chi_2\,,\\
&\mathrm{Patch} V_3: \quad \rho_1^{(3)} = \chi_1 - \chi_3\,, \quad \rho_2^{(3)} = \chi_2 - \chi_3\;.
\end{aligned} \tag{6.57}$$

For each coordinate patch, the coordinate along the S^1 fiber will be a linear combination of the χ_i angles, $\psi = \alpha_i \chi_i$. With the normalisation $\sum_i \alpha_i = 1$, we find the $U(1)_{\mathrm{KK}}$ connection in each patch to be:

$$\begin{aligned}
\mathrm{C}^{(1)} &= \frac{1}{4}(1 - 4\alpha_2 - \cos(2\theta) - 2\cos(2\phi)\sin(\theta)^2) d\rho_1^{(1)} + \frac{1}{2}(1 - 2\alpha_3 + \cos(2\theta)) d\rho_2^{(1)}\,,\\
\mathrm{C}^{(2)} &= \frac{1}{2}(1 - 2\alpha_3' + \cos(2\theta)) d\rho_1^{(2)} + \frac{1}{4}(1 - 4\alpha_1' - \cos(2\theta) + 2\cos(2\phi)\sin(\theta)^2) d\rho_2^{(2)}\,,\\
\mathrm{C}^{(3)} &= \frac{1}{4}(1 - 4\alpha_1'' - \cos(2\theta) + 2\cos(2\phi)\sin(\theta)^2) d\rho_1^{(3)}\\
&\quad + \frac{1}{4}(1 - 4\alpha_2'' - \cos(2\theta) - 2\cos(2\phi)\sin(\theta)^2) d\rho_2^{(3)}\;.
\end{aligned} \tag{6.58}$$

Let us note that the patch V_i is not defined at $z_i = 0$, at which point the differential $d\chi_i$ is ill-defined. Then, imposing continuity for the connection and well-definiteness on every coordinate patch (that is, the absence of 'Dirac string' singularities), we should pick the following coordinates along the S^1 fiber:

$$\psi^{(1)} = \chi_1\,, \qquad \psi^{(2)} = \chi_2\,, \qquad \psi^{(3)} = \chi_3\;. \tag{6.59}$$

In this way, we find the following transformations between angles as we change coordinate patches of the $\mathbb{P}^2$ base:

$$\begin{pmatrix} \rho_1^{(2)} \\ \rho_2^{(2)} \\ \psi^{(2)} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho_1^{(1)} \\ \rho_2^{(1)} \\ \psi^{(1)} \end{pmatrix}, \qquad \begin{pmatrix} \rho_1^{(3)} \\ \rho_2^{(3)} \\ \psi^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \rho_1^{(1)} \\ \rho_2^{(1)} \\ \psi^{(1)} \end{pmatrix}. \tag{6.60}$$

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In the toric description of S^5, the five-sphere is a T^3 fibration over a triangle [49]. Moreover, the Ω-background parameters $\tau_1^{(i)}$ and $\tau_2^{(i)}$ on each patch $V_l \cong \mathbb{C}^2$ can be interpreted as complex structure parameters for the tori $T^2 \subset T^3$ spanned by the angular coordinates $(\rho_1^{(i)}, \psi_i)$ and $(\rho_2^{(i)}, \psi_i)$, respectively. The $SL(3,\mathbb{Z})$ transformation matrices (6.60) then suggest the following gluing rules for the Nekrasov partition functions:

$$\begin{aligned}
V_1 \;&: & \tau_1\,, \qquad & & \tau_2\,, \qquad & & \boldsymbol{a}\,, \\
V_2 \;&: & \tau_1^* = \frac{-\tau_1+\tau_2}{\tau_1+1}\,, \qquad & & \tau_2^* = \frac{-\tau_1}{\tau_1+1}\,, \qquad & & \boldsymbol{a}^* = \frac{\boldsymbol{a}}{\tau_1+1}\,, \\
V_3 \;&: & \widetilde{\tau}_1 = \frac{-\tau_2}{\tau_2+1}\,, \qquad & & \widetilde{\tau}_2 = \frac{\tau_1-\tau_2}{\tau_2+1}\,, \qquad & & \widetilde{\boldsymbol{a}} = \frac{\boldsymbol{a}}{\tau_2+1}\,,
\end{aligned} \tag{6.61}$$

which generalises (6.48).

Lens spaces. It is also instructive to consider lens spaces $S^5/\mathbb{Z}_p$, as a simple generalisation of the above. The lens space $L(p; q_1, q_2, q_3)$ can be defined as a quotient of $S^5 \subset \mathbb{C}^3$ by the $\mathbb{Z}_p$ action generated by:

$$(z_1, z_2, z_3) \mapsto \left(e^{2\pi i \frac{q_1}{p}} z_1, e^{2\pi i \frac{q_2}{p}} z_2, e^{2\pi i \frac{q_3}{p}} z_3\right)\,, \tag{6.62}$$

with $q_i, p \in \mathbb{Z}$ and q_i coprime to p. For generic values of q_i, however, these five-manifolds are not fibrations over $\mathbb{P}^2$, but rather over singular quotients of $\mathbb{P}^2$, as one can see by considering the induced action on the $\mathbb{P}^2$ coordinates (6.57). In this paper, we restrict our attention to principal circle bundles over four-manifolds, which corresponds to the case $q_i = 1$. We denote the resulting lens space by $L(p;1)$. It is simply a principal circle bundle:

$$S^1 \to L(\mathfrak{p};1) \to \mathbb{P}^2\,, \tag{6.63}$$

with $\mathfrak{p} = p$. We can then derive the gluing rules in the same way as for the round S^5. One finds:

$$Z_{L(\mathfrak{p};1)}(\boldsymbol{a})_{\mathfrak{m}} = \prod_{l=1}^{\chi(\mathbb{P}^2)} Z_{\mathbb{C}^2\times S^1}\left(\frac{\boldsymbol{a}+\tau_1^{(l)}\mathfrak{n}_l+\tau_2^{(l)}\mathfrak{n}_{l+1}}{\gamma^{(l)}}\,, \frac{\tau_1^{(l)}}{\gamma^{(l)}}\,, \frac{\tau_2^{(l)}}{\gamma^{(l)}}\right)\,, \tag{6.64}$$

where $\tau_i^{(l)}$ are the equivariant parameters appearing in the $\mathbb{P}^2 \times S^1$ gluing (6.48), while the denominators γ are given by:

$$\gamma^{(l)} = (1,\, p\tau_1+1,\, p\tau_2+1)\ . \tag{6.65}$$

In (6.64) we also allowed for background fluxes, as in (6.37). In the non-equivariant limit (and turning on ε as before), this expression reproduces exactly the one expected from (5.36).

6.2.3 Fibrations over toric Kähler 4-manifolds

Having discussed fibrations over $\mathbb{P}^2$, it is natural to consider the generalisation to any toric $\mathcal{M}_4$. Here, we first derive the gluing formula for $\mathcal{M}_4 = \mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, which has an explicitly known Kähler metric. We then conjecture a gluing formula in the general case.

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The $\mathcal{M}_4 = \mathbb{F}_0$ case. Consider circle fibrations over $\mathbb{F}_0$, with Chern numbers $\mathfrak{p} = (\mathfrak{p}_1, \mathfrak{p}_2)$, such that the metric of such a space is given by:

$$ds^2 = \sum_{i=1}^{2} \left(d\theta_i^2 + \sin^2\theta_i d\phi_1^2 \right) + \left(d\psi + \frac{1}{2} \sum_{i=1}^{2} \mathfrak{p}_i (\pm 1 + \cos\theta_i) d\phi_i \right)^2 , \tag{6.66}$$

with $\theta_i \in [0, \pi)$, $\phi_i \in [0, 2\pi)$ and $\psi \in [0, 2\pi)$. This space has four coordinate patches, corresponding to the two patches of each of the $\mathbb{P}^1$ spaces. We proceed as before, by finding the well-defined coordinates on each patch. Defining:

$$\gamma^{(l)} = (1,\, \mathfrak{p}_1\tau_1 + 1,\, \mathfrak{p}_1\tau_1 + \mathfrak{p}_2\tau_2 + 1,\, \mathfrak{p}_2\tau_2 + 1)\, , \tag{6.67}$$

we then propose that the partition function on non-trivial fibrations over $\mathbb{F}_0$ is given by:

$$Z_{\mathbb{F}_0^{(\mathfrak{p})}}(\boldsymbol{a}) = \prod_{l=1}^{\chi(\mathbb{F}_0)} Z_{\mathbb{C}^2\times S^1} \left(\frac{\boldsymbol{a} + \tau_1^{(l)}\mathfrak{n}_l + \tau_2^{(l)}\mathfrak{n}_{l+1}}{\gamma^{(l)}} , \frac{\tau_1^{(l)}}{\gamma^{(l)}} , \frac{\tau_2^{(l)}}{\gamma^{(l)}} \right) , \tag{6.68}$$

where, as before, $\tau_i^{(l)}$ are the equivariant parameters appearing in the $\mathbb{F}_0 \times S^1$ case and $\mathfrak{n}_l$ are fluxes associated with the toric divisors.

General toric Kähler surfaces. We would like to generalise the above result to any principal circle bundle over a toric Kähler surface $\mathcal{M}_4$. The prescription used for non-trivial fibrations over $\mathbb{P}^2$ and $\mathbb{F}_0$ involved the coordinates $(\rho_1^{(l)}, \rho_2^{(l)})$ along the coordinate patches V_{σ_l} of the base four-manifold, as well as the coordinate along the fiber $\psi^{(l)}$. As explained in the previous sections, a non-trivial fibration can be viewed as a non-trivial flux for a $U(1)_{\mathrm{KK}}$ background symmetry on $\mathcal{M}_4$. We then propose that the denominators $\gamma^{(l)}$ should be given by:

$$\gamma^{(l)} = 1 + \tau_1^{(l)} p_l + \tau_2^{(l)} p_{l+1}\, , \tag{6.69}$$

where p_l are $U(1)_{\mathrm{KK}}$ fluxes associated with the non-compact toric divisors D_l, such that:

$$\sum_{k=1}^{\chi-2} \mathfrak{p}_k [\mathrm{S}_k] = - \sum_{l=1}^{\chi} p_l D_l\, , \tag{6.70}$$

as in (6.39). Thus, the CB partition function on non-trivial fibrations over $\mathcal{M}_4$ with Chern numbers $\mathfrak{p}$ should generalise to:

$$\mathbf{Z}_{\mathcal{M}_5} = \sum_{\mathfrak{n}_l} \oint d\boldsymbol{a} \prod_{l=1}^{\chi(\mathcal{M}_4)} Z_{\mathbb{C}^2\times S^1} \left(\frac{\boldsymbol{a} + \tau_1^{(l)}\mathfrak{n}_l + \tau_2^{(l)}\mathfrak{n}_{l+1}}{\gamma^{(l)}}, \frac{\tau_1^{(l)}}{\gamma^{(l)}}, \frac{\tau_2^{(l)}}{\gamma^{(l)}} \right) , \tag{6.71}$$

naturally generalising the Nekrasov conjecture [42]. (Here, as in the original conjecture, the precise form of the contour integration and of the sum over fluxes remain to be determined.)

Given the factorised integrand $Z_{\mathcal{M}_5}$ in (6.71) with the non-trivial Ω-background, we can again check our general formalism by taking the non-equivariant limit, generalising the formula (6.40). For every toric Fano four-manifold $\mathcal{M}_4$ in (6.36), using (6.15), we find the following expression:

$$\begin{aligned}\log Z_{\mathcal{M}_5}(\boldsymbol{a}) \approx \log Z_{\mathcal{M}_4\times S^1}(\boldsymbol{a}) - 2\pi i \left(\sum D_l\right)\cdot\left(\sum p_l D_l\right)\left(H(\boldsymbol{a}) - \boldsymbol{a}^I\frac{\partial H(\boldsymbol{a})}{\partial \boldsymbol{a}^I}\right)\\ + \left(\sum p_l D_l\right)\cdot\left(\sum \mathfrak{n}_l^I D_l\right)\log \mathscr{K}_I(\boldsymbol{a}) + \frac{1}{2}\left(\sum p_l D_l\right)\cdot\left(\sum p_l D_l\right)\log\mathscr{F}(\boldsymbol{a})\,,\end{aligned} \tag{6.72}$$

where $Z_{\mathcal{M}_4\times S^1}$ is given by (6.40) and $\mathscr{F}$, $\mathscr{K}$ are precisely the quantities defined in (5.34) and (5.35), respectively. Then, using the relation (6.70), as well as the substitution $H\to \varepsilon^I\partial_I\mathcal{F}$ for the extended topological twist, we recover the complete master formula (5.36) for the CB partition function on $\mathcal{M}_5$.

We should note that the proposal (6.71) appears slightly different from the results above for $\mathbb{P}^2$ and $\mathbb{F}_0$, though all the formulas agree perfectly in the non-equivariant limit. For instance, for $\mathbb{P}^2$, the gluing (6.71) uses:

$$\gamma^{(l)}_{\rm new} = \left(1+p_1\tau_1+p_2\tau_2,\ 1-(p_2+p_3)\tau_1+p_2\tau_2,\ 1+p_1\tau_1-(p_1+p_3)\tau_2\right), \tag{6.73}$$

where $\mathfrak{p}=-(p_1+p_2+p_3)$, while previously we derived:

$$\gamma^{(l)} = \left(1,\ \mathfrak{p}\tau_1+1,\ \mathfrak{p}\tau_2+1\right)\,. \tag{6.74}$$

However, setting $p_1=p_2=0$, the two expressions become identical. Similar comments hold true in general. It might be the case that the individual fluxes p_l (and $\mathfrak{n}_l$) have an intrinsic meaning on the Ω-deformed $\mathcal{M}_5$, which we did not explore. Our main motivation, here, was to provide a strong consistency check for our formulas for the fibering operator in the DW-twisted theory, hence our main focus was on the non-equivariant limit.

6.3 The spin/charge relation for the E_n theories

Finally, let us further comment on the spin/charge relation implied by the constraint (5.50). In appendix C, we explicitly compute the refined GV invariants of the so-called toric E_n 5d SCFTs, which are rank-one SCFTs that can be deformed to a 5d $\mathcal{N}=1$ $SU(2)$ gauge theory with $N_f=n-1$ fundamental hypermultiplets. Consider first the E_1 theory, corresponding to the $SU(2)_0$ gauge theory. In this case, perturbatively in the 5d gauge-theory limit, we only have the massive W-boson, of spin $(0,\frac{1}{2})$, which satisfies the condition

$$\frac{1}{2}+j_l+j_r+\boldsymbol{\beta}(\varepsilon)\in\mathbb{Z}\,, \tag{6.75}$$

with charge $\boldsymbol{\beta}=(0,1)$ in the basis corresponding to the two factors $\mathbb{P}_b\times\mathbb{P}_f$ of the local $\mathbb{F}_0$ geometry in M-theory. Hence we need to have $\varepsilon^{I=2} \mod 1 = 0$ in this basis. Similarly, looking at the first instanton particle, $\boldsymbol{\beta}=(1,0)$, we have $\varepsilon^{I=1} \mod 1 = 0$. (The SCFT has a symmetry exchanging the two charges, $\boldsymbol{\beta}=(m,n)\leftrightarrow(m,n)$.) Hence, by consistency, we should have $\frac{1}{2}+j_l+j_r\in\mathbb{Z}$ for any other particle in the spectrum, of any charge. For instance, we have the states:

$$N^{(1,d)}_{j_l,j_r} = \delta_{j_l,0}\delta_{j_r,d+\frac{1}{2}}\,, \qquad\qquad d\in\mathbb{Z}_{\geq 0}\,, \tag{6.76}$$

which capture the one-instanton correction in the $SU(2)$ gauge-theory interpretation. The spin/charge relation should similarly hold for any higher-spin state. This is indeed the case, at least to the order that we have checked it. (See table 1 in appendix.)

Similarly, consider the local dP_2 geometry, corresponding to the E_2 theory, which can be obtained from the $\mathbb{F}_0$ geometry by blowing up a curve $\mathcal{C}_m$. The M2-brane wrapped on $\mathcal{C}_m$ gives a hypermultiplet (a particle of spin $(0,0)$), hence we should choose $\varepsilon^{I=3} \mod 1 = \frac{1}{2}$, in the natural basis $(\mathcal{C}^I) = (\mathbb{P}_b, \mathbb{P}_f, \mathcal{C}_m)$. We thus have the constraint that, for any BPS particle of charge $\boldsymbol{\beta} = (m, n, p)$ in E_2 theory, we should have $\frac{1}{2} + j_l + j_r + \frac{1}{2}p \in \mathbb{Z}$. This is indeed the case (see appendix C.2). Similarly, for the E_3 theory that arises at the local dP_3 geometry, we have a basis $(\mathcal{C}^I) = (\mathbb{P}_b, \mathbb{P}_f, \mathcal{C}_{m_1}, \mathcal{C}_{m_2})$ and we should choose $\varepsilon^{I=3} \mod 1 = \frac{1}{2}$ and $\varepsilon^{I=4} \mod 1 = \frac{1}{2}$. Given the charges $\boldsymbol{\beta} = (m, n, p, q)$, we then need $\frac{1}{2}+j_l+j_r+\frac{1}{2}(p+q) \in \mathbb{Z}$, which is the case (see appendix C.3). Another interesting example is the E_0 SCFT, realised as the local $\mathbb{P}^2$ geometry in M-theory [105]. It has a single charge, $\boldsymbol{\beta} = d$, and no gauge theory interpretation. We should then have $\frac{1}{2} + j_l + j_r + \frac{1}{2}d \in \mathbb{Z}$, which is indeed the case (see [114, table 3]).

Hence, all these models can be coupled consistently to our $\mathcal{M}_5$ background with the extended DW twist on $\mathcal{M}_4$. It would be interesting to understand whether the spin/charge relation must always hold, *a priori*, in any 5d SCFT.

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A Geometry and supersymmetry conventions

In this appendix, we state our conventions for 4d and 5d geometry and we recall various basic facts about 4d $\mathcal{N} = 2$ and 5d $\mathcal{N} = 1$ supersymmetry in flat space. We are working in Euclidean signature throughout.

A.1 Four-dimensional conventions

A.1.1 Flat-space conventions

Consider Euclidean flat space, $\mathbb{R}^4$, with the real Cartesian coordinates x^μ. The rotation group is $SO(4)$. In order to define spinors, we consider its universal cover:

$$\mathrm{Spin}(4) \cong SU(2)_l \times SU(2)_r \; . \tag{A.1}$$

In 4d, we mostly deal with Weyl spinors, ψ_α and $\widetilde{\psi}^{\dot\alpha}$, which transform in the representation with spin $(j_l, j_r) = (\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, respectively, in the standard Wess-and-Bagger notation [115]. In particular, we raise and lower spinor indices with the tensors $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$ and $\epsilon^{\dot\alpha\dot\beta}$, $\epsilon_{\dot\alpha\dot\beta}$ such that $\epsilon^{12} = \epsilon^{\dot 1\dot 2} = \epsilon_{21} = \epsilon_{\dot 2\dot 1} = 1$. We choose the Euclidean σ-matrices:[29]

$$\sigma^\mu = (\sigma^i, i\mathbf{1})\,, \qquad \widetilde\sigma^\mu = (-\sigma^i, i\mathbf{1})\;. \tag{A.2}$$

The Dirac spinor Ψ and the γ-matrices are then given by:

$$\Psi_{\mathbf{a}} = \begin{pmatrix} \psi_\alpha \\ \widetilde\chi^{\dot\alpha} \end{pmatrix}\,, \qquad (\gamma^\mu)_{\mathbf{a}}{}^{\mathbf{b}} = \begin{pmatrix} 0 & \sigma^\mu \\ -\widetilde\sigma^\mu & 0 \end{pmatrix}\,, \tag{A.3}$$

with Dirac indices $\mathbf{a}, \mathbf{b} = 1, \cdots, 4$. Note that:

$$\sigma^\mu\widetilde\sigma^\nu + \sigma^\nu\widetilde\sigma^\mu = -2\delta^{\mu\nu}\,, \qquad \widetilde\sigma^\mu\sigma^\nu + \widetilde\sigma^\nu\sigma^\mu = -2\delta^{\mu\nu}\,, \tag{A.4}$$

and therefore $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$. One also defines the matrices:

$$\sigma^{\mu\nu} \equiv \frac{1}{4}\left(\sigma^\mu\widetilde\sigma^\nu - \sigma^\nu\widetilde\sigma^\mu\right)\,, \qquad \widetilde\sigma^{\mu\nu} \equiv \frac{1}{4}\left(\widetilde\sigma^\mu\sigma^\nu - \widetilde\sigma^\nu\sigma^\mu\right)\,, \tag{A.5}$$

which generate $SU(2)_l$ and $SU(2)_r$ rotation on ψ and $\widetilde\chi$, respectively. They are respectively anti-self-dual (ASD) and self-dual (SD):

$$\frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\sigma_{\rho\lambda} = -\sigma^{\mu\nu}\,, \qquad \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\widetilde\sigma_{\rho\lambda} = \widetilde\sigma^{\mu\nu}\;. \tag{A.6}$$

We also extensively use the complex coordinates:

$$z^1 = x^1 + ix^2\,, \qquad z^2 = x^3 + ix^4\,, \tag{A.7}$$

on $\mathbb{C}^2 \cong \mathbb{R}^4$.

A.1.2 Curved-space conventions

On $\mathcal{M}_4$ a Riemannian manifold, with the metric:

$$ds^2(\mathcal{M}_4) = g_{\mu\nu}\, dx^\mu dx^\nu = \delta_{ab}\, \widehat e^a_\mu \widehat e^b_\nu\, dx^\mu dx^\nu\,, \tag{A.8}$$

we use the notation x^μ for any set of local coordinates. Here, we also introduced the veirbein $\widehat e^a$, with the frame indices $a = 1, \cdots, 4$. On a Kähler manifold, we also pick a complex frame $e^a, e^{\bar a}$ (with indices $a = 1, 2$):[30]

$$e^1 = \widehat e^1 + i\widehat e^2\,, \quad e^2 = \widehat e^3 + i\widehat e^4\,, \quad e^{\bar 1} = \widehat e^1 - i\widehat e^2\,, \quad e^{\bar 2} = \widehat e^3 - i\widehat e^4\;. \tag{A.9}$$

[29] Here σ^i are the Pauli matrices and $\mathbf{1}$ is the 2×2 identity matrix. Therefore:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad \sigma^4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\;.$$

.

[30] We use the same a for the distinct indices in $\widehat e^a$ $(a = 1, \cdots, 4)$ and e^a $(a = 1, 2)$. That should cause no confusion.

We note that the complex structure then takes the form:

$$J^a{}_b = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \tag{A.10}$$

in the frame basis $\{\hat{e}^a\}$. Note also that the fully-antisymmetric Levi-Civita tensor, $\epsilon^{\mu\nu\rho\sigma}$, takes the values:

$$\epsilon^{1234} = \epsilon_{1234} = 1 , \qquad \epsilon^{12\bar{1}\bar{2}} = 4 , \quad \epsilon_{12\bar{1}\bar{2}} = \frac{1}{4} , \tag{A.11}$$

in the real and complex frames, respectively. Spinors are locally defined with respect to the frame $\{\hat{e}^a\}$, using the conventions of subsection A.1.1. The covariant derivative of Weyl spinors reads:

$$\nabla_\mu \psi = (\partial_\mu - \frac{1}{2}\omega_{\mu ab}\sigma^{ab})\psi , \qquad \nabla_\mu \widetilde{\chi} = (\partial_\mu - \frac{1}{2}\omega_{\mu ab}\widetilde{\sigma}^{ab})\widetilde{\chi} . \tag{A.12}$$

Here, the spin connection ω_μ is defined in terms of the veirbein $\hat{e}^a_\mu$ and the Levi-Civita connection $\Gamma^\lambda_{\mu\nu}$:

$$\omega_{\mu a}{}^b = \hat{e}^\nu_a \hat{e}^b_\rho \, \Gamma^\rho_{\mu\nu} - \hat{e}^b_\nu \partial_\mu \hat{e}^\nu_a . \tag{A.13}$$

This is equivalent to the statement that the covariant derivative of the veirbein vanishes:

$$\nabla_\mu \hat{e}^a_\nu \equiv \partial_\mu \hat{e}^a_\nu + \Gamma^\rho_{\mu\nu}\hat{e}^a_\rho - \omega_{\mu b}{}^a \hat{e}^b_\nu = 0 . \tag{A.14}$$

The curvature tensor of the spin connection is given by:

$$R_{\mu\nu a}{}^b(\omega) = \partial_\mu \omega_{\nu a}{}^b - \partial_\nu \omega_{\mu a}{}^b + \omega_{\nu a}{}^c \omega_{\mu c}{}^b - \omega_{\mu a}{}^c \omega_{\nu c}{}^b . \tag{A.15}$$

This is simply the ordinary Riemann tensor with two frame indices $R_{\mu\nu a}{}^b = \hat{e}^\rho_a \hat{e}^b_\lambda R_{\mu\nu\rho}{}^\lambda$. Finally, the Ricci scalar is defined as:

$$R = \hat{e}^a_\mu \hat{e}^\nu_b \, R^\mu{}_{\nu a}{}^b(\omega) . \tag{A.16}$$

A.1.3 4d $\mathcal{N} = 2$ supersymmetry

Consider 4d $\mathcal{N} = 2$ supersymmetry on $\mathbb{R}^4$ (in Euclidean signature). The supersymmetry algebra reads:

$$\begin{gathered} \{Q^I_\alpha, \bar{Q}_{\dot{\beta}J}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \, \delta^I{}_J , \\ \{Q^I_\alpha, Q^J_\beta\} = 2\varepsilon_{\alpha\beta}\epsilon^{IJ}\widetilde{Z} , \qquad \{\widetilde{Q}_{\dot{\alpha}I}, \widetilde{Q}_{\dot{\beta}J}\} = 2\varepsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{IJ} Z , \end{gathered} \tag{A.17}$$

with $I = 1, 2$ the $SU(2)_R$ index and Z the complex central charge. We denote any $SU(2)_R$ doublet $\mathbf{2}$ by φ^I, and its conjugate $\bar{\mathbf{2}}$ by φ_I. The $SU(2)_R$ indices are raised and lowered according to:

$$\varphi_I = \epsilon_{IJ}\varphi^J , \qquad \varphi^I = -\epsilon^{IJ}\varphi_J . \tag{A.18}$$

Note that $\epsilon_{12} = \epsilon^{12} = 1$ here, in our convention for $SU(2)_R$. Under the maximal R-symmetry of the $\mathcal{N}=2$ supersymmetry algebra, $U(2) \cong SU(2)_R \times U(1)_r$, the supercharges transform as:

$$\begin{array}{c|c|c|c} & SU(2)_R \times U(1)_r & T_3^{SU(2)_R} & U(1)_R^{\mathcal{N}=1} \\ \hline Q^I_\alpha & (\mathbf{2})_{-1} & \pm\frac{1}{2} & -\delta^{I1} \\ \widetilde{Q}_{\dot\alpha J} & (\bar{\mathbf{2}})_1 & \mp\frac{1}{2} & +\delta_{J1} \\ \hline Z & (\mathbf{1})_2 & 0 & \end{array} \tag{A.19}$$

The central charge Z breaks the $U(1)_r$ symmetry explicitly. It is sometimes useful to decompose the $\mathcal{N}=2$ supersymmetry multiplets with respect to $\mathcal{N}=1$ multiplets. We will choose the $\mathcal{N}=1$ subalgebra generated by:

$$Q = Q^{I=1}\,, \qquad\qquad \widetilde{Q} = \widetilde{Q}_{J=1}\,, \tag{A.20}$$

with the $\mathcal{N}=1$ $U(1)_R$ symmetry indicated in (A.19).

Let us present the supersymmetry transformations for the vector multiplet and the charged hypermultiplet, for a general supersymmetry transformation:

$$\delta = i\xi_I Q^I + i\widetilde{\xi}^I \widetilde{Q}_I\,, \tag{A.21}$$

with any constant (commuting) Weyl spinors ξ and $\widetilde{\xi}$.

The vector multiplet. The vector multiplet has components:

$$\mathcal{V}_{\mathcal{N}=2} = \left(A_\mu, \phi, \widetilde{\phi}, \lambda^I, \widetilde{\lambda}_I, D_{IJ}\right)\,, \tag{A.22}$$

in the Wess-Zumino gauge. This includes the auxiliary fields $D_{IJ} = D_{JI}$, which transform as a triplet of $SU(2)_R$. Note the default positions for the $SU(2)_R$ indices in (A.22). The field A_μ is a gauge field for some gauge group G, and all the other fields transform into the adjoint representation of $\mathfrak{g} = \mathrm{Lie}(G)$. The Euclidean $\mathcal{N}=2$ super-Yang-Mills action reads:

$$\begin{aligned}\mathscr{L}_{\rm SYM} &= \frac{1}{g^2}\,\mathrm{tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\widetilde{\phi}D^\mu\phi + i\widetilde{\lambda}_I\widetilde{\sigma}^\mu D_\mu\lambda^I + \frac{1}{4}D_{IJ}D^{IJ}\right.\\ &\qquad\left. + \frac{1}{2}[\widetilde{\phi},\phi]^2 - \frac{i}{\sqrt{2}}\lambda_I[\widetilde{\phi},\lambda^I] + \frac{i}{\sqrt{2}}\widetilde{\lambda}^I[\phi,\widetilde{\lambda}_I]\right)\,.\end{aligned} \tag{A.23}$$

Here, 'tr' denotes the trace in some fundamental representation of $\mathfrak{g}$, which we normalise as

$$\mathrm{tr}(T^A T^B) = \delta^{AB}\,, \tag{A.24}$$

with T^A the generators of $\mathfrak{g}$ in this representation. A more canonical convention would be to take the trace in the adjoint,

$$\mathscr{L}_{\rm SYM} = \frac{1}{g^2 h^\vee}\,\mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \cdots\right)\,, \tag{A.25}$$

with the standard normalisation in terms of the dual Coxeter number $h^\vee$ of $\mathfrak{g}$,

$$\mathrm{Tr}(T^A_{\rm adj}T^B_{\rm adj}) = h^\vee\delta^{AB}\,. \tag{A.26}$$

We use 'tr' defined in (A.24) throughout this paper, which has the advantage of being valid for any semi-simple G, including the $U(1)$ factors, but we could as easily formally substitute $\mathrm{tr} \to \frac{1}{h^\vee}\,\mathrm{Tr}$ for each simple factor. The supersymmetry transformations of the vector multiplet are given by

$$
\begin{aligned}
\delta\phi &= \sqrt{2}\xi_I \lambda^I\,,\\
delta\widetilde{\phi} &= \sqrt{2}\widetilde{\xi}^I \widetilde{\lambda}_I\,,\\
\delta A_\mu &= i\xi^I \sigma_\mu \widetilde{\lambda}_I - i\widetilde{\xi}_I \widetilde{\sigma}_\mu \lambda^I\,,\\
\delta\lambda^I &= -i\xi^I[\widetilde{\phi},\phi] - iD^{IJ}\xi_J + \sigma^{\mu\nu}\xi^I F_{\mu\nu} + i\sqrt{2}\sigma^\mu \widetilde{\xi}^I D_\mu \phi\,,\\
\delta\widetilde{\lambda}_I &= -i\widetilde{\xi}_I[\widetilde{\phi},\phi] - iD_{IJ}\widetilde{\xi}^J - \widetilde{\sigma}^{\mu\nu}\widetilde{\xi}_I F_{\mu\nu} + i\sqrt{2}\widetilde{\sigma}^\mu \xi_I D_\mu \widetilde{\phi}\,,\\
\delta D^{IJ} &= -\xi^I\sigma^\mu D_\mu \widetilde{\lambda}^J - \xi^J \sigma^\mu D_\mu \widetilde{\lambda}^I - \sqrt{2}\xi^I[\widetilde{\phi},\lambda^J] - \sqrt{2}\xi^J[\widetilde{\phi},\lambda^I]\\
&\quad - \widetilde{\xi}^I\widetilde{\sigma}^\mu D_\mu \lambda^J - \widetilde{\xi}^J \widetilde{\sigma}^\mu D_\mu \lambda^I + \sqrt{2}\widetilde{\xi}^I[\phi,\widetilde{\lambda}^J] + \sqrt{2}\widetilde{\xi}^J[\phi,\widetilde{\lambda}^I]\,.
\end{aligned}
\tag{A.27}
$$

Here, the gauge-covariant derivatives act on any adjoint-valued field φ as $D_\mu\varphi = \partial_\mu\varphi - i[A_\mu,\varphi]$. Note that the supersymmetry transformations (A.27) hold off-shell.

Upon projection onto the $\mathcal{N}=1$ subalgebra (A.20), the vector multiplet decomposes into an $\mathcal{N}=1$ vector multiplet $\mathcal{V}_{\mathcal{N}=1} = (A_\mu, \lambda, \widetilde{\lambda}, D)_{\mathcal{N}=1}$ and an $\mathcal{N}=1$ chiral multiplet $\Phi = (\phi,\psi,F)$, $\widetilde{\Phi} = (\widetilde{\phi},\widetilde{\psi},\widetilde{F})$, with:

$$
\lambda^I = (\psi,\lambda)_{\mathcal{N}=1}\,,\quad \widetilde{\lambda}_I = (\widetilde{\psi},\widetilde{\lambda})_{\mathcal{N}=1}\,,\quad D_{IJ} = \begin{pmatrix} \sqrt{2}i\widetilde{F} & D + [\widetilde{\phi},\phi] \\ D + [\widetilde{\phi},\phi] & \sqrt{2}iF \end{pmatrix}_{\mathcal{N}=1}\,. \tag{A.28}
$$

The hypermultiplet. Consider a hypermultiplet $\mathcal{H}$ valued in the representation $\mathfrak{R}$ of $\mathfrak{g}$. The physical field components are:

$$
\mathcal{H} = \left(q^I\,,\, \widetilde{q}_I\,,\, \eta\,,\, \widetilde{\eta}\,,\, \chi\,,\, \widetilde{\chi}\right)\,. \tag{A.29}
$$

Note that the fields q^I, η and $\widetilde{\chi}$ are valued in $\mathfrak{R}$, while $\widetilde{q}_I$, $\widetilde{\eta}$ and χ are valued in the conjugate representation, $\bar{\mathfrak{R}}$. The Lagrangian for the hypermultiplet coupled to a vector multiplet then reads:

$$
\begin{aligned}
\mathscr{L}_{\mathcal{H}} &= D_\mu \widetilde{q}_I D^\mu q^I + i\widetilde{\eta}\widetilde{\sigma}^\mu D_\mu \eta + i\chi\sigma^\mu D_\mu \widetilde{\chi} + \widetilde{q}_I \left(\{\widetilde{\phi},\phi\}\delta^I{}_J + D^I{}_J\right) q^J\\
&\quad + \sqrt{2}i\chi\phi\eta - \sqrt{2}i\widetilde{\eta}\widetilde{\phi}\widetilde{\chi} + \sqrt{2}i\widetilde{q}_I\lambda^I\eta - \sqrt{2}i\widetilde{\eta}\widetilde{\lambda}_I q^I\\
&\quad + \sqrt{2}i\epsilon_{IJ}\chi\lambda^I q^J + \sqrt{2}i\epsilon^{IJ}\widetilde{q}_I\widetilde{\lambda}_J\widetilde{\chi}\,.
\end{aligned}
\tag{A.30}
$$

Here the trace over gauge indices is left implicit. The on-shell $\mathcal{N}=2$ supersymmetry transformations for the hypermultiplet read:

$$
\begin{aligned}
\delta q^I &= \sqrt{2}\xi^I\eta + \sqrt{2}\widetilde{\xi}^I\widetilde{\chi}\,,\\
\delta\widetilde{q}_I &= \sqrt{2}\xi_I\chi - \sqrt{2}\widetilde{\xi}_I\widetilde{\eta}\,,\\
\delta\eta &= 2i\xi_I\widetilde{\phi}q^I - i\sqrt{2}\sigma^\mu\widetilde{\xi}_I D_\mu q^I\,,\\
\delta\chi &= 2i\xi^I\widetilde{q}_I\widetilde{\phi} + i\sqrt{2}\sigma^\mu\widetilde{\xi}^I D_\mu\widetilde{q}_I\,,\\
\delta\widetilde{\eta} &= i\sqrt{2}\widetilde{\sigma}^\mu\xi^I D_\mu\widetilde{q}_I - 2i\widetilde{\xi}^I\widetilde{q}_I\phi\,,\\
\delta\widetilde{\chi} &= i\sqrt{2}\widetilde{\sigma}^\mu\xi_I D_\mu q^I + 2i\widetilde{\xi}_I\phi q^I\,.
\end{aligned}
\tag{A.31}
$$

Focussing on the right-chiral supersymmetries, we have the supersymmetry algebra:

$$\{\delta_{\widetilde{\xi}}, \delta_{\widetilde{\xi}'}\}\varphi = 2i\sqrt{2}(\widetilde{\xi}_I \widetilde{\xi}'^I)\phi\varphi\,, \tag{A.32}$$

with ϕ acting on the field φ in the appropriate gauge representation ($\mathfrak{R}$ or $\widetilde{\mathfrak{R}}$). It is satisfied upon imposing the equations of motion for the fermions:

$$\begin{aligned} &i\widetilde{\sigma}^\mu D_\mu \eta - \sqrt{2}i\widetilde{\phi}\widetilde{\chi} - \sqrt{2}i\widetilde{\lambda}_I q^I = 0\,, \quad i\widetilde{\sigma}^\mu D_\mu \chi - \sqrt{2}i\widetilde{\eta}\widetilde{\phi} + \sqrt{2}i\epsilon^{IJ}\widetilde{q}_I\widetilde{\lambda}_J = 0\,,\\ &i\sigma^\mu D_\mu \widetilde{\eta} + \sqrt{2}i\chi\phi + \sqrt{2}i\widetilde{q}_I\lambda^I = 0\,, \quad i\sigma^\mu D_\mu \widetilde{\chi} + \sqrt{2}i\phi\eta + \sqrt{2}i\epsilon_{IJ}\lambda^I q^J = 0\,. \end{aligned} \tag{A.33}$$

Note also that, in terms of the $\mathcal{N}=1$ subalgebra (A.20), we have two chiral multiplets:

$$Q = (q, \psi_Q)_{\mathcal{N}=1}\,, \qquad Q' = (q', \psi_{Q'})_{\mathcal{N}=1}\,, \tag{A.34}$$

transforming in the representations $\mathfrak{R}$ and $\bar{\mathfrak{R}}$ of $\mathfrak{g}$, respectively, and their charge conjugate anti-chiral multiplets:

$$\widetilde{Q} = (\widetilde{q}, \widetilde{\psi}_Q)_{\mathcal{N}=1}\,, \qquad \widetilde{Q}' = (\widetilde{Q}', \widetilde{\psi}_{Q'})_{\mathcal{N}=1} \tag{A.35}$$

In the $\mathcal{N}=2$ notation, we have:

$$\begin{aligned} &q^I = (\widetilde{q}', q)^{\mathcal{N}=1}\,, \qquad \widetilde{q}_I = (q', \widetilde{q})^{\mathcal{N}=1}\,,\\ &\eta = \psi_Q\,, \quad \widetilde{\eta} = \widetilde{\psi}_Q\,, \quad \chi = \psi_{Q'}\,, \quad \widetilde{\chi} = \widetilde{\psi}_{Q'}\,. \end{aligned} \tag{A.36}$$

Off-shell formulation on Kähler Manifolds. The $\mathcal{N}=2$ kinetic term (A.30) is the sum of the standard $\mathcal{N}=1$ kinetic term and of the $\mathcal{N}=1$ superpotential contribution $W = \sqrt{2}iq'\phi q$.

It is well known that an off-shell formulation that realizes all eight supercharges for the hypermultiplet requires an infinite number of fields. Furthermore, the off-shell prescription that is obtained from the $\mathcal{N}=1$ language, in terms of the auxiliary fields of $\mathcal{N}=1$ chiral multiplets, leads to an action that is not '$\mathcal{Q}$-exact' in the usual Donaldson-Witten twist [83].

Here, we give an explicit off-shell realisation of the two specific supersymmetries relevant to the DW twist. (For another approach valid for a single generic supercharge, see e.g. [84, 116].) We introduce the spinors $h_\alpha, \widetilde{h}_\alpha$ as auxiliary bosonic fields such that, for the two supercharges corresponding to the supersymmetry parameters

$$\widetilde{\xi}^{\dot{\alpha}}_{(1)I} = \delta^{\dot{\alpha}1}\delta_{I1}\,, \qquad \widetilde{\xi}^{\dot{\alpha}}_{(2)I} = \delta^{\dot{\alpha}2}\delta_{I2}\,, \tag{A.37}$$

we have:

$$\begin{aligned} &\delta_1 q^I = \sqrt{2}\widetilde{\xi}^I_{(1)}\widetilde{\chi}\,, && \delta_2 q^I = \sqrt{2}\widetilde{\xi}^I_{(2)}\widetilde{\chi}\,,\\ &\delta_1 \widetilde{q}_I = -\sqrt{2}\widetilde{\xi}^{(1)}_I\widetilde{\eta}\,, && \delta_2 \widetilde{q}_I = -\sqrt{2}\widetilde{\xi}^{(2)}_I\widetilde{\eta}\,,\\ &\delta_1 \eta_\alpha = -i\sqrt{2}\left(\sigma^\mu\widetilde{\xi}^{(1)}_I\right)_\alpha D_\mu q^I + h_\alpha\,, && \delta_2 \eta_\alpha = -i\sqrt{2}\left(\sigma^\mu\widetilde{\xi}^{(2)}_I\right)_\alpha D_\mu q^I\,,\\ &\delta_1 \chi_\alpha = i\sqrt{2}\left(\sigma^\mu\widetilde{\xi}^I_{(1)}\right)_\alpha D_\mu \widetilde{q}_I\,, && \delta_2 \chi_\alpha = i\sqrt{2}\left(\sigma^\mu\widetilde{\xi}^I_{(2)}\right)_\alpha D_\mu \widetilde{q}_I + \widetilde{h}_\alpha,\\ &\delta_1 \widetilde{\eta}^{\dot{\alpha}} = -2i\widetilde{\xi}^{I\dot{\alpha}}_{(1)}\widetilde{q}_I\phi\,, && \delta_2 \widetilde{\eta}^{\dot{\alpha}} = -2i\widetilde{\xi}^{I\dot{\alpha}}_{(2)}\widetilde{q}_I\phi\,,\\ &\delta_1 \widetilde{\chi}^{\dot{\alpha}} = 2i\widetilde{\xi}^{\dot{\alpha}}_{(1)I}\phi q^I\,, && \delta_2 \widetilde{\chi}^{\dot{\alpha}} = 2i\widetilde{\xi}^{\dot{\alpha}}_{(2)I}\phi q^I\,, \end{aligned} \tag{A.38}$$

with the variations of the auxiliary fields given in terms of the fermion equations of motion as:

$$\begin{aligned} &\delta_1 h_\alpha = 0\,, && \delta_2 h_\alpha = 2i\kappa\sqrt{2}\phi\eta_\alpha + 2i\kappa\left(\sigma^\mu D_\mu\widetilde{\chi}\right)_\alpha + 2i\kappa\sqrt{2}\epsilon_{IJ}\lambda^I_\alpha q^J\,, \\ &\delta_1\widetilde{h}_\alpha = -2i\kappa\sqrt{2}\chi_\alpha\phi - 2i\kappa\left(\sigma^\mu D_\mu\widetilde{\eta}\right)_\alpha - 2i\kappa\sqrt{2}\widetilde{q}_I\lambda^I_\alpha\,, && \delta_2\widetilde{h}_\alpha = 0\;. \end{aligned} \tag{A.39}$$

Here κ is the scalar constructed from the DW supersymmetry parameters:

$$\kappa = \widetilde{\xi}^I_{(1)}\widetilde{\xi}_{(2)I} = 1\;. \tag{A.40}$$

Note that the differential forms $h^{0,1}$ and $\widetilde{h}^{1,0}$ introduced in section 2.3 can be expressed in terms of the above auxiliary fields, in the local frame basis, as:

$$h^{0,1} = -h_2 e^{\bar{1}} - h_1 e^{\bar{2}}\,, \qquad \widetilde{h}^{1,0} = \widetilde{h}_1 e^1 + \widetilde{h}_2 e^2\;. \tag{A.41}$$

For this choice of supercharges, the kinetic Lagrangian is $\mathcal{Q}$-exact and can be written as:

$$\mathscr{L}_{\mathcal{H}} = \frac{1}{2}(\delta_1+\delta_2)\left(-\widetilde{h}\eta - i\sqrt{2}\chi\sigma^\mu\widetilde{\xi}_I D_\mu q^I + i\sqrt{2}\widetilde{\eta}\widetilde{\xi}_I\widetilde{\phi}q^I + 2i\widetilde{q}_I\widetilde{\xi}^J\widetilde{\lambda}^I q_J\right)\,, \tag{A.42}$$

where we defined $\widetilde{\xi} = \widetilde{\xi}^{(1)} + \widetilde{\xi}^{(2)}$.

A.2 Five-dimensional conventions

A.2.1 Flat-space conventions

We use five-dimensional conventions that naturally reduce to our four-dimensional conventions upon circle reduction along the fifth coordinate. In flat space, we use the coordinates $x^M = (x^\mu, x^5)$, with the index $M = (\mu, 5)$ $(\mu = 1, \cdots, 4)$. The five-dimensional spin group is:

$$\mathrm{Spin}(5) \cong \mathrm{Usp}(4)\;. \tag{A.43}$$

The 5d Dirac spinor $\Psi = \Psi_{\mathbf{a}}$ transforms in the $\bar{\mathbf{4}}$ of USp(4), with $\mathbf{a} = 1, \cdots, 4$ the Dirac index. The 5d γ-matrices, γ^M, are chosen to be:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\widetilde{\sigma}^\mu & 0 \end{pmatrix}\,, \qquad \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}\,, \tag{A.44}$$

with the natural index positions $\gamma^M = (\gamma^M{}_{\mathbf{a}}{}^{\mathbf{b}})$. The Dirac representation matrices are:

$$(\Sigma^{MN})_{\mathbf{a}}{}^{\mathbf{b}} = \frac{i}{4}[\gamma^M, \gamma^N]_{\mathbf{a}}{}^{\mathbf{b}}\;. \tag{A.45}$$

The Dirac indices are raised with:

$$\Omega^{\mathbf{ab}} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot\alpha\dot\beta} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\,, \tag{A.46}$$

such that $\Psi^{\mathbf{a}} \equiv \Omega^{\mathbf{ab}}\Psi_{\mathbf{b}}$. In these conventions, the 5d Dirac spinor simply reduces to the 4d Dirac spinor upon dimensional reduction along x^5. Denoting by $\widetilde{\Psi}$ the hermitian conjugate to Ψ in Lorentzian signature, we have:

$$(\Psi_{\mathbf{a}}) = \begin{pmatrix} \psi_\alpha \\ \widetilde{\chi}^{\dot\alpha} \end{pmatrix}\,, \qquad (\widetilde{\Psi}^{\mathbf{a}}) = \begin{pmatrix} \chi^\alpha\,, & -\widetilde{\psi}_{\dot\alpha} \end{pmatrix}\,. \tag{A.47}$$

Given two Dirac spinors Ψ and Ψ', we have the five-dimensional scalar $\Psi\Psi' \equiv \Psi^{\mathbf{a}}\Psi'_{\mathbf{a}} = \Omega^{\mathbf{ab}}\Psi_{\mathbf{b}}\Psi'_{\mathbf{a}} = \psi\psi' + \widetilde{\chi}\widetilde{\chi}'$, which is a sum of two four-dimensional scalars.

The 5d Dirac representation (i.e. the $\bar{\mathbf{4}}$ of Spin(5)) is pseudo-real but not real, therefore we cannot impose the Majorana condition. Nonetheless, given a pair of spinors Ψ^I ($I = 1, 2$), we can impose the Majorana-Weyl (MW) condition:[31]

$$\widetilde{\Psi}_I^{\mathbf{a}} = \epsilon_{IJ}\Omega^{\mathbf{ab}}\Psi^J_{\mathbf{b}}\,. \tag{A.48}$$

In particular, the five-dimensional supercharge $\mathcal{Q} = (\mathcal{Q}^I)$ is a Majorana-Weyl spinor, which reduces to the two distinct Majorana spinors Q^I ($I = 1, 2$) in 4d.

A.2.2 Curved-space conventions

As explained in section 3.1, we are interested in five-manifolds that are fibered over a Kähler four-manifold $\mathcal{M}_4$. In particular, we pick a distinguished 'fifth dimension'. We choose the complex frame (with the fifth direction along a real covector η) and a metric adapted to the fibration structure:

$$(e^{\mathbf{A}}) = (e^1, e^{\bar 1}, e^2, e^{\bar 2}, \hat{e}^5 \equiv \eta)\,, \qquad ds^2 = e^1 e^{\bar 1} + e^2 e^{\bar 2} + \eta^2\,. \tag{A.49}$$

We also denote by x^M ($M = 1, \cdots, 5$) any local coordinates. The spin connection and curvature tensors are defined exactly as in 4d. The covariant derivative on 5d Dirac spinors takes the form:

$$\nabla_M \Psi \equiv (\partial_\mu - \frac{i}{2}\omega_{M\mathbf{AB}}\Sigma^{\mathbf{AB}})\Psi\,, \tag{A.50}$$

with Σ^{MN} defined in (A.45), and $\mathbf{A}, \mathbf{B}$ denoting 5d frame indices.

A.2.3 5d $\mathcal{N} = 1$ supersymmetry

Here we discuss 5d $\mathcal{N} = 1$ supersymmetry in flat space. The five-dimensional supersymmetry algebra reads:

$$\{\mathcal{Q}^I_{\mathbf{a}}, \mathcal{Q}^J_{\mathbf{b}}\} = 2\epsilon^{IJ}\left(\gamma^M_{\mathbf{ab}}P_M - i\Omega_{\mathbf{ab}}Z_{5\text{d}}\right)\,, \tag{A.51}$$

with $\mathcal{Q}^I$ the Majorana-Weyl supercharge, and $Z_{5\text{d}}$ a real central charge. Upon dimensional reduction along the fifth dimension, (A.51) reproduces (A.17) with:

$$\mathcal{Q}^I_{\mathbf{a}} = \begin{pmatrix} Q^I_\alpha \\ -\epsilon^{IJ}\widetilde{Q}^{\dot\alpha}_J \end{pmatrix}\,, \qquad Z = P_5 + iZ_{5\text{d}}\,, \qquad \widetilde{Z} = P_5 - iZ_{5\text{d}}\,. \tag{A.52}$$

[31] Here $\widetilde{\Psi}$ is the Hermitian conjugate of Ψ in Lorentzian signature, which involves conjugation of the $SU(2)$ representation (the index I). In 4d notation, we simply have $\Psi^I = \begin{pmatrix} \psi^I \\ -\epsilon^{IJ}\widetilde{\psi}_J \end{pmatrix}$ for a 5d Majorana-Weyl spinor.

We denote the supersymmetry variation by:

$$\delta = -i\zeta_I \mathcal{Q}^I \ . \tag{A.53}$$

The commuting-spinor parameter ζ_I is related to the 4d parameters according to:

$$\zeta_I = \begin{pmatrix} -\xi_I \\ \widetilde{\xi}_I \end{pmatrix} \ . \tag{A.54}$$

5d $\mathcal{N} = 1$ vector multiplet. The five-dimensional vector multiplet consists of a gauge field A_M, a real scalar field σ, a Majorana-Weyl spinor Λ_I $(I = 1, 2)$ and a triplet of auxiliary scalar fields $D_{IJ} = D_{JI}$:

$$\mathcal{V}_{\text{5d}} = \Big(A_M\,,\, \sigma\,,\, \Lambda_I\,,\, D^{IJ}\Big) \ . \tag{A.55}$$

The 5d $\mathcal{N} = 1$ super-Yang-Mills action reads:

$$\begin{aligned} \mathscr{L}_{\text{SYM}} = \frac{1}{g_{\text{5d}}^2} \operatorname{tr}\Big(&\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}D_M\sigma D^M\sigma + \frac{i}{2}\Lambda^I\gamma^M D_M\Lambda_I \\ &+ \frac{i}{2}\Lambda^I[\sigma, \Lambda_I] + \frac{1}{4}D_{IJ}D^{IJ}\Big) \ . \end{aligned} \tag{A.56}$$

One easily checks that it reduces to the 4d Lagrangian (A.23) upon dimensional reduction on a circle. The 5d $\mathcal{N} = 1$ supersymmetry transformations are:

$$\begin{aligned} \delta A_M &= i\zeta_I\gamma_M\Lambda^I\,, \\ \delta\sigma &= -\zeta_I\Lambda^I\,, \\ \delta\Lambda_I &= -i\Sigma^{MN}\zeta_I\, F_{MN} + i\gamma^M\zeta_I D_M\sigma - iD_{IJ}\zeta^J\,, \\ \delta D^{IJ} &= \zeta^I\gamma^M D_M\Lambda^J + \zeta^J\gamma^M D_M\Lambda^I + \zeta^I[\sigma, \Lambda^J] + \zeta^J[\sigma, \Lambda^I] \ . \end{aligned} \tag{A.57}$$

This reduces to (A.27) in 4d. The dimensional reduction is straightforward to carry out explicitly. The gauge field A_M splits as (A_μ, A_5), with A_5 a real scalar, and the 4d complex scalar $\phi, \widetilde{\phi}$ is given by:

$$\phi = \frac{1}{\sqrt{2}}\,(\sigma + iA_5)\ , \qquad \widetilde{\phi} = \frac{1}{\sqrt{2}}\,(\sigma - iA_5)\ . \tag{A.58}$$

The auxiliary fields D_{IJ} are taken to be the same in 5d and 4d, and the gauginos are related as:

$$\Lambda_I = \begin{pmatrix} \epsilon_{IJ}\lambda^J \\ \widetilde{\lambda}_I \end{pmatrix} \ . \tag{A.59}$$

One can check that the SUSY algebra closes up to translations and gauge transformations. Picking any two constant supersymmetry parameters ξ_I and ζ_I (taken as commuting MW spinors), let us define the following bilinears:

$$K^M = -\xi^I\gamma^M\zeta_I\,, \qquad \kappa = \xi^I\zeta_I \ . \tag{A.60}$$

One finds, in particular (see e.g. [84]):

$$
\begin{aligned}
\{\delta_\xi, \delta_\zeta\}\sigma &= -2iK^M D_M\sigma\,, \\
\{\delta_\xi, \delta_\zeta\}A_M &= -2iK^N F_{NM} + 2\kappa D_M\sigma\,, \\
\{\delta_\xi, \delta_\zeta\}\Lambda_I &= -2iK^M D_M\Lambda_I + 2i\kappa\,[\sigma, \Lambda_I]\ , \\
\{\delta_\xi, \delta_\zeta\}D_{IJ} &= -2iK^M D_M D_{IJ} + 2i\kappa\,[\sigma, D_{IJ}]\ .
\end{aligned}
\tag{A.61}
$$

The supersymmetric Lagrangian (A.56) is 'almost' $\mathcal{Q}$-exact, like in 4d. For any supersymmetry parameter ζ_I, we have:

$$
\begin{aligned}
\mathscr{L}_{\text{SYM}} = \frac{1}{g_{5d}^2}\,\frac{1}{\zeta^K\zeta_K}\,\delta\left(-\frac{1}{4}\zeta^I\gamma^{MN}\Lambda_I F_{MN} - \frac{i}{2}\zeta_J\Lambda_I D^{IJ} - \frac{i}{2}\zeta^I\gamma^M\Lambda_I D_M\sigma\right) + \\
+\frac{1}{8g_{5d}^2}\,\frac{\zeta^I\gamma_R\zeta_I}{\zeta^J\zeta_J}\,\epsilon^{MNPQR}F_{MN}F_{PQ}\ .
\end{aligned}
\tag{A.62}
$$

The second line reduces to the instanton density in 4d.

5d $\mathcal{N}=1$ hypermultiplet. Consider the 5d charged hypermultiplet:

$$
\mathcal{H}_{5\text{d}} = \left(q^I\,,\,\widetilde{q}_I\,,\,\Psi\,,\,\widetilde{\Psi}\right)\ . \tag{A.63}
$$

The free-field Lagrangian for the hypermultiplet coupled to a vector multiplet reads:

$$
\begin{aligned}
\mathscr{L}_{\mathcal{H}} \;=\; & D_M\widetilde{q}_I D^M q^I + i\widetilde{\Psi}\gamma^M D_M\Psi + \widetilde{q}_I\left(\sigma^2\delta^I{}_J + D^I{}_J\right)q^J - i\widetilde{\Psi}\sigma\Psi \\
& - \sqrt{2}i\widetilde{\Psi}\Lambda_I q^I - \sqrt{2}i\widetilde{q}_I\Lambda^I\Psi\ .
\end{aligned}
\tag{A.64}
$$

The on-shell supersymmetry transformations are:

$$
\begin{aligned}
\delta q^I &= \sqrt{2}\zeta^I\Psi\,, \\
\delta\widetilde{q}_I &= -\sqrt{2}\zeta_I\widetilde{\Psi}\,, \\
\delta\Psi &= \sqrt{2}i\gamma^M\zeta_I D_M q^I + \sqrt{2}i\zeta_I\sigma q^I\,, \\
\delta\widetilde{\Psi} &= \sqrt{2}i\zeta^I\gamma^M D_M\widetilde{q}_I - \sqrt{2}i\zeta^I\widetilde{q}_I\sigma\,,
\end{aligned}
\tag{A.65}
$$

with the fields (q, Ψ) transforming in some representation $\mathfrak{R}$ of the gauge group and the fields $(\widetilde{q}, \widetilde{\Psi})$ in the complex conjugate representation $\bar{\mathfrak{R}}$. In our conventions, expressions of the type σf refer to contractions of the type $\sigma^a T^a_{ij} f^j$, where T^a_{ij} are generators for the representation under which the field f transforms, with indices i, j, while a are adjoint representation indices. We use the generators of the representation $\mathfrak{R}$ for all fields of the five-dimensional hypermultiplet. Thus, the covariant derivatives become:

$$
D_M q^I = \partial_M q^I - iA_M q^I\,, \qquad D_M\widetilde{q}_I = \partial_M\widetilde{q}_I + i\widetilde{q}_I A_M\ . \tag{A.66}
$$

The kinetic Lagrangian left invariant by the above transformations is:

$$
\begin{aligned}
\mathscr{L}_{\mathcal{H}} = & D_M\widetilde{q}_I D^M q^I + i\widetilde{\Psi}\gamma^M D_M\Psi + \widetilde{q}_I\sigma\sigma q^I + \widetilde{q}_I D^I{}_J q^J - i\widetilde{\Psi}\sigma\Psi \\
& + i\sqrt{2}\widetilde{\Psi}\Lambda^I q_I - i\sqrt{2}\widetilde{q}_I\Lambda^I\Psi\ .
\end{aligned}
\tag{A.67}
$$

When reducing to 4d, the Lagrangian (A.64) and the variations (A.65) reproduce (A.30) and (A.31), respectively, with the identification:

$$\Psi = \begin{pmatrix} -\eta \\ \widetilde{\chi} \end{pmatrix} , \qquad \widetilde{\Psi} = \begin{pmatrix} \chi \, , \widetilde{\eta} \end{pmatrix} \ . \tag{A.68}$$

The equations of motion for the fermions read:

$$\begin{aligned} i\gamma^M D_M \Psi - i\sigma\Psi + i\sqrt{2}\Lambda^I q_I = 0\,, \\ -iD_M\widetilde{\Psi}\gamma^M - i\widetilde{\Psi}\sigma - i\sqrt{2}\widetilde{q}_I\Lambda^I = 0\ . \end{aligned} \tag{A.69}$$

It is straightforward to first show that:

$$\{\delta_\xi, \delta_\zeta\} q^I = -2iK^M D_M q^I + 2i\kappa\sigma q^I\,, \qquad \{\delta_\xi,\delta_\zeta\}\widetilde{q}_I \ = -2iK^M D_M \widetilde{q}_I - 2i\kappa\widetilde{q}_I\sigma\,, \tag{A.70}$$

and using the equations of motion we then find that:

$$\{\delta_\xi, \delta_\zeta\}\Psi = -2iK^M D_M\Psi + 2i\kappa\sigma\Psi\,, \qquad \{\delta_\xi,\delta_\zeta\}\widetilde{\Psi} \ = -2iK^M D_M\widetilde{\Psi} - 2i\kappa\widetilde{\Psi}\sigma\ . \tag{A.71}$$

For the off-shell formulation, let us focus on two supercharges with the corresponding Killing spinors being the uplift of the four dimensional spinors $\delta^{\dot{\alpha}\dot{1}}\delta_{I1}$ and $\delta^{\dot{\alpha}\dot{2}}\delta_{I2}$. In this setup, the five-dimensional spinors of interest are:

$$(\zeta_{(1)I\mathbf{a}}) = \begin{pmatrix} 0 \\ \delta^{\dot{\alpha}\dot{1}}\delta_{I1} \end{pmatrix} , \qquad (\zeta_{(2)I\mathbf{a}}) = \begin{pmatrix} 0 \\ \delta^{\dot{\alpha}\dot{2}}\delta_{I2} \end{pmatrix} \ . \tag{A.72}$$

We introduce the auxiliary fields (five-dimensional commuting spinors) $h_{\mathbf{a}}, \widetilde{h}^{\mathbf{a}}$, with components $(h_1, h_2, 0, 0)$, such that the SUSY variations become:

$$\begin{aligned} \delta_1 q^I &= \sqrt{2}\,\zeta^I_{(1)}\Psi\,, \\ \delta_2 q^I &= \sqrt{2}\,\zeta^I_{(2)}\Psi\,, \\ \delta_1\widetilde{q}_I &= -\sqrt{2}\,\zeta_{(1)I}\widetilde{\Psi}\,, \\ \delta_2\widetilde{q}_I &= -\sqrt{2}\,\zeta_{(2)I}\widetilde{\Psi}\,, \\ \delta_1\Psi_{\mathbf{a}} &= i\sqrt{2}(\gamma^M\zeta_{(1)I})_{\mathbf{a}} D_M q^I + i\sqrt{2}\zeta_{(1)I\mathbf{a}}\sigma q^I + h_{\mathbf{a}}\,, \\ \delta_2\Psi_{\mathbf{a}} &= i\sqrt{2}(\gamma^M\zeta_{(2)I})_{\mathbf{a}} D_M q^I + i\sqrt{2}\zeta_{(2)I\mathbf{a}}\sigma q^I\,, \\ \delta_1\widetilde{\Psi}^{\mathbf{a}} &= i\sqrt{2}(\zeta^I_{(1)}\gamma^M)^{\mathbf{a}} D_M\widetilde{q}_I - i\sqrt{2}\zeta^{I\mathbf{a}}_{(1)}\widetilde{q}_I\sigma\,, \\ \delta_2\widetilde{\Psi}^{\mathbf{a}} &= i\sqrt{2}(\zeta^I_{(2)}\gamma^M)^{\mathbf{a}} D_M\widetilde{q}_I - i\sqrt{2}\zeta^{I\mathbf{a}}_{(2)}\widetilde{q}_I\sigma + \widetilde{h}^{\mathbf{a}}\,, \end{aligned} \tag{A.73}$$

together with:

$$\begin{aligned} &\delta_1 h_{\mathbf{a}} = 0\,, \\ &\delta_2 h_{\mathbf{a}} \approx \ 2i\kappa\sigma\Psi_{\mathbf{a}} - 2i\kappa\left(\gamma^M D_M\Psi\right)_{\mathbf{a}} - 2i\sqrt{2}\kappa\Lambda^I_{\mathbf{a}} q_I\,, \\ &\delta_1\widetilde{h}^{\mathbf{a}} \approx -2i\kappa\widetilde{\Psi}\sigma - 2i\kappa\left(D_M\widetilde{\Psi}\gamma^M\right)^{\mathbf{a}} - 2i\sqrt{2}\kappa\widetilde{q}_I\Lambda^{I\mathbf{a}}\,, \\ &\delta_2\widetilde{h}^{\mathbf{a}} = 0\ . \end{aligned} \tag{A.74}$$

The $\approx$ symbol is used to emphasise that these equations should be interpreted as equalities only when restricted to the non-zero components of h and $\widetilde{h}$, i.e. for the indices $\mathbf{a} \in \{\mathbf{1}, \mathbf{2}\}$. For the variations to be correct for any $\mathbf{a}$, one needs to introduce some additional terms that can be worked out from $\delta\delta\Psi_{\mathbf{a}}$. Note also that we restrict the definitions of κ and K^M to the two spinors $\zeta_{(1)}$ and $\zeta_{(2)}$. In particular, for the Killing spinors (A.72) and with our choice of frame, we have

$$K^M \partial_M = \partial_5 \,, \qquad \kappa = 1 \,. \tag{A.75}$$

The off-shell Lagrangian becomes:

$$\begin{aligned} \mathscr{L}_{\mathcal{H}} = & D_M \widetilde{q}_I D^M q^I + i\widetilde{\Psi}\gamma^M D_M \Psi + \widetilde{q}_I \sigma\sigma q^I + \widetilde{q}_I D^I{}_J q^J - i\widetilde{\Psi}\sigma\Psi \\ & + i\sqrt{2}\widetilde{\Psi}\Lambda^I q_I - i\sqrt{2}\widetilde{q}_I \Lambda^I \Psi + \frac{1}{2}\widetilde{h}^{\mathbf{a}} h_{\mathbf{a}} \,, \end{aligned} \tag{A.76}$$

which can be shown to be $\mathcal{Q}$-exact,

$$\mathscr{L}_{\mathcal{H}} = \frac{1}{2\kappa}(\delta_1 + \delta_2)\left(\widetilde{h}\Psi - i\sqrt{2}\,\widetilde{\Psi}\gamma^M \zeta_I \, D_M q^I + i\sqrt{2}\,\widetilde{\Psi}\zeta_I \, \sigma q^I + 2i\,\widetilde{q}_I \zeta^J \Lambda^I q_J\right) \,. \tag{A.77}$$

Here we used the notation $\zeta = \zeta_{(1)} + \zeta_{(2)}$.

B More about one-loop determinants

The material of this appendix complements the discussion of section 4.

B.1 Hypermultiplet mode cancellation

Let us consider the one-loop determinant for the hypermultiplet on $\mathcal{M}_4$, as in section 4.1. Here, for completeness, we would like to explicitly display the mode cancellation between fermions and bosons. Recall that the perturbative contribution to the partition function on $\mathcal{M}_4$ comes from:

$$\frac{\det\left(\Delta_{\text{fer}}\right)}{\det\left(\Delta_{\text{bos}}\right)} = \frac{\det(\mathbb{L}^{0,1})}{\det(\mathbb{L}^{0,0})\det(\mathbb{L}^{0,2})} \,, \tag{B.1}$$

where $\mathbb{L} = -i\sqrt{2}\phi$. A different approach to computing this one-loop determinant is to count the modes of $\mathbb{L}$, which will be related to the index of the twisted Dolbeault operator as discussed in the main text. To show this, one starts with a fermionic eigenmode with eigenvalue M_{fer}:

$$\Delta_{\text{fer}}\Psi = M_{\text{fer}}\Psi \,, \tag{B.2}$$

where $\Psi = (\eta^{0,1}, \widetilde{\chi}^{0,0}, \widetilde{\chi}^{0,2})$ and Δ_{fer} is given by (4.5). Then, combining the first and third equations obtained from (B.2), one finds a bosonic mode for $Q^{0,0}$, with the bosonic eigenvalue given by:

$$M_{\text{bos}} = M_{\text{fer}}\left(-2M_{\text{fer}} - 2\sqrt{2}(\phi - 2\widetilde{\phi})\right) \,. \tag{B.3}$$

In fact, the same mode can also be constructed by starting with the fermionic eigenvalue:

$$-M_{\text{fer}} - \frac{i}{\sqrt{2}}(\phi - 2\widetilde{\phi}) \,, \tag{B.4}$$

and thus the two fermionic modes are paired with one bosonic mode. This construction relies on the map:

$$\widetilde{\chi}^{0,0} \longrightarrow Q^{0,0}\,, \tag{B.5}$$

being independent of $\widetilde{\chi}^{0,2}$. Similarly, the bosonic modes for $Q^{0,2}$ are constructed solely from $\widetilde{\chi}^{0,2}$, with the fermionic eigenvalues:

$$M_{\text{fer}}\,, \qquad M_{\text{fer}} - \frac{i}{2\sqrt{2}}(2\phi + \widetilde{\phi})\,, \tag{B.6}$$

being mapped to the bosonic mode with:

$$M_{\text{fer}}\left(2M_{\text{fer}} + \frac{i}{\sqrt{2}}(2\phi + \widetilde{\phi})\right)\,. \tag{B.7}$$

In both constructions, however, there are unpaired modes that correspond to degrees of freedom for $\eta^{0,1}$. From the eigenvalue equation (B.2), these modes are given by:

$$\star\,\partial_A \star \eta^{0,1} = 0\,, \qquad \star\bar{\partial}_A \eta^{0,1} = 0\,, \tag{B.8}$$

for the two constructions, respectively. For the reverse map, fermionic modes can be built from bosonic modes in two ways, that is:

$$\widetilde{\chi}^{0,0} = cQ^{0,0}\,, \qquad \eta^{0,1} = c\frac{2(\sqrt{2}\widetilde{\phi} + iM_{\text{fer}})}{2\phi\widetilde{\phi} - M_{\text{bos}}}\,\bar{\partial}_A Q^{0,0}\,, \tag{B.9}$$

or, alternatively:

$$\widetilde{\chi}^{0,2} = cQ^{0,2}\,, \qquad \eta^{0,1} = c\frac{\sqrt{2}\widetilde{\phi} - 4iM_{\text{fer}}}{2\phi\widetilde{\phi} - 4M_{\text{bos}}} \star \partial_A Q^{0,2}\,, \tag{B.10}$$

for some constant c. The map is again 1 (boson) to 2 (fermions), with the eigenvalues being the same as before. In this case, the pairing is not complete if the two fermionic modes built from these maps are not independent. This occurs for:

$$\bar{\partial}_A Q^{0,0} = 0\,, \qquad \star\partial_A Q^{0,2} = 0\,, \tag{B.11}$$

for the two constructions, respectively. This analysis confirms that the one-loop determinant reduces to (4.12).

B.2 Vector multiplet one-loop determinant

In this appendix, we compute the one-loop determinant for a W-boson. (We closely follow a similar computation in [117].) Consider a non-abelian 4d $\mathcal{N} = 2$ vector multiplet on a Kähler manifold $\mathcal{M}_4$ and introduce the usual BRST ghosts $c, \widetilde{c}$ and auxiliary fields b valued in the adjoint of $\mathfrak{g}$. Then, the standard BRST transformations read:

$$\begin{aligned} sA_\mu &= D_\mu c\,, \qquad s\varphi_b = i[c,\varphi_b]\,, \quad s\varphi_f = i\{c,\varphi_f\}\,, \\ sc &= \frac{i}{2}\{c,c\}\,, \qquad s\widetilde{c} = -b\,, \qquad sb = 0\,, \end{aligned} \tag{B.12}$$

with $\varphi_{b,f}$ the bosonic and fermionic fields in the $\mathcal{N}=2$ vector multiplet and s the BRST symmetry generator. It follows that s is nilpotent and, furthermore:

$$\{s, \delta_i\} = 0\,, \qquad i = 1, 2\,, \tag{B.13}$$

for the two supersymmetry transformations defined in (2.32). We then define the modified supersymmetry transformations:

$$\delta_i' = \delta_i + s\,, \tag{B.14}$$

which still satisfy the supersymmetry algebra (2.35):

$$(\delta_i')^2 = 0\,, \qquad \{\delta_1', \delta_2'\} = \{\delta_1, \delta_2\} = 2\sqrt{2}\delta_{g(\phi)}\,, \tag{B.15}$$

where $\delta_{g(\phi)}$ is a gauge transformation with parameter ϕ. Note that the BRST transformation of the vector multiplet fields is just a gauge transformation, so the Yang-Mills lagrangian is automatically invariant under this action. In the absence of supersymmetry, the ghost and gauge-fixing contributions can be recovered from a s-exact term. In our case, we have instead:

$$\mathscr{L}_{\text{gf}} = (\delta_1' + \delta_2')F_{\text{gf}}\,, \tag{B.16}$$

for some function F_{gf}, which ensures that the action preserves supersymmetry. We choose the following conventions:

$$\mathscr{L}_{\text{gf}} = \frac{1}{2}(\delta_1' + \delta_2')\left(\widetilde{c}\left(G_{\text{gf}} + \frac{1}{2}\xi_{\text{gf}} b\right)\right) = s\left(\widetilde{c}\left(G_{\text{gf}} + \frac{1}{2}\xi_{\text{gf}} b\right)\right) + \frac{1}{2}\widetilde{c}\,(\delta_1 + \delta_2)G_{\text{gf}}\,, \tag{B.17}$$

which, upon integrating out the auxiliary field b, reads:

$$\mathscr{L}_{\text{gf}} = \frac{1}{2\xi_{\text{gf}}}G_{\text{gf}}^2 - \widetilde{c}\,(sG_{\text{gf}}) + \frac{1}{2}\widetilde{c}\,(\delta_1 + \delta_2)G_{\text{gf}}\,. \tag{B.18}$$

The gauge-fixing function will typically include a term of the form:

$$G_{\text{gf}} \supset D_\mu A^\mu\,, \tag{B.19}$$

leading to the kinetic terms in the Lagrangian:

$$\mathscr{L}_{\text{gf}} = \frac{1}{2\xi_{\text{gf}}}G_{\text{gf}}^2 + D_\mu \widetilde{c}\, D^\mu c + \dots\,. \tag{B.20}$$

We will set the gauge-fixing function to:

$$G_{\text{gf}} = D_\mu A^\mu + i\xi_{\text{gf}}[\phi, \widetilde{\phi}]\,, \tag{B.21}$$

and work in a background where ϕ and $\widetilde{\phi}$ are constant: $\phi = \hat{\phi}$, $\widetilde{\phi} = \hat{\widetilde{\phi}}$. In the Feynman gauge $\xi_{\text{gf}} = 1$, expanding at second order in fluctuations around these constant values leads to diagonal kinetic terms between A_μ and ϕ, $\widetilde{\phi}$, while the ghost one-loop determinant cancels completely that of ϕ and $\widetilde{\phi}$, since the Lagrangian contains the terms:

$$\mathscr{L}_{\text{SYM}} + \mathscr{L}_{\text{gf}} \supset D_\mu \widetilde{\phi} D^\mu \phi + D_\mu \widetilde{c}\, D^\mu c\,. \tag{B.22}$$

For the kinetic terms of the gauge field, we get contributions from:

$$\mathscr{L}_{\text{SYM}} + \mathscr{L}_{\text{gf}} \supset \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu A^\mu)^2 - [A_\mu, \tilde{\hat{\phi}}][A^\mu, \hat{\phi}]\,, \tag{B.23}$$

where the last term comes from expanding the $D_\mu\widetilde{\phi}D^\mu\phi$ kinetic term. In the local frame, in the complex basis, we have:

$$\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -2F_{1\bar{1}}^2 + 4F_{12}F_{\bar{1}\bar{2}} - 4F_{1\bar{2}}F_{2\bar{1}} - 2F_{2\bar{2}}^2\,, \tag{B.24}$$

with the first two terms in (B.23) combining to:

$$\mathscr{L}_{\text{kin}} \supset -2A_{\bar{j}}D_\mu D^\mu A_j - [A_\mu, \tilde{\hat{\phi}}][A^\mu, \hat{\phi}]\,, \tag{B.25}$$

where the remaining terms correspond to cubic or higher degree terms in the gauge field, and $j \in \{1, 2\}$. Note that this is almost identical to the bosonic kinetic term of the free hypermultiplet in (4.4), which can be rewritten in terms of the untwisted fields as:

$$\mathscr{L}_{\text{bos}}^{\text{hyper}} = -\widetilde{q}_I\left(D_\mu D^\mu + 2\phi\widetilde{\phi}\right)q^I\,, \tag{B.26}$$

in the supersymmetric background described by (4.1) and (4.2).

Similarly, for the gauginos before the topological twist, the kinetic terms are encoded in:

$$\mathscr{L}_{\text{SYM}} \supset i\widetilde{\lambda}_I\widetilde{\sigma}^\mu D_\mu\lambda^I - \frac{i}{\sqrt{2}}\lambda_I[\tilde{\hat{\phi}}, \lambda^I] + \frac{i}{\sqrt{2}}\widetilde{\lambda}^I[\hat{\phi}, \widetilde{\lambda}_I]\,, \tag{B.27}$$

After topological twisting, we group the gauginos into the formal variables $\widetilde{\Psi} = (\Lambda^{1,0}, \widetilde{\Lambda}^{0,0}_{(1)}, \widetilde{\Lambda}^{2,0})$ and $\Psi = (-\Lambda^{0,1}, \widetilde{\Lambda}^{0,0}_{(2)}, \widetilde{\Lambda}^{0,2})$, such that the kinetic terms of the Lagrangian can be expressed as:

$$\mathscr{L}_{\text{kin}} \supset \star\left(\widetilde{\Psi} \wedge \star\Delta_{\text{fer}}\Psi\right)\,, \tag{B.28}$$

where:

$$\Delta_{\text{fer}} = \begin{pmatrix} -\frac{i\sqrt{2}}{4}\,[\tilde{\hat{\phi}}, \cdot\,] & i\,\bar{\partial}_A & \frac{i}{2}\star\partial_A \\ -\,i\star\partial_A\star & i\sqrt{2}\,[\hat{\phi}, \cdot\,]\text{dvol} & 0 \\ -\frac{i}{2}\star\bar{\partial}_A & 0 & \frac{i\sqrt{2}}{4}\,[\hat{\phi}, \cdot\,] \end{pmatrix}\,. \tag{B.29}$$

Note that this is the same as the kinetic operator for the hypermultiplet in (4.5), up to an irrelevant sign in one of the diagonal terms.

One can now work in the Cartan-Weyl basis E_α, H_a, with α denoting the non-vanishing roots and H_a the Cartan elements, by expanding every field as $\varphi = \sum_a \varphi_a H_a + \sum_\alpha \varphi_\alpha E_\alpha$. The computation is similar to the 2d argument presented in [117]. From this analysis, we conclude that, at one-loop, the W-bosons enter the topologically twisted partition function on $\mathcal{M}_4$ exactly as a twisted hypermultiplet with higher-spin $(j_l, j_r) = (0, \frac{1}{2})$, in agreement with the discussion in section 4.4.

B.3 Index theorems: review and computations

In this appendix, we review the Atiyah-Singer index theorem (see e.g. [118] for an introduction aimed at physicists), and we work out some indices that are useful for our purposes.

Given (E, D) an elliptic complex over an n-dimensional compact manifold M without boundary:

$$\ldots \xrightarrow{D_{i-2}} \Gamma(M, E_{i-1}) \xrightarrow{D_{i-1}} \Gamma(M, E_i) \xrightarrow{D_i} \Gamma(M, E_{i+1}) \longrightarrow \ldots\,, \tag{B.30}$$

the Atiyah-Singer theorem states that the index (i.e. the Euler characteristic) of this complex is determined as follows:

$$\mathrm{ind}(E, D) = (-1)^{\frac{n(n+1)}{2}} \int_M \mathrm{ch}\left(\bigoplus_i (-1)^i E_i\right) \frac{\mathrm{Td}\left(TM^{\mathbb{C}}\right)}{e(TM)}\,. \tag{B.31}$$

Here Td denotes the Todd class, ch the Chern character and e the Euler class. TM is the tangent bundle of M, while $TM^{\mathbb{C}}$ is the complexified version of it.

Two quantities that appear often in the context of 4-manifolds are the Euler characteristic χ and the signature σ. They are the indices of the de Rham complex and of the signature complex, respectively. They are given by:

$$\chi = \sum_{i=0}^{4} (-1)^i b_i = \int_M e(TM)\,, \qquad \sigma = b_2^+ - b_2^- = \frac{1}{3}\int_M p_1(TM)\,, \tag{B.32}$$

where b_i are the Betti numbers of M and p_1 is the first Pontryagin class. For complex 4-manifolds, we also have $e(TM) = c_2(TM)$ in terms of the second Chern class c_2, where in the latter the tangent bundle TM is viewed as a complex vector bundle.

Dolbeault Complex. Let us consider the Dolbeault complex of a complex four-manifold M. The Dolbeault complex is the elliptic complex for the $\bar\partial$ operator, with $E_i = \Omega^{0,i} \equiv \bigwedge^i T^*M^-$, where $\Omega^{0,1} \cong T^*M^-$ is the anti-holomorphic cotangent bundle spanned by $\{d\bar z^\mu\}$. The analytical index is given by the alternating sum of Hodge numbers:

$$\mathrm{ind}(\bar\partial) = \sum_{i=0}^{2} (-1)^i h^{0,i} = \chi_h\,, \tag{B.33}$$

called the holomorphic Euler characteristic χ_h, and the index theorem gives us:

$$\mathrm{ind}(\bar\partial) = \int_M \mathrm{Td}(TM^+)\,, \tag{B.34}$$

with TM^+ the tangent bundle spanned by $\{\partial/\partial z^\mu\}$. This integral evaluates to:

$$\mathrm{ind}(\bar\partial) = \int_M \frac{1}{12}\left(c_1(TM^+) \wedge c_1(TM^+) + c_2(TM^+)\right) = \frac{\chi + \sigma}{4}\,, \tag{B.35}$$

which gives us the relation $\chi_h = \frac{\chi+\sigma}{4}$. One can also consider the tensor product bundles $\Omega^{0,i} \otimes V$, for some holomorphic vector bundle V over M, leading to the twisted Dolbeault complex:

$$\ldots \xrightarrow{\bar\partial_V} \Omega^{0,i-1}(M) \otimes V \xrightarrow{\bar\partial_V} \Omega^{0,i}(M) \otimes V \xrightarrow{\bar\partial_V} \Omega^{0,i+1}(M) \otimes V \xrightarrow{\bar\partial_V} \ldots\,. \tag{B.36}$$

Applying the splitting principle for the Chern character,

$$\mathrm{ch}\left(\bigoplus_i (-1)^i \Omega^{0,i}(M) \otimes V\right) = \mathrm{ch}(V) \sum_i (-1)^i \mathrm{ch}\left(\Omega^{0,i}(M)\right), \tag{B.37}$$

together with the computation for the Dolbeault complex, the index theorem specialises to the Hirzebruch-Riemann-Roch theorem:

$$\mathrm{ind}(\bar{\partial}_V) = \int_M \mathrm{Td}(TM^+)\mathrm{ch}(V) = \chi(M, V)\ . \tag{B.38}$$

Here, $\chi(M, V)$ is also often called the holomorphic Euler characteristic.

Dirac Complex. Consider a spin bundle $\mathbf{S}$ over an n-dimensional (n even) orientable manifold M. The spin bundle splits as $S^+ \oplus S^-$. The Dirac operator, defined as $\not{D} = i\gamma^\mu \nabla_\mu$, is in fact an elliptic operator, and it also splits as:

$$\not{D} = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}. \tag{B.39}$$

In this case, the analytical index, $\mathrm{ind}(D^+) = n_+ - n_-$, counts the difference between the numbers of positive and negative chirality modes. It can be shown that the topological index can be solely expressed in terms of the $\hat{A}$-roof genus, which is a characteristic class containing only $4j$-forms. Thus, the index vanishes unless $n = 0 \mod 4$. For 4-dimensional manifolds, it reads:

$$\mathrm{ind}(\not{D}) = \int_M \hat{A}(TM) = -\frac{1}{24}\int_M p_1(TM) = -\frac{1}{8}\sigma(M)\ . \tag{B.40}$$

More generally, spinors can transform in some representation of a group G. They are then sections of $S(M) \otimes E$, where E is an associated vector bundle to the principal G-bundle over M, in an appropriate representation. The twisted Dirac operator $\not{D}_E$ now contains both the spin connection and a gauge connection on E, while the index theorem gives us:

$$\mathrm{ind}(\not{D}_E) = n_+ - n_- = \int_M \hat{A}(TM)\mathrm{ch}(E)\ . \tag{B.41}$$

Relation between the twisted Dirac and Dolbeault complexes. On a complex surface, we have a formal equivalence between the indices of the twisted Dirac and Dolbeault complexes, by tensoring with the square root of the canonical line bundle, $\mathcal{K}^{\frac{1}{2}}$:

$$\mathrm{ind}(\bar{\partial}_V) = \mathrm{ind}(\not{D}_E)\,, \qquad V = \mathcal{K}^{\frac{1}{2}} \otimes E\,, \tag{B.42}$$

where the index theorem gives us:

$$\mathrm{ind}(\bar{\partial}_V) = -\frac{\mathrm{rk}(E)}{8}\sigma + \int_M \mathrm{ch}(E)\ . \tag{B.43}$$

In particular, in the case of the extended topological twist of the hypermultiplet (as introduced in section 2.3), we have $V = \mathcal{K}^{\varepsilon+\frac{1}{2}} \otimes L$, where L is a well-defined $U(1)$ bundle with flux $c_1(L) = \mathfrak{m}$, and one finds:

$$\mathrm{ind}\left(\bar{\partial}_{\mathcal{K}^{\varepsilon+\frac{1}{2}}\otimes L}\right) = -\frac{\sigma}{8} + \frac{1}{2}\varepsilon^2(2\chi + 3\sigma) + \int_M c_1(L) \wedge \left(c_1(L) + 2\varepsilon\, c_1(\mathcal{K})\right)\ . \tag{B.44}$$

Recall that the standard topological twist corresponds to $\varepsilon = 0$, while in general, on a non-spin complex surface, we need to consider the extended DW twist with $\varepsilon \neq 0$.

Let us also collect a few identities that are useful for computing indices. Firstly, the relevant contributions from the $\widehat{A}$ and Todd classes read:

$$\begin{aligned} \hat{A}(TM) &= 1 - \frac{1}{24} p_1(TM)\,, \\ \mathrm{Td}(TM^+) &= 1 + \frac{1}{2} c_1(TM^+) + \frac{1}{12}\left(c_1(TM^+)^2 + c_2(TM^+)\right) \,. \end{aligned} \tag{B.45}$$

We have the relations $c_1(TM^+) = -c_1(\mathcal{K})$ and $c_2(TM^+) = c_2(\Omega^{0,1})$, as well as:

$$\int_M c_1(K)^2 = 2\chi + 3\sigma\,, \qquad \int_M c_2(\Omega^{0,1}) = \chi\,. \tag{B.46}$$

Finally, the Chern character of any holomorphic vector bundle E expands as:

$$\mathrm{ch}(E) = \mathrm{rk}(E) + c_1(E) + \frac{1}{2} c_1(E)^2 - c_2(E)\,. \tag{B.47}$$

The higher-spin particle index. Finally, let us compute the index of the Dolbeault complex $\bar{\partial}_V$ twisted by:

$$V = \mathcal{K}^{\frac{1}{2}+\varepsilon} \otimes S^{2j_l}(S_-) \otimes S^{2j_r}(S_+) \otimes F\,, \tag{B.48}$$

over a Kähler four-manifold $\mathcal{M}_4$, where F is some holomorphic vector bundle. This is index relevant for computing the contribution of a massive 5d particle of spin (j_l, j_r) (see section 4.4). The symmetric powers of the spin bundles $S_\pm$ can be formally expanded as:

$$S^{2j_l}(S_-) \cong \mathcal{K}^{j_l} \otimes S^{2j_l}(\Omega^{0,1})\,, \qquad S^{2j_r}(S_+) \cong \oplus_{m_r=-j_r}^{j_r} \mathcal{K}^{m_r}\,. \tag{B.49}$$

Their Chern characters are given by:[32]

$$\begin{aligned} \mathrm{ch}\left(S^{2j_l}(S_-)\right) &= (2j_l+1)\left(1 + \frac{j_l(j_l+1)}{6}(c_1(\mathcal{K})^2 - 4c_2(\Omega^{0,1}))\right)\,, \\ \mathrm{ch}\left(S^{2j_r}(S_+)\right) &= (2j_r+1)\left(1 + \frac{j_r(j_r+1)}{6} c_1(\mathcal{K})^2\right)\,. \end{aligned} \tag{B.50}$$

Then, we immediately find:

$$\begin{aligned} \mathrm{ind}(\bar{\partial}_V) = \mathrm{rk}(F)(2j_l+1)(2j_r+1)\Bigg[&-\frac{\sigma}{8} + \frac{1}{2}\varepsilon^2(2\chi+3\sigma) - \frac{2}{3} j_l(j_l+1)\chi \\ &+ \frac{j_l(j_l+1) + j_r(j_r+1)}{6}(2\chi+3\sigma)\Bigg] \\ &+ (2j_l+1)(2j_r+1)\left[\varepsilon \int_M c_1(F) c_1(\mathcal{K}) + \int_M \mathrm{ch}(F)\right]\,, \end{aligned} \tag{B.51}$$

which gives us the result (4.56).

[32] To compute the Chern character of the k-symmetric power of any vector bundle E with Chern roots x_i, we can use the fact that:

$$\sum_k \mathrm{ch}(S^k(E)) t^k = \prod_i \frac{1}{1 - t e^{x_i}}\,.$$

C E_n SCFTs and Gopakumar-Vafa invariants

Five-dimensional SCFTs are engineered in M-theory compactifications on Calabi-Yau threefold canonical singularities $\mathbf{X}$. Their partition function in the Ω-background is computed using the refined topological vertex and is fully determined by the numbers $N^{\beta}_{j_l,j_r}$ of massive particles of spin (j_l, j_r) for a fixed Kähler class $\beta \in H_2(\widetilde{\mathbf{X}}, \mathbb{Z})$. The simplest five-dimensional SCFTs are the so-called Seiberg E_n theories, originally studied in [5, 105, 106], which we shall also focus on here.

In this appendix, we list some of the refined GV invariants for the local del Pezzo and Hirzebruch geometries. We first consider the local $\mathbb{F}_0$ threefold to streamline the computation. The GV invariants for this can be found in [58], as well as in [114]. For this computation, we use the fact that the refined topological string free energy can be expressed as:

$$\boldsymbol{F}(Q,\tau_1,\tau_2) = \sum_{j_l,j_r\geq 0}\sum_{\beta}\sum_{w=1}^{\infty}(-1)^{2j_l+2j_r}N^{\beta}_{j_l,j_r}\frac{\chi_{j_l}\left((q_1/q_2)^{\frac{w}{2}}\right)\chi_{j_r}\left((q_1q_2)^{\frac{w}{2}}\right)}{w\left(q_1^{\frac{w}{2}}-q_1^{-\frac{w}{2}}\right)\left(q_2^{\frac{w}{2}}-q_2^{-\frac{w}{2}}\right)}Q^{w\beta}\,, \tag{C.1}$$

where Q are the Kähler parameters, $q_k = e^{2\pi i\tau_k}$, for $k = 1, 2$, and χ_j are $SU(2)$ characters:

$$\chi_j(q) = \frac{q^{2j+1}-q^{-2j-1}}{q-q^{-1}}\,. \tag{C.2}$$

Note that this expression corresponds to the ordinary DW twist, setting $\varepsilon = 0$. We will use this result together with the Nekrasov partition functions to compute the GV invariants $N^{\beta}_{j_l,j_r}$. We will also comment on the perturbative part of the partition function by using the results obtained from the SW geometry.

Let us also note that the prepotential can be resummed to:

$$\mathcal{F} = -\frac{1}{(2\pi i)^3}\sum_{\beta} d_{\beta}\mathrm{Li}_3(Q^{\beta})\,, \qquad d_{\beta} = \sum_{j_l,j_r} c_0^{(j_l,j_r)}N^{\beta}_{j_l,j_r}\,, \tag{C.3}$$

with $c_0^{(j_l,j_r)} = (-1)^{2(j_l+j_r)}(2j_l+1)(2j_r+1)$. Similarly, the gravitational couplings become:

$$\mathcal{A} = \frac{1}{2\pi i}\sum_{\beta} d^{\mathcal{A}}_{\beta}\log(1-Q^{\beta})\,, \qquad \mathcal{B} = \frac{1}{2\pi i}\sum_{\beta} d^{\mathcal{B}}_{\beta}\log(1-Q^{\beta})\,, \tag{C.4}$$

with:

$$d^{\mathcal{A},\mathcal{B}}_{\beta} = \sum_{j_l,j_r} c^{(j_l,j_r)}_{\mathcal{A},\mathcal{B}}N^{\beta}_{j_l,j_r}\,. \tag{C.5}$$

Here, $c^{(j_l,j_r)}_{\mathcal{A}}$ and $c^{(j_l,j_r)}_{\mathcal{B}}$ are given by (6.33) for a particle of spin (j_l, j_r). Restoring the extended topological twist, the factors H and $\mathcal{G}$ appearing in the non-equivariant limit (6.10) become:

$$H = -\frac{1}{(2\pi i)^2}\varepsilon\sum_{\beta} d_{\beta}\,\mathrm{Li}_2(Q^{\beta})\,, \qquad \mathcal{G} = \frac{1}{2\pi i}\varepsilon^2\sum_{\beta} d_{\beta}\log(1-Q^{\beta})\,. \tag{C.6}$$

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C.1 Five-dimensional SCFTs: the E_1 theory

In this subsection, we shall focus on the E_1 SCFT. We also offer an alternative way of using the fibering operator and the gluings of Nekrasov partition functions, by expressing the result in terms of instanton corrections.

Recall first the proposal of [19] that the gravitational couplings $\mathcal{A}$ and $\mathcal{B}$ can be obtained directly from the Seiberg-Witten geometry, in analogy with the four-dimensional prescription of [26, 43, 54]. Given any rank-one Seiberg-Witten geometry, with CB parameter U, we thus have:

$$A(U) = \alpha \left(\frac{dU}{da}\right) , \qquad B(U) = \beta \left(\Delta^{phys}\right)^{\frac{1}{8}} , \tag{C.7}$$

with a the VEV of the 5d vector multiplet, Δ^{phys} the 'physical-discriminant' and α, β some constant prefactors determined in [19]. Let us also point out that, in the conventions of [19], the physical discriminant is equivalent to the discriminant of the Seiberg-Witten curve, up to a numerical prefactor.

Focusing on the E_1 SCFT, which is the UV completion of the 5d $\mathcal{N} = 1$ $SU(2)_0$ gauge theory, its Seiberg-Witten curve reads:

$$\begin{aligned} g_2(U) &= \frac{1}{12}\Big(U^4 - 8(1+\lambda)U^2 + 16\left(1-\lambda+\lambda^2\right)\Big) , \\ g_3(U) &= -\frac{1}{216}\Big(U^6 - 12(1+\lambda)U^4 + 24\left(2+\lambda+2\lambda^2\right)U^2 - 32\left(2-3\lambda-3\lambda^2+2\lambda^3\right)\Big) , \end{aligned} \tag{C.8}$$

with the discriminant given by:

$$\Delta(U) = \lambda^2\Big(U^4 - 8(1+\lambda)U^2 + 16(1-\lambda)^2\Big) . \tag{C.9}$$

Here λ is the inverse gauge coupling, which, in the geometric engineering limit, becomes the instanton counting parameter as it gets identified with the dynamically generated scale Λ^4 of the 4d $SU(2)$ gauge theory [119]. The complexified Kähler parameters of the IIA geometry are:

$$Q_f \equiv Q^2 = e^{2\pi i t_f} = e^{4\pi i a} , \qquad Q_b = e^{2\pi i t_b} = e^{2\pi i(2a+\mu_0)} , \qquad Q_b = \lambda Q_f , \tag{C.10}$$

with the periods $t_{f,b}$ corresponding to $D2$-branes wrapping the $\mathbb{P}^1$ curves $\mathcal{C}_{b,f}$ of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Thus, the Kähler classes are labelled by pairs of integers $\boldsymbol{\beta} = (d_1, d_2)$ corresponding to the degrees of (Q_b, Q_f). Of course, for this particular geometry, there is a symmetry exchanging the two integers. We will denote by $Q^{\boldsymbol{\beta}} = Q_b^{d_1} Q_f^{d_2}$, for $\boldsymbol{\beta} = (d_1, d_2)$, which should not be confused with the notation $Q_f = Q^2 = e^{4\pi i a}$.

As λ plays the role of the instanton counting parameter, the n^{th} instanton contribution of the four-dimensional theory is recovered from all particles of Kähler classes (n, d_2), for all $d_2 \in \mathbb{Z}$. Note that this gauge-theory point of view is not the most natural one from a geometric perspective, but it serves as a computational tool.

The simplest Gopakumar-Vafa invariants for the E_1 theory are:

$$N^{(1,d)}_{j_l, j_r} = N^{(d,1)}_{j_l, j_r} = \delta_{j_l,0}\delta_{j_r, d+\frac{1}{2}} , \tag{C.11}$$

β	$j_l \backslash j_r$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	$\frac{11}{2}$	6	$\frac{13}{2}$	7	$\frac{15}{2}$	8	$\frac{17}{2}$	9	$\frac{19}{2}$
(2,2)	0						1		1												
	1/2									1											
(2,3)	0						1		1		2										
	1/2									1		1									
	1												1								
(2,4)	0						1		1		2		2								
	1/2									1		1		2							
	1												1		1						
	3/2															1					
(2,5)	0						1		1		2		2		3						
	1/2									1		1		2		2					
	1												1		1		2				
	3/2															1		1			
	2																		1		
(3,3)	0				1		1		3		3		4								
	1/2							1		2		3		3		1					
	1										1		2		3						
	3/2													1		1					
	2																1				
(3,4)	0		1		1		3		4		7		6		7		1		1		
	1/2					1		2		4		6		8		7		2			
	1								1		2		5		6		7		1		
	3/2											1		2		4		4		1	
	2														1		2		3		
	5/2																	1		1	
	3																				1

Table 1. Refined GV invariants $N^{\boldsymbol{\beta}}_{j_l,j_r}$ for local $\mathbb{F}_0$ geometry.

Following the above discussion, the perturbative contribution to the prepotential will be thus determined by the particles in the class $\boldsymbol{\beta} = (0,1)$, which amount for a single W-boson, namely a particle of spin $(j_l, j_r) = (0, \frac{1}{2})$. This leads to:

$$\mathcal{F}_{\text{pert}} = \frac{1}{(2\pi i)^3} 2\text{Li}_3(Q_f)\,, \qquad \mathcal{A}_{\text{pert}} = \mathcal{B}_{\text{pert}} = -\frac{1}{4\pi i}\log(1-Q_f)\ . \tag{C.12}$$

Note that, from the Seiberg-Witten geometry computation (C.7), there is an additional $\log(Q)$ term for both $\mathcal{A}$ and $\mathcal{B}$. These terms depend on the choice of quantisation scheme, as pointed out in [19, 59]. We list some of the higher degree invariants in table 1.

Partition function. Given the above refined GV invariants, let us now consider the CB partition function of the E_1 theory on $\mathcal{M}_5$. Knowing the expressions for $\mathcal{F}, \mathcal{A}$ and $\mathcal{B}$, or, alternatively, for the GV invariants $N_{\boldsymbol{\beta}}^{j_l,j_r}$, the result should follow immediately. We will express the result in terms of the instanton counting parameter λ.

From the topological string partition function — or, equivalently, from the Nekrasov instanton partition function of the 5d $SU(2)_0$ gauge theory — we have the following closed-form expression for the flat-space partition function:

$$Z^{\text{1-inst}}_{\mathbb{C}^2\times S^1}(a,\tau_1,\tau_2) = \frac{qp(1+qp)}{(1-q)(1-p)(1-Qqp)(1-Q^{-1}qp)}\,, \tag{C.13}$$

See [19] for our conventions on the instanton partition function. Note that the above result holds in the $\varepsilon = 0$ case. The extended topological twist can be easily recovered by shifting $a \to a + \varepsilon(\tau_1+\tau_2)$, as explained in (6.9). We then note that:

$$\begin{aligned} &\mathcal{F}_{\text{1-inst}}(a) = \frac{1}{(2\pi i)^3}\frac{2Q}{(1-Q)^2}\,, \\ &\mathcal{A}_{\text{1-inst}}(a) = \frac{1}{2\pi i}\frac{Q(1+6Q+Q^2)}{2(1-Q)^4}\,, \qquad \mathcal{B}_{\text{1-inst}}(a) = \frac{1}{2\pi i}\frac{Q(1+10Q+Q^2)}{2(1-Q)^4}\,, \end{aligned} \tag{C.14}$$

while $H^{\text{1-inst}}$ vanishes for the DW twist. Recall that the 1-instanton contribution comes from the states associated to the Kähler classes $\boldsymbol{\beta} = (1,k)$, for all $k \in \mathbb{N}$. This is a sum over an infinite number of particles, which is, in principle, difficult to compute as the GV invariants are only known up to some finite order. Such sums were shown to reproduce the 4d instanton corrections in [119] using an assumption on the growth of the coefficients $d_{\boldsymbol{\beta}}$. For the 1-instanton case, no such assumption is needed due to (C.11), which leads to $d_{(1,k)} = -2(k+1)$, while:

$$d^{\mathcal{A}}_{(1,k)} = -\frac{1}{6}(1+k)(1+2k)(3+2k)\,, \qquad d^{\mathcal{B}}_{(1,k)} = -\frac{1}{2}(1+k)(1+4k+2k^2)\,. \tag{C.15}$$

As a result, expanding the polylogarithms in the instanton counting parameter, one recovers (C.14) after summing over all classes $\boldsymbol{\beta} = (1,k)$, for $k \in \mathbb{N}$. Similar expressions to (C.14) can be obtained for higher instanton corrections from the instanton partition function (see e.g. [19] for a recent discussion) and it can be checked that those are reproduced by the GV invariants.

Consider now the five-sphere partition function in the absence of background fluxes, for which we shall analyze explicitly the 1-instanton correction. Using the gluing (6.61), the contribution to the (logarithm of the) five-sphere partition function reads, in the non-equivariant limit:

$$\begin{aligned} &\lim_{\tau_{1,2}\to 0}\lambda^2\left(\log Z^{\text{1-inst}}_{\mathbb{C}^2\times S^1}(a,\tau_1,\tau_2) + \log Z^{\text{1-inst}}_{\mathbb{C}^2\times S^1}(a^*,\tau_1^*,\tau_2^*) + \log Z^{\text{1-inst}}_{\mathbb{C}^2\times S^1}(\tilde{a},\tilde{\tau}_1,\tilde{\tau}_2)\right) = \\ &\lambda^2\left(-\frac{2Q(1+7Q+Q^2)}{(1-Q)^4}\right) + \lambda^2\left(\frac{1}{4\pi^2}\frac{2Q}{(1-Q)^2} - \frac{Q(1+Q)}{\pi i(1-Q)^3}a - \frac{Q(1+4Q+Q^2)}{(1-Q)^4}a^2\right)\,. \end{aligned} \tag{C.16}$$

The first term can be reorganised as:

$$-2\pi i\Big(\chi(\mathbb{P}^2)\mathcal{A}_{\text{1-inst}}(a) + \sigma(\mathbb{P}^2)\mathcal{B}_{\text{1-inst}}(a)\Big)\,, \tag{C.17}$$

being the contribution to the $\mathbb{P}^2 \times S^1$ partition function. Moreover, the remaining term is due to the fibering operator, giving us $\frac{1}{2}\log \mathscr{F}(\boldsymbol{a})$, with $\mathscr{F}$ as defined in (5.34), as one can easily check using the 1-instanton correction to the prepotential in (C.14). It does agree with the general form of the fibering operator for $\varepsilon = 0$. For a generic 5d SCFTs, one needs to consider the extended topological twist with $\varepsilon \neq 0$, in general, which is easily done as explained in the main text. For the E_1 theory, we can consistently choose $\varepsilon = 0$.

C.2 Local dP_2 geometry

For the E_2 theory, the SW curve in the parametrisation of [19] is given by:

$$\sqrt{\lambda}\left(1+\frac{M_1}{w}\right)+t\left(\frac{1}{w}+w-2U\right)+\sqrt{\lambda}\,t^2=0\ . \tag{C.18}$$

The instanton corrections to the prepotential agree with Nekrasov instanton counting results upon identifying $M_1 = -e^{-2\pi i\mu}$, with μ the five-dimensional mass parameter (see [19] for our conventions on the instanton partition function). Inspired by the topological vertex formalism, let us define $Q_m = e^{-2\pi i\mu}e^{-2\pi i a} = e^{-2\pi i t_m}$. Here $t_m = a+\mu$ is the complexified Kähler parameter associated to the exceptional curve $\mathcal{C}_m$, resulting from blowing-up of $\mathbb{F}_0$ at a single generic point:[33]

$$t_m = \int_{\mathcal{C}_m} (B+iJ)\ . \tag{C.19}$$

The perturbative contributions to the prepotential obtained from the Seiberg-Witten curve can be expressed in a compact form as:

$$(2\pi i)^3\mathcal{F}_{\rm pert} = 2\mathrm{Li}_3(Q_f) - \mathrm{Li}_3\left(Q_f Q_m\right) - \mathrm{Li}_3\left(Q_m^{-1}\right)\ . \tag{C.20}$$

In the basis (Q_b, Q_f, Q_m), we find that the prepotential is reproduced by the states:

$$\begin{aligned}
(0,1,0):&\quad N_{j_l,j_r} = \delta_{j_l,0}\ \delta_{j_r,\frac{1}{2}}\ ,\\
(0,0,-1):&\quad N_{j_l,j_r} = \delta_{j_l,0}\ \delta_{j_r,0}\ ,\\
(0,1,1):&\quad N_{j_l,j_r} = \delta_{j_l,0}\ \delta_{j_r,0}\ .
\end{aligned} \tag{C.21}$$

Let us note that the only state contributing to the perturbative part of $\mathcal{A}$ is the $(0,1,0)$ term, which is in perfect agreement with the SW geometry results, namely:

$$2\pi i\,\mathcal{A}_{\rm pert} = -\frac{1}{2}\log(1-Q_f)\ . \tag{C.22}$$

On the other hand, the $\mathcal{B}$ gravitational correction receives contributions from all of the above states, leading to:

$$2\pi i\,\mathcal{B}_{\rm pert} = -\frac{1}{2}\log(1-Q_f) - \frac{1}{8}\log\left(1-Q_f Q_m\right) - \frac{1}{8}\log\left(1-Q_m^{-1}\right)\ . \tag{C.23}$$

[33] In [19], when viewing dP_n as a blow-up of $\mathbb{F}_0$ at $n-1$ generic points, these exceptional curves were denoted by E_i, for $i = 1,\ldots,n-1$.

The '1-instanton' GV invariants in the basis (Q_b, Q_f, Q_m) are given by:

$$\begin{aligned}
(1,n,0): \quad & N_{j_l,j_r} = \delta_{j_l,0}\ \delta_{j_r,n+\frac{1}{2}}\,, \\
(1,n,1): \quad & N_{j_l,j_r} = \delta_{j_l,0}\ \delta_{j_r,n}\,, \\
(1,n,d): \quad & N_{j_l,j_r} = 0\,, \quad \text{for} \quad d>2\,.
\end{aligned} \tag{C.24}$$

Let us also comment on the limits to E_1 (i.e. local $\mathbb{F}_0$) and $\widetilde{E}_1$ (i.e. local dP_1). From the instanton counting results, one can check that for the n-instanton contribution the terms of order $(M_1)^0$ and $(M_1)^n$ reproduce the E_1 and $\widetilde{E}_1$ contributions, respectively (and thus, the two geometries are recovered in the $M_1 \to 0$ and $M_1 \to \infty$ limits, respectively). In terms of the GV invariants, this means that the $(n,m,0)$ invariants should precisely reproduce the $\mathbb{F}_0$ invariants listed in table 1. We list some of the low-degree GV invariants for the local dP_2 that are not $\mathbb{F}_0$ invariants in table 2. Note also that the (n,m,n) invariants do coincide with the dP_1 invariants computed in [58].

C.3 Local dP_3 geometry

For the E_3 theory we choose the following parametrisation [19]:

$$\sqrt{\lambda}\left(1+\frac{M_1}{w}\right)+t\left(\frac{1}{w}+w-2U\right)+\sqrt{\lambda}\,t^2(1+M_2 w)=0\,. \tag{C.25}$$

As before, the instanton corrections agree with Nekrasov instanton counting results upon identifying $M_i = -e^{-2\pi i\mu_i}$. As for E_2, we define $Q_{m_i} = e^{-2\pi i\mu_i}e^{-2\pi i a}$. The perturbative contributions to the prepotential obtained from the Seiberg-Witten curve are:

$$(2\pi i)^3\mathcal{F}_{\rm pert} = 2\mathrm{Li}_3(Q_f) - \sum_{i=1}^{2}\left(\mathrm{Li}_3\left(Q_f Q_{m_i}\right)+\mathrm{Li}_3\left(Q_{m_i}^{-1}\right)\right)\,. \tag{C.26}$$

Let us also note the symmetry in $Q_{m_1} \leftrightarrow Q_{m_2}$. For this reason, we avoid writing down redundant invariants. In the basis $(Q_b, Q_f, Q_{m_1}, Q_{m_2})$, the states that reproduce the above prepotential are:

$$\begin{aligned}
(0,1,0,0): N_{j_l,j_r} &= \delta_{j_l,0}\ \delta_{j_r,\frac{1}{2}}\,, \\
(0,0,-1,0): N_{j_l,j_r} &= \delta_{j_l,0}\ \delta_{j_r,0}\,, \\
(0,1,1,0): N_{j_l,j_r} &= \delta_{j_l,0}\ \delta_{j_r,0}\,.
\end{aligned} \tag{C.27}$$

The only state contributing to $\mathcal{A}$ at this order is the $(0,1,0,0)$ state, in perfect agreement with the SW geometry results. The $\mathcal{B}$ gravitational correction receives contributions from all of the above states, as was the case in the E_2 theory. The only 1-instanton GV invariants are given by:

$$\begin{aligned}
(1,n,0,0): \quad & N_{j_l,j_r} = \delta_{j_l,0}\ \delta_{j_r,n+\frac{1}{2}}\,, \\
(1,n,1,0): \quad & N_{j_l,j_r} = \delta_{j_l,0}\ \delta_{j_r,n}\,, \\
(1,n,1,1): \quad & N_{j_l,j_r} = \delta_{j_l,0}\ \delta_{j_r,n-\frac{1}{2}}\,.
\end{aligned} \tag{C.28}$$

For the 2-instanton GV invariants let us first note that some of these will be the same as the E_1, $\widetilde{E}_1$ invariants, namely:

$$(2,n,0,0)_{dP_3} \cong (2,n)_{\mathbb{F}_0}\,, \qquad (2,n,2,0)_{dP_3} \cong (2,n,0,2)_{dP_3} \cong (2,n)_{dP_1}\,, \tag{C.29}$$

$\boldsymbol{\beta}$	$j_l \backslash j_r$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	$\frac{11}{2}$	6	$\frac{13}{2}$	7	$\frac{15}{2}$	8	$\frac{17}{2}$	9	$\frac{19}{2}$
(2,1,1)	0					1															
(2,2,1)	0					1		2													
	1/2								1												
(2,3,1)	0					1		2		3											
	1/2								1		2										
	1											1									
(2,4,1)	0					1		2		3		4									
	1/2								1		2		3								
	1											1		2							
	3/2														1						
(2,5,1)	0					1		2		3		4		5							
	1/2								1		2		3		4						
	1											1		2		3					
	3/2														1		2				
	2																	1			
(2,2,2)	0						1														
(2,3,2)	0						1		1												
	1/2									1											
(2,4,2)	0						1		1		2										
	1/2									1		1									
	1												1								
(2,5,2)	0						1		1		2		2								
	1/2									1		1		2							
	1												1		1						
	3/2															1					
(2,6,2)	0						1		1		2		2		3						
	1/2									1		1		2		2					
	1												1		1		2				
	3/2															1		1			
	2																		1		
(3,1,1)	0							1													
(3,2,1)	0					1		2		3											
	1/2								1		2										
	1											1									

Table 2. Refined GV invariants $N^{\boldsymbol{\beta}}_{j_l,j_r}$ for local dP_2 geometry.

β	$j_l\backslash j_r$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	$\frac{11}{2}$	6	$\frac{13}{2}$	7	$\frac{15}{2}$
(2,1,1,1)	0				1												
(2,2,1,1)	0				1		3										
	1/2							1									
(2,3,1,1)	0				1		3		5								
	1/2							1		3							
	1										1						
(2,4,1,1)	0				1		3		5		7						
	1/2							1		3		5					
	1										1		3				
	3/2													1			
(2,5,1,1)	0				1		3		5		7		9				
	1/2							1		3		5		7			
	1										1		3		5		
	3/2													1		3	
	2																1

Table 3. Refined GV invariants $N^{\beta}_{j_l,j_r}$ for local dP_3 geometry.

where the subscript indicates the geometry. In fact, all these also appear as dP_2 invariants, together with the $(2,n,1,0)_{dP_3} \cong (2,n,0,1)_{dP_3}$ invariants. Furthermore, the GV invariants $(2,n,2,1)_{dP_3} \cong (2,n,1,2)_{dP_3}$ also correspond to the dP_2 invariants $(2,n-1,1)_{dP_2}$, while the $(2,n,2,2)_{dP_3}$ invariants correspond to the $(2,n-2)_{\mathbb{F}_0}$ invariants. We list some of the new invariants in table 3 (again, due to the $Q_{m_1} \leftrightarrow Q_{m_2}$ symmetry we do not write down explicitly some of these).

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