

Stable maps to Looijenga pairs built from the plane

van Garrel, Michel

License:

None: All rights reserved

Document Version

Peer reviewed version

Citation for published version (Harvard):

van Garrel, M 2023, 'Stable maps to Looijenga pairs built from the plane', *Proceedings of the Nottingham Algebraic Geometry Seminar*.

[Link to publication on Research at Birmingham portal](#)

General rights

Unless a licence is specified above, all rights (including copyright and moral rights) in this document are retained by the authors and/or the copyright holders. The express permission of the copyright holder must be obtained for any use of this material other than for purposes permitted by law.

- Users may freely distribute the URL that is used to identify this publication.
- Users may download and/or print one copy of the publication from the University of Birmingham research portal for the purpose of private study or non-commercial research.
- User may use extracts from the document in line with the concept of 'fair dealing' under the Copyright, Designs and Patents Act 1988 (?)
- Users may not further distribute the material nor use it for the purposes of commercial gain.

Where a licence is displayed above, please note the terms and conditions of the licence govern your use of this document.

When citing, please reference the published version.

Take down policy

While the University of Birmingham exercises care and attention in making items available there are rare occasions when an item has been uploaded in error or has been deemed to be commercially or otherwise sensitive.

If you believe that this is the case for this document, please contact UBIRA@lists.bham.ac.uk providing details and we will remove access to the work immediately and investigate.

STABLE MAPS TO LOOIJENGA PAIRS BUILT FROM THE PLANE

MICHEL VAN GARREL

ABSTRACT. Choosing a normal crossings anticanonical divisor of \mathbb{P}^2 leads to four log Calabi–Yau surfaces, three of which are Looijenga pairs. In this survey article, I describe how to count rational curves in these that intersect each component of the divisor in one point of maximal tangency.

CONTENTS

1. Introduction	1
2. Genus 0 log Gromov-Witten invariants of maximal tangency	4
3. Tropical correspondence and the invariants $N_d(\mathbb{P}^2, D^{\text{toric}})$	4
4. Degeneration and the invariants $N_d(\mathbb{P}^2, D_1 + D_2)$	6
5. Scattering and the invariants $N_d(\mathbb{P}^2, D_3)$	11
References	12

1. INTRODUCTION

A log Calabi–Yau surface (Y, D) is a surface Y with a choice of normal crossings divisor D such that the complement is Calabi–Yau. (Y, D) is a Looijenga pair if in addition, D has a dimension 0 stratum. Starting with $Y = \mathbb{P}^2$, D is of degree 3 and there are 4 ways of building a log Calabi–Yau surface: $D = D^{\text{toric}}$, the toric divisor given as a union of 3 lines; $D = D_1 + D_2$, the union of a line D_1 and a conic D_2 not tangent to D_1 ; $D = D_3$ a nodal cubic; and $D = E$ a smooth cubic. All but the last are Looijenga pairs.

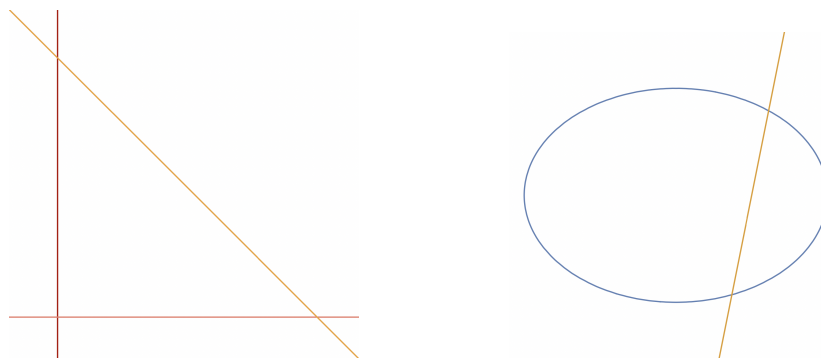


FIGURE 1.1. The toric divisor D^{toric} and the divisor $D_1 + D_2$ formed of a line and a conic.

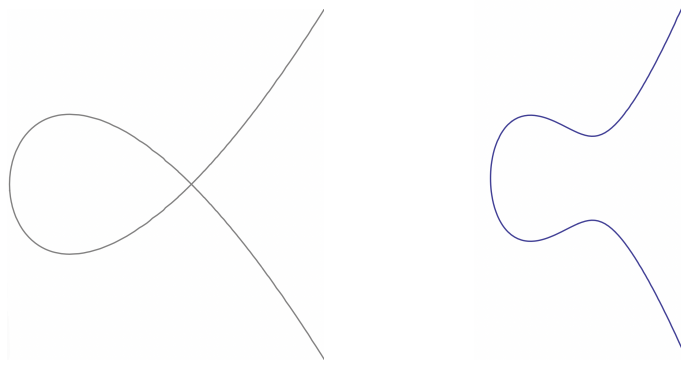


FIGURE 1.2. The divisors formed by resp. a nodal cubic D_3 and a smooth cubic E .

Denote by $l \leq 3$ the number of irreducible components of D and let $\mathbb{P}^1[l]$ be the complement of l points in \mathbb{P}^1 . Up to automorphism, we have that $\mathbb{P}^1[1] = \mathbb{A}^1$, $\mathbb{P}^1[2] = \mathbb{C}^*$ and $\mathbb{P}^1[3] = \mathbb{C}^* \setminus \{1\}$. A surprisingly intricate question is to study the geometry of $Y \setminus D$ by counting pseudo-holomorphic maps

$$\mathbb{P}^1[l] \longrightarrow Y \setminus D$$

of degree $d \in \mathbb{Z}_{>0}$ that pass through $l - 1$ fixed points. For example, in degree 1 and for $D = E$, there are 9 such \mathbb{A}^1 -curves corresponding to the 9 inflection lines of E .

The modern formulation of this count is as the genus 0 maximal tangency log Gromov–Witten invariant $N_d(\mathbb{P}^2, D)$ [1, 20, 33]. These invariants, more generally the punctured invariants of [3], play a crucial role in the mirror constructions of intrinsic mirror symmetry [34, 35]. In this survey, I describe the methods used to compute the $N_d(\mathbb{P}^2, D)$. For the case of $D = D_1 + D_2$, I provide a full proof, which may serve as an extensive introduction to the scattering diagram methods of [15], see also [17] for an introduction emphasizing the links with physics.

For the toric $(\mathbb{P}^2, D^{\text{toric}})$, we can make use of tropical correspondence results [39, 40, 42], which turn $N_d(\mathbb{P}^2, D^{\text{toric}})$ into a count of weighted tropical curves. A special case of [14, Theorem 3.2] then states that

$$N_d(\mathbb{P}^2, D^{\text{toric}}) = d^2, \tag{1.1}$$

reproven in Proposition 3.3 below.

In the case of a line and a conic, $N_d(\mathbb{P}^2, D_1 + D_2)$ is obtained by degeneration. The result of [15, Example 3.1], or Proposition 4.3 below, is that

$$N_d(\mathbb{P}^2, D_1 + D_2) = \binom{2d}{d}. \tag{1.2}$$

The invariants $N_d(\mathbb{P}^2, D_3)$ are encoded by the wall-crossing function of the central ray of a local scattering diagram. The terms of this function are conjectured in [31, Equation (1.3)], which is proven in [43]. Applying [32], the invariants are given by the equality of power series expansions

$$\sum_{d=1}^{\infty} d N_d(\mathbb{P}^2, D_3) x^d = 3 \log \left(\sum_{k=0}^{\infty} \frac{1}{3k+1} \binom{4k}{k} x^k \right). \tag{1.3}$$

The structure of the invariants of (\mathbb{P}^2, E) carries similarities with the structure of invariants of K3 surfaces [16, 22] and thus sometimes (\mathbb{P}^2, E) is called a log K3 surface. The computation of the $N_d(\mathbb{P}^2, E)$ is outside of the scope of this survey. It has a long and rich history. It was studied for the first time in the landmark [44], who stated and provided evidence for 3 influential conjectures regarding (\mathbb{P}^2, E) . The first one is the first appearance of the *log/local correspondence*, which expresses the $N_d(\mathbb{P}^2, E)$ as the genus 0 local Gromov–Witten invariants of the local surface given as the total space $\text{Tot}(\mathcal{O}_{\mathbb{P}^2}(-E))$. This conjecture was first proven in [28], subsequently generalised in [24], and then extended in [7, 13–16, 41, 45]. The second conjecture is a BPS version of the log/local correspondence for (\mathbb{P}^2, E) , that is, a correspondence between the integer-valued counts that govern the genus 0 log and local theories. It was proven through the succession of works [11, 12, 18, 36].

The article [44] also introduces log mirror mirror symmetry of (\mathbb{P}^2, E) . The author builds a mirror family and computes the periods on the mirror family. The third conjecture then states that the second order period written in canonical coordinates is a generating function of the $N_d(\mathbb{P}^2, E)$. Building on [18, 35, 36], a geometric proof of this prediction is given in all dimensions in the upcoming work [25], see also [26] for a self-contained proof for the case of (\mathbb{P}^2, E) .

1.1. Methods. The approach I take in this survey is to degenerate to *toric models* for the Looijenga pairs. These are other – *toric* – Looijenga pairs $(\overline{\mathbb{P}^2}, \overline{D})$ such that (\mathbb{P}^2, D) is a logarithmic modification of a blow-up of $(\overline{\mathbb{P}^2}, \overline{D})$ in smooth points of \overline{D} . Concretely this is a diagram

$$\begin{array}{ccc} & (\widetilde{\mathbb{P}^2}, \widetilde{D}) & \\ \varphi \swarrow & & \searrow \pi \\ (\mathbb{P}^2, D) & & (\overline{\mathbb{P}^2}, \overline{D}) \end{array}$$

such that φ is a sequence of blow-ups in singular points of (the total transform of) D and π is a toric model, i.e. a sequence of blow-ups in smooth points of (the proper transform of) \overline{D} . By [29, Proposition 1.3] such toric models always exist. By [5] φ preserves the log Gromov–Witten invariants.

The next step is to perform a sequence of standard degenerations to $(\widetilde{\mathbb{P}^2}, \widetilde{D})$. Applying the degeneration formula [2, 4, 19, 37, 38], the outcome is that the invariants $N_d(\mathbb{P}^2, D)$ are computed in terms of the log Gromov–Witten invariants of $(\overline{\mathbb{P}^2}, \overline{D})$, which are calculated by tropical correspondence, and the log Gromov–Witten invariants of some standard pieces, which are known. This very general algorithm was formalised in terms of scattering diagram algorithms in [32].

One goal of this note is to unwrap the constructions of [29, 30, 32] for the examples of $(\mathbb{P}^2, D_1 + D_2)$ and (\mathbb{P}^2, D_3) . For a full treatment, the reader is invited to consult [15].

There are other approaches to computing these invariants, all interesting in their own right. Working at the level of moduli spaces these include [6, 21–24, 27, 41] and working with Givental-style mirror symmetry [7, 8, 45, 46].

ACKNOWLEDGEMENTS

I thank Dhruv Ranganathan for originally suggesting to study the example of $D_1 + D_2$ leading to [13, 15]. I thank the Nottingham Online Algebraic Geometry Seminar, in particular Al Kasprzyk and Livia Campo, for giving me the opportunity to present these results. I thank my collaborators Pierrick Bousseau, Andrea Brini, Jinwon Choi, Navid Nabijou, Helge Ruddat, Yannik Schüler and Bernd Siebert for various discussions linking to several aspects of this paper. I thank Fenglong You for discussions about parallel methods for computing these invariants and Tim Gräfnitz for discussions on [36].

2. GENUS 0 LOG GROMOV-WITTEN INVARIANTS OF MAXIMAL TANGENCY

For each Looijenga pair (\mathbb{P}^2, D) , endow \mathbb{P}^2 with the divisorial log structure determined by D . The log structure is used as a combinatorial tool to impose tangency conditions along the components of D . For a map

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^2 \tag{2.1}$$

this is achieved by requiring that the pullback map on log structures at marked points is given by multiplication by the tangency imposed at that point. Compactifying the space of maps as in (2.1), [1, 20, 33] construct the moduli space $\overline{M}_m^{\log}(\mathbb{P}^2, D, d)$ of genus 0 maximally tangent basic stable log maps with m marked points, and its virtual fundamental class

$$[\overline{M}_m^{\log}(\mathbb{P}^2, D, d)]^{\text{vir}} \in \mathbb{H}_{2(l+m-1)}(\overline{M}_m^{\log}(\mathbb{P}^2, D, d)).$$

I suppress in the notation the l marked points that carry a tangency and keep track of the m interior points that do not carry a tangency condition. The notion of basicness/minimality selects a universal log structure needed for algebraicity and compactness of the moduli space.

For $j = 1, \dots, m$, the moduli space admits evaluation maps at the j th marked points

$$\text{ev}_j : \overline{M}_m^{\log}(\mathbb{P}^2, D, d) \longrightarrow \mathbb{P}^2.$$

Imposing passing through a fixed point in \mathbb{P}^2 corresponds to capping the virtual fundamental class with $\text{ev}_j^*([\text{pt}])$. Doing this $l - 1$ times leads to the invariant

$$N_d(\mathbb{P}^2, D) := \int_{[\overline{M}_{l-1}^{\log}(\mathbb{P}^2, D, d)]^{\text{vir}}} \prod_{j=1}^{l-1} \text{ev}_j^*([\text{pt}]).$$

Other insertions may be imposed. For example, for the class toric varieties for which each toric divisor is nef, [14] computes all invariants with point and psi classes.

Example 2.1. Fixing a point P , there are two lines in the plane that pass through P , intersect D_1 in one point of order 1 and intersect D_2 in one point of order 2, see Figure 2.1.

3. TROPICAL CORRESPONDENCE AND THE INVARIANTS $N_d(\mathbb{P}^2, D^{\text{toric}})$

The Looijenga pair $(\mathbb{P}^2, D^{\text{toric}})$ is toric, thus is its own toric model, and we can compute $N_d(\mathbb{P}^2, D^{\text{toric}})$ by enumerating weighted tropical curves in the fan Ξ of $(\mathbb{P}^2, D^{\text{toric}})$.

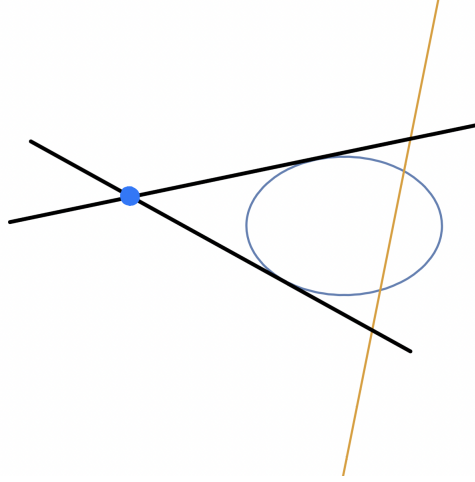


FIGURE 2.1. The two lines passing through a point and meeting a conic, resp. a line, in a point of order 2, resp. 1.

Definition 3.1. A degree d genus 0 maximal tangency tropical curve Γ in Ξ is a graph Γ that may have unbounded edges such that:

- (1) The vertices v of Γ are elements of the support of Ξ separated by edges e , which are line segments of rational slope $\in \mathbb{Z}^2$ that carry weights $w(e) \geq 1$.
- (2) Balancing condition: For a vertex v of Γ with adjacent edge e , denote by $u_{(v,e)}$ its primitive outgoing vector. Weighted by w , these sum to 0:

$$\sum_{v \in e} w(e) u_{(v,e)} = 0.$$

- (3) Writing $D^{\text{toric}} = T_1 + T_2 + T_3$ for the toric divisors T_i , Γ has exactly one unbounded edge of weight d parallel to the ray of Ξ corresponding to T_i , and no other unbounded edges.
- (4) The first Betti number $b_1(\Gamma) = 0$.

For our purposes, a tropical curve can be thought of as keeping track of the tangency behaviour of stable log maps.

Definition 3.2. For a trivalent vertex $v \in \Gamma$ with outgoing edges e_1, e_2, e_3 , the weight of v is

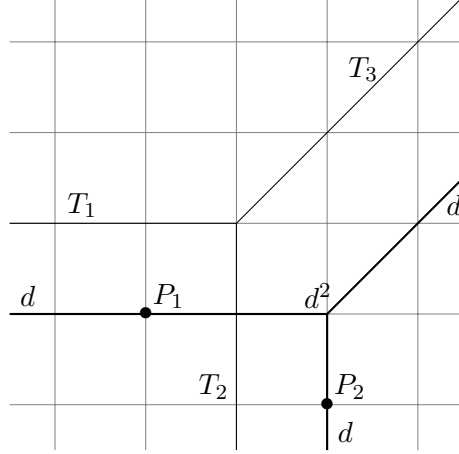
$$w(v) := w(e_i) w(e_j) \left| \det \begin{pmatrix} u_{(v,e_i)} & u_{(v,e_j)} \end{pmatrix} \right|$$

where $i \neq j$ and $\{i, j\} \subset \{1, 2, 3\}$. This is well-defined by the balancing condition. The weight of Γ , $w(\Gamma)$, is given by the products of the weights of each trivalent vertex.

Proposition 3.3. Fixing two general points $P_1, P_2 \in \Xi$, there is only one genus 0 degree d maximally tangent tropical curve in Ξ that passes through P_1 and P_2 . It carries weight d^2 and hence

$$N_d(\mathbb{P}^2, D^{\text{toric}}) = d^2.$$

Proof. The only such tropical curve is given in the following diagram. It has weight d^2 . The result follows from the tropical correspondence [39, 40, 42].



□

4. DEGENERATION AND THE INVARIANTS $N_d(\mathbb{P}^2, D_1 + D_2)$

4.1. The toric model. Denote by p one of the two points of intersection of D_1 and D_2 and denote by L the line tangent to D_2 at p . I describe a log modification that is obtained by successively blowing up points which are dimension 0 strata of the (total transform) of the divisor $D = D_1 + D_2$. I choose D_1 and L to be toric divisors and describe a series of modifications that turn D_2 into a toric divisor as well (of a different surface). Every blow-up and blow-down is toric.

Convention 4.1. I denote by the same letter divisors on different surfaces which are related by strict transforms under blow-ups and blow-downs. This process alters the self-intersection numbers.

Blow up p which leads to the exceptional divisor F_1 . Then blow up the intersection of F_1 with D_2 and write F_2 for the exceptional divisor. This gives a log Calabi-Yau surface $(\widetilde{\mathbb{P}^2}, \widetilde{D})$, where \widetilde{D} is the total transform of $D_1 + D_2$, see Figure 4.1, and where H denotes the pullback of the hyperplane class of \mathbb{P}^2 .

Proposition 4.2. The Chow group $\text{CH}_1(\widetilde{\mathbb{P}^2})$ is generated by $[H]$, $[F_1]$ and $[F_2]$ with relations

$$[D_1] = [L] + [F_2], \quad [H] = [D_1] + [F_1] + [F_2], \quad [D_2] = [H] + [L],$$

$$[H] \cdot [D_1] = [H] \cdot [L] = 1, \quad [L]^2 = -1, \quad [H]^2 = 1, \quad [D_1]^2 = 0, \quad [D_2]^2 = 2, \quad [D_2] \cdot [L] = 0.$$

Proof. Immediate either by construction or by the toric relations determined by the fan. □

The toric model $(\overline{\mathbb{P}^2}, \overline{D})$ then is given by blowing down the strict transform of L , which is a (-1) -curve, resulting in $\overline{\mathbb{P}^2} = \mathbb{F}_2$. This leads to the fan on the left-hand side of Figure 4.2. Then $\overline{D} = D_1 + F_1 + F_2 + D_2$ with F_1 the (-2) -curve, D_2 a 2-curve, and D_1, F_2 the toric fibres of the bundle projection $\mathbb{F}_2 \rightarrow \mathbb{P}^1$. Applying the fan automorphism

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$$

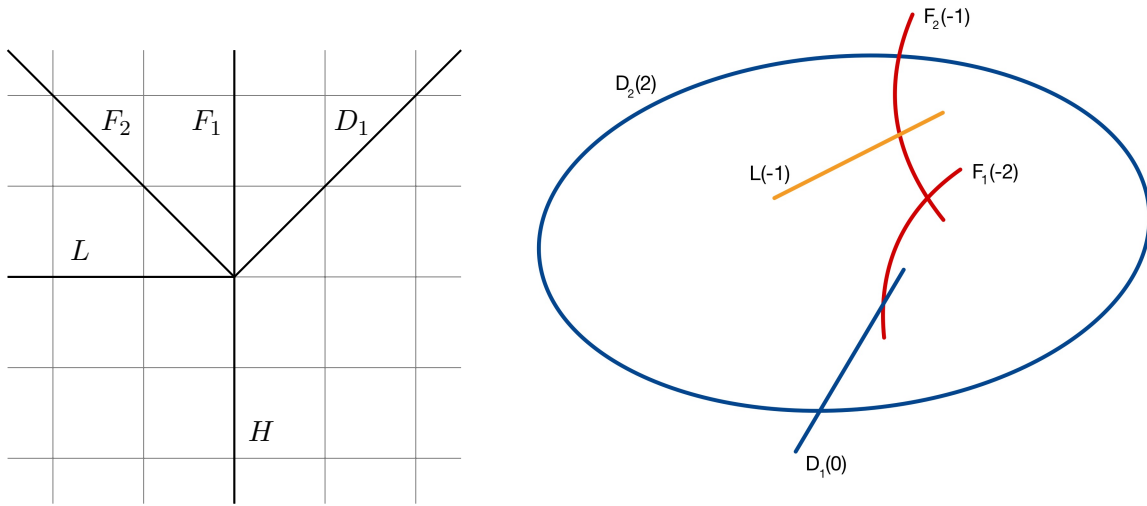


FIGURE 4.1. The log modification $(\widetilde{\mathbb{P}^2}, \widetilde{D})$.

we obtain the fan on the right hand side of Figure 4.2. In addition, I label the boundary divisors with their self-intersections and I keep track of L by adding a \times on the ray of F_2 indicating that we blow up a smooth point of F_2 in order to obtain a log Calabi–Yau surface logarithmically equivalent to $(\mathbb{P}^2, D_1 + D_2)$.

N.B.: Through this process, D_2 became a *toric* divisor.

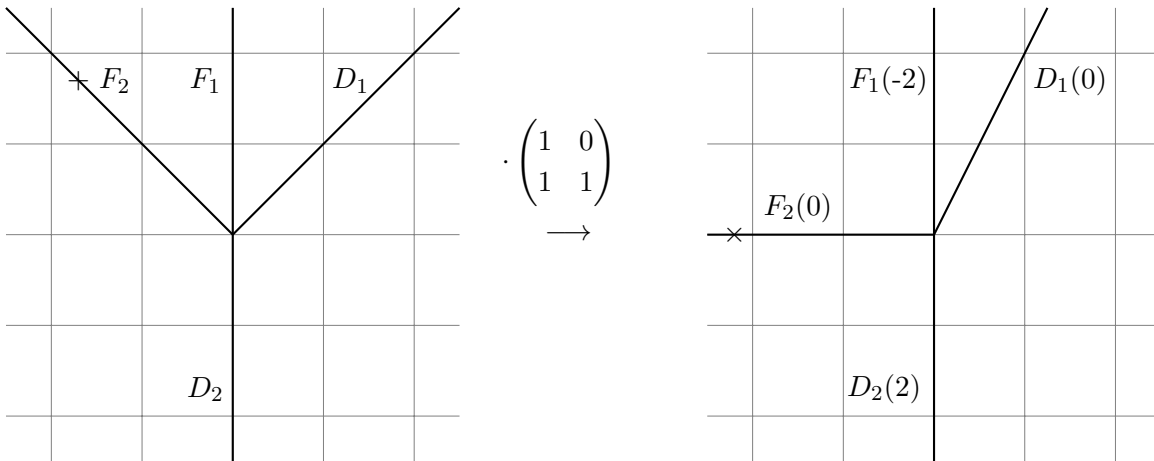


FIGURE 4.2. The toric model of $(\mathbb{P}^2, D_1 + D_2)$.

Remark 4.1. Note that the fan of a smooth toric surface is uniquely determined by the self-intersections of its toric divisors. Indeed, if ρ_1, ρ_2, ρ_3 are primitive generators of successive rays in the fan corresponding to toric divisors T_1, T_2, T_3 , then $\rho_1 + (T_2)^2 \rho_2 + \rho_3 = 0$. The collection of these equations determines the fan up to multiplication by an element of $\text{SL}(2, \mathbb{Z})$.

4.2. Calculation by degeneration. By invariance of log Gromov-Witten invariants under log modifications, $N_d^{\log}(\mathbb{P}^2, D_1 + D_2)$ is the genus 0 degree $d[H]$ log Gromov-Witten invariant of $(\widetilde{\mathbb{P}^2}, \widetilde{D})$, passing through one interior point, meeting D_1 in one point of tangency d and meeting D_2 in one point of tangency $2d$ and with no other conditions.

I describe a suitable degeneration of $(\widetilde{\mathbb{P}^2}, \widetilde{D})$ into two components. One component, $(\overline{\mathbb{P}^2}, \overline{D})$, is toric and thus tropical correspondence results apply. L specialises into the other component, which however will be simple enough so that we can compute its Gromov-Witten invariants.

Start with the trivial family $\mathbb{F}_2 \times \mathbb{A}^1$. The blow up (degeneration to the normal cone)

$$\mathcal{X}' := \text{Bl}_{\mathbb{F}_2 \times \{0\}}(\mathbb{F}_2 \times \mathbb{A}^1)$$

has general fibre \mathbb{F}_2 and special fibre $\mathbb{F}_2 \cup_{\mathbb{F}_2} \mathbb{F}_0$, where $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}(N_{\mathbb{F}_2/\mathbb{F}_2} \oplus \mathcal{O}_{\mathbb{F}_2})$. Denote by H_1, H_2 the two effective curve classes generating $H_2(\mathbb{F}_0, \mathbb{Z})$. \mathbb{F}_2 is glued to \mathbb{F}_0 by identifying F_2 with one of the toric divisors of \mathbb{F}_0 of class H_1 . Denote by \mathcal{F} the strict transform of $F_2 \times \mathbb{A}^1$. Choose a section \mathcal{S} of $\mathcal{F} \rightarrow \mathbb{A}^1$ whose image lies in the smooth locus (interior) of F_2 . Then

$$\mathcal{X} := \text{Bl}_{\mathcal{S}} \mathcal{X}'$$

has general fibre $(\widetilde{\mathbb{P}^2}, \widetilde{D})$ and special fibre $\mathbb{F}_2 \cup_{\mathbb{F}_2} Y$, where $Y = \text{Bl}_{pt}(\mathbb{F}_0)$ is the blow-up of \mathbb{F}_0 at a smooth point of the other toric divisor in class H_1 . If \mathcal{L} is the exceptional divisor of the family \mathcal{X} , by abuse of notation I denote by L the fibre-wise exceptional divisors.

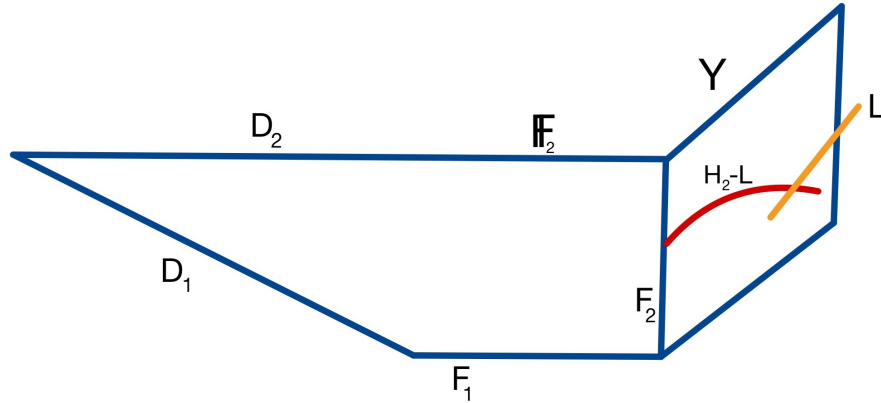


FIGURE 4.3. The two components of the central fibre.

The degeneration formula is a virtual way of keeping track of counts of curves in the degeneration. The first step is to understand the behavior of curves as they specialise from the generic fibre to the special fibre. Start with a curve C in \mathbb{P}^2 , or more generally an element in Chow $\alpha \in \text{CH}_*(\mathbb{P}^2)$. Then [9,10] construct the prelog Chow ring $\text{CH}_*^{\text{prelog}}(\mathbb{F}_2 \cup_{\mathbb{F}_2} Y)$ along with a *specialisation morphism*

$$\sigma : \text{CH}_*(\widetilde{\mathbb{P}^2}) \rightarrow \text{CH}_*^{\text{prelog}}(\mathbb{F}_2 \cup_{\mathbb{F}_2} Y).$$

The ring $\text{CH}_*^{\text{prelog}}(\mathbb{F}_2 \cup_{\mathbb{F}_2} Y)$ is defined as the quotient of tuples $(\alpha_1, \alpha_2) \in \text{CH}_*(\mathbb{F}_2) \oplus \text{CH}_*(Y)$ that satisfy $\alpha_1|_{F_2} = \alpha_2|_{F_2}$ under the identification of classes on both components which are equal as elements of $\text{CH}_*(\mathbb{F}_2 \cup_{\mathbb{F}_2} Y)$.

Proposition 4.3. The prelog Chow ring $\mathrm{CH}_*^{\mathrm{prelog}}(\mathbb{F}_2 \cup_{F_2} Y)$ is generated by the classes

- $[(\mathbb{F}_2, Y)]$ in degree 2;
- $[(D_2, H_2)]$, $[(F_2, 0)] = [(0, H_1)]$ and $[(0, L)]$ in degree 1;
- $[(\mathrm{pt}, 0)] = [(0, \mathrm{pt})]$ in degree 0,

and with intersection products

$$[(D_2, H_2)] \cdot [(F_2, 0)] = 1, \quad [(D_2, H_2)] \cdot [(0, L)] = [(F_2, 0)] \cdot [(0, L)] = 0.$$

Proof. The degrees 2 and 0 are clear. In the case of curves, $\mathrm{CH}_1(\mathbb{F}_2)$ is generated by D_2 and F_2 . $\mathrm{CH}_1(Y)$ is generated by H_1 , H_2 and L . Moreover, the gluing $\mathbb{F}_2 \cup_{F_2} Y$ identifies the classes F_2 and H_1 . Then the classes are determined by the tuples that agree on the intersection between both components. Moreover, the intersections can be computed by choosing representatives on each of the components, see [10, Definition 2.2]. \square

Proposition 4.4. The specialisation of $[H] \in \mathrm{CH}_1(\widetilde{\mathbb{P}^2})$ is

$$\sigma([H]) = [(D_2, H_2 - L)] \in \mathrm{CH}_1^{\mathrm{prelog}}(\mathbb{F}_2 \cup_{F_2} Y).$$

Proof. By Proposition 4.2, $[H] = [D_1] + [F_1] + [F_2]$. By construction

$$\begin{aligned} \sigma([D_1]) &= [(F_2, 0)], & \sigma([F_1]) &= [(F_1, H_2)] = [(D_2, H_2)] - 2[(F_2, 0)], \\ \sigma([F_2]) &= [(0, [H_1] - [L])] = [(F_2, 0)] - [(0, L)]. \end{aligned}$$

Since σ is a ring homomorphism, we thus have that

$$\sigma([H]) = \sigma([D_1]) + \sigma([F_1]) + \sigma([F_2]) = [(F_2 + D_2 - 2F_2 + F_2, H_2 - L)] = [(D_2, H_2 - L)].$$

\square

Remark 4.2. One may explicitly check that σ is a ring homomorphism, at least with respect to intersections with the class $[H]$.

Let $d \geq 1$. Assume we have two genus 0 curves C_1 and C_2 in \mathbb{F}_2 and Y respectively, of classes $[C_1] = d[D_2]$ and $[C_2] = d([H_2] - [L])$. Then $C_1 \cup C_2$ can be smoothed to the generic fibre if and only if C_1 and C_2 have the same intersection pattern with F_2 , i.e. if and only if C_1 and C_2 meet F_2 at the same points with the same intersection multiplicities. This is called the *pre-deformability* condition.

Proposition 4.3.

$$N_d(\mathbb{P}^2, D_1 + D_2) = \binom{2d}{d}.$$

Proof. The degeneration formula [2, 4, 19, 37, 38] states that virtually counting genus 0 curves C of class $d[H] \in \mathrm{CH}_1(\widetilde{\mathbb{P}^2})$ satisfying a list of properties is the same as virtually counting genus 0 curves C_1 and C_2 of class $([C_1], [C_2]) = d[(D_2, H_2 - L)] \in \mathrm{CH}_1^{\mathrm{prelog}}(\mathbb{F}_2 \cup_{F_2} Y)$ that satisfy the pre-deformability condition as well as the specialisation of the list of properties. For us the list of properties is to meet D_1 , resp. D_2 , in one point of tangency d , resp. $2d$, and to pass through a point in the interior of $\widetilde{\mathbb{P}^2}$. Furthermore, the degeneration formula has an additional multiplicity

factor that keeps track of the fact that several different curves in the generic fibre may specialise to the same curve in the special fibre.

As (-1) -curves admit no infinitesimal deformations, there is only one curve in Y of class $H_2 - L$. It is the strict transform of the fibre in class H_2 of \mathbb{F}_0 that meets the point $\mathcal{S}(0) \in \mathbb{F}_0$. As a consequence, the only contributions coming from Y are unions of multiple covers over this fibre.

The precise expression of the degeneration formula (see [32, Proposition 5.3]) is that

$$N_d(\mathbb{P}^2, D_1 + D_2) = \sum_{m \vdash d} \prod_{\ell=1}^d \frac{\ell^{m_\ell}}{m_\ell!} \left(\frac{(-1)^{\ell-1}}{\ell^2} \right)^{m_\ell} N_m(\mathbb{F}_2).$$

- The sum is over the partitions $m = (m_1, \dots, m_d)$ of d so that $d = \sum_{\ell=1}^d \ell m_\ell$ with $m_\ell \geq 0$ and the convention that $0! = 1$. This keeps track of the intersection profile of C_1 , equivalently C_2 , along F_2 . In this notation, C_1 , equivalently C_2 , has m_ℓ points of intersection multiplicity ℓ for $0 \leq \ell \leq d$ (m_ℓ may be zero).
- Since $H_2 - L$ is a (-1) -curve, C_2 decomposes as a union of multiple covers of degree ℓ over $H_2 - L$, each meeting F_2 in multiplicity ℓ at the unique point of intersection of F_2 with $H_2 - L$. In order to lead to a genus 0 connected curve (more precisely a genus 0 connected stable map), these are glued to a single *connected* curve (more precisely a connected stable map) C_1 in \mathbb{F}_2 . For each m , there are $m_1 + \dots + m_d$ multiple covers over $H_2 - L$ that are glued to C_1 according to the ramification profile determined by m .
- The factors ℓ^{m_ℓ} are the multiplicities coming from the degeneration formula.
- The $1/m_\ell!$ are the symmetry factors corresponding to the permutations of the m_ℓ multiple covers of degree ℓ over $H_2 - L$. This adjusts for the over-counting that resulted from counting each constellation separately.
- The $(-1)^{\ell-1}/\ell^2$ are the Y -contributions coming from degree ℓ covers of the fibre of class $H_2 - L$, see [32, Proposition 5.2].
- $N_m(\mathbb{F}_2)$ is the genus 0 log Gromov–Witten invariant of \mathbb{F}_2 of class $d[D_2]$, counting rational curves that pass through one fixed point in general position in \mathbb{F}_2 and have
 - one point of contact of order $2d$ with D_2 ,
 - one point of contact of order d with D_1 ,
 - for every $1 \leq \ell \leq d$, m_ℓ contact points of contact order ℓ with F_2 , at the same fixed position on F_2 corresponding to the intersection of F_2 with the unique fibre of Y of class $H_2 - L$.

The invariant $N_m(\mathbb{F}_2)$ is computed tropically. See Figure 4.4 for $d = 2$ and $m = (2, 0)$.

There is only one tropical curve contributing to $N_m(\mathbb{F}_2)$. It has $m_1 + \dots + m_\ell + \dots + m_d$ infinite edges coming from *fixed* F_2 -directions, grouped according to ℓ . The n th such curve, for $0 < m_1 + \dots + m_{\ell-1} < n \leq m_1 + \dots + m_\ell \leq d$ carries weight ℓ corresponding to the intersection multiplicity at that point. The tropical curve has one infinite edge for each of D_1 and D_2 , of weights d and $2d$. Requiring the curve to pass through a point in the fan, all other edges and

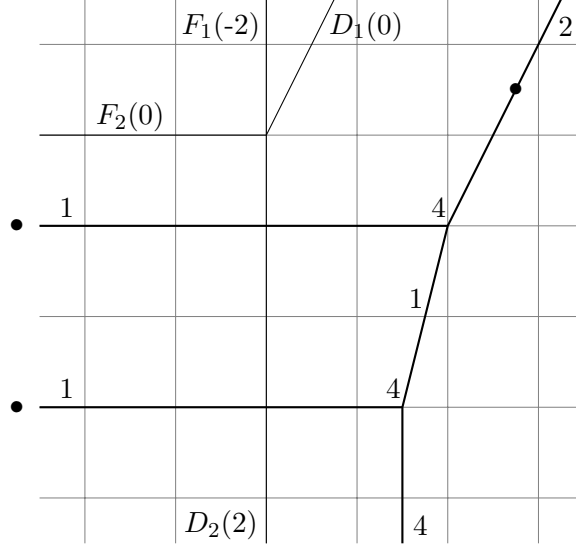


FIGURE 4.4. For $d = 2$, the only tropical curve with $m = (m_1, m_2) = (2, 0)$. The numbers indicate the weights of the edges and vertices. The bullets indicate that we fix a point in the interior of \mathbb{F}_2 and twice the same point on F_2 .

multiplicities are determined by the balancing condition. Then

$$N_m(\mathbb{F}_2) = \prod_{\ell=1}^d \left(\frac{1}{\ell} (2d\ell) \right)^{m_\ell} = \prod_{\ell=1}^d (2d)^{m_\ell}.$$

The factors $2d\ell$ are the multiplicities of the vertices. The factors $1/\ell$ come from the fact that the positions of the contact points with \mathbb{F}_2 are fixed. Finally, we obtain

$$N_d(\mathbb{P}^2(1, 4)) = \sum_{m \vdash d} \prod_{\ell=1}^d \frac{1}{m_\ell!} \left(2d \frac{(-1)^{\ell-1}}{\ell} \right)^{m_\ell}.$$

This is the coefficient of x^d in the power series expansion of $\exp(2d \log(1+x)) = (1+x)^{2d}$, hence the result. \square

5. SCATTERING AND THE INVARIANTS $N_d(\mathbb{P}^2, D_3)$

We come to the Looijenga pair formed by the nodal cubic D_3 . Denote by P the point of self-intersection. First we get a toric model. To do so, blow up P leading to an exceptional divisor E , and keep track of the two tangent lines L_1 and L_2 . Blow up the intersections of E with the proper transforms of both L_1 and L_2 and denote the two new exceptional divisors by F_1 and F_2 .

The proper transforms of L_1 and L_2 are now (-1) -curves, which we may both blow down to obtain the Hirzebruch surface \mathbb{F}_3 . This determines a scattering diagram [32], which is obtained from the fan of the Hirzebruch surface by adding two focus-focus singularities on the two toric fibres.

The two walls that meet are directed by vectors v_1 and v_2 that have $|\det(v_1, v_2)| = 3 \neq 1$. As a consequence, there is infinite scattering as described e.g. in [31]. Each wall carries a wall-crossing function, which identifies the ring of functions on each side of the wall. Scattering is a process of

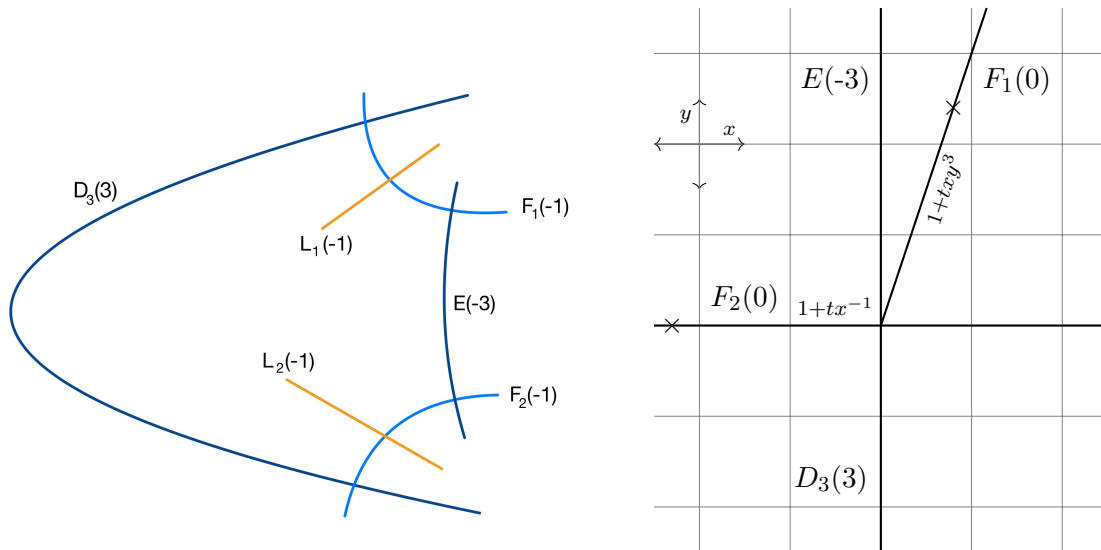


FIGURE 5.1. The toric model of (\mathbb{P}^2, D_3) .

inductively adding rays to the scattering diagram in order to make it consistent order by order. Consistency means that the sequence of automorphisms of a loop around the origin gives the identity up to a given order.

The rays are wall-crossing functions whose coefficients are log Gromov–Witten invariants as shown in [32]. The invariant $N_d(\mathbb{P}^2, D_3)$ has one point of contact with D_3 . As a consequence, by the correspondence results of [32] between wall-crossing functions and log Gromov–Witten invariants, $N_d(\mathbb{P}^2, D_3)$ is encoded by the wall-crossing function of the central ray of the scattering diagram leading to (1.3).

REFERENCES

- [1] D. ABRAMOVICH AND Q. CHEN, Stable logarithmic maps to Deligne–Faltings pairs II, *Asian J. Math.*, 18 (2014), 465–488.
- [2] D. ABRAMOVICH, Q. CHEN, M. GROSS, AND B. SIEBERT, Decomposition of degenerate Gromov–Witten invariants, *Compos. Math.*, 156 (2020), 2020–2075.
- [3] ———, Punctured logarithmic maps. Preprint available at [arXiv:2009.07720](https://arxiv.org/abs/2009.07720).
- [4] D. ABRAMOVICH AND B. FANTECHI, Orbifold techniques in degeneration formulas, *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)*, 16 (2016), 519–579.
- [5] D. ABRAMOVICH AND J. WISE, Birational invariance in logarithmic Gromov–Witten theory, *Compos. Math.*, 154 (2018), 595–620.
- [6] L. BARROTT AND N. NABIJOU, Tangent curves to degenerating hypersurfaces, *J. Reine Angew. Math.*, 793 (2022), 185–224.
- [7] L. BATTISTELLA, NABIJOU, H. N. TSENG, AND F. YOU, The local-orbifold correspondence for simple normal crossings pairs, *J. Inst. Math. Jussieu* (2022).
- [8] L. BATTISTELLA AND N. NABIJOU, Relative quasimaps and mirror formulae, *Int. Math. Res. Not.*, 2021 (2021), 7885–7931.
- [9] C. BÖHNING, H.-C. GRAF VON BOTHMER, AND M. VAN GARREL, Prelog Chow groups of self-products of degenerations of cubic threefolds, *Eur. J. Math.*, 8 (2022), 260–290.

- [10] ———, Prelog Chow rings and degenerations, *Rend. Circ. Mat. Palermo (2), Special Issue: Rationality Problems*, to appear (2023). Preprint available at [arXiv:1911.08930](https://arxiv.org/abs/1911.08930).
- [11] P. BOUSSEAU, Scattering diagrams, stability conditions, and coherent sheaves on \mathbb{P}^2 , *J. Algebr. Geom.*, 31 (2022), 593–686.
- [12] ———, A proof of N. Takahashi’s conjecture on genus zero Gromov–Witten theory of (\mathbb{P}^2, E) , *Duke Math. J.*, to appear (2023). Preprint available at [1909.02992](https://arxiv.org/abs/1909.02992).
- [13] P. BOUSSEAU, A. BRINI, AND M. VAN GARREL, Stable maps to Looijenga pairs: orbifold examples, *Lett. Math. Phys.*, 111 (2021), 37.
- [14] ———, On the log–local principle for the toric boundary, *Bull. Lond. Math. Soc.*, 54 (2022), 161–181.
- [15] ———, Stable maps to Looijenga pairs, *Geom. Topol.*, to appear (2023). Preprint available at [arXiv:2011.08830](https://arxiv.org/abs/2011.08830).
- [16] P. BOUSSEAU, H. FAN, S. GUO, AND L. WU, Holomorphic anomaly equation for (\mathbb{P}^2, E) and the Nekrasov–Shatashvili limit of local \mathbb{P}^2 , *Forum Math. Pi*, 9 (2021), 57.
- [17] A. BRINI, Enumerative geometry of surfaces and topological strings. Preprint available at [arXiv:2211.11037](https://arxiv.org/abs/2211.11037).
- [18] M. CARL, M. PUMPERLA, AND B. SIEBERT, A tropical view on Landau–Ginzburg models. Preprint available at [2205.07753](https://arxiv.org/abs/2205.07753).
- [19] Q. CHEN, The degeneration formula for logarithmic expanded degenerations, *J. Algebr. Geom.*, 23 (2014), 341–392.
- [20] ———, Stable logarithmic maps to Deligne–Faltings pairs I, *Ann. of Math. (2)*, 180 (2014), 455–521.
- [21] J. CHOI, M. VAN GARREL, S. KATZ, AND N. TAKAHASHI, Local BPS invariants: enumerative aspects and wall-crossing, *Int. Math. Res. Not.*, 2020 (2020), 5450–5475.
- [22] ———, Log BPS numbers of log Calabi–Yau surfaces, *Trans. Am. Math. Soc.*, 374 (2021), 687–732.
- [23] ———, Sheaves of maximal intersection and multiplicities of stable log maps, *Sel. Math., New Ser.*, 27 (2021), 51.
- [24] M. VAN GARREL, T. GRABER, AND H. RUDDAT, Local Gromov–Witten invariants are log invariants, *Adv. Math.*, 350 (2019), 860–876.
- [25] M. VAN GARREL, H. RUDDAT, AND B. SIEBERT, Enumerative period integrals of Landau–Ginzburg models via wall structures, *to appear* (2023).
- [26] ———, Intrinsic enumerative mirror symmetry: Takahashi’s log mirror symmetry for (\mathbb{P}^2, E) revisited, *to appear* (2023).
- [27] M. VAN GARREL, T. W. H. WONG, AND G. ZAIMI, Integrality of relative BPS state counts of toric del Pezzo surfaces, *Commun. Number Theory Phys.*, 7 (2013), 671–687.
- [28] A. GATHMANN, Relative Gromov–Witten invariants and the mirror formula, *Math. Ann.*, 325 (2003), 393–412.
- [29] M. GROSS, P. HACKING, AND S. KEEL, Mirror symmetry for log Calabi–Yau surfaces I, *Publ. Math. Inst. Hautes Études Sci.*, 122 (2015), 65–168.
- [30] M. GROSS, P. HACKING, AND B. SIEBERT, *Theta functions on varieties with effective anti-canonical class*. Mem. Am. Math. Soc., 1367. Providence, RI: American Mathematical Society (AMS), 2022.
- [31] M. GROSS AND R. PANDHARIPANDE, Quivers, curves, and the tropical vertex, *Port. Math. (N.S.)*, 67 (2010), 211–259.
- [32] M. GROSS, R. PANDHARIPANDE, AND B. SIEBERT, The tropical vertex, *Duke Math. J.*, 153 (2010), 297–362.
- [33] M. GROSS AND B. SIEBERT, Logarithmic Gromov–Witten invariants, *J. Amer. Math. Soc.*, 26 (2013), 451–510.
- [34] ———, Intrinsic Mirror Symmetry. Preprint available at [arXiv:1909.07649](https://arxiv.org/abs/1909.07649).
- [35] ———, The canonical wall structure and intrinsic mirror symmetry, *Invent. Math.*, 229 (2022), 1101–1202.
- [36] T. GRÄFNITZ, Tropical correspondence for smooth del Pezzo log Calabi–Yau pairs, *J. Algebraic Geom.*, 31 (2022), 687–749.
- [37] B. KIM, H. LHO, AND H. RUDDAT, The degeneration formula for stable log maps, *Manuscripta Math.* (2021).
- [38] J. LI, A degeneration formula of GW-invariants, *J. Differential Geom.*, 60 (2002), 199–293.

- [39] T. MANDEL AND H. RUDDAT, Descendant log Gromov–Witten invariants for toric varieties and tropical curves, *Trans. Amer. Math. Soc.*, 373 (2020), 1109–1152.
- [40] G. MIKHALKIN, Enumerative tropical algebraic geometry in \mathbb{R}^2 , *J. Amer. Math. Soc.*, 18 (2005), 313–377.
- [41] N. NABLIJOU AND D. RANGANATHAN, Gromov-Witten theory with maximal contacts, *Forum Math. Sigma*, 10 (2022), 34.
- [42] T. NISHINO AND B. SIEBERT, Toric degenerations of toric varieties and tropical curves, *Duke Math. J.*, 135 (2006), 1–51.
- [43] M. REINEKE, Cohomology of quiver moduli, functional equations, and integrality of Donaldson-Thomas type invariants., *Compos. Math.*, 147 (2011), 943–964.
- [44] N. TAKAHASHI, Log mirror symmetry and local mirror symmetry, *Comm. Math. Phys.*, 220 (2001), 293–299.
- [45] H-H. TSENG AND F. YOU, A mirror theorem for multi-root stacks and applications, *Sel. Math. New Ser.*, 29 (2023).
- [46] F. YOU, The proper Landau–Ginzburg potential, intrinsic mirror symmetry and the relative mirror map. Preprint available at 2209.15371.

UNIVERSITY OF BIRMINGHAM, SCHOOL OF MATHEMATICS, B15 2TT, BIRMINGHAM, UNITED KINGDOM

Email address: `m.vangarrel@bham.ac.uk`