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DOI:
10.1002/mana.202000267

Received: 1 June 2020 | Revised: 18 February 2022 | Accepted: 24 February 2022

DOI: 10.1002/mana.202000267

ORIGINAL ARTICLE

MATHEMATISCHE NACHRICHTEN

Branes on the singular locus of the Hitchin system via Borel and other parabolic subgroups

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Funding information
Fundação de Amparo à Pesquisa do Estado de São Paulo; Fundação para a Ciência e a Tecnologia; H2020 European Research Council

Abstract

We study the mirror symmetry on the singular locus of the Hitchin system at two levels. First, by covering it by (supports of) (BBB)-branes, corresponding to Higgs bundles reducing their structure group to the Levi subgroup of some parabolic subgroup P, whose conjectural dual (BAA)-branes we describe. Heuristically speaking, the latter are given by Higgs bundles reducing their structure group to the unipotent radical of P. Second, when P is a Borel subgroup, we are able to construct a family of hyperholomorphic bundles on the (BBB)-brane and study the variation of the dual under this choice. We give evidence of both families of branes being dual under mirror symmetry via an integral functor induced by Fourier–Mukai in the moduli stack of Higgs bundles.

KEYWORDS
Higgs bundles, mirror symmetry

MSC (2020)
14J33, 14D21

1 | INTRODUCTION

1.1 | Brief description

In this paper, we study the action of mirror symmetry on the singular locus of the moduli space M_n of Higgs bundles. We proceed first by describing hyperholomorphic subvarieties covering $\mathrm{M}_n^{\mathrm{sing}}$, those become (BBB)-branes after specifying a hyperholomorphic bundle on them. Then, we construct complex Lagrangian subvarieties, supporting (BAA)-branes after being equipped with a flat bundle, and we conjecture that behind these constructions stands a pair of mirror dual branes. Each of the previous pairs of branes is naturally associated with a parabolic subgroup of $\mathrm{GL}(n,\mathbb{C})$. When this parabolic is the Borel subgroup, we find ourselves over the locus of totally reducible spectral curves, namely, those with a maximal number of irreducible components, cf. Equation (3.1). A more complete analysis is possible in this case and we are able to construct families of flat (hence hyperholomorphic) bundles giving rise to (BBB)-branes. These (BBB)-branes only intersect Hitchin fibers associated with coarse compactified Jacobians where no Fourier–Mukai transform has been defined. We then consider the Fourier–Mukai transform between the associated stacks and prove that it restricts to a transform whose source is the support of the (BBB)-branes associated with the Borel subgroup. Our biggest contribution is the description of the behavior of these (BBB)-branes under such a transform, showing that it returns a sheaf supported on the complex Lagrangian subvarieties we have previously described.

1.2 | Mathematical background and motivation

Hitchin introduced in [35] Higgs bundles over a smooth projective curve X and soon it was noted that their moduli space M_n carries a very interesting geometry [35, 52, 59, 60]. In particular, M_n can be endowed with a hyperkähler structure $(g, \Gamma_1, \Gamma_2, \Gamma_3)$ [16, 20, 35, 59, 60] and fibers over a vector space $h : \mathrm{M}_n \to \mathrm{H}$ with Lagrangian tori as generic fibers [36]. A natural generalization is to consider Higgs bundles for complex reductive Lie groups other than $\mathrm{GL}(n, \mathbb{C})$. After the work of [18, 19, 32], the moduli spaces of Higgs bundles for two Langlands dual groups equipped with the afore-mentioned fibrations become *SYZ mirror partners* (as defined by [32] based on work by [61]) and mirror symmetry is expected to be implemented by a Fourier–Mukai transform relative to the fibers of the Hitchin fibration. In this paper, we focus in the case of $\mathrm{GL}(n, \mathbb{C})$, which is Langlands self-dual.

Branes in the Higgs moduli space were introduced in [41] and have since attracted great attention. A (BBB)-*brane* in M_n is given by a pair $(N, \mathbf{F}, \nabla_{\mathbf{F}})$, where $N \subset \mathrm{M}_n$ is a hyperholomorphic subvariety and $(\mathbf{F}, \nabla_{\mathbf{F}})$ a hyperholomorphic sheaf on N. This means that the connection $\nabla_{\mathbf{F}}$ on the sheaf $\mathbf{F}$ is of type (1,1) with respect to all three complex structures $\Gamma_1, \Gamma_2, \Gamma_3$. Additionally, a (BAA)–*brane* is a pair (S, W, ∇_W) where $S \subset \mathrm{M}_n$ is a subvariety which is complex Lagrangian with respect to the holomorphic symplectic form in the complex structure Γ_1, and (W, ∇_W) is a flat bundle over S. It is conjectured in [41] that mirror symmetry interchanges (BBB)–branes with (BAA)–branes. This context has motivated many authors to construct (BBB) and (BAA)–branes [7, 8, 10, 11, 13, 14, 25–27, 34, 38, 39]. Papers such as [25, 26, 39] go a step further by giving evidence of the duality between certain (BBB) and (BAA)–branes, however focusing on the smooth locus of the Hitchin system.

Mirror symmetry is more obscure over singular Hitchin fibers, since it involves autoduality of compactified Jacobians of singular curves. Such autoduality was stated via Fourier–Mukai equivalences by Arinkin [5, 6] in the case of integral curves, and by Melo, Rapagnetta and Viviani [48, 49] in the case of fine compactified Jacobians. Kass [42] extended the autoduality to the case of coarse compactified Jacobians, which is the one that concerns us, although his construction does not provide a Fourier–Mukai transform.

Our main motivation is to extend the study of mirror symmetry for branes to the locus of singular Hitchin fibers. This has been addressed also in some papers that appeared after the first preprint of the present one. In [24], written by the authors along with Gothen and Oliveira, some pair of (BBB) and (BAA)–branes are considered, noting that the (BBB)–branes play a crucial role in the topological mirror symmetry [32]. These branes are dense over Hitchin fibers associated with integral curves so Arinkin's Fourier–Mukai transform [5, 6] is enough to study, in this case, the behavior of these branes under the mirror symmetry. Branco [14] studied the intersection of certain branes with the locus of Hitchin fibers associated with non-reduced curves. In this case, the mirror symmetry is discussed in geometrical terms, by dualizing a certain abelian variety inside the non-reduced Hitchin fibers. It is noteworthy to mention the work of Hausel, Mellit and Pei [31], who showed that the pair of branes described by Hitchin in [38] satisfies an agreement of certain topological invariants. This gives strong evidence for the duality of these branes, as proposed in [38], where such duality was only checked over the locus of smooth Hitchin fibers.

1.3 | Our work

We start by constructing a family of (BBB)–branes and complex Lagrangian subvarieties (support of (BAA)–branes) indexed by a topologically trivial line bundle $\mathscr{L} \to X$. Both lie over the locus of singular Hitchin fibers given by totally reducible spectral curves and both constructions involve the Borel subgroup $B < \mathrm{GL}(n, \mathbb{C})$.

We shall consider Car, the locus of Higgs bundles whose structure group reduces to the Cartan subgroup $C < B$, as the support of our (BBB)–brane. It is well known that this subvariety is naturally hyperholomorphic (being given by reduction of the structure group to a reductive subgroup), the novel point of this piece of work is the construction of different flat (hence hyperholomorphic) bundles, constructed from a chosen line bundle $\mathscr{L} \to X$. Our (BBB)–brane $\mathbf{Car}(\mathscr{L})$ consists of Car equipped with this bundle. The image of Car under the Hitchin fibration $h(\mathrm{Car})$ is the locus totally reducible spectral curves $\overline{X}_b$, making Schaub's spectral correspondence [55] explicit over this subset of the singular locus.

We define as well a complex Lagrangian subvariety $\mathrm{Uni}(\mathscr{L})$ consisting of Higgs bundles whose structure group reduces to B, and whose associated graded bundle is constant and depends on $\mathscr{L}$. Thus, this complex Lagrangian subvariety depends on $\mathscr{L} \to X$, and, heuristically speaking, parameterizes Higgs bundles that reduce their structure group to the unipotent radical of B. After specifying a flat bundle over $\mathrm{Uni}(\mathscr{L})$, we shall obtain a (BAA)–brane.

To study the behavior of $\mathbf{Car}(\mathscr{L})$ and $\mathrm{Uni}(\mathscr{L})$ under mirror symmetry one would like to transform $\mathbf{Car}(\mathscr{L})$ under a Fourier–Mukai transform. These branes are supported on $h(\mathrm{Car})$, included in the locus of (singular) reducible curves. Then, Car and $\mathrm{Uni}(\mathscr{L})$ only intersect Hitchin fibers $h^{-1}(b) \cong \overline{\mathrm{Jac}}(\overline{X}_b)$ that are coarse compactified Jacobians, not fine, and therefore a full Fourier–Mukai transform is not known to exist, not even after restricting ourselves to the open subset of the Cartan locus whose associated spectral curves are nodal. Nevertheless, it is possible to construct a Poincaré sheaf over the moduli stack of torsion-free sheaves over reducible nodal curves although it is yet not known whether the associated integral functor is a derived equivalence or not. The restriction of this stacky Poincaré sheaf to the support of the stacky version of $\mathbf{Car}(\mathscr{L})$ and the Jacobian can be lifted to a sheaf on the corresponding schemes. We then define the associated integral functor

$$\Phi^{\mathrm{Car}} : D^b\Big(\mathrm{Car} \cap \overline{\mathrm{Jac}}(\overline{X}_b)\Big) \longrightarrow D^b\Big(\mathrm{Jac}(\overline{X}_b)\Big).$$

Our main result (Corollary 6.4) consists on checking that this functor relates the generic loci of both branes.

Theorem 1. *There is an equality*

$$\mathrm{supp}\Big(\Phi^{\mathrm{Car}}\Big(\mathbf{Car}(\mathscr{L})|_{\overline{\mathrm{Jac}}(\overline{X}_b)}\Big)\Big) = \mathrm{Uni}(\mathscr{L}) \cap \mathrm{Jac}(\overline{X}_b).$$

We finish by discussing how this construction can be generalized to a large class of branes in the moduli space M_n of rank n Higgs bundles covering the whole singular locus. In the (BBB)-case, the support of these branes correspond to the image of $M_{r_1} \times \cdots \times M_{r_s}$, or equivalently, the locus of those Higgs bundles reducing its structure group to the Levi subgroup $\mathrm{GL}(r_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(r_s, \mathbb{C})$. We observe that these subvarieties cover the singular locus of N_n. The (BAA)–brane is given by a complex Lagrangian subvariety constructed in a similar way as before, but substituting the Borel subgroup with the parabolic subgroup associated to the partition $n = r_1 + \cdots + r_s$. As in the case of the Borel group, we are able to identify the spectral correspondence over the nodal locus.

A word should be said about the possible applications of the present piece of work. The branes hereby described are used in a crucial way in [24] to prove that certain branes are of type (BAA). On the other hand, the analysis of spectral data corresponding to reducible spectral curves furnishes a useful tool to study the geometry of these loci.

1.4 | Structure of the paper

The greater completeness of the analysis for the Borel case is the first reason for the choice of the structure of the paper, presenting first this case, then the case of a general parabolic subgroup. The second reason for this choice is of a more prosaic nature and is linked to the complications in the geometry of these singular loci. Indeed, the singular locus consists of several submanifolds which are nested into one another. The smallest, contained in all the others, is precisely the locus of singular points over totally reducible spectral curves. Thus, a good understanding of the singular locus requires as a first step a good understanding of the singular locus over totally reducible spectral curves.

This paper is organized as follows. Section 2.1 gives the necessary background on Higgs moduli spaces and the Hitchin system. In Section 2.2, we address the construction of the Poincaré sheaf over the moduli stack of torsion-free rank 1 sheaves on nodal reduced curves. This construction is a natural generalization of that of [6] and makes part of unpublished work of Arinkin and Pantev [53]. The detailed description of this construction is included in Section 2.2 for the sake of completeness of our paper.

In Section 3, we study the locus of singular Hitchin fibers associated with totally reducible spectral curves. We prove that the preimage of this locus under h coincides with the locus of Higgs bundles whose structure group reduces to the Borel subgroup (Proposition 3.2) and describe the associated spectral data (Propositions 3.7 and 3.12).

We provide the construction of the (BBB)-brane $\mathbf{Car}(\mathscr{L})$ in Section 4. We consider the Cartan locus, Car, given by those Higgs bundles, whose structure group reduces to the Cartan subgroup $\mathrm{C} \cong (\mathbb{C}^\times)^n < \mathrm{GL}(n, \mathbb{C})$. The Cartan locus is given by the image of $c : \mathrm{Sym}^n(\mathrm{M}_1) \hookrightarrow \mathrm{M}_n$, where M_1 is the rank one Higgs moduli space. Also, we prove that the choice of a topologically trivial line $\mathscr{L}$ bundle on X yields a hyperholomorphic bundle on Car. This produces the (BBB)-brane $\mathbf{Car}(\mathscr{L})$ (cf. Proposition 4.3). Finally, we analyze the restriction of the brane $\mathbf{Car}(\mathscr{L})$ to a generic Hitcin fiber (Proposition 4.4), which is crucial to study the behavior of $\mathbf{Car}(\mathscr{L})$ under the mirror symmetry.

Section 5 addresses the construction and description of the complex Lagrangian subvariety $\mathrm{Uni}(\mathscr{L})$, supporting a (BAA)-brane. $\mathrm{Uni}(\mathscr{L})$ is defined as the subvariety of the locus of all the Higgs bundles reducing to the Borel subgroup B, whose underlying vector bundle project to a certain C-bundle determined by $\mathscr{L}$. Then, we prove that $\mathrm{Uni}(\mathscr{L})$ is isotropic by gauge considerations, closed and half-dimensional, hence Lagrangian (Theorem 5.6). We finish this section by studying the spectral data of the points of $\mathrm{Uni}(\mathscr{L})$ in Proposition 5.7.

We have at this point a description of the generic restriction of $\mathbf{Car}(\mathscr{L})$ and $\mathrm{Uni}(\mathscr{L})$ to a generic Hitchin fiber. In this case, the generic Hitchin fibers are isomorphic to the coarse compactified Jacobian of reduced but reducible curves. We study in Section 6 the transformation of the first under a Fourier–Mukai integral functor. To deal with the lack of a Poincaré sheaf over coarse compactified Jacobians, we consider the Poincaré sheaf over the associated moduli stack that we reviewed in Section 2.2 and observe in Proposition 6.1 that its restriction to Car and the Jacobian provides a sheaf $\mathscr{P}^{\mathrm{Car}}$. It is then natural to study the behavior of $\mathbf{Car}(\mathscr{L})$ under the Fourier–Mukai integral functor constructed with $\mathscr{P}^{\mathrm{Car}}$, which we do. We obtain that the generic restriction of $\mathbf{Car}(\mathscr{L})$ to a Hitchin fiber is sent to a sheaf over $\mathrm{Uni}(\mathscr{L})$ (Corollary 6.4). This leads us to conjecture that the (BBB)-brane $\mathbf{Car}(\mathscr{L})$ is dual under mirror symmetry to a (BAA)-brane supported on $\mathrm{Uni}(\mathscr{L})$.

In Section 7, we adapt the above results to arbitrary parabolic subgroups. Given a partition $n = r_1 + \cdots + r_s$ we consider the associated parabolic subgroup $\mathrm{P}_{\bar{r}} < \mathrm{GL}(n, \mathbb{C})$ with Levi subgroup $\mathrm{L}_{\bar{r}} < \mathrm{P}_{\bar{r}}$. In Section 7.1, we consider the subvariety $\mathrm{M}_{\bar{r}}$ of M_n, consisting of Higgs bundles whose structure group reduces to $\mathrm{L}_{\bar{r}}$, and describe the intersection with generic Hitchin fibers (Proposition 7.5). The variety $\mathrm{M}_{\bar{r}}$ is a complex subscheme for Γ_1, Γ_2, and Γ_3, hence the support of a (BBB)-brane. By varying the partition $\bar{r}$, we produce families of branes covering the strictly semistable locus of M_n. On the other hand, in Section 7.2 we consider $\mathrm{Uni}^{\bar{r}}(E_1, \ldots, E_s)$, consisting of Higgs bundles with structure group reducing to $\mathrm{P}_{\bar{r}}$ and fixed associated graded bundle $\bigoplus_{i=1}^{s} E_i$. We prove that under the right conditions on $\overline{E}$, this is a Lagrangian submanifold (Theorem 7.14), and so a choice of flat bundle on it produces a (BAA)-brane. The imposed hypotheses are related to the existence of a hyperholomorphic bundle on the hypothetical dual $\mathrm{M}_{\bar{r}}$ (see Remark 7.10). A look at the spectral data of both $\mathrm{M}_{\bar{r}}$ and $\mathrm{Uni}^{\bar{r}}(\overline{E})$, as well as the comparison with the case $P_{(1,\ldots,1)}$, indicates the existence of a duality.

2 | PRELIMINARIES

2.1 | Higgs bundles and their moduli

Let X be a smooth projective curve over $\mathbb{C}$. A Higgs bundle over X is a pair (E, φ) given by a holomorphic vector bundle E over X and a Higgs field $\varphi \in H^0(X, \mathrm{End}(E) \otimes K)$, which is a holomorphic section of the endomorphisms bundle twisted by the canonical bundle K of X [35, 58–60].

A Higgs bundle (E, Φ) of trivial degree is stable (resp. semistable) if every Φ-invariant subbundle $F \subset E$ has negative (resp. non-positive) degree, and it is polystable if it is semistable and decomposes as a direct sum of stable Higgs bundles. The moduli space of rank n and degree 0 semistable Higgs bundles on X was constructed in [35, 52, 59, 60]. We review this construction in the following paragraphs.

Fix a topological bundle $\mathbb{E}$ of degree 0 on X and consider the space $\mathscr{A}$ of holomorphic structures on $\mathbb{E}$. This is an affine space modeled on $\Omega^{0,1}(X, \mathrm{ad}(\mathbb{E}))$, whose cotangent bundle is

$$T^*\mathscr{A} = \mathscr{A} \times \Omega^0(X, \mathrm{ad}(\mathbb{E}) \otimes K),$$

where we have identified $\mathrm{ad}(\mathbb{E})$ and its dual by means of the Killing form (rather, a non-degenerate extension of it to the center, to which we will henceforth refer as Killing form). Given a Hermitian metric h on $\mathbb{E}$ let us denote its Chern connection by ∇_h. We consider the following conditions for pairs:

1. There exists a Hermitian metric h such that $\nabla_h^2 + [\varphi, \varphi^{*_h}] = 0$,
2. $\overline{\partial}_A(\varphi) = 0$,
3. $\partial_{A,h}(\varphi^{*,h}) = 0$.

Observe that condition (2) implies that the pair determines a Higgs bundle and in that case (3) is automatically satisfied for any choice of metric h. We shall denote by $(T^*\mathscr{A})_H$ the subset of solutions to (2) (and, therefore, to (3)). Condition (1)

is known as the Hitchin equation and it follows from [35, 59, 60] that a Higgs bundle is polystable if and only if (1) holds, so we will write $(T^*\mathscr{A})_H^{\mathrm{pst}}$ for the locus of pairs satisfying simultaneously (1) and (2) (hence (3) as well). Note that we have $(T^*\mathscr{A})_H^{\mathrm{st}} \subset (T^*\mathscr{A})_H^{\mathrm{pst}} \subset (T^*\mathscr{A})_H^{\mathrm{sst}}$, where st and sst stand for stable and semistable Higgs bundles. These loci are all preserved by the action of the complex gauge group,

$$\mathscr{G} = \Omega^0(X, \mathrm{Aut}(\mathbb{E})),$$

and $(T^*\mathscr{A})_H^{\mathrm{sst}}$ and $(T^*\mathscr{A})_H^{\mathrm{pst}}$ classify semistable and closed orbits, respectively. The moduli space of semistable Higgs bundles over X of rank n and trivial degree is identified with

$$M_n \cong (T^*\mathscr{A})_H /\!\!/ \mathscr{G} = (T^*\mathscr{A})_H^{\mathrm{pst}}/\mathscr{G}, \tag{2.1}$$

where the double quotient denotes the Geometric Invariant Theory (GIT) quotient. This is a quasi-projective variety of dimension

$$\dim M_n = 2n^2(g-1) + 2, \tag{2.2}$$

whose points represent isomorphism classes of polystable Higgs bundles and the smooth locus is given by the locus of stable Higgs bundles [60]. The geometry of M_n is surprisingly rich. In particular, it can be equipped with a hyperkähler structure and becomes an integrable system by means of the Hitchin fibration.

We shall first study the hyperkähler structure of M_n. Let us fix a particular Hermitian metric h_0 on the topological bundle $\mathbb{E}$, this choice determines a Hermitian metric η on $T^*\mathscr{A}$. Let

$$\mathscr{G}_0 = \Omega^0(X, \mathrm{Aut}(\mathbb{E}, h_0)),$$

be the *unitary gauge group* of automorphisms of $\mathbb{E}$ preserving the metric h_0. We can see that η is preserved by $\mathscr{G}_0$. Also, one can naturally define three complex structures $\widetilde{\Gamma}_1$, $\widetilde{\Gamma}_2$, and $\widetilde{\Gamma}_3$ on $T^*\mathscr{A}$ satisfying the quaternionic relations, together with a hyperkähler metric preserved by $\mathscr{G}_0$. This action defines a moment map μ_i associated with each of the complex structures $\widetilde{\Gamma}_i$, and one can see that η is hyperkähler with respect to them. One can see that the vanishing of μ_1 coincides with Equation (1), the vanishing of μ_2 with Equation (2) and the vanishing of μ_3 with Equation (3). Therefore, the moduli space of Higgs bundles is identified with the hyperholomorphic quotient,

$$M_n \cong \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/\mathscr{G}_0,$$

as it follows from [35, 59, 60]. The complex structures $\widetilde{\Gamma}_i$ descend to complex structures Γ_i in the quotient and so does the hyperkähler metric η, defining a hyperkähler structure on M_n. Observe that natural the complex structure in M_n obtained by the identification (2.1) coincides with Γ_1. Additionally, [16, 20] proved that the moduli space of rank n flat connections on the C^∞ vector bundle $\mathbb{E}$ over X of degree 0 is isomorphic to the above hyperkähler quotient equipped with the complex structure Γ_2.

The hyperkähler structure defined on M_n induces a holomorphic 2-form $\Omega_1 = \omega_2 + \mathrm{i}\omega_3$ on M_n, where ω_2 and ω_3 are the Kähler forms associated with Γ_2 and Γ_3. We next give the expression of Ω_1 by means of the gauge theoretic construction of M_n. Let $(\partial_A, \varphi) \in (T^*\mathscr{A})_H^{\mathrm{st}}$, and consider two tangent vectors

$$(\dot{A}_i, \dot{\varphi}_i) \in T_{(\partial_A,\varphi)}T^*\mathscr{A} \qquad i = 1, 2$$

we have

$$\Omega_1\big((\dot{A}_1, \dot{\varphi}_1), (\dot{A}_2, \dot{\varphi}_2)\big) = \int_X \dot{A}_1 \dot{\wedge} \dot{\varphi}_2 - \dot{A}_2 \dot{\wedge} \dot{\varphi}_1, \tag{2.3}$$

where to define the wedge product $\dot{\wedge}$, we identity $\Omega^{0,1}(X, \mathrm{ad}(\mathbb{E})) \cong (\Omega^0(X, \mathrm{ad}(\mathbb{E})) \otimes \Omega_X^{0,1})$ and $\Omega^0(\mathrm{ad}(\mathbb{E}) \otimes K) \cong (\Omega^0(\mathrm{ad}(\mathbb{E}) \otimes \Omega_X^{1,0})$, and for $Z_i \otimes \omega_i$, $i = 1, 2$, $Z_i \in \Omega^0(X, \mathrm{ad}(\mathbb{E}))$, $\omega_i \in \Omega^1(X)$, we set

$$(Z_1 \otimes \omega_1)\dot{\wedge}(Z_2 \otimes \omega_2) = \langle Z_1, Z_2 \rangle \otimes \omega_1 \wedge \omega_2$$

with $\langle\ ,\ \rangle$ being the Killing form.

We recall now the Hitchin fibration and spectral construction given in [9, 36]. Let $(q_1, \dots, q_n)$ be a basis of the algebra $\mathbb{C}[\mathfrak{gl}(n, \mathbb{C})]^{\mathrm{GL}(n,\mathbb{C})}$ of regular functions on $\mathfrak{gl}(n, \mathbb{C})$ invariant under the adjoint action of $\mathrm{GL}(n, \mathbb{C})$. We choose them so that $\deg(q_i) = i$. The Hitchin map is defined by

$$\begin{array}{rccc} h: & \mathrm{M}_n & \longrightarrow & H := \bigoplus_{i=1}^n H^0(X, K^i) \\ & (E, \varphi) & \longmapsto & (q_1(\varphi), \dots, q_n(\varphi)). \end{array}$$

It is a surjective proper morphism [36, 52] endowing the moduli space with the structure of an algebraically completely integrable system. In particular, its generic fibers are abelian varieties and every fiber is a compactified Jacobian [55, 60]. To describe these, consider the total space $|K|$ of the canonical bundle and the obvious algebraic surjection $\pi : |K| \to X$. We note that the pullback bundle $\pi^*K \to |K|$ admits a tautological section λ. Given an element $b \in \mathrm{H}$, with $b = (b_1, \dots, b_n)$, we construct the *spectral curve* $\overline{X}_b \subset |K|$ by considering the vanishing locus of the section of π^*K^n

$$\lambda^n + \pi^* b_1 \lambda^{n-1} + \cdots + \pi^* b_{n-1}\lambda + \pi^* b_n. \tag{2.4}$$

The restriction of $\pi : |K| \to X$ to X_b is a ramified degree n cover that which by abuse of notation we also denote by

$$\pi : \overline{X}_b \longrightarrow X.$$

Since the canonical divisor of the symplectic surface $|K|$ is zero and $\overline{X}_b$ belongs to the linear system $|nK|$, one can compute the arithmetic genus of $\overline{X}_b$,

$$g\left(\overline{X}_b\right) = 1 + n^2(g-1). \tag{2.5}$$

By Riemann–Roch, the rank n bundle $\pi_* \mathcal{O}_{\overline{X}_b}$ is has degree

$$\deg(\pi_* \mathcal{O}_{\overline{X}_b}) = -(n^2 - n)(g-1).$$

Given a torsion-free rank one sheaf $\mathcal{F}$ over $\overline{X}_b$ of degree δ, where

$$\delta := n(n-1)(g-1), \tag{2.6}$$

we have that $E_{\mathcal{F}} := \pi_* \mathcal{F}$ is a vector bundle on X of rank n and degree 0. Since π is an affine morphism, the natural $\mathcal{O}_{|K|}$-module structure on $\mathcal{F}$, given by understanding $\mathcal{F}$ as a sheaf supported on $|K|$, corresponds to a $\pi_* \mathcal{O}_{|K|} = \mathrm{Sym}^\bullet(K^*)$-module structure on $E_{\mathcal{F}}$. Such structure on $E_{\mathcal{F}}$ is equivalent to a Higgs field

$$\varphi_{\mathcal{F}} : E_{\mathcal{F}} \longrightarrow E_{\mathcal{F}} \otimes K. \tag{2.7}$$

As expected, one has that

$$h((E_{\mathcal{F}}, \varphi_{\mathcal{F}})) = b.$$

A stability notion may be defined for a torsion-free sheaf $\mathcal{F}$ of rank one on the curve $\overline{X}_b$. If $\overline{X}_b$ is reduced and irreducible (integral) then $\mathcal{F}$ is automatically stable. For reduced but reducible curves, [55, Théorème 3.1] gives an easy characterization of semistability, modulo some corrections pointed out in [15, Remark 4.2] and [17, Section 2.4]. A torsion-free rank one

sheaf $\mathcal{F}$ on $\overline{X}_b$ of degree δ is stable (resp. semi-stable) if and only if for every closed sub-scheme $Z \subset \overline{X}_b$ pure of dimension one has that

$$\deg_Z \mathcal{F}_Z > (n_Z^2 - n_Z)(g-1) \quad (\text{resp. } \geq), \tag{2.8}$$

where $\mathcal{F}_Z := \mathcal{F}|_Z/\mathrm{Tor}(\mathcal{F}|_Z)$ and $n_Z = \mathrm{rk}(\pi_*\mathcal{O}_Z)$. One can easily check that every line bundle is stable so the Jacobian $\mathrm{Jac}^\delta(\overline{X}_b)$ is contained inside the moduli space of semistable torsion-free rank 1 degree δ sheaves on $\overline{X}_b$. Furthermore, the former is projective (see [59]) what explains that we refer to it as the *compactified Jacobian* and denote it by $\overline{\mathrm{Jac}}^\delta(\overline{X}_b)$.

The previous construction provides a one-to-one correspondence between rank 1 torsion-free sheaves over a certain spectral curve and Higgs bundles over the corresponding point of the Hitchin base. Furthermore, stability is preserved under such correspondence.

Theorem 2.1 [55, 60]. *A torsion-free rank one sheaf $\mathcal{F}$ on the spectral curve $\overline{X}_b$ is stable (resp. semistable, polystable) if and only if the corresponding Higgs bundle $(E_{\mathcal{F}}, \varphi_{\mathcal{F}})$ on X is stable (resp. semistable, polystable). Hence, the Hitchin fiber over $b \in \mathrm{H}$ is isomorphic to the moduli space of semistable torsion-free rank one sheaves of degree $\delta = (n^2-n)(g-1)$ over $\overline{X}_b$,*

$$h^{-1}(b) \cong \overline{\mathrm{Jac}}^{\,\delta}\left(\overline{X}_b\right).$$

For the case of trivial degree, one can construct a section of the Hitchin fibration, named *Hitchin section*, associated with any line bundle $\mathcal{J} \in \mathrm{Jac}^{\delta/n}(X)$. This section is constructed by assigning to each $b \in B$ the Higgs bundle whose spectral data are the line bundle $\pi^*\mathcal{J}$ over the spectral curve $\overline{X}_b$. In other words, we have a morphism

$$\begin{array}{rccc} \Sigma_{\mathcal{J}} : & H_n & \longrightarrow & M_n \\ & b & \longmapsto & (E_{(\mathcal{J},b)} := \pi_*\pi^*\mathcal{J}, \varphi_{(\mathcal{J},b)}), \end{array} \tag{2.9}$$

where $\varphi_{(\mathcal{J},b)} = \varphi_{E_{(\mathcal{J},b)}}$ as defined in Equation (2.7). One can check that the push-forward of the trivial sheaf of any spectral curve is $\bigoplus_{i=0}^{n-1} K^{-i}$, applying the projection formula one has

$$E_{(\mathcal{J},b)} \cong \mathcal{J} \otimes \pi_*\mathcal{O}_{\overline{X}_b} \cong \mathcal{J} \otimes \left(\bigoplus_{i=0}^{n-1} K^{-i}\right) \tag{2.10}$$

for all $b \in \mathrm{H}_n$.

When studying mirror symmetry beyond the generic locus, one is quickly brought to considering the moduli stack of Higgs bundles. We thus finish this section with some elements about the geometry of the moduli stack $\mathfrak{M}_n$ of Higgs bundles of rank n and trivial degree over the smooth projective curve X, and its relation with the moduli space M_n.

Let us recall that the stack $\mathfrak{M}_n$ contains an open set $\mathfrak{M}_n^{\mathrm{sst}}$ of semistable objects.

Theorem 2.2 [2]. *The moduli space M_n is a good moduli space for $\mathfrak{M}_n^{\mathrm{sst}}$ in the sense of [1]. That is, there exists a quasi-compact morphism*

$$\Psi : \mathfrak{M}_n^{\mathrm{sst}} \longrightarrow M_n$$

such that the push-forward functor is exact and induces an isomorphism of sheaves $\Psi_*\mathcal{O}_{\mathfrak{M}} \cong \mathcal{O}_{\mathrm{M}}$.

The notion of a good moduli space recovers the usual properties of good quotients of finite-dimensional varieties by group actions [51, 56]. In particular, Ψ is surjective and universally closed, and M_n has the quotient topology.

The proof of Theorem 2.2 combines a number of results: Alper proved that the stack of bundles has a good moduli space [1, Theorem 13.6]. In [33, Section 1.F], Heinloth explained how the classical stability notion for bundles can be seen in terms of Θ-stability (notion developed also independently by Halpern–Leistner [29]). As explained in [2, Section 6], one may deduce a similar result for Higgs bundles, so $\mathfrak{M}_n^{\mathrm{sst}}$ are Hilbert–Mumford semistable objects for a suitable line bundle. Theorem C in [6] implies the existence of a good moduli space for $\mathfrak{M}_n^{\mathrm{sst}}$.

2.2 | Arinkin's Poincaré sheaf and Fourier–Mukai transform

Arinkin constructed a Poincaré sheaf [6] on the compactified Jacobian of an integral curve with planar singularities, yielding a Fourier–Mukai transform between these spaces and their duals. This was generalized by Melo, Rapagnetta and Viviani [48, 49] to any fine compactified Jacobian of a reduced curve. The universal sheaf for the fine compactified Jacobian is a crucial piece in Arinkin's construction and, because of this, no Poincaré sheaf has been constructed for coarse compactified Jacobians which is the situation that concern us in this paper. Nevertheless, Arinkin's methods adapt naturally to moduli stacks as we will review in this section. The construction of a Poincaré sheaf over the moduli stack of torsion-free rank 1 sheaves over a reducible planar curve makes part of unpublished work by Arinkin and Pantev [53], where they conjecture that the associated Fourier–Mukai transform gives rise to self-duality of the moduli stack. A sketch of the construction appears in the preprints [45, 46].

Here, we restrict to the case of nodal curves. We do so because for these curves the construction of the Poincaré sheaf is considerably simpler than in the case of an arbitrary reducible curve (see [6, Section 4.3]).

Let $\overline{X}$ be a connected reduced curve with at most nodal singularities and pick an ample line bundle $\mathcal{O}_{\overline{X}}(1)$ on it. Let $\overline{\mathfrak{Jac}}^{\delta}(\overline{X})$ be the moduli stack of rank 1 torsion-free sheaves over $\overline{X}$ and denote by $\mathfrak{U} \to \overline{X} \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})$ the associated universal sheaf. Denote also by $\mathfrak{Jac}^{\delta}(\overline{X})$ the substack of those sheaves that are invertible (i.e., line bundles), and by $\mathfrak{U}^0 \to \overline{X} \times \mathfrak{Jac}^{\delta}(\overline{X})$ the restriction of the universal bundle to it.

Recall that the Hilbert scheme is a fine moduli space represented by a universal subscheme $\mathcal{Z}_N \subset \overline{X} \times \mathrm{Hilb}^N(\overline{X})$. Write $\mathcal{I}_Z$ for the ideal sheaf associated with the zero-dimensional subscheme $Z \subset \overline{X}$ and $\mathcal{I}_{\mathcal{Z}_N} \to \overline{X} \times \mathrm{Hilb}^N(\overline{X})$ for the ideal sheaf associated with the universal subscheme. Since $\overline{X}$ is a nodal curve, we have that $\mathcal{I}_Z^{\vee}$ is a torsion-free sheaf. One can use the universal subscheme $\mathcal{Z}_m := \mathcal{Z}_{N_m}$ to construct the associated *Abel–Jacobi map*

$$\begin{array}{rccc} \alpha_m : & \mathrm{Hilb}^{N_m}(\overline{X}) & \longrightarrow & \overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \\ & Z & \longmapsto & \mathcal{I}_Z^{\vee} \otimes \mathcal{O}_{\overline{X}}(-m), \end{array}$$

where $N_m = m \deg \mathcal{O}_{\overline{X}}(1) + \delta$. Note that α_m is given by

$$\mathcal{I}_{\mathcal{Z}_m}^{\vee} \otimes q_m^* \mathcal{O}_{\overline{X}}(-m) \to \overline{X} \times \mathrm{Hilb}^{N_m}(\overline{X}), \tag{2.11}$$

where q_m denotes the projection $\overline{X} \times \mathrm{Hilb}^{N_m}(\overline{X}) \to \overline{X}$. Denote by $\mathrm{Hilb}^{N_m}(\overline{X})'$ the open subset of $\mathrm{Hilb}^{N_m}(\overline{X})$ given by those zero-dimensional subschemes $Z \subset S$ that can be embedded in a smooth curve. Define W_m to be the open subset of $\mathrm{Hilb}^{N_m}(\overline{X})'$ given by those subschemes Z, whose ideal sheaf $\mathcal{I}_Z$ satisfies the condition $H^1(\overline{X}, \mathcal{I}_Z^{\vee}) = 0$. For any positive integer r, we set $W^r := \bigsqcup_{m=r}^{\infty} W_m$ and $\alpha^r := \prod_{m=r}^{\infty} \alpha_m|_{W_m}$.

The following is well known although it appears in the literature [4, 6, 47] in different forms than how we present it here.

Proposition 2.3. *Let $\overline{X}$ be a connected reduced curve with at most nodal singularities. For any r, the Abel–Jacobi map induces a smooth atlas*

$$\alpha^r : W^r \to \overline{\mathfrak{Jac}}^{\delta}(\overline{X})$$

for the Artin stack $\overline{\mathfrak{Jac}}^{\delta}(\overline{X})$. Using this atlas, the universal sheaf is $\{\mathcal{U}_m \to \overline{X} \times W_m\}_{m=r}^{\infty}$ where the $\mathcal{U}_m$ are given by restricting the sheaves (2.11) to $\overline{X} \times W_m$.

Now, we construct the Poincaré bundle over the product $\overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \times \mathfrak{Jac}^{\delta}(\overline{X})$. Given a flat morphism $f : Y \to S$ whose geometric fibers are curves, for any S-flat sheaf $\mathcal{E}$ on Y, we can construct the determinant of cohomology $\mathcal{D}_f(\mathcal{E})$ (see, for instance, (see [43] and [22, Section 6.1])), which is an invertible sheaf on S constructed locally as the determinant of complexes of free sheaves locally quasi-isomorphic to $Rf_*\mathcal{E}$. Consider the triple product $\overline{X} \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \times \mathfrak{Jac}^{\delta}(\overline{X})$ and denote by f_{ij} the projection to the product of the ith and jth factors. We define the Poincaré bundle $\mathfrak{P} \to \overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \times$

$\mathfrak{Jac}^{\delta}(\overline{X})$ as the invertible sheaf

$$\mathfrak{P} = \mathscr{D}_{f_{23}}\left(f_{12}^{*}\mathfrak{U} \otimes f_{13}^{*}\mathfrak{U}^{0}\right) \otimes \mathscr{D}_{f_{23}}\left(f_{13}^{*}\mathfrak{U}^{0}\right)^{-1} \otimes \mathscr{D}_{f_{23}}\left(f_{12}^{*}\mathfrak{U}\right)^{-1}. \tag{2.12}$$

Given a degree δ line bundle J over $\overline{X}$, denote by $\mathfrak{P}_J := \mathfrak{P}|_{\overline{\mathfrak{Jac}}^{\delta}(\overline{X})\times\{J\}}$ the restriction of $\mathfrak{P}$ to the slice corresponding to J. In fact, if we consider the obvious projections $f_1 : \overline{X} \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \to \overline{X}$ and $f_2 : \overline{X} \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \to \overline{\mathfrak{Jac}}^{\delta}(\overline{X})$, one has (see [48, Lemma 5.1] for instance) that

$$\mathfrak{P}_J = \mathscr{D}_{f_2}(\mathfrak{U} \otimes f_1^{*}J) \otimes \mathscr{D}_{f_2}(f_1^{*}J)^{-1} \otimes \mathscr{D}_{f_2}(\mathfrak{U})^{-1}. \tag{2.13}$$

Remark 2.4. If $\overline{X}$ is a smooth irreducible curve, rank 1 torsion-free sheaves over it are simple line bundles so

$$\overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \cong \mathfrak{Jac}^{\delta}(\overline{X}) \cong \left[\mathrm{Jac}^{\delta}(\overline{X})/\mathbb{C}^{*}\right],$$

and $\mathfrak{P}$ pulls-back to a bundle $\mathscr{P} \to \mathrm{Jac}^{\delta}(\overline{X}) \times \mathrm{Jac}^{\delta}(\overline{X})$ under the projection $\mathrm{Jac}^{\delta}(\overline{X}) \to [\mathrm{Jac}^{\delta}(\overline{X})/\mathbb{C}^{*}]$. The integral functor associated with $\mathscr{P}$ is a derived equivalence of categories [50], the Fourier–Mukai transform.

One can reverse the roles of $\mathfrak{Jac}^{\delta}(\overline{X})$ and $\overline{\mathfrak{Jac}}^{\delta}(\overline{X})$ in Equation (2.12) to obtain a Poincaré bundle over $\mathfrak{Jac}^{\delta}(\overline{X}) \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})$ which coincides with the one defined in Equation (2.12) over $\mathfrak{Jac}^{\delta}(\overline{X}) \times \mathfrak{Jac}^{\delta}(\overline{X})$. We then see that the Poincaré bundle extends naturally to a bundle over

$$\left(\overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})\right)^{\sharp} := \left(\overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \times \mathfrak{Jac}^{\delta}(\overline{X})\right) \cup \left(\mathfrak{Jac}^{\delta}(\overline{X}) \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})\right)$$

that we denote by $\mathfrak{P}^{\sharp}$. Following [6], it is possible to extend $\mathfrak{P}^{\sharp}$ even further to a Cohen–Macaulay sheaf over $\overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})$, as we will see below.

First, we need some definitions. Consider the projection to the Hilbert scheme of its associated universal scheme $h_m : \mathscr{X}_m \to \mathrm{Hilb}^{N_m}(\overline{X})$, the coherent sheaf of algebras $\mathscr{A}_m := h_{m,*}\mathcal{O}_{\mathscr{X}_m}$ over $\mathrm{Hilb}^{N_m}(\overline{X})$ and denote by $\mathscr{A}_m^{*}$ the subsheaf of invertible elements. Consider p_1 to be the projection of $\mathrm{Hilb}^{N_m}(\overline{X}) \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})$ to the first factor and take the pull-back $p_1^{-1}\mathscr{A}_m^{*}$. Given a sheaf, we use the subindex $p_1^{-1}(\mathscr{A}_m^{*})$ to denote the maximal quotient of the sheaf, where $p_1^{-1}(\mathscr{A}_m^{*})$ acts via the norm character.

Consider also the triple product $\overline{X} \times W_m \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})$ and denote by g_{ij} the projections to the ith and jth factors. Following [6], we define the sheaf over $W_m \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})$

$$\overline{\mathfrak{P}}_m := \left(\bigwedge^{N_m} g_{23,*}(g_{12}^{*}\mathcal{O}_{\mathscr{X}_m} \otimes g_{13}^{*}\mathfrak{U})\right)_{p_1^{-1}(\mathscr{A}_m^{*})} \otimes \left(\bigwedge^{N_m} g_{23,*}(g_{12}^{*}\mathcal{O}_{\mathscr{X}_m})\right)^{-1}. \tag{2.14}$$

The following is an immediate adaptation of [6].

Proposition 2.5. *The sheaves $\overline{\mathfrak{P}}_m \to W_m \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})$ are Cohen–Macaulay and flat over $\overline{\mathfrak{Jac}}^{\delta}(\overline{X})$ for all positive integer m.*

Proof. Up to a base change, the construction of Equation (2.14) coincides with Arinkin's definition of the sheaf Q' after making the substitution of the fine compactified Jacobian (of an integral curve) and its universal sheaf by the moduli stack of torsion-free sheaves (on a nodal curve) and its associated universal sheaf. After the same substitution, one can also adapt Arinkin's construction of another sheaf Q which he shows to be isomorphic to Q' in [6, Proposition 4.5]. The proof of [6, Proposition 4.5] relies entirely on a result [6, Lemma 3.6] concerning isospectral Hilbert schemes of surfaces,

so [6, Proposition 4.5] extends to our case and both constructions coincide here as well. Using the construction of $\overline{\mathfrak{P}}_m$ associated with Q and [6, Lemma 2.1 and Proposition 4.2], we have that $\overline{\mathfrak{P}}_m$ is a Cohen–Macaulay sheaf, flat over $\overline{\mathfrak{Jac}}^{\delta}(\overline{X})$. Note that [6, Lemma 2.1] is a statement for Cohen–Macaulay sheaves in general and [6, Proposition 4.2] works for any reduced curve and any rank 1 torsion-free sheaf on it, so both are valid in our case. □

This construction recovers the Poincaré bundle.

Proposition 2.6. *$\mathfrak{P}$ and $\overline{\mathfrak{P}}_m|_{W_m\times\mathfrak{Jac}^{\delta}(\overline{X})}$ are isomorphic up to the twisting by a line bundle over $\mathfrak{Jac}^{\delta}(\overline{X})$.*

Proof. Since the $\mathcal{U}_m$ are defined as (the restriction to $W_m\times\mathrm{Hilb}^{N_m}(\overline{X})$ of) Equation (2.11), in terms of the Abel–Jacobi atlas from Proposition 2.3, $\mathfrak{P}$ reads

$$\mathfrak{P}\cong\mathcal{D}_{g_{23}}(g_{12}^*\mathcal{I}_{\mathcal{X}_m}^{\vee}\otimes g_{12}^*q_m^*\mathcal{O}_{\overline{X}}(-m)\otimes g_{13}^*\mathfrak{U}_0)\otimes\mathcal{D}_{g_{23}}(g_{13}^*\mathfrak{U}_0)^{-1}\otimes\mathcal{D}_{g_{23}}(g_{12}^*\mathcal{I}_{\mathcal{X}_m}^{\vee}\otimes g_{12}^*q_m^*\mathcal{O}_{\overline{X}}(-m))^{-1}.$$

We recall that W_m is a subset of those subschemes Z such that the first cohomology space of its ideal sheaf is trivial, $H^1(\overline{X},\mathcal{I}_Z)=0$. It then follows that $R^1g_{23,*}(g_{12}^*\mathcal{O}_{\mathcal{X}_m})$ vanishes and $R^0g_{23,*}(g_{12}^*\mathcal{O}_{\mathcal{X}_m})$ is locally free of rank N_m. Under these conditions, the second term in the tensorization of the right-hand side of Equation (2.14) equals the determinant in cohomology,

$$\bigwedge^{N_m}g_{23,*}(g_{12}^*\mathcal{O}_{\mathcal{X}_m})\cong\det R^0g_{23,*}(g_{12}^*\mathcal{O}_{\mathcal{X}_m})\cong\mathcal{D}_{g_{23}}(g_{12}^*\mathcal{O}_{\mathcal{X}_m}).$$

Also, $g_{13}^*\mathfrak{U}$ is a line bundle over $W_m\times\mathfrak{Jac}^{\delta}(\overline{X})$. This implies, for large m, that $R^1g_{23,*}(g_{12}^*\mathcal{O}_{\mathcal{X}_m}\otimes g_{13}^*\mathfrak{U})$ vanishes and $R^0g_{23,*}(g_{12}^*\mathcal{O}_{\mathcal{X}_m}\otimes g_{13}^*\mathfrak{U})$ is locally free of rank N_m. Then,

$$\bigwedge^{N_m}g_{23,*}(g_{12}^*\mathcal{O}_{\mathcal{X}_m}\otimes g_{13}^*\mathfrak{U}_0)\cong\det R^0g_{23,*}(g_{12}^*\mathcal{O}_{\mathcal{X}_m}\otimes g_{13}^*\mathfrak{U})\cong\mathcal{D}_{g_{23}}(g_{12}^*\mathcal{O}_{\mathcal{X}_m}\otimes g_{13}^*\mathfrak{U}_0)$$

is a line bundle on which $p_1^{-1}(\mathcal{A}_m^*)$ acts via the norm character. Therefore, we have seen that

$$\overline{\mathfrak{P}}_m|_{W_m\times\mathfrak{Jac}^{\delta}(\overline{X})}\cong\mathcal{D}_{g_{23}}(g_{12}^*\mathcal{O}_{\mathcal{X}_m}\otimes g_{13}^*\mathfrak{U}_0)\otimes\mathcal{D}_{g_{23}}(g_{12}^*\mathcal{O}_{\mathcal{X}_m})^{-1}.$$

From the short exact sequence

$$0\to g_{12}^*\mathcal{O}_{\overline{X}\times\mathrm{Hilb}^{N_m}(\overline{X})}\to g_{12}^*\mathcal{I}_{\mathcal{X}_m}^{\vee}\to g_{12}^*\mathcal{O}_{\mathcal{X}_m}\to 0,$$

and the additivity property of the determinant in cohomology, one can deduce

$$\mathcal{D}_{g_{23}}(g_{12}^*\mathcal{O}_{\mathcal{X}_m}\otimes g_{13}^*\mathfrak{U}_0)\cong\mathcal{D}_{g_{23}}(g_{12}^*\mathcal{I}_{\mathcal{X}_m}^{\vee}\otimes g_{13}^*\mathfrak{U}_0)\otimes\mathcal{D}_{g_{23}}(g_{13}^*\mathfrak{U}_0)^{-1}$$

and

$$\mathcal{D}_{g_{23}}(g_{12}^*\mathcal{O}_{\mathcal{X}_m})\cong\mathcal{D}_{g_{23}}(g_{12}^*\mathcal{I}_{\mathcal{X}_m}^{\vee}).$$

Therefore,

$$\overline{\mathfrak{P}}_m|_{W_m\times\mathfrak{Jac}^{\delta}(\overline{X})}\cong\mathcal{D}_{g_{23}}(g_{12}^*\mathcal{I}_{\mathcal{X}_m}^{\vee}\otimes g_{13}^*\mathfrak{U}_0)\otimes\mathcal{D}_{g_{23}}(g_{13}^*\mathfrak{U}_0)^{-1}\otimes\mathcal{D}_{g_{23}}(g_{12}^*\mathcal{I}_{\mathcal{X}_m}^{\vee})^{-1}.$$

Thanks to this description of $\overline{\mathfrak{P}}_m|_{W_m\times\mathfrak{Jac}^{\delta}(\overline{X})}$ and the description of $\mathfrak{P}$ given at the beginning of the proof, the result follows from [49, Claim after Equation (4.18)]. □

The following theorem was explained to us by T. Pantev, who proved it in collaboration with D. Arinkin. Since the proof is not published, we include one here.

Theorem 2.7 (D. Arinkin and T. Pantev). *Let $\overline{X}$ be a connected reduced curve with at most nodal singularities. For r large enough, the $\{\overline{\mathfrak{P}}_m \to W_m \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})\}_{m=r}^{\infty}$ descend to a Cohen–Macaulay sheaf $\overline{\mathfrak{P}}$ over $\overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})$, that extends $\mathfrak{P}$ up to a twist.*

Proof. Thanks to Proposition 2.6 one has that the set of restrictions $\{\overline{\mathfrak{P}}_m|_{W_m \times \mathfrak{Jac}^{\delta}(\overline{X})}\}_{m=r}^{\infty}$ descend to a bundle over the product of stacks $\overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \times \mathfrak{Jac}^{\delta}(\overline{X})$. Let W_m^{ℓ} denote that subset of $W_m \subset \mathrm{Hilb}^{N_m}(\overline{X})$ given by those subschemes whose ideal sheaf is invertible. One can proceed analogously as we did in the proof of Proposition 2.6 and show that the restriction $\{\overline{\mathfrak{P}}_m|_{W_m^{\ell} \times \mathfrak{Jac}^{\delta}(\overline{X})}\}_{m=r}^{\infty}$ descend to a bundle over the product of stacks $\mathfrak{Jac}^{\delta}(\overline{X}) \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})$. Therefore, the restriction of the $\overline{\mathfrak{P}}_m$ to $\left(W_m \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})\right)^{\sharp} := \left(W_m \times \mathfrak{Jac}^{\delta}(\overline{X})\right) \cup \left(W_m^{\ell} \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})\right)$ descend to a bundle over $\left(\overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})\right)^{\sharp}$ that we denote by $\overline{\mathfrak{P}}_m^{\sharp}$.

We now recall that $i : \left(\overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})\right)^{\sharp} \hookrightarrow \overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})$ has codimension at least 2. Thanks to Proposition 2.5, we have that $\overline{\mathfrak{P}}$ is Cohen –Macaulay. Then, it follows that

$$\overline{\mathfrak{P}}_m \cong i^* \overline{\mathfrak{P}}_m^{\sharp} \tag{2.15}$$

so the collection $\{\overline{\mathfrak{P}}_m^{\sharp}\}_{m=r}^{\infty}$ descend to a bundle on $\left(\overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})\right)^{\sharp}$. Thanks to Equation (2.15), one has that

$$\overline{\mathfrak{P}} \cong i^* \overline{\mathfrak{P}}^{\sharp}.$$

Therefore, $\{\overline{\mathfrak{P}}_m\}_{m=\ell}^{\infty}$ descend to a sheaf over $\overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \times \overline{\mathfrak{Jac}}^{\delta}(\overline{X})$. The rest of the proof is straightforward. □

When our curve $\overline{X}$ is irreducible any rank 1 torsion-free sheaf is stable and simple. Therefore, the moduli stack of torsion-free sheaves on a curve is the quotient stack associated with the fine compactified Jacobian $\overline{\mathrm{Jac}}^{\delta}(\overline{X})$ quotiented by the trivial action of $\mathbb{C}^*$,

$$\overline{\mathfrak{Jac}}^{\delta}(\overline{X}) \cong \left[\overline{\mathrm{Jac}}^{\delta}(\overline{X})/\mathbb{C}^*\right].$$

Let us denote by $\overline{\mathcal{P}} \to \overline{\mathrm{Jac}}^{\delta}(\overline{X}) \times \overline{\mathrm{Jac}}^{\delta}(\overline{X})$ the pull-back of the Poincaré sheaf $\mathfrak{P}$ under the obvious projection $\overline{\mathrm{Jac}}^{\delta}(\overline{X}) \to [\overline{\mathrm{Jac}}^{\delta}(\overline{X})/\mathbb{C}^*]$, and one can consider the integral functor given by it,

$$\begin{array}{rccc} \overline{\Phi}: & D^b\left(\overline{\mathrm{Jac}}^{\delta}(\overline{X})\right) & \longrightarrow & D^b\left(\overline{\mathrm{Jac}}^{\delta}(\overline{X})\right) \\ & \mathcal{E}^{\bullet} & \longmapsto & R\pi_{2,*}(\pi_1^*\mathcal{E}^{\bullet} \otimes \overline{\mathcal{P}}). \end{array} \tag{2.16}$$

The Poincaré sheaf $\overline{\mathcal{P}}$ was first obtained by [23] for compactified Jacobians of irreducible nodal curves. Arinkin [6] extended this construction to any irreducible reduced planar curve, showing also that Equation (2.16) is a derived equivalence. Although his result does not extend to the context under consideration, we include it for the sake of completeness:

Theorem 2.8 [6]. *If $\overline{X}$ is an irreducible reduced planar curve, then the Fourier–Mukai integrable functor $\overline{\Phi}$ provides an equivalence of categories.*

The integral functor associated with $\overline{\mathfrak{P}}$ is an eigenfunctor of the derived category of sheaves over the moduli stack of torsion-free rank 1 sheaves over a reducible planar curve. It is being studied by Arinkin and Pantev [53] whether this provides an equivalence or not.

3 | TOTALLY REDUCIBLE SPECTRAL CURVES

3.1 | The locus of totally reducible spectral curves and the Borel subgroup

We start by studying the Hitchin fibers associated with spectral curves that are totally reducible.

Recall from Section 2.1 that, for any $b \in \mathrm{H}$, the associated spectral curve $\overline{X}_b$ is the $n : 1$ cover of the base curve X given by the vanishing of Equation (2.4). If $\overline{X}_b$ is totally reducible, then, by definition, one can rewrite Equation (2.4) as

$$\prod_{i=1}^{n}(\lambda - \pi^*\alpha_i), \tag{3.1}$$

where $\alpha_i \in H^0(X, K)$. In view of this, consider the symmetric product

$$V := \mathrm{Sym}^n\big(H^0(X, K)\big). \tag{3.2}$$

Hence

$$\dim V = ng. \tag{3.3}$$

There is an injection into the Hitchin base

$$\begin{array}{ccc} V & \hookrightarrow & H \\ (\alpha_1, \dots, \alpha_n)_{\mathfrak{S}} & \longmapsto & (q_1(\alpha_1, \dots, \alpha_n), \dots, q_n(\alpha_1, \dots, \alpha_n)). \end{array} \tag{3.4}$$

In the above: $(\alpha_1, \dots, \alpha_n)_{\mathfrak{S}}$ denotes the orbit of $(\alpha_1, \dots, \alpha_n)$ under the nth symmetric group $\mathfrak{S}$, and $q_i(\alpha_1, \dots, \alpha_n)$ is the evaluation of q_i on the diagonal Higgs field with entries α_i. Note that the q_i being invariant under the adjoint action, this depends only on the orbit $(\alpha_1, \dots, \alpha_n)_{\mathfrak{S}}$.

Seen inside the Hitchin base, V describes the locus of totally reducible spectral curves.

Lemma 3.1. *V parameterizes all spectral curves that are totally reducible. Let $v \in V$ be given by $v = (\alpha_1, \overset{m_1}{\dots}, \alpha_1, \dots, \alpha_\ell, \overset{m_\ell}{\dots}, \alpha_\ell)_{\mathfrak{S}}$, where $\sum_{i=1}^{\ell} m_i = n$ and $\alpha_i \neq \alpha_j$ if $i \neq j$. Then, its corresponding spectral curve is*

$$\overline{X}_v = \bigcup_{i=1}^{\ell} X_i^{m_i}, \tag{3.5}$$

where each $X_i^{m_i}$ is a curve of multiplicity m_i whose reduced subscheme is $X_i := \alpha_i(X)$, isomorphic to X.

Proof. This follows easily from Equation (3.1). □

Fix a Borel subgroup $B < \mathrm{GL}_n(\mathbb{C})$ containing C, so that $B = C \ltimes U$, where $U = [B, B]$ is the unipotent radical of B. Let us consider the subvariety given by those Higgs bundles whose structure group reduces to B,

$$\mathrm{Bor} := \left\{ (E, \varphi) \in M_n \,\middle|\, \begin{array}{l} \exists\, \sigma \in H^0(X, E/B), \\ \varphi \in H^0(X, E_\sigma(\mathfrak{b}) \otimes K) \end{array} \right\},$$

where $E_\sigma := \sigma^* E$ is the principal B-bundle on X associated with the section $\sigma \in H^0(X, E/\mathrm{B})$.

We can see that Bor coincides with the preimage under the Hitchin map of the locus of totally reducible spectral curves.

Proposition 3.2. *One has the following*

$$M_n \times_{\mathrm{H}} V = \mathrm{Bor}. \tag{3.6}$$

Proof. We first see that $\mathrm{Bor} \subset M_n \times_{\mathrm{H}} V$. This is a consequence of the following fact: given the Jordan–Chevalley decomposition of $x = x_s + x_n \in \mathfrak{gl}_n(\mathbb{C})$ into a semisimple x_s and a nilpotent piece x_n, the invariant polynomials q_i defining the Hitchin fibration evaluate independently of the nilpotent part, namely $q_i(x) = q_i(x_s)$.

For the other inclusion, one has to prove that any Higgs bundle $(E, \varphi) \in M_n \times_{\mathrm{H}} V$ admits a full-flag decomposition.

Denote by $\mathcal{F}$ the torsion-free sheaf over the spectral curve $\overline{X}_v$ associated with (E, φ) under the spectral correspondence. Recall that $\overline{X}_v$ is described in Equation (3.5) and, using this notation, define

$$Y_i := \bigcup_{j=1}^{i} X_j^{m_j}, \qquad Z_i := \bigcup_{k=i+1}^{\ell} X_k^{m_k}. \tag{3.7}$$

We consider the restriction of $\mathcal{F}$ to $\mathcal{F}|_{Z_i}$ and denote its kernel by $\mathcal{F}_i$,

$$0 \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}|_{Z_i} \longrightarrow 0. \tag{3.8}$$

Since $\mathcal{F}_i$ is a subsheaf of $\mathcal{F}$, it gives the Higgs subbundle $(E_i, \varphi_i) \subset (E, \varphi)$ under the spectral correspondence. Since $\mathcal{F}_{i-1}$ is a subsheaf of $\mathcal{F}_i$ we have that $(E_{i-1}, \varphi_{i-1}) \subset (E_i, \varphi_i)$ so we obtain a filtration

$$0 \subset (E_1, \varphi_1) \subset \cdots \subset (E_\ell, \varphi_\ell) = (E, \varphi). \tag{3.9}$$

Note that a full-flag filtration for each of the $(F_i, \phi_i) := (E_i, \varphi_i)/(E_{i-1}, \varphi_{i-1})$ will induce a full-flag filtration of (E, φ).

Note that the eigenvalues of ϕ_i are all equal to α_i. Set $F_{i,1} = \ker(\phi_i - \alpha_i \otimes \mathbf{1}_{F_i})$ and let $\phi_{i,1}$ be the restriction to $F_{i,1}$. Set $(F_i', \phi_i') = (F_i, \phi_i)/(F_{i,1}, \phi_{i,1})$ and take $F_{i,2}' = \ker(\phi_i' - \alpha_i \otimes \mathbf{1}_{F_i'})$ and $\phi_{i,2}' = \phi_i'|_{F_{i,2}'}$. Note that $(F_{i,2}', \phi_{i,2}') \subset (F_i', \phi_i')$ lifts to a subbundle $(F_{i,2}, \phi_{i,2})$ of (F_i, ϕ_i) which contains $(F_{i,1}, \phi_{i,1})$. Repeating this procedure one gets a filtration

$$0 \subset (F_{i,1}, \phi_{i,1}) \subset \cdots \subset (F_{i,s}, \phi_{i,s}) = (F_i, \phi_i),$$

where each quotient $(F_{i,j}, \phi_{i,j})/(F_{i,j-1}, \phi_{i,j-1})$ is isomorphic to a Higgs bundle of the form $(G_{i,j}, \alpha \otimes \mathbf{1}_{G_{i,j}})$.

Given an ample line bundle $\mathcal{O}_X(1)$, one has that, for sufficiently high $N > 0$, that $\mathcal{O}_X(-N)$ is a subbundle of $G_{i,j}$, and the same is valid for the quotient $G_{i,j}/\mathcal{O}_X(-N)$. Hence, one can always construct a full-flag filtration for each of the $G_{i,j}$. This provides a full-flag filtration for all the (F_i, ϕ_i), hence a full-flag filtration for (E, φ). □

Remark 3.3. Note that Proposition 3.2 generalizes to the corresponding moduli stacks as stability plays no role on its proof.

Remark 3.4. The full-flag filtration of the Higgs bundle (E, φ) determines the reduction to the Borel subgroup $\sigma \in H^0(X, E/\mathrm{B}$ with $\varphi \in H^0(X, E_\sigma(\mathfrak{b} \otimes K))$. Note that, in general, one cannot give a canonical such a full-flag filtration.

In the remaining of the section, we will focus on an open subset of V. Denote the big diagonal of V by

$$\Delta := \{(\alpha_1, \dots, \alpha_n)_{\mathfrak{S}} \in V \text{ such that } \alpha_i = \alpha_j \text{ for some } i, j\}$$

and its complement in V by

$$V^{\mathrm{red}} := V \setminus \Delta.$$

Let us provide a description of the spectral curves parameterized by V^{red}.

Lemma 3.5. *V^{red} is a dense open subset of V parameterizing reduced, totally reducible, and nodal spectral curves. Furthermore, for any $v \in V^{\mathrm{red}}$ given by $(\alpha_1, \dots, \alpha_n)_{\mathfrak{S}_n}$, the spectral curve $\overline{X}_v$ is reduced and has the following decomposition into*

irreducible components,

$$\overline{X}_v = \bigcup_{i=1}^{n} X_i, \tag{3.10}$$

with $X_i = \alpha_i(X) \cong X$. It is a singular curve with singularity divisor of length $|D| = (n^2 - n)(g - 1) = \delta$. Its normalization, $\overline{X}_v$, is isomorphic to

$$\widetilde{X}_v \cong \bigsqcup_{i=1}^{n} X_i \cong \bigsqcup_{i=1}^{n} X, \tag{3.11}$$

and the normalization morphism,

$$\nu : \widetilde{X}_v \to \overline{X}_v, \tag{3.12}$$

is the identity restricted to each of the X_i.

Proof. Δ is a closed subset of V of codimension 1, hence V^{red} is open and dense. When $v \in V \setminus \Delta$, Equation (3.5) implies that $\overline{X}_v$ is the union of n different reduced and irreducible curves X_i all isomorphic to X. It then follows that $\overline{X}_v$ is reduced and its normalization is as described in Equation (3.11). The description of the normalization morphism follows form the description of the spectral curve given in Equation (3.10). The length of D can be obtained after an easy computation using Riemann–Roch. □

For any two α_i and α_j with $i \neq j$, denote the divisor $D_{ij} = \alpha_i(X) \cap \alpha_j(X)$. Consider also the following subset of V^{red},

$$V^{\mathrm{nod}} := \left\{ \begin{array}{l} (\alpha_1, \dots, \alpha_n)_{\mathfrak{S}} \in V^{\mathrm{red}} \text{such that for every } i < j < k \\ \text{(a) there is no multiple point on } D_{ij}, \text{ and} \\ \text{(b)} D_{ij} \cap D_{ik} \text{ is empty.} \end{array} \right\}.$$

Lemma 3.6. *V^{nod} is a dense open subset of V parameterizing reduced, totally reducible, and nodal spectral curves. For any $v \in V^{\mathrm{nod}}$ given by $(\alpha_1, \dots, \alpha_n)_{\mathfrak{S}_n}$, the singularity divisor D of the spectral curve $\overline{X}_v$ is*

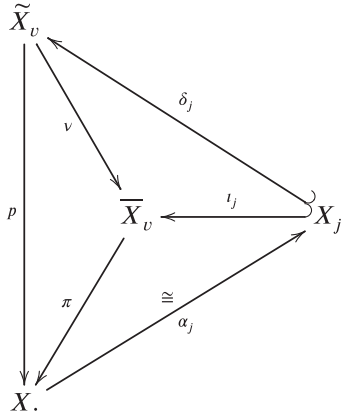
$$D := \bigcup_{i,j} D_{ij}$$

and consists only of simple points.

Proof. Since conditions (a) and (b) are open and generic, V^{nod} is a dense open subset of V^{red}. It then follows from Lemma 3.5 that V^{nod} is dense within V too and the first statement follows.

Recall the description of $\overline{X}_v$ given in Lemma 3.5. Take two irreducible components of $\overline{X}_v$, X_i, and X_j, intersecting each other at D_{ij}. Note that D coincides with the set of intersection points and recall that we have imposed the condition $D_{ij} \cap D_{ik} = \emptyset$ if $j \neq k$ in the definition of V^{nod}, so D is the union of the D_{ij}. □

Using the notation of Lemma 3.5, consider the following morphisms:



(3.13)

We have seen in Remark 3.4 that the reduction to the Borel subgroup cannot be defined canonically for an arbitrary Higgs bundle in Bor. However, for those Higgs bundles lying over $v \in V^{\mathrm{nod}}$, one can fix such a reduction after choosing an ordering for the components of v.

Proposition 3.7. *Let $v = (\alpha_1, \dots, \alpha_n)_{\mathfrak{S}_n} \in V^{\mathrm{nod}}$ and let $(E, \varphi) \in h^{-1}(v)$. For any ordering $J = (\alpha_{j_1}, \dots, \alpha_{j_n})$ of the set $\{\alpha_1, \dots, \alpha_n\}$, one can chose canonically a filtration*

$$(E_J)_\bullet : 0 \subsetneq (E_1, \varphi_1) \subsetneq \cdots \subsetneq (E_n, \varphi_n) = (E, \varphi),$$

such that the Higgs field induced by φ on E_i/E_{i-1} is α_{j_i}. Furthermore, if the associated spectral datum associated with (E, φ) is a line bundle over the spectral curve, $L \in \mathrm{Jac}^{\delta}(\overline{X}_v)$, then

$$E_i/E_{i-1} \cong (\alpha_{j_i}^* \iota_{j_i}^* L) \otimes K^{i-n}.$$

Proof. Using the ordering J set $Y_i = \bigcup_{k=1}^{i} X_{j_k}, Z_i = \bigcup_{k=i+1}^{n} X_{j_k}$ as in Equation (3.7). After the choice of J, the filtration for the spectral data given in Equation (3.8) is canonical and so is the filtration (3.9) of (E, φ). Since $v \in V^{\mathrm{nod}}$, Equation (3.9) is a full-flag filtration what proves the first statement.

For the second statement, recall that the filtration of L is defined by the subsheaves $L_i = L \otimes \mathcal{I}_{\overline{X}, Z_i}$ where $\mathcal{I}_{\overline{X}, Z_i}$ denotes the ideal defining the subscheme $Z_i \subset \overline{X}$. Now, $\mathcal{I}_{\overline{X}, Z_i} \cong \mathcal{O}_{Y_i} \otimes \mathcal{I}_{Y_i, Z_i \cap Y_i}$, thus

$$L_i \cong L|_{Y_i} \otimes \mathcal{I}_{Y_i, Z_i \cap Y_i}.$$

Note that

$$0 \longrightarrow L_i/L_{i-1} \longrightarrow L|_{Z_{i-1}} \longrightarrow L|_{Z_i} \longrightarrow 0$$

is exact, so that

$$\begin{aligned} L_i/L_{i-1} &\cong L|_{Z_i} \otimes \mathcal{I}_{Z_{i-1}, Z_i} \\ &\cong L|_{Z_i} \otimes \mathcal{O}_{X_i} \otimes \mathcal{I}_{X_i, Z_i \cap X_i} \\ &\cong L|_{X_i}(-\sum_{k=i+1}^{n} D_{ik}). \end{aligned}$$

Now, the push-forward of

$$0 \longrightarrow L_{i-1} \longrightarrow L_i \longrightarrow L_i/L_{i-1} \longrightarrow 0$$

gives under the spectral correspondence

$$(E_i, \varphi_i)/(E_{i-1}, \varphi_{i-1}) \cong \left(\alpha_{j_i}^* \iota_{j_i}^* L\left(-\sum_{k=i+1}^{n} D_{ik}\right), \alpha_{j_i}\right),$$

where we abuse notation by identifying the divisor D_{jk} and its image under π. Naturally, $K \cong \mathcal{O}_X(D_{jk})$, which yields the result. □

3.2 | Totally reducible nodal spectral curves and their desingularization

We study in this section the relation between the Hitchin fibers associated with totally reducible spectral curves with only nodal singularities and their partial and complete desingularizations.

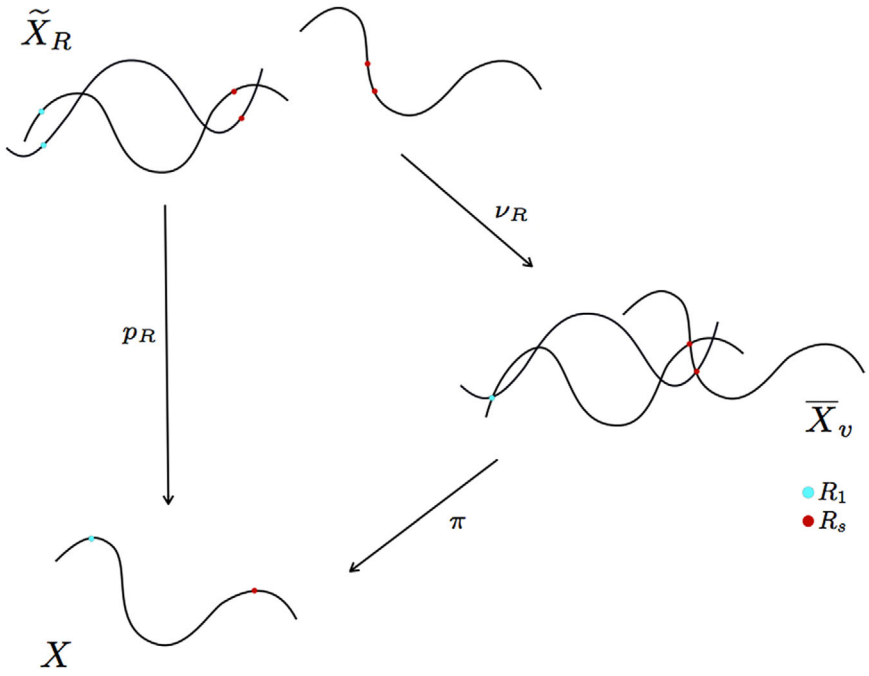


FIGURE 1 Partial desingularization along R.

We first recall some well-known facts about rank one torsion-free sheaves on a reduced connected nodal curve $\overline{X}$ with divisor of singularities D. We start by studying the particular case of line bundles which admit a simple description in terms of their pullback to partial (and complete) desingularization. Consider $R \subset D$ a subdivisor of the singular divisor of the reduced curve $\overline{X}$, and let

$$\nu_R : \widetilde{X}_R \to \overline{X} \tag{3.14}$$

be the partial desingularization at R (see Figure 1). Note that $\nu_D : \widetilde{X}^D \to \overline{X}$ is just the normalization map ν that appeared in Equation (3.12). Denote by

$$\begin{array}{rccc} \hat{\nu}_R : & \mathrm{Jac}(\overline{X}) & \longrightarrow & \mathrm{Jac}(\widetilde{X}_R) \\ & L & \longmapsto & \nu_R^* L \end{array}$$

the pullback map. The fibers of this map are described in the following lemma due to Grothendieck [28, Proposition 21.8.5], that we reproduce adapted to our notation.

Lemma 3.8 [28]. *For any subdivisor $R \subset D$ of the singular divisor of the reduced nodal curve $\overline{X}$, the pullback map $\hat{\nu}_R$ is a smooth fibration with fiber $(\mathbb{C}^\times)^{|R|-n_R+1}$, where n_R is the number of connected components of $\widetilde{X}_R$.*

One can give the following geometrical interpretation of Lemma 3.8: line bundles on reduced nodal curves can be described in terms of line bundles on each of the n_R irreducible components of the (partial) desingularization, together with $|R|$ gluing data (i.e., an element of $\mathbb{C}^\times$ identifying the two local components of the nodal point) for each of the intersection points, taking into account the identification given by scalar automorphisms on each of the components.

In the case of $R = D$, we have that $\widetilde{X}_R = \widetilde{X}_v$ is the normalization of the spectral curve and ν_R coincides with the normalization map ν. One has the following description adapted to that case.

Corollary 3.9. *The pullback map*

$$\begin{array}{rccc} \hat{\nu} : & \mathrm{Jac}(\overline{X}_v) & \longrightarrow & \mathrm{Jac}(\widetilde{X}_v) \\ & L & \longmapsto & \nu^* L \end{array} \tag{3.15}$$

is a smooth fibration with fiber $(\mathbb{C}^\times)^{n(n-1)(g-1)-(n-1)}$.

It can be checked that that the degree of a line bundle L on a connected nodal curve $\overline{X}$ with irreducible components X_i is given by the sum of the degrees of the line bundles obtained by restricting to each of the components, $\deg L = \sum_i \deg L|_{X_i}$. In view of this, we refer to the *multidegree* of a line bundle L on $\overline{X}$ as the degree on each of the connected components of $\widetilde{X}$. In other words, the multidegree of $\hat{\nu}(L) = \nu^* L$ over the disconnected curve $\widetilde{X}$.

A rank one torsion-free sheaf on $\overline{X}$ is either a line bundle or a push-forward of a line bundle on a partial desingularization ν_R of $\overline{X}$ (see [57], for instance). Consider $L \in \mathrm{Jac}(\widetilde{X}_R)$ be given by the line bundles L_i on each connected component $\widetilde{X}_{R,i}$ of $\widetilde{X}_R$. Geometrically, the (rank one torsion-free coherent) sheaf $\nu_{R,*}L$ on $\overline{X}$ is obtained by considering n_R-tuples of $L_i \to \widetilde{X}_{R,i}$, together with identifications at all points $x \in D \setminus R$. One can also check that

$$\deg(\nu_{R,*}L) = \deg(L) + |R|. \tag{3.16}$$

We now study in more detail the spectral curves parameterized by V^{nod} and their corresponding Hitchin fibers. Let us first fix some notation. Recall that, for $v \in V^{\mathrm{nod}}$ given by $(\alpha_1, \dots, \alpha_n)_{\mathfrak{S}_n}$ we denote the associated spectral curve by $\overline{X}_v$. After Lemmas 3.5 and 3.6, $\overline{X}_v = \bigcup_{i=1}^n X_i$, where $X_i = \alpha_i(X) \cong X$ and be the divisor of singularities D has length δ and it is given by the union of the two-by-two intersection of the smooth irreducible components. For any subdivisor $R \subset D$ consider the partial desingularization along R,

$$\begin{array}{ccc} \widetilde{X}_R & \xrightarrow{\nu_R} & \overline{X}_v \\ & \searrow^{p_R} & \downarrow \pi \\ & & X. \end{array} \tag{3.17}$$

Consider the decomposition $\widetilde{X}_R = \bigsqcup_{i=1}^{n_R} \widetilde{X}_{R,i}$ into connected components and denote as $\overline{X}_{R,i} = \nu_R(\widetilde{X}_{R,i})$. Therefore, one has the decomposition $R = R_1 \sqcup \cdots \sqcup R_{n_R} \sqcup R_s$ such that

$$\nu_{R,i} : \widetilde{X}_{R,i} \longrightarrow \overline{X}_{R,i}$$

is a partial desingularization of $\overline{X}_{R,i}$ along a non-separating divisor R_i, and R_s is the separating divisor in R (i.e., the divisor along which connected components are to appear after desingularization). Denote by $p_{R,i}$ the restriction of p_R to the corresponding connected component. For each irreducible component $X_j = \alpha_j(X) \cong X$ of $\overline{X}_v$, and its corresponding connected component $\widetilde{X}_j \cong X_j \cong X$ of the normalization $\widetilde{X}_v$, consider the commuting diagram

$$\begin{array}{ccccc} \widetilde{X}_{R,i} & & \xleftarrow{\quad\widetilde{\iota}_j\quad} & & \widetilde{X}_j \\ & \searrow^{\nu_{R,i}} & & & \downarrow \cong\ \nu^j_{R,i} \\ p_{R,i} \downarrow & & \overline{X}_{R,i} & \xleftarrow{\quad\iota_j\quad} & X_j \\ & \swarrow_{\pi} & & \nearrow_{\alpha_j} \cong & \\ X. & & & & \end{array} \tag{3.18}$$

We then see that $\widetilde{X}_j \cong X$ are the irreducible components of $\widetilde{X}_{R,i}$ and denote by C_i the index set of these components, hence $\widetilde{X}_{R,i} = \bigcup_{j \in C_i} \widetilde{X}_j$ has $|C_i|$ irreducible components. Write $\tilde{D}_i \subset \widetilde{X}_{R,i}$ for the singular divisor of $\widetilde{X}_{R,i}$ and observe that it coincides with the ramification divisor of $p_{R,i} : \widetilde{X}_{R,i} \longrightarrow X$. Observe as well that

$$D_i := \nu_{R,i}(\tilde{D}_i) = \sum_{j,k \in C_i} D_{jk} - R_i \subset D \tag{3.19}$$

and

$$D = \sum_i (D_i + R_i) + R_s. \tag{3.20}$$

We provide in the following lines a description of the Jacobians over $\widetilde{X}_R$. Choose an ordering $(\widetilde{X}_{R,1}, \dots, \widetilde{X}_{R,n_R})$ of the connected components of $\widetilde{X}_R$ and, with respect to it, denote

$$\mathrm{Jac}^{\overline{\eta}}(\widetilde{X}_R) \cong \mathrm{Jac}^{\eta_1}(\widetilde{X}_{R,1}) \times \cdots \times \mathrm{Jac}^{\eta_{n_R}}(\widetilde{X}_{R,n_R})$$

for each multidegree $\overline{\eta}$, and set $|\overline{\eta}| = \sum_{i=1}^{n_R} \eta_i$. Consider the decomposition

$$\mathrm{Jac}^{\eta}(\widetilde{X}_R) \cong \bigcup_{|\overline{\eta}|=\eta} \mathrm{Jac}^{\overline{\eta}}(\widetilde{X}_R). \tag{3.21}$$

Let also

$$\mathrm{Jac}^{\eta_i}(\widetilde{X}_{R,i}) = \bigcup_{\sum d_i^j = \eta_i} \mathrm{Jac}^{(d_i^1, \dots, d_i^{|C_i|})}(\widetilde{X}_{R,i}),$$

be the decomposition in terms of the multidegree associated with the irreducible components.

With the notation being settled, we now study push-forward of line bundles under ν_R. Recall that every rank one torsion-free sheaf on $\overline{X}_v$ is either of this form or a line bundle.

Lemma 3.10. *Let $v \in V^{\mathrm{nod}}$. Only if*

$$\eta_i = \sum_{k=1}^{|C_i|} d_i^k = |D_i|, \tag{3.22}$$

one has that the push-forward map

$$\begin{array}{rccc} \check{\nu}_R : & \mathrm{Jac}^{(d_1^1, \dots, d_1^{|C_1|})}(\widetilde{X}_{R,1}) \times \cdots \times \mathrm{Jac}^{(d_{n_R}^1, \dots, d_{n_R}^{|C_{n_R}|})}(\widetilde{X}_{R,n_R}) & \longrightarrow & \overline{\mathrm{Jac}}^{\delta}(\overline{X}_v) \\ & L & \longmapsto & \nu_{R,*}L, \end{array} \tag{3.23}$$

is well defined and an injection. Furthermore, when $R_s \neq \emptyset$, the Higgs bundles whose corresponding spectral data is in the image of $\check{\nu}_R$ are strictly polystable.

Proof. Assume first that $R_s = \emptyset$ hence $n_R = 1$ so $\widetilde{X}_R$ is connected. In that case, $\nu_{R,*}L$ is stable. Otherwise, as any destabilizing subsheaf of $\nu_{R,*}L$ will come from a destabilizing subsheaf of L and this would imply that L is unstable. But L is a line bundle so it is forcely stable. One also has that $\nu_{R,*}L \ncong \nu_{R,*}L'$ if $L \ncong L$ so it only remains to prove that the degree $\nu_{R,*}(L)$ is $\delta = |D|$. Note that this follows from Equations (3.16) and (3.20), since Equation (3.22) is equivalent to $\eta = |D_1|$ as $\widetilde{X}_R$ is connected.

Now, we study the case where $R_s \neq \emptyset$, so $\widetilde{X}_R$ has $n_R > 1$ connected components. Denote $\widetilde{\iota}_k^* L = L_k$, where the notation is as in Equation (3.18). Note that

$$\pi_* \nu_{R,*} L = p_{R,*} L = \bigoplus_{i=1}^{n_R} p_{R,i,*} L_i,$$

where the notation is as in Equation (3.18). Note that the direct sum is invariant by the Higgs field, since the Higgs field is equivalent to a $\pi_* \mathcal{O}_{\overline{X}_v}$ module structure on $\pi_* \nu_{R,*} L$, and the latter factors through a $\pi_* \nu_{R,*} \mathcal{O}_{\widetilde{X}_R}$-module structure. This proves that the Higgs bundle associated with L is decomposable. Note that, as before, $\nu_{R,i,*} L_i$ is stable as L_i is a line bundle,

hence stable. Therefore, it must happen that

$$\deg p_{R,i,*}L_i = \deg \pi_{i,*}\nu_{R,i,*}L_i = 0 \tag{3.24}$$

for the Higgs bundle to be polystable. Note that we have used $p_{R,i} = \pi_i \circ \nu_{R,i}$.

Given that $\overline{X}_{R,i}$ is a totally reducible nodal spectral curve with $|C_i|$ irreducible components, arguing as in Lemma 3.1 (compare with Equation (2.6)) we find that Equation (3.24) is equivalent to

$$\deg \nu_{R,i,*}L_i = (|C_i|^2 - |C_i|)(g-1) = \left| \sum_{j,k \in C_i} D_{jk} \right|.$$

Now, considering

$$0 \longrightarrow \nu_{R,i}^* \mathcal{O}_{\overline{X}_{R,i}} \longrightarrow \mathcal{O}_{\widetilde{X}_{R,i}} \longrightarrow \mathcal{O}_{R_i} \longrightarrow 0,$$

we have that

$$\left| \sum_{j,k \in C_i} D_{jk} \right| = \deg \nu_{R,i,*}L_i = \deg L_i + |R_i|,$$

which together with Equation (3.19) implies that Equation (3.24) is equivalent to Equation (3.22). In that case Equation (3.23) is well defined and it is injective since, as before, we have that $\nu_{R,i,*}L_i \ncong \nu_{R,i,*}L_i'$ whenever L_i and L_i' are not isomorphic. □

As a corollary of Lemma 3.10, one can derive the following well-known fact when $R = D$. Hence, after Lemma 3.5 the normalization $\widetilde{X}_\upsilon = \widetilde{X}_R$ of $\overline{X}$ decomposes into n connected components, each of them isomorphic to the base curve X.

Corollary 3.11. *The push-forward map*

$$\begin{array}{rccc} \check{\nu}: & \mathrm{Jac}^{(0,\dots,0)}(\widetilde{X}_\upsilon) & \longrightarrow & \overline{\mathrm{Jac}}^{\delta}(\overline{X}_\upsilon) \\ & L & \longmapsto & \nu_* L, \end{array} \tag{3.25}$$

is well defined and an injection. Furthermore, $\check{\nu}\left(\mathrm{Jac}^{(0,\dots,0)}(\widetilde{X}_\upsilon)\right)$ *classifies those strictly polystable Higgs bundles that decompose into direct sum of line Higgs bundles.*

In Proposition 3.7, we provided a description of the dense open subset of the Hitchin fiber over $\upsilon \in V^{\mathrm{nod}}$ corresponding to line bundles. Recalling that every torsion-free sheaf is given by the push-forward of a line bundle under a partial normalization ν_R, we complete in the following lines the description initiated in Proposition 3.7 of Higgs bundles lying over V^{nod}.

Proposition 3.12. *Take any* $\upsilon \in V^{\mathrm{nod}}$ *given by* $\upsilon = (\alpha_1, \dots, \alpha_n)_{\mathfrak{S}_n}$ *and suppose that the multidegree* $\overline{d}$ *satisfies Equation (3.22). One has the following,*

1. *Assume* $R_s \neq \emptyset$. *Then, the Higgs bundles corresponding to spectral data in* $\check{\nu}_R\left(\mathrm{Jac}^{\overline{d}}(\widetilde{X}_R)\right)$ *admit a reduction of their structure group to* $B_1 \times \cdots \times B_{n_R} \subset B$, *where* B_i *is the Borel subgroup of* $\mathrm{GL}(|C_i|, \mathbb{C})$.
2. *Consider the Higgs bundle* $(E, \varphi) = \bigoplus_{k=1}^{n_R}(E_k, \varphi_k)$ *in* $h^{-1}(\upsilon) \cap \check{\nu}_R\left(\mathrm{Jac}^{\overline{d}}(\widetilde{X}_R)\right)$. *Suppose that the spectral data of* (E, Φ) *are* $\nu_{R,*}L$, *where* L *is a line bundle over* $\widetilde{X}_R$. *Then, for any ordering* $J_k = (\alpha_{j_1}, \dots, \alpha_{j_{|C_k|}})$ *of* C_k, *one can chose canonically a filtration for* (E_k, φ_k), *for all* $k \in \{1, \dots, n_R\}$,

$$(E_{J_k})_\bullet : 0 \subsetneq (E_{k,1}, \varphi_1) \subsetneq \cdots \subsetneq (E_{k,|C_k|}, \varphi_{k,|C_k|}) = (E_k, \varphi_k)$$

such that

$$(E_{k,i},\varphi_{k,i})/(E_{k,i-1},\varphi_{k,i-1}) = \left(L|_{\widetilde{X}_{j_i}} \otimes \mathcal{O}\left(-\sum_{i'\geq i+1} \widetilde{X}_{j_i}\cap \widetilde{X}_{j_{i'}}\right), \alpha_{j_i}\right)$$

where we abuse notation by identifying the subdivisors $\widetilde{X}_{j_i}\cap\widetilde{X}_{j_{i'}} \subset D_{i'}$ *Equation (3.19) and their images under* $p_{i'}$*, and* $L|_{\widetilde{X}_{j_i}}$ *with its pullback under* $\alpha_{j_i}\circ(\nu_{R,k}^{j_i})^{-1}$.

Proof.

(1) Follows from Proposition 3.2 and Lemma 3.10.
(2) To simplify notation, take the orderings $((\alpha_1,\dots,\alpha_{|C_1|}),\dots(\alpha_{|C_{n_R-1}|},\dots,\alpha_n))$. The reasoning that follows adapts just the same way to any other choice of orderings. The statement is proven as Proposition 3.7, taking the following remarks into account:

First note that the subscheme $Z_i \subset \overline{X}_v$ appearing in the proof of Proposition 3.7 is the image of its partial desingularization $\widetilde{Z}_i \subset \widetilde{X}_R$, on which the filtration will be given on each of the connected components. This restricts the proof to line bundles over connected curves $\widetilde{X}_R$.

By the previous remark, we may assume that $\widetilde{X}_R$ is connected and J is an ordering for $\{\alpha_1,\dots,\alpha_n\}$. We obtain a full flag in the same way as in the proof of Proposition 3.7, the difference with this case being that the ideal

$$\mathcal{I}_{\widetilde{Z}_{i-1},\widetilde{Z}_i} \cong \mathcal{O}_{\widetilde{X}_i}(-\widetilde{X}_i\cap\widetilde{Z}_i)$$

depends on the ordering (and R) and so does

$$\widetilde{X}_i\cap\widetilde{Z}_i = \sum_{i'\geq i+1}\widetilde{X}_i\cap\widetilde{X}_{i'}.$$

□

4 | A (BBB)-BRANE FROM THE CARTAN SUBGROUP

In this section, we construct a (BBB)-brane of M_n, which is, by definition (cf. [41]), a pair $(N,(\mathbf{F},\nabla_{\mathbf{F}}))$ given by:

- A hyperholomorphic subvariety $N\subset M_n$, that is, a subvariety which is holomorphic with respect to the three complex structures Γ_1, Γ_2, and Γ_3.
- A hyperholomorphic sheaf $(\mathbf{F},\nabla_{\mathbf{F}})$ supported on N, that is, a sheaf $\mathbf{F}$ equipped with a connection whose curvature $\nabla_{\mathbf{F}}$ is of type (1,1) in the complex structures Γ_1, Γ_2, and Γ_3.

Remark 4.1. A flat connection is trivially of type (1,1) in any complex structure.

The embedding of the Cartan subgroup $\mathrm{C}\cong(\mathbb{C}^\times)^n$ into $\mathrm{GL}(n,\mathbb{C})$ induces the *Cartan locus* of the moduli space of semistable Higgs bundles

$$\mathrm{Car} = \left\{(E,\varphi)\in M_n \,\middle|\, \begin{array}{l}\exists\, s\in H^0(X,E/C),\\ \varphi\in H^0(X,E_s(\mathfrak{c})\otimes K).\end{array}\right\},$$

where $\mathbf{c}=\mathrm{Lie}(C)$ and E_s is the principal C-bundle on X constructed from the section s. Observe that Car is the image of the injective morphism

$$c:\mathrm{Sym}^n(M_1)\longrightarrow M_n,$$

which is hyperholomorphic, so Car is a hyperholomorphic subvariety.

Now, we address the construction of the hyperholomorphic sheaf on Car for any topologically trivial line bundle $\mathscr{L}\to X$. Since a flat bundle is hyperholomorphic and the morphism c is a hyperholomorphic morphism, it will suffice to construct a flat bundle on $\mathrm{Sym}^n(M_1)$ and take its direct image under c.

After fixing a point $x_0 \in X$ we get an embedding $X \hookrightarrow \mathrm{Jac}^0(X)$. Consider our initial line bundle $\mathscr{L} \to X$, and let $\nabla_{\mathscr{L}}$ be a flat connection on it. Denote by $(\check{\mathscr{L}}, \check{\nabla}_{\mathscr{L}})$ the unique flat line bundle in $\mathrm{Jac}^0(X)$ that restricts to $(\mathscr{L}, \nabla_{\mathscr{L}})$. From a flat line bundle on $\mathrm{Jac}^0(X)$ one can define a flat line bundle on $\mathrm{Sym}^n(\mathrm{Jac}^0(X))$ as we explain in the following lemma.

Lemma 4.2. *Let $(\check{\mathscr{L}}, \check{\nabla}_{\mathscr{L}})$ be a flat line bundle on $\mathrm{Jac}^0(X)$. Consider*

$$\pi_i : (\mathrm{Jac}^0(X))^{\times n} \to \mathrm{Jac}^0(X)$$

the projection onto the ith factor. Let

$$\check{\mathscr{L}}^{\boxtimes n} := \bigotimes_{i=1}^{n} \pi_i^* \check{\mathscr{L}}$$

and

$$\check{\nabla}_{\mathscr{L}}^{\boxtimes n} := \sum_{i=1}^{n} \pi_i^* \check{\nabla}_{\check{\mathscr{L}}} \otimes \bigotimes_{j \neq i} \mathbf{1}_{\pi_j^* \check{\mathscr{L}}}.$$

Then, $\left(\check{\mathscr{L}}^{\boxtimes n}, \check{\nabla}_{\mathscr{L}}^{\boxtimes n}\right)$ is a flat bundle that descends to a flat bundle $\left(\check{\mathscr{L}}^{(n)}, \check{\nabla}_{\mathscr{L}}^{(n)}\right)$ on $\mathrm{Sym}^n(\mathrm{Jac}^0(X))$.

Proof. The bundle $\check{\mathscr{L}}^{\boxtimes n}$ is invariant by the action of $\mathfrak{S}_n$ and moreover the natural linearization action derived from the one on the bundle $\bigoplus_{i=1}^n \check{\mathscr{L}}$ satisfies that over point $p \in (\mathrm{Jac}^0(X))^{\times n}$ with nontrivial centralizer $Z_p \subset \mathfrak{S}$, the centralizer Z_p acts trivially on $\check{\mathscr{L}}_p^{\boxtimes n}$. It follows from Kempf's descent lemma that $\check{\mathscr{L}}^{\boxtimes n}$ descends to a line bundle $\check{\mathscr{L}}^{(n)}$ on $\mathrm{Sym}^n(\mathrm{Jac}^0(X))$

$$\check{\mathscr{L}}^{(n)} := \left(q_* \check{\mathscr{L}}^{\boxtimes n}\right)^{\mathfrak{S}_n},$$

where q denotes the projection $\mathrm{Jac}^0(X)^{\times n} \to \mathrm{Sym}^n(\mathrm{Jac}^0(X))$.

Note that $\check{\nabla}_{\mathscr{L}}^{\boxtimes n}$ is flat since the $\pi_i^* \check{\nabla}_{\mathscr{L}}$ are flat and for any two $i \neq j$, one has that $\pi_i^* \check{\nabla}_{\mathscr{L}}$ and $\pi_j^* \check{\nabla}_{\mathscr{L}}$ commute. By equivariance with respect to the action of the symmetric group $\mathfrak{S}_n$, it descends to a flat connection $\check{\nabla}_{\mathscr{L}}^{(n)}$ on $\check{\mathscr{L}}^{(n)}$. □

Recall that the moduli space of topologically trivial rank 1 Higgs bundles fibers over the Jacobian, $\mathrm{M}_1 \longrightarrow \mathrm{Jac}^0(X)$. This fibration extends to the symmetric product

$$p : \mathrm{Sym}^n(\mathrm{M}_1) \longrightarrow \mathrm{Sym}^n\left(\mathrm{Jac}^0(X)\right).$$

Then, the flat line bundle $(\check{\mathscr{L}}^{(n)}, \check{\nabla}^{(n)})$ gives a flat line bundle $p^* \left(\check{\mathscr{L}}^{(n)}, \check{\nabla}_{\mathscr{L}}^{(n)}\right)$ on $\mathrm{Sym}^n(\mathrm{M}_1)$ and further a hyperholomorphic sheaf

$$(\mathbf{L}, \nabla_{\mathbf{L}}) = c_* p^* (\check{\mathscr{L}}^{(n)}, \check{\nabla}_{\mathscr{L}}^{(n)})$$

on the Cartan locus Car. Consider the pair

$$\mathbf{Car}(\mathscr{L}) := (\mathrm{Car}, (\mathbf{L}, \nabla_{\mathbf{L}})).$$

The above discussion implies the following.

Proposition 4.3. $\mathbf{Car}(\mathscr{L})$ *is a* (BBB)*-brane on* M_n*, which we call* Cartan (BBB)*-brane associated with the line bundle* $\mathscr{L} \to X$.

Note that the image of the Cartan locus under the Hitchin map coincides with the locus of totally reducible spectral curves,

$$h(\mathrm{Car}) \cong \mathrm{V}.$$

We finish this section with a description of the intersection of the Cartan locus with a generic Hitchin fiber associated with a nodal curve. Recall that the push-forward map $\check{\nu}$ is an injective morphism as we have seen in Lemma 3.10.

Proposition 4.4. *For any $v \in \mathrm{V}^{\mathrm{nod}}$, one has*

$$h^{-1}(v) \cap \mathrm{Car} = \check{\nu}\left(\mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)\right) \cong \mathrm{Jac}^{\overline{0}}(\widetilde{X}_v).$$

Consider the isomorphism

$$m : \mathrm{Jac}^{\overline{0}}(\widetilde{X}_v) \cong \mathrm{Jac}^0(X)^{\times n} \tag{4.1}$$

induced by the ordering $(X_1, X_2, \dots, X_n)$ of the connected components of $\widetilde{X}$. One has that under the isomorphism m:

1. *The spectral datum $L \in \check{\nu}\left(\mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)\right)$ corresponding to $\bigoplus_{i=1}^n (L_i, \alpha_i) \in \mathrm{Car}$ is taken to $(L_1, \dots, L_n) \in \mathrm{Jac}^{\overline{0}}(\widetilde{X})^{\times n}$. Namely, $L = \nu_* F = \bigoplus_j (\iota_j)_* L_j$ where ι_j is as in Equation (3.13) and $F \in \mathrm{Jac}(\widetilde{X})$ restricts to $F|_{X_j} = L_j$.*
2. *The restriction of $\mathbf{L} \to \mathrm{Car}$ to $h^{-1}(v) \cap \mathrm{Car}$ corresponds to $\check{\mathscr{L}}^{\boxtimes n} \to \mathrm{Jac}^0(X)^{\times n}$ defined in Lemma 4.2.*

Proof.

(1) By construction, a Higgs bundle in Car decomposes as a direct sum of line bundles,

$$(E, \varphi) \cong \bigoplus_{i=1}^n (L_i, \alpha_i).$$

After Corollary 3.11, $\check{\nu}(\mathrm{Jac}^{\overline{0}}) \subset h^{-1}(v) \cap \mathrm{Car}$. Now, let $L \in \overline{\mathrm{Jac}}^{\delta}(\overline{X}_v)$ be the spectral datum corresponding to and element $(E, \varphi) \in h^{-1}(v) \cap \mathrm{Car}$. It is easy to see that the Higgs bundle is totally decomposable if and only if its $\pi_* \mathcal{O}_{\overline{X}_v}$-module structure factors through a $\pi_* \nu_* \mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X^{\oplus n}$-module structure. Hence, $L = \nu_* F$ for some $F \in \mathrm{Jac}(\widetilde{X})$. Corollary 3.11 finishes the proof, as the only possible multidegree is $(0, \dots, 0)$.

(2) In order to prove the second statement, note that the isomorphism (4.1) is totally determined by a choice of an ordering of the connected components of $\widetilde{X}$, in this case $(X_1, \dots, X_n)$. Now, the choice of such an ordering induces an embedding $j : (\mathrm{Jac}^0(X))^{\times n} \hookrightarrow \mathrm{Sym}^n(\mathrm{Jac}^0(X))$ making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Jac}^0(X))^{\times n} & \xrightarrow{m} & h^{-1}(u) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ \mathrm{Sym}^n(\mathrm{M}_1) & \xrightarrow{c} & \mathrm{Car} \\ {\scriptstyle p}\downarrow & & \\ \mathrm{Sym}^n(\mathrm{Jac}^0(X)), & & \end{array} \qquad q = p \circ j$$

with $q = p \circ j$ being the usual quotient map. We need to check that

$$m^* i^* \mathbf{L} \cong \check{\mathscr{L}}^{\boxtimes n}.$$

But, since the above diagram commutes and c is an injection, the left-hand side is equal to $j^* c^* \mathbf{L} = j^* c^* c_* p^* \check{\mathscr{L}}^{(n)} \cong j^* p^* \check{\mathscr{L}}^{(n)} \cong q^* \check{\mathscr{L}}^{(n)}$ and the statement follows by the construction of $\check{\mathscr{L}}^{(n)}$. □

5 | (BAA)-BRANES FROM THE UNIPOTENT RADICAL OF THE BOREL SUBGROUP

Recall from Section 2.1 that M_n is a hyperkähler scheme with $((\Gamma_1,\omega_1),(\Gamma_2,\omega_2),(\Gamma_3,\omega_3))$ being its Kähler structures. After [41], a (BAA)-*brane* on M_n is a pair $(W,(\mathcal{G},\nabla_{\mathcal{G}}))$, with:

- W being a complex Lagrangian subvariety of M_n for the holomorphic symplectic form $\Omega_1=\omega_2+i\omega_3$.
- $(\mathcal{G},\nabla_{\mathcal{G}})$ being a flat bundle supported on W.

Starting from the line bundle $\mathscr{L}\to \mathrm{Jac}^0(X)$, we construct in this section a complex Lagrangian subvariety $\mathrm{Uni}(\mathscr{L})$ of the moduli space of Higgs bundles, mapping to the Cartan locus $V\subset H$ of the Hitchin base. As we have seen, $\mathrm{Uni}(\mathscr{L})$ is the support of a (BAA)-brane after specifying a flat vector bundle on it.

Recall that we have fixed a point $x_0\in X$. Denote by $\hat{\mathscr{L}}$ our topologically trivial line bundle $\mathscr{L}\to X$ tensored $\delta/n=(n-1)(g-1)$ times by $\mathcal{O}_X(x_0)$,

$$\hat{\mathscr{L}}:=\mathscr{L}\otimes\mathcal{O}_X(x_0)^{(n-1)(g-1)}. \tag{5.1}$$

Having in mind Proposition 3.2, we define the subvariety of $M_n\times_{\mathrm{H}}V$,

$$\mathrm{Uni}(\mathscr{L})=\left\{(E,\varphi)\in \mathrm{Bor}\;\middle|\;\begin{array}{l}\exists\,\sigma\in H^0(X,E/\mathrm{B}),\\ \varphi\in H^0(X,E_\sigma(\mathfrak{b})\otimes K),\\ E_{\mathrm{C}}:=E_\sigma/\mathrm{U}\cong(\hat{\mathscr{L}}\otimes K^{\otimes 1-n})\boxplus\cdots\boxplus(\hat{\mathscr{L}}\otimes K^{-1})\boxplus\hat{\mathscr{L}}.\end{array}\right\}. \tag{5.2}$$

Proposition 5.1. $\mathrm{Uni}(\mathscr{L})$ *is closed in* M_n.

Proof. Recall that we denoted by $\mathfrak{M}_n$ the moduli stack of rank n and degree 0 Higgs bundles and its semistable locus by $\mathfrak{M}_n^{\mathrm{sst}}\subset\mathfrak{M}_n$. Recall as well that Theorem 2.2 (see also the discussion following it) states that M_n is a good moduli space for $\mathfrak{M}_n^{\mathrm{sst}}$ and there is a morphism

$$\Psi:\mathfrak{M}_n^{\mathrm{sst}}\longrightarrow M_n$$

which induces the quotient topology.

Let us denote by $\mathfrak{Bor}$ the moduli stack of B-Higgs bundles, that is, the moduli stack classifying pairs $(E_{\mathrm{B}},\varphi_{\mathrm{B}})$, where E_{B} is a holomorphic B-bundle and φ_{B} is an element of $H^0(X,E_{\mathrm{B}}(\mathfrak{b})\otimes K)$. By extension of the structure group $\mathrm{B}\hookrightarrow\mathrm{GL}(n,\mathbb{C})$, one gets a morphism

$$\mathfrak{i}:\mathfrak{Bor}\to\mathfrak{M}_n.$$

Recalling Theorem 2.2, and the definition of Bor, we see that the restriction of $\mathfrak{i}(\mathfrak{Bor})$ to the semistable locus $\mathfrak{M}_n^{\mathrm{sst}}$ of $\mathfrak{M}_n$ surjects to Bor. Also, one can construct the following projection:

$$\begin{array}{rccc}\mathfrak{j}: & \mathfrak{Bor} & \longrightarrow & \mathrm{Jac}(X)^n\\ & (E_{\mathrm{B}},\varphi_{\mathrm{B}}) & \longmapsto & E_{\mathrm{C}}=E_{\mathrm{B}}/\mathrm{U}.\end{array}$$

Both $\mathfrak{i}$ and $\mathfrak{j}$ are algebraic morphisms hence smooth. Consider the substack of $\mathfrak{M}_n$ given by

$$\mathfrak{Uni}(\mathscr{L}):=\mathfrak{i}\big(\mathfrak{j}^{-1}((\hat{\mathscr{L}}\otimes K^{\otimes 1-n})\boxplus\cdots\boxplus(\hat{\mathscr{L}}\otimes K^{-1})\boxplus\hat{\mathscr{L}})\big).$$

Again, thanks to Theorem 2.2 and the construction of $\mathrm{Uni}(\mathscr{L})$, we have that the restriction to the semistable locus, $\mathfrak{Uni}(\mathscr{L})^{\mathrm{sst}}:=\mathfrak{Uni}(\mathscr{L})\cap\mathfrak{M}_n^{\mathrm{sst}}$, surjects to $\mathrm{Uni}(\mathscr{L})$. Note that $\mathfrak{j}^{-1}((\hat{\mathscr{L}}\otimes K^{\otimes 1-n})\boxplus\cdots\boxplus(\hat{\mathscr{L}}\otimes K^{-1})\boxplus\hat{\mathscr{L}})$ is a closed substack of $\mathfrak{Bor}$ as it is the preimage of a closed point, then $\mathfrak{Uni}(\mathscr{L})$ is closed inside $\mathfrak{i}(\mathfrak{Bor})$. We now observe that it is enough to prove that $\mathfrak{i}(\mathfrak{Bor})$ is closed in $\mathfrak{M}_n$ as this would imply that $\mathfrak{Uni}(\mathscr{L})$ is closed in $\mathfrak{M}_n$. Now, by Theorem 2.2 the previous discussion implies that $\mathfrak{Uni}(\mathscr{L})^{\mathrm{sst}}$ is closed inside $\mathfrak{M}_n^{\mathrm{sst}}$, and thus maps onto a closed subset, proving the statement.

Now, universal closedness of $\mathfrak{i}(\mathfrak{Bor})$ follows from the valuative criterion, as the image of $\mathfrak{Bor}$ has a universal bundle $(\mathfrak{E}, \mathbf{\Phi})$ admitting a reduction of the structure group to B. Given a discrete valuation ring R with fraction field k, properness of $\mathrm{GL}_n(\mathbb{C})/\mathrm{B}$ ensures that the existence of a reduction of the structure group over $\mathrm{Spec}(k)$ extends uniquely to $\mathrm{Spec}(R)$. This proves the valuative criterion for the bundle. Now, assume that the universal Higgs field defines a B-equivariant morphism

$$\phi : \mathfrak{E}_{\mathrm{B}}|_{\mathrm{Spec}(k)} \longrightarrow \mathfrak{b} \otimes K,$$

where $\mathfrak{E}_{\mathrm{B}}$ denotes the universal bundle together with a reduction to B. Since ϕ extends to $\phi' : \mathfrak{E}|_{\mathrm{Spec}(R)} \longrightarrow \mathfrak{gl}(n, \mathbb{C}) \otimes K$, closedness of $\mathfrak{b} \subset \mathfrak{gl}(n, \mathbb{C})$ étale local triviality of $\mathfrak{E}|_{\mathrm{Spec}(R)}$ do the rest. □

In order to prove that $\mathrm{Uni}(\mathscr{L})$ is an isotropic submanifold of (M_n, Ω_1) we first give a description of it in gauge theoretic terms. Let $\mathbb{E}$ denote the topologically trivial rank n vector bundle; choose a reduction of the structure group to B (which always exists), and let $\mathbb{E}_{\mathrm{B}}$ be the corresponding principal B-bundle, so that $\mathbb{E} \cong \mathbb{E}_{\mathrm{B}}(\mathrm{GL}(n, \mathbb{C}))$. Define $\mathbb{E}_{\mathrm{C}} = \mathbb{E}_{\mathrm{B}}/\mathrm{U}$. It follows from Equation (5.2) that

$$\mathrm{Uni}(\mathscr{L}) = \left\{ (\overline{\partial}_A, \varphi) \in \mathrm{M}_n \;\middle|\; \begin{array}{l} \exists g \in \mathscr{G} \text{ satisfying} \\ 1)\ g \cdot \overline{\partial} = \overline{\partial}_{\mathrm{C}} + N, \text{where} \\ \quad N \in \Omega^{0,1}(X, \mathbb{E}_{\mathrm{B}}(\mathfrak{n})), \\ \quad (\mathbb{E}_{\mathrm{C}}, \overline{\partial}_{\mathrm{C}}) = (\hat{\mathscr{L}} \otimes K^{\otimes 1-n}) \boxplus \cdots \boxplus (\hat{\mathscr{L}} \otimes K^{-1}) \boxplus \hat{\mathscr{L}}; \\ 2)\ g \cdot \varphi \in \Omega^0(X, \mathbb{E}_{\mathrm{B}}(\mathfrak{b}) \otimes K). \end{array} \right\}. \tag{5.3}$$

Remark 5.2. Both Car and $\mathrm{Uni}(\mathscr{L})$ are subvarieties of $M_n \times_{\mathrm{H}} V$, but they do not intersect, as the elements of $\mathrm{Car} \cap \mathrm{Uni}(\mathscr{L})$ would have underlying bundle of the form E_{C} in Equation (5.2), which is unstable, and totally decomposable Higgs field, conditions which yield unstable Higgs bundles.

Proposition 5.3. *The complex subvariety* $\mathrm{Uni}(\mathscr{L})$ *of* M_n *is isotropic with respect to the symplectic form* Ω_1 *defined in Equation (2.3).*

Proof. It is enough to prove the statement for open subset of stable points in $\mathrm{Uni}(\mathscr{L})$. We will check that this subset is non-empty in Proposition 5.7.

So let $(E, \varphi) \in \mathrm{Uni}(\mathscr{L})$ be a stable point. By Equation (5.3), a vector $(\dot{A}, \dot{\varphi}) \in T_{(E,\varphi)}\mathrm{M}_n$ satisfies that, up to the adjoint action of the gauge Lie algebra,

$$(\dot{A}, \dot{\varphi}) \in \Omega^{0,1}(X, \mathbb{E}_{\mathrm{B}}(\mathfrak{n})) \times \Omega^0(X, \mathbb{E}_{\mathrm{B}}(\mathfrak{b}) \otimes K).$$

The result follows from gauge invariance of the symplectic form Ω_1 and the fact that $\mathfrak{n} \subset \mathfrak{b}^{\perp}$, where orthogonality is taken with respect to the Killing form. □

We now give a description of the spectral data of the Higgs bundles corresponding to the points of $\mathrm{Uni}(\mathscr{L})$. We will focus on the open subset of those Higgs bundles whose spectral data are a line bundle. This will allow us to show that this subvariety is mid-dimensional, and, after Proposition 5.3, Lagrangian.

Proposition 5.4. *Let* $\hat{\mathscr{L}}$ *be defined as in Equation (5.1). For every* $v \in V^{\mathrm{nod}}$*, one has the following identification inside* $h^{-1}(v)$*,*

$$\mathrm{Uni}(\mathscr{L}) \cap \mathrm{Jac}^{\delta}(\overline{X}_v) = \left\{ L \in \mathrm{Jac}^{\delta}(\overline{X}_v) \text{ such that } \nu^* L = p^* \hat{\mathscr{L}} \cong \left(\hat{\mathscr{L}}, \ldots, \hat{\mathscr{L}}\right) \right\}. \tag{5.4}$$

Furthermore, Higgs bundles described in Equation (5.4) are stable.

Proof. Thanks to Proposition 3.7, we have that the spectral datum L of any $(E,\varphi) \in \mathrm{Uni}(\mathscr{L}) \cap \mathrm{Jac}^{\delta}(\overline{X}_{\upsilon})$ satisfies

$$\hat{\mathscr{L}} = \alpha_i^* \iota_i^* L.$$

Now, since any line bundle on $\widetilde{X}_{\upsilon}$ is totally determined by its restriction to all the connected components, it is enough to check that $j_i^* p^* \hat{\mathscr{L}} = j_i^* \nu^* L$, which follows from commutativity of the arrows in Equation (3.13) and the fact that $\alpha_i : X \to X_i$ is an isomorphism. This concludes the proof. □

The description of the spectral data given in Proposition 5.4 allows us to study the dimension of $\mathrm{Uni}(\mathscr{L})$, which turns up to be one half of $\dim \mathrm{M}_n$.

Proposition 5.5. *The complex subvariety* $\mathrm{Uni}(\mathscr{L})$ *of* M_n *has dimension*

$$\dim \mathrm{Uni}(\mathscr{L}) = n^2(g-1) + 1 = \frac{1}{2} \dim M_n.$$

Proof. First, we observe that $\mathrm{Uni}(\mathscr{L})$ is a fibration over V and recall that $\dim V = ng$. By Proposition 5.7, over the dense open subset $V^{\mathrm{nod}} \subset \mathrm{V}$, the fiber of $\mathrm{Uni}(\mathscr{L})|_{\mathrm{V}^{\mathrm{nod}}} \to V^{\mathrm{nod}}$ at υ has a dense open subset

$$\hat{\nu}^{-1}(\hat{\mathscr{L}}, \dots, \hat{\mathscr{L}}) \subset \overline{\mathrm{Jac}}^{\delta}(\overline{X}_{\upsilon}) \cong h^{-1}(\upsilon),$$

where we recall the pull-back map described in Equation (3.15). Now, by Corollary 3.9,

$$\hat{\nu}^{-1}(\hat{\mathscr{L}}, \dots, \hat{\mathscr{L}}) \cong (\mathbb{C}^{\times})^{\delta - n + 1}.$$

By smoothness of the point, the Hitchin fiber is transverse to the (local) Hitchin section, so

$$\begin{aligned} \dim \mathrm{Uni}(\mathscr{L})|_{\mathrm{V}^{\mathrm{nod}}} &= \dim V^{\mathrm{nod}} + \dim \hat{\nu}^{-1}\left(\hat{\mathscr{L}}, \dots, \hat{\mathscr{L}}\right) \\ &= ng + \delta - n + 1 \\ &= ng + (n^2 - n)(g-1) - n + 1 \\ &= n^2(g-1) + 1, \end{aligned}$$

which is half of the dimension of M_n, as we recall from Equation (2.2). This finishes the proof since by Proposition 5.3, $\mathrm{Uni}(\mathscr{L})$ is isotropic, so its dimension cannot be greater than $\frac{1}{2} \dim M_n$. □

Finally, we can state the main result of the section.

Theorem 5.6. *The complex subvariety* $\mathrm{Uni}(\mathscr{L})$ *of* M_n *is a closed complex Lagrangian with respect to* Ω_1.

Proof. This is clear after Propositions 5.1, 5.3, and 5.5. □

Thanks to Proposition 3.12, we have at hand a description of every point in the Hitchin fibers over V^{nod}. Hence, we can study the intersection of these fibers with $\mathrm{Uni}(\mathscr{L})$ as we will do in the remaining of the section. Before stating the result we need some extra definitions. Let $\upsilon = (\alpha_1, \dots, \alpha_n)_{\mathfrak{S}_n}$ in V^{nod} giving the spectral curve $\overline{X}_{\upsilon}$ with singular divisor $D \subset \overline{X}_{\upsilon}$, and let $R \subset D$ be a subdivisor. We have seen that $\overline{X}_{\upsilon}$ has n irreducible components $X_i = \alpha_i(X)$ and recall that we have set $D_{ij} = X_i \cap X_j$. For each ordering $J = (\alpha_{j_1}, \dots, \alpha_{j_n})$ of the set $\{\alpha_1, \dots, \alpha_n\}$, define the divisors

$$B_{J,i} := \sum_{i' \geq i+1} D_{j_i j_{i'}} \cap R. \tag{5.5}$$

Set also

$$b_{J,i} := |B_{J,i}|.$$

Proposition 5.7. *Let $\hat{\mathscr{L}}$ be defined as in Equation (5.1) and let $v \in \mathrm{V}^{\mathrm{nod}}$ with spectral curve $\overline{X}_v$ and divisor of singularities D. Chose $R \subset D$ and consider the associated desingularization $\widetilde{X}_R$ of $\overline{X}_v$. Then, for any n-tuple of integers $\overline{d} = (d_1, \dots, d_n)$, we have the following identifications inside $h^{-1}(v)$,*

$$\mathrm{Uni}(\mathscr{L}) \cap \check{\nu}_R\left(\mathrm{Jac}^{\overline{d}}(\widetilde{X}_R)\right) = \left\{ L \in \mathrm{Jac}^{\overline{d}}(\widetilde{X}_R) \;\middle|\; \begin{array}{l} \exists J = (\alpha_{j_1}, \dots, \alpha_{j_n}) \text{ ordering of } \{\alpha_1, \dots, \alpha_n\} \\ \text{such that, for all } 1 \le i \le n, \text{ we have:} \\ a)\, d_i = \delta - b_{J,i} \text{ and} \\ b)\, L|_{X_{j_i}} \cong \hat{\mathscr{L}} \otimes \mathcal{O}(B_{J,i}). \end{array} \right\}, \tag{5.6}$$

when $R_s = \emptyset$ and $\overline{d}$ satisfies b) for some ordering J, and

$$\mathrm{Uni}(\mathscr{L}) \cap \check{\nu}_R\left(\mathrm{Jac}^{\overline{d}}(\widetilde{X}_R)\right) = \emptyset,$$

in contrary case.

Proof. Recall the notation of Proposition 3.2. Take $(E, \varphi) \in h^{-1}(v)$ where $v \in V^{\mathrm{nod}}$ is given by $(\alpha_1, \dots, \alpha_n)_{\mathfrak{S}_n}$. Note that $(E, \varphi) \in \mathrm{Uni}(\mathscr{L})$ if and only there exists an ordering $J = (\alpha_{j_1}, \dots, \alpha_{j_n})$ and a filtration

$$0 = (E_0, \varphi_0) \subsetneq (E_1, \varphi_1) \subsetneq \cdots \subsetneq (E_n, \varphi_n) = (E, \varphi)$$

such that

$$(E_i, \varphi_i)/(E_{i-1}, \varphi_{i-1}) \cong (\hat{\mathscr{L}} \otimes K^{i-1}, \alpha_{j_i}).$$

The statement then follows from Proposition 3.12, noting that

$$\nu_R^*\left(K^{i-n} \otimes \mathcal{O}(-B_{J,i})\right) = \mathcal{O}\left(\sum_{i' \ge i+1} \widetilde{X}_{j_i} \cap \widetilde{X}_{j_{i'}}\right).$$

□

6 | DUALITY

In this section, we discuss about the duality under mirror symmetry of the (BBB)-brane $\mathbf{Car}(\mathscr{L})$, and a (BAA)-brane supported on $\mathrm{Uni}(\mathscr{L})$. Ideally, we would like to transform them under a Fourier–Mukai transform between coarse compactified Jacobians of reducible curves. Since such a tool is unavailable, we will make use of the integral functor Φ between the corresponding moduli stacks. Since the Cartan locus Car and the Jacobian $\mathrm{Jac}^{\delta}(\overline{X})$ are both fine moduli spaces, we will restrict the Poincaré sheaf $\overline{\mathfrak{P}}$ to Car on one side and $\mathrm{Jac}^{0}(\overline{X})$ on the other, obtaining an integral functor Φ^{Car} between their derived categories of sheaves. As we will see in this section, Φ^{Car} sends our (BBB)-brane $\mathbf{Car}(\mathscr{L})$ to the trivial sheaf supported on $\mathrm{Uni}(\mathscr{L})$ what provides evidence of a duality statement between them. A note of warning should be added here: ongoing work by Arinkin and Pantev [53] shows that the integral functor Φ on the stack of Higgs bundles over totally reducible spectral curves need not preserve semistability [53]. We do not see this phenomenon occurring here, as we pick the target of Φ^{Car} to be the Jacobian, although this should be taken into account when studying the transform of $\mathbf{Car}(\mathscr{L})$ under the whole integral functor Φ.

Recall that in our case, the normalization $\widetilde{X}_v$ is the disjoint union $\bigsqcup_i X_i$ of copies of the base curve X, which is smooth. Then, the direct product of Jacobians $\prod_i \mathrm{Jac}^0(X_i)$ is the moduli space classifying line bundles of multidegree $\overline{0}$, which is a fine moduli space with universal line bundle $\widetilde{\mathcal{U}}$. The restriction of each X_i is a line bundle over an irreducible smooth

curve, hence simple. It then follows that the associated moduli stack is

$$\mathfrak{Jac}^{\overline{0}}(\widetilde{X}_v) \cong \left[\mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)/(\mathbb{C}^*)^{\times n}\right] \cong \prod_{i=1}^{n}\left[\mathrm{Jac}^{\overline{0}}(X_i)/\mathbb{C}^*\right],$$

where each $\mathbb{C}^*$ acts trivially. Recall also that the restriction of the Cartan locus Car to the Hitchin fiber associated with $\overline{X}_v$ is $\check{\nu}\left(\mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)\right)$. Note that this is a fine moduli space with universal sheaf

$$\mathcal{U}^{\mathrm{Car}} := (\nu \times \check{\nu})_* \widetilde{\mathcal{U}} \longrightarrow \overline{X}_v \times \check{\nu}\left(\mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)\right).$$

We consider the substack $\check{\nu}\left(\mathfrak{Jac}^{\overline{0}}(\widetilde{X}_v)\right)$ of $\mathfrak{Jac}^{\delta}(\overline{X}_v)$. By all of the above, we have that

$$\check{\nu}\left(\mathfrak{Jac}^{\overline{0}}(\widetilde{X}_v)\right) \cong \left[\check{\nu}\left(\mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)\right)/(\mathbb{C}^*)^{\times n}\right],$$

and the restriction of the universal sheaf $\mathfrak{U}|_{\overline{X}_v \times \check{\nu}\left(\mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)\right)}$ pulls-back to $\mathcal{U}^{\mathrm{Car}}$ under the obvious projection

$$\check{\nu}\left(\mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)\right) \longrightarrow \left[\check{\nu}\left(\mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)\right)/(\mathbb{C}^*)^{\times n}\right]. \tag{6.1}$$

It follows from a result of Mumford (see, for instance, [12, Theorem 2, Section 8.2]) that the Jacobian of degree δ line bundles over a reduced curve $\overline{X}_v$ is a fine moduli space $\mathrm{Jac}^{\delta}(\overline{X}_v)$ with universal line bundle $\mathcal{U}^0 \to \overline{X}_v \times \mathrm{Jac}^{\delta}(\overline{X}_v)$. Since line bundles are simple, one has that the corresponding moduli stack is the quotient stack

$$\mathfrak{Jac}^{\delta}(\overline{X}_v) \cong \left[\mathrm{Jac}^{\delta}(\overline{X}_v)/\mathbb{C}^*\right],$$

for the trivial action of $\mathbb{C}^*$. One trivially has that $\mathcal{U}^0$ is the pull-back of $\mathfrak{U}^0$ under the projection

$$\mathrm{Jac}^{\delta}(\overline{X}_v) \longrightarrow \left[\mathrm{Jac}^{\delta}(\overline{X}_v)/\mathbb{C}^*\right]. \tag{6.2}$$

With $\mathcal{U}^0$ and $\mathcal{U}^{\mathrm{Car}}$ we already have all the ingredients for the following definition, analogous to Equation (2.12), of a Poincaré bundle over $\check{\nu}\left(\mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)\right) \times \mathrm{Jac}^{\delta}(\overline{X}_v)$,

$$\mathcal{P}^{\mathrm{Car}} := \mathcal{D}_{f_{23}}\left(f_{12}^*\mathcal{U}^{\mathrm{Car}} \otimes f_{13}^*\mathcal{U}^0\right)^{-1} \otimes \mathcal{D}_{f_{23}}\left(f_{13}^*\mathcal{U}^0\right) \otimes \mathcal{D}_{f_{23}}\left(f_{12}^*\mathcal{U}^{\mathrm{Car}}\right), \tag{6.3}$$

where the f_{ij} are the corresponding projections from $\overline{X}_v \times \check{\nu}\left(\mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)\right) \times \mathrm{Jac}^{\delta}(\overline{X}_v)$ to the product of the ith and jth factors.

We can see that $\mathcal{P}^{\mathrm{Car}}$ is obtained from the restriction of the Poincaré sheaf $\overline{\mathfrak{P}}$ to the Cartan locus and the Jacobian of $\overline{X}_v$.

Proposition 6.1. *The sheaf $\mathcal{P}^{\mathrm{Car}}$ is the pull-back of $\overline{\mathfrak{P}}|_{\check{\nu}\left(\mathfrak{Jac}^{\overline{0}}(\widetilde{X}_v)\right)\times\mathfrak{Jac}^{\delta}(\overline{X})}$ under the product of morphisms (6.1) and (6.2).*

Proof. Since $\overline{\mathfrak{P}}$ extends $\mathfrak{P} \to \overline{\mathfrak{Jac}}^{\delta}(\overline{X}_v) \times \mathfrak{Jac}^{\delta}(\overline{X}_v)$, we have from Equation (2.12) that

$$\begin{aligned}\overline{\mathfrak{P}}|_{\check{\nu}\left(\mathfrak{Jac}^{\overline{0}}(\widetilde{X}_v)\right)\times\mathfrak{Jac}^{\delta}(\overline{X})} &\cong \mathfrak{P}|_{\check{\nu}\left(\mathfrak{Jac}^{\overline{0}}(\widetilde{X}_v)\right)\times\mathfrak{Jac}^{\delta}(\overline{X})} \\ &\cong \mathscr{D}_{f_{23}}\left(f_{12}^{*}\mathfrak{U}|_{\overline{X}_v\times\check{\nu}\left(\mathfrak{Jac}^{\overline{0}}(\widetilde{X}_v)\right)} \otimes f_{13}^{*}\mathfrak{U}^{0}\right)\otimes \mathscr{D}_{f_{23}}\left(f_{13}^{*}\mathfrak{U}^{0}\right)^{-1} \\ &\otimes \mathscr{D}_{f_{23}}\left(f_{12}^{*}\mathfrak{U}|_{\overline{X}_v\times\check{\nu}\left(\mathfrak{Jac}^{\overline{0}}(\widetilde{X}_v)\right)}\right)^{-1}.\end{aligned}$$

Then, the result follows from the observation that $\mathscr{U}^{\mathrm{Car}}$ is the pull-back of $\check{\nu}\left(\mathfrak{Jac}^{\overline{0}}(\widetilde{X}_v)\right)$ under Equation (6.1), and $\mathscr{U}^{0}$ is the pull-back of $\mathfrak{U}^{0}$ under Equation (6.2). □

Let us consider the integral functor associated with $\mathscr{P}^{\mathrm{Car}}$,

$$\begin{array}{rccc}\Phi^{\mathrm{Car}}: & D^{b}\left(\check{\nu}\left(\mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)\right)\right) & \longrightarrow & D^{b}\left(\mathrm{Jac}^{\delta}(\overline{X}_v)\right) \\ & \mathscr{E}^{\bullet} & \longmapsto & R\pi_{2,*}(\pi_1^{*}\mathscr{E}^{\bullet}\otimes\mathscr{P}^{\mathrm{Car}}),\end{array} \tag{6.4}$$

where π_1 and π_2 to be, respectively, the projection from $\check{\nu}\left(\mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)\right)\times \mathrm{Jac}^{\delta}(\overline{X}_v)$ to the first and second factors.

Recall that our (BBB)-brane $\mathbf{Car}(\mathscr{L})$ is given by the hyperholomorphic bundle $\mathbf{L}$ supported on Car. By Proposition 4.4, over the dense open subset V^{nod} of the Cartan locus of the Hitchin base $V = h(\mathrm{Car}) \subset H$, the hyperholomorphic sheaf $\mathbf{L}$ restricted to a certain Hitchin fiber $\overline{\mathrm{Jac}}^{\delta}(\overline{X}_v)$ is $\check{\nu}_{*}\check{\mathscr{L}}^{\boxtimes n}$, supported on $\check{\nu}(\mathrm{Jac}^{\overline{\delta}}(\widetilde{X}_v))$. The main result of this section is the study of the behavior of $\check{\nu}_{*}\check{\mathscr{L}}^{\boxtimes n}$ under φ^{Car}, but first we need some technical results.

Fix x_0 and take the line bundle $\mathscr{O}(x_0)^{(n-1)(g-1)}$. Denote

$$\tau : \mathrm{Jac}^{\overline{0}}(\widetilde{X}) \xrightarrow{\cong} \mathrm{Jac}^{\overline{\delta}}(\widetilde{X})$$

the isomorphism given, on each of the components, by tensorization by the previous line bundle. We can define a Poincaré bundle $\widetilde{\mathscr{P}} \to \mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)\times \mathrm{Jac}^{\overline{\delta}}(\widetilde{X}_v)$.

Consider the projections to the first and second factors

$$\begin{array}{ccccc} & & \mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)\times \mathrm{Jac}^{\overline{\delta}}(\widetilde{X}_v) & & \\ & \swarrow^{\widetilde{\pi}_1} & & \searrow^{\widetilde{\pi}_2} & \\ \mathrm{Jac}^{\overline{0}}(\widetilde{X}_v) & & & & \mathrm{Jac}^{\overline{\delta}}(\widetilde{X}_v),\end{array}$$

and, using $\widetilde{\mathscr{P}}$, one can construct another Fourier–Mukai integral functor

$$\begin{array}{rccc}\widetilde{\Phi}: & D^{b}(\mathrm{Jac}^{\overline{0}}(\widetilde{X}_v)) & \longrightarrow & D^{b}(\mathrm{Jac}^{\overline{\delta}}(\widetilde{X}_v)) \\ & \mathscr{E}^{\bullet} & \longmapsto & R\widetilde{\pi}_{2,*}(\widetilde{\pi}_1^{*}\mathscr{E}^{\bullet}\otimes\widetilde{\mathscr{P}}).\end{array}$$

Note that $\widetilde{\Phi}$ is governed by the usual Fourier–Mukai transform on each of the $\mathrm{Jac}^{0}(X_i)$. We need the following lemma in order to describe the interplay between Φ^{Car} and $\widetilde{\Phi}$.

Lemma 6.2. *One has that*

$$(\check{\nu}\times \mathbf{1}_{\mathrm{Jac}})^{*}\mathscr{P}^{\mathrm{Car}} \cong (\mathbf{1}_{\widetilde{\mathrm{Jac}}}\times \hat{\nu})^{*}\widetilde{\mathscr{P}}.$$

Proof. Note that $(\check{\nu}\times \mathbf{1}_{\mathrm{Jac}})^{*}\,\mathscr{P}^{\mathrm{Car}}$ is a family of line bundles over $\mathrm{Jac}^{\overline{0}}(\widetilde{X})$ parameterized by $\mathrm{Jac}^{\delta}(\overline{X}_{\upsilon})$. Since $\widetilde{\mathscr{P}} \to \mathrm{Jac}^{\overline{0}}(\widetilde{X}_{\upsilon})\times \mathrm{Jac}^{\overline{0}}(\widetilde{X}_{\upsilon})$ is a universal family for these objects, there exists a map

$$t : \mathrm{Jac}^{\delta}(\overline{X}_{\upsilon}) \longrightarrow \mathrm{Jac}^{\overline{0}}(\widetilde{X}_{\upsilon}),$$

such that

$$(\check{\nu}\times \mathbf{1}_{\mathrm{Jac}})^{*}\mathscr{P}^{\mathrm{Car}} \cong (\mathbf{1}_{\widetilde{\mathrm{Jac}}}\times t)^{*}\widetilde{\mathscr{P}}.$$

Recall the description of $\mathscr{P}_{J}$ given in Equation (2.13) for each $J \in \mathrm{Jac}^{\delta}(\overline{X}_{\upsilon})$. Recall as well the projections $f_1 : \overline{X}_{\upsilon}\times \overline{\mathrm{Jac}}^{\delta}(\overline{X}_{\upsilon}) \to \overline{X}_{\upsilon}$ and $f_2 : \overline{X}_{\upsilon}\times \overline{\mathrm{Jac}}^{\delta}(\overline{X}_{\upsilon}) \to \overline{\mathrm{Jac}}^{\delta}(\overline{X}_{\upsilon})$, and consider the following commuting Cartesian diagram

$$\begin{array}{ccc} \overline{X}_{\upsilon}\times \mathrm{Jac}^{\overline{0}}(\widetilde{X}_{\upsilon}) & \xrightarrow{\mathbf{1}_{\widetilde{X}}\times\check{\nu}} & \overline{X}_{\upsilon}\times \overline{\mathrm{Jac}}^{\delta}(\overline{X}_{\upsilon}) \\ \downarrow{\scriptstyle f_2'} & & \downarrow{\scriptstyle f_2} \\ \mathrm{Jac}^{\overline{0}}(\widetilde{X}_{\upsilon}) & \xrightarrow{\check{\nu}} & \overline{\mathrm{Jac}}^{\delta}(\overline{X}_{\upsilon}) \end{array}$$

We know from [22, Proposition 44 (1)] that the determinant of cohomology commutes with base change, that is,

$$\check{\nu}^{*}\mathscr{D}_{f_2} = \mathscr{D}_{f_2'}(\mathbf{1}_{\overline{X}}\times \check{\nu})^{*}. \tag{6.5}$$

Consider the obvious projection $\widetilde{f}_2 : X_{\gamma}\times \mathrm{Jac}^{\overline{0}}(\widetilde{X}_{\upsilon}) \to \mathrm{Jac}^{\overline{0}}(\widetilde{X}_{\upsilon})$. Since the following diagram commutes,

$$\begin{array}{ccc} \widetilde{X}_{\upsilon}\times \mathrm{Jac}^{\overline{0}}(\widetilde{X}_{\upsilon}) & \xrightarrow{\nu\times\mathbf{1}_{\widetilde{\mathrm{Jac}}}} & \overline{X}_{\upsilon}\times \mathrm{Jac}^{\overline{0}}(\widetilde{X}_{\upsilon}) \\ & \searrow^{\widetilde{f}_2} & \downarrow{\scriptstyle f_2'} \\ & & \mathrm{Jac}^{\overline{0}}(\widetilde{X}_{\upsilon}), \end{array}$$

the definition of the determinant of cohomology ensures that

$$\mathscr{D}_{f_2'}(\nu\times \mathbf{1}_{\widetilde{\mathrm{Jac}}})_{*} \cong \mathscr{D}_{\widetilde{f}_2}. \tag{6.6}$$

One also has that the following diagrams commute:

$$\begin{array}{ccc} \overline{X}_{\upsilon}\times \mathrm{Jac}^{0}(X_{\gamma}) & \xrightarrow{\mathbf{1}_{\overline{X}}\times\check{\nu}} & \overline{X}_{\upsilon}\times \overline{\mathrm{Jac}}^{\delta}(\overline{X}_{\upsilon}) \\ & \searrow^{f_1'} & \downarrow{\scriptstyle f_1} \\ & & \overline{X}_{\upsilon}, \end{array} \tag{6.7}$$

and

$$\begin{array}{ccccc} & & \widetilde{X}_{\upsilon}\times \mathrm{Jac}^{\overline{0}}(\widetilde{X}_{\upsilon}) & & \\ & \swarrow^{\nu\circ\widetilde{f}_1} & \downarrow{\scriptstyle (\nu\times\mathbf{1}_{\widetilde{\mathrm{Jac}}})} & \searrow^{\widetilde{f}_2} & \\ \overline{X}_{\upsilon} & \xleftarrow{f_1'} & \overline{X}_{\upsilon}\times \mathrm{Jac}^{\overline{0}}(\widetilde{X}_{\upsilon}) & \xrightarrow{f_2'} & \mathrm{Jac}^{\overline{0}}(\widetilde{X}_{\upsilon}). \end{array}$$

As a consequence, one has that $f_1'((f_2')^{-1}(U)) = \nu\widetilde{f}_1(\widetilde{f}_2^{-1}(U))$ for every open subset $U \subset \mathrm{Jac}^{\overline{0}}(\widetilde{X}_\upsilon)$. It then follows from the definition of pull-back and push-forward that, for any $J \in \mathrm{Jac}^{\delta}(\overline{X}_\upsilon)$,

$$\begin{aligned}(f_2')_*(f_1')^*J(U) &= \lim_{W \supseteq f_1'((f_2')^{-1}(U))} J(W)\\ &= \lim_{W \supseteq f_1'((f_2')^{-1}(U))} J(W)\\ &= (\widetilde{f}_2)_*(\nu\circ\widetilde{f}^1)^*J(U),\end{aligned}$$

so $(f_2')_*(f_1')^* = (\widetilde{f}_2)_*(\nu\circ\widetilde{f}_1)^*$ and therefore,

$$\mathscr{D}_{f_2'}(f_1')^* \cong \mathscr{D}_{\widetilde{f}_2}\widetilde{f}_1^*\nu^*. \tag{6.8}$$

Recalling the definition of $\mathscr{U}^{\mathrm{Car}}$ as $(\nu\times\check{\nu})_*\widetilde{\mathscr{U}}$, we observe that

$$(\mathbf{1}_{\overline{X}}\times\check{\nu})^*\mathscr{U}^{\mathrm{Car}} \cong (\nu\times\mathbf{1}_{\widetilde{\mathrm{Jac}}})_*\widetilde{\mathscr{U}}. \tag{6.9}$$

Using the projection formula and Equations (6.5)–(6.9), we have that, for any $J \in \mathrm{Jac}^{\delta}(\overline{X}_\upsilon)$,

$$\begin{aligned}\widetilde{\mathscr{P}}_{t(J)} &\cong \check{\nu}^*\mathscr{P}_J^{\mathrm{Car}}\\ &\cong \check{\nu}^*\Big(\mathscr{D}_{f_2}\big(\mathscr{U}^{\mathrm{Car}}\otimes f_1^*J\big)^{-1}\otimes\mathscr{D}_{f_2}(f_1^*J)\otimes\mathscr{D}_{f_2}\big(\mathscr{U}^{\mathrm{Car}}\big)\Big)\\ &\cong \check{\nu}^*\mathscr{D}_{f_2}\big(\mathscr{U}^{\mathrm{Car}}\otimes f_1^*J\big)^{-1}\otimes\check{\nu}^*\mathscr{D}_{f_2}(f_1^*J)\otimes\check{\nu}^*\mathscr{D}_{f_2}\big(\mathscr{U}^{\mathrm{Car}}\big)\\ &\cong \mathscr{D}_{f_2'}\big((\mathbf{1}_{\overline{X}}\times\check{\nu})^*\big(\mathscr{U}^{\mathrm{Car}}\otimes f_1^*J\big)\big)^{-1}\otimes\mathscr{D}_{f_2'}\big((\mathbf{1}_{\overline{X}}\times\check{\nu})^*(f_1^*J)\big)\otimes\mathscr{D}_{f_2'}\big((\mathbf{1}_{\overline{X}}\times\check{\nu})^*\mathscr{U}^{\mathrm{Car}}\big)\\ &\cong \mathscr{D}_{f_2'}\big((\mathbf{1}_{\overline{X}}\times\check{\nu})^*\mathscr{U}^{\mathrm{Car}}\otimes(f_1')^*J\big)^{-1}\otimes\mathscr{D}_{f_2'}\big((f_1')^*J)\big)\otimes\mathscr{D}_{f_2'}\big((\mathbf{1}_{\overline{X}}\times\check{\nu})^*\mathscr{U}^{\mathrm{Car}}\big)\\ &\cong \mathscr{D}_{f_2'}\big((\nu\times\mathbf{1}_{\widetilde{\mathrm{Jac}}})_*\widetilde{\mathscr{U}}\otimes(f_1')^*J\big)^{-1}\otimes\mathscr{D}_{f_2'}\big((f_1')^*J)\big)\otimes\mathscr{D}_{f_2'}\big((\nu\times\mathbf{1}_{\widetilde{\mathrm{Jac}}})_*\widetilde{\mathscr{U}}\big)\\ &\cong \mathscr{D}_{f_2'}\big((\nu\times\mathbf{1}_{\widetilde{\mathrm{Jac}}})_*\big(\widetilde{\mathscr{U}}\otimes\widetilde{f}_1^*\nu^*J\big)\big)^{-1}\otimes\mathscr{D}_{f_2'}\big((f_1')^*J)\big)\otimes\mathscr{D}_{f_2'}\big((\nu\times\mathbf{1}_{\widetilde{\mathrm{Jac}}})_*\widetilde{\mathscr{U}}\big)\\ &\cong \mathscr{D}_{\widetilde{f}_2}(\widetilde{\mathscr{U}}\otimes\widetilde{f}_1^*\nu^*J)^{-1}\otimes\mathscr{D}_{\widetilde{f}_2}\big(\widetilde{f}_1^*\nu^*J\big)\otimes\mathscr{D}_{\widetilde{f}_2}(\widetilde{\mathscr{U}})\\ &\cong \widetilde{\mathscr{P}}_{\nu^*J}\\ &\cong \widetilde{\mathscr{P}}_{\hat{\nu}(J)}.\end{aligned}$$

This implies that $t = \hat{\nu}$, thus completing the proof. □

We can now study the image of $\check{\nu}_*(\check{\mathscr{L}}^{\boxtimes n})$ under Equation (6.4).

Proposition 6.3. *One has the isomorphism*

$$\Phi^{\mathrm{Car}}(\check{\nu}_*(\check{\mathscr{L}}^{\boxtimes n})) \cong \hat{\nu}^*\widetilde{\Phi}(\check{\mathscr{L}}^{\boxtimes n}),$$

and furthermore, $\hat{\nu}^*\widetilde{\Phi}(\check{\mathscr{L}}^{\boxtimes n})$ is a complex supported on degree g given by $\hat{\nu}^*\mathscr{O}_{(\check{\mathscr{L}}^{\boxtimes n})}$.

Proof. Let us also consider the following maps:

$$\begin{array}{ccc} & \overline{\mathrm{Jac}^0}(\widetilde{X}_v) \times \mathrm{Jac}^\delta(\overline{X}_v) & \\ {\scriptstyle \pi_1'}\swarrow & & \searrow{\scriptstyle \pi_2'} \\ \overline{\mathrm{Jac}^0}(\widetilde{X}_v) & & \mathrm{Jac}^\delta(\overline{X}_v), \end{array}$$

and observe that

- $\pi_2' = \pi_2 \circ (\check{\nu} \times \mathbf{1}_{\mathrm{Jac}})$,
- $\pi_1' = \widetilde{\pi}_1 \circ (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \hat{\nu})$,
- $\pi_1 \circ (\check{\nu} \times \mathbf{1}_{\mathrm{Jac}}) = \check{\nu} \circ \pi_1'$, and
- $\widetilde{\pi}_2 \circ (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \hat{\nu}) = \hat{\nu} \circ \pi_2'$.

Recalling Lemma 6.2, that $\check{\nu}$ is an injection and that $\hat{\nu}$ is flat by Lemma 3.8, one has the following,

$$\begin{aligned} \Phi^{\mathrm{Car}}(\check{\nu}_*(\check{\mathscr{L}}^{\boxtimes n})) =& R\pi_{2,*}\left(\pi_1^*\check{\nu}_*(\check{\mathscr{L}}^{\boxtimes n}) \otimes \mathscr{P}^{\mathrm{Car}}\right) \\ \cong& R\pi_{2,*}\left(R(\check{\nu} \times \mathbf{1}_{\mathrm{Jac}})_*(\pi_1')^*(\check{\mathscr{L}}^{\boxtimes n}) \otimes \mathscr{P}^{\mathrm{Car}}\right) \\ \cong& R\pi_{2,*}R(\check{\nu} \times \mathbf{1}_{\mathrm{Jac}})_*\left((\pi_1')^*(\check{\mathscr{L}}^{\boxtimes n}) \otimes (\check{\nu} \times \mathbf{1}_{\mathrm{Jac}})^*\mathscr{P}^{\mathrm{Car}}\right) \\ \cong& R\pi_{2,*}R(\check{\nu} \times \mathbf{1}_{\mathrm{Jac}})_*\left((\pi_1')^*(\check{\mathscr{L}}^{\boxtimes n}) \otimes (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \hat{\nu})^*\widetilde{\mathscr{P}}\right) \\ \cong& R\pi_{2,*}'\left((\pi_1')^*(\check{\mathscr{L}}^{\boxtimes n}) \otimes (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \hat{\nu})^*\widetilde{\mathscr{P}}\right) \\ \cong& R\pi_{2,*}'\left((\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \hat{\nu})^*\widetilde{\pi}_1^*(\check{\mathscr{L}}^{\boxtimes n}) \otimes (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \hat{\nu})^*\widetilde{\mathscr{P}}\right) \\ \cong& R\pi_{2,*}'(\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \hat{\nu})^*\left(\widetilde{\pi}_1^*(\check{\mathscr{L}}^{\boxtimes n}) \otimes \widetilde{\mathscr{P}}\right) \\ \cong& \hat{\nu}^* R\widetilde{\pi}_{2,*}\left(\widetilde{\pi}_1^*(\check{\mathscr{L}}^{\boxtimes n}) \otimes \widetilde{\mathscr{P}}\right) \\ \cong& \hat{\nu}^*\widetilde{\Phi}(\check{\mathscr{L}}^{\boxtimes n}). \end{aligned}$$

Finally, recalling that the usual Fourier–Mukai transform on $\mathrm{Jac}^0(X) \times \mathrm{Jac}^{\delta/n}(X)$ sends the line bundle $\check{\mathscr{L}}$ to the (complex supported on degree g given by) sky-scraper sheaf $\mathcal{O}_{\hat{\mathscr{L}}}$, we have that $\Phi^{\mathrm{Car}}(\check{\nu}_*\check{\mathscr{L}}^{\boxtimes n})$ is (the complex supported on degree g given by)

$$\hat{\nu}^*\widetilde{\Phi}(\check{\mathscr{L}}^{\boxtimes n}) \cong \hat{\nu}^*\mathcal{O}_{(\hat{\mathscr{L}}^{\boxtimes n})},$$

and the proof is complete. □

Recalling Proposition 5.7, we arrive to the main result of the section, which shows that our (BBB)-brane $\mathbf{Car}(\mathscr{L})$ and our (BAA)-brane $\mathbf{Uni}(\mathscr{L})$ are related under the Fourier–Mukai integral functor Φ^{Car}.

Corollary 6.4. *For every $v \in V^{\mathrm{nod}}$, the support of the image under Φ^{Car} of the* (BBB)*-brane* $\mathbf{Car}(\mathscr{L})$ *restricted to a Hitchin fiber $h^{-1}(v)$, is the support of our* (BAA)*-brane* $\mathbf{Uni}(\mathscr{L})$ *restricted to the open subset of the (dual) Hitchin fiber given by the locus of invertible sheaves,*

$$\mathrm{supp}\left(\Phi^{\mathrm{Car}}\left(\check{\nu}_*(\check{\mathscr{L}}^{\boxtimes n})\right)\right) = \mathrm{Uni}(\mathscr{L}) \cap \mathrm{Jac}^\delta(\overline{X}_v).$$

Remark 6.5. Corollary 6.4 points at a duality between **Car**($\mathscr{L}$) and **Uni**($\mathscr{L}$). The piece of work [24] has provided evidence for this fact via a Fourier–Mukai transform. Indeed, when X is an unramified cover of a smooth curve Y, there exist submanifolds of **Car**($\mathscr{L}$) and (unions of) **Uni**($\mathscr{L}$) covering two Fourier–Mukai dual branes on the moduli space of Higgs bundles on Y.

7 | PARABOLIC SUBGROUPS AND BRANES ON THE SINGULAR LOCUS

Cartan branes are the simplest example of branes supported on the singular locus M_n^{sing} of the moduli space of Higgs bundles. In this section, we first study the other hyperholomorphic subvarieties covering the singular locus, and, in second place, we construct Lagrangian subvarieties paired to them.

7.1 | Levi subgroups and the singular locus

Consider the n-tuple of positive integers

$$\bar{r} = \left(r_1, \overset{m_1}{\ldots}, r_1, \ldots, r_s, \overset{m_s}{\ldots}, r_s\right)$$

where $0 < r_1 < \cdots < r_s$ and set $|\bar{r}| = \sum_{\ell=1}^{s} m_\ell r_\ell$ and $m_{\bar{r}} = \sum_{\ell=1}^{s} m_\ell$. Any maximal rank reductive subgroup of $\mathrm{GL}(n,\mathbb{C})$ is conjugate to

$$\mathrm{L}_{\bar{r}} := \mathrm{GL}(r_1,\mathbb{C}) \times \overset{m_1}{\ldots} \times \mathrm{GL}(r_1,\mathbb{C}) \times \cdots \times \mathrm{GL}(r_s,\mathbb{C}) \times \overset{m_s}{\ldots} \times \mathrm{GL}(r_s,\mathbb{C}),$$

where $|\bar{r}| = n$. Denote by $M_{\bar{r}} \subset \mathrm{M}_n$ the image of the moduli space $\mathrm{M}_{\mathrm{L}_{\bar{r}}}$ of $\mathrm{L}_{\bar{r}}$-Higgs bundles. Note that $M_{\bar{r}}$ is the image of the injective morphism,

$$c_{\bar{r}} : \mathrm{Sym}^{m_1}(M_{r_1}) \times \ldots \cdots \times \mathrm{Sym}^{m_s}(M_{r_s}) \longrightarrow M_n.$$

Remark 7.1. In particular, $\mathrm{Car} = M_{(1,\overset{n}{\ldots},1)}$ for $\bar{r} = (1, \overset{n}{\ldots}, 1)$.

The same arguments as in the case of Cartan subgroups show that this is a complex subscheme in all three complex structures of M_n.

Proposition 7.2. *Fix $\bar{r}$ with $|\bar{r}| = n$, and consider $M_{\bar{r}} \subset M_n$. This subvariety is complex in all three complex structures $\Gamma_1, \Gamma_2, \Gamma_3$, and therefore hyperholomorphic.*

The union of these subvarieties covers the singular locus of the moduli space of Higgs bundles.

Proposition 7.3 [60], Section 11. *The singular locus is the locus of strictly polystable bundles,*

$$M_n^{\mathrm{sing}} = \bigcup_{|\bar{r}|=n} M_{\bar{r}}.$$

Denote

$$H_{\bar{r}} := \mathrm{Sym}^{m_1}(H_{r_1}) \times \cdots \times \mathrm{Sym}^{m_s}(H_{r_s})$$

and, relating the invariant polynomials of $\mathrm{L}_{\bar{r}}$ with those of $\mathrm{GL}(n,\mathbb{C})$, construct an injective morphism

$$H_{\bar{r}} \longrightarrow H_n.$$

Note that the image $h(M_{\bar{r}})$ under the Hitchin map of $M_{\bar{r}}$ coincides with the image of $H_{\bar{r}}$ under this morphism. Write H_r^{sm} for the locus of smooth spectral curves in the Hitchin base and set

$$V_{\bar{r}} := \mathrm{Sym}^{m_1}(H_{r_1}^{\mathrm{sm}}) \times \cdots \times \mathrm{Sym}^{m_s}(H_{r_s}^{\mathrm{sm}}).$$

Every point $\beta \in V_{\overline{r}}$ is of the form $\beta = (\beta^1, \dots, \beta^s)$, being $\beta^\ell \in \mathrm{Sym}^{m_\ell}(H_{r_\ell})$ given by $\beta^\ell = (b_1^\ell, \dots, b_{m_\ell}^\ell)_{\mathfrak{S}}$ with $b_i^\ell = (b_{i1}^\ell, \dots, b_{ir_i}^\ell)$ and $b_{ij}^\ell \in H^0(X, K^j)$.

Denote by Δ_r the big diagonal of $\mathrm{Sym}^r(H_r)$ and set

$$V_{\overline{r}}^{\mathrm{red}} := \left(\mathrm{Sym}^{m_1}(H_{r_1}^{\mathrm{sm}}) \setminus \Delta_{r_1}\right) \times \cdots \times \left(\mathrm{Sym}^{m_s}(H_{r_s}^{\mathrm{sm}}) \setminus \Delta_{r_s}\right).$$

Proceeding as in Lemmas 3.1 and 3.5, one can prove that, for every $\beta \in V_{\overline{r}}^{\mathrm{red}}$, the corresponding spectral curve $\overline{X}_\beta$ is reduced with $m_{\overline{r}}$ irreducible components $\overline{X}_{b_1^1}, \dots, \overline{X}_{b_{m_1}^1}, \dots, \overline{X}_{b_1^s}, \dots, \overline{X}_{b_{m_s}^s}$, which are in turn spectral curves for $b_i^\ell \in H_{r_i}$. Observe that the corresponding r_i-to-1 spectral covers $\pi_i^\ell : \overline{X}_{b_i^\ell} \to X$ coincide with the restriction of $\pi : \overline{X}_\beta \to X$ to each of the irreducible components, so that

$$\begin{array}{ccc} \overline{X}_{b_i^\ell} & \xrightarrow{\iota_i^\ell} & \overline{X}_\beta \\ & \searrow^{\pi_i^\ell} & \downarrow \pi \\ & & X \end{array}$$

commutes. We consider the nodal locus $V_{\overline{r}}^{\mathrm{nod}} \subset V_{\overline{r}}$, consisting of spectral curves with smooth irreducible components intersecting only in nodal points. Note that $V_{\overline{r}}^{\mathrm{nod}}$ is dense within $V_{\overline{r}}$ and the latter is dense in $H_{\overline{r}}$.

Lemma 7.4. *Let $\beta \in V_{\overline{r}}$. Then $D_{i,i'}^{\ell,\ell'} = \overline{X}_{b_i^\ell} \cap \overline{X}_{b_{i'}^{\ell'}}$ is a divisor linearly equivalent to $K^{r_i r_{i'}}$, thus of length $2r_i r_{i'}(g-1)$.*

Moreover, if $\beta \in V_{\overline{r}}^{\mathrm{nod}}$, then the divisor of singularities of $\overline{X}_\beta$ has simple points, and is given by the union $D = \bigcup_{\ell<\ell', i<i'} D_{i,i'}^{\ell,\ell'}$, and the normalization is $\nu_\beta : \widetilde{X}_\beta = \overline{X}_{b_1^1} \sqcup \cdots \sqcup \overline{X}_{b_{m_1}^1} \sqcup \cdots \sqcup \overline{X}_{b_1^s} \sqcup \cdots \sqcup \overline{X}_{b_{m_s}^s} \to \overline{X}_\beta$.

Proof. To see the first statement, deform the plane curve $\overline{X}_{b_i}$ to $\lambda^{r_i} = 0$. Then, the intersection with $\overline{X}_{b_{i'}}$ is the vanishing locus of a section of $\pi^* K^{r_{i'}}$ along X with multiplicity r_i. The second and third statements are obvious. □

The following proposition is proved as Proposition 4.4.

Proposition 7.5. *Let $\beta \in V_{\overline{r}}^{\mathrm{nod}}$, and let $\delta_i = (r_i^2 - r_i)(g-1)$. Then*

$$h^{-1}(\beta) \cap M_{\overline{r}} = \check{\nu}\left(\mathrm{Jac}^{\overline{\delta}}(\widetilde{X}_\beta)\right),$$

where $\overline{\delta} = (\delta_1, \overset{m_1}{\dots}, \delta_1, \dots, \delta_s, \overset{m_s}{\dots}, \delta_s)$.

7.2 | Parabolic subgroups and complex Lagrangian subvarieties

Let $P_{\overline{r}}$ be the parabolic subgroup whose Levi subgroup is $L_{\overline{r}}$. Recall that the corresponding unipotent radical is $U_{\overline{r}} = [P_{\overline{r}}, P_{\overline{r}}]$, and one has the identification $P_{\overline{r}} = L_{\overline{r}} \ltimes U_{\overline{r}}$. In this section, we construct Lagrangian subvarieties associated with the choice of the parabolic subgroup of the form $P_{\overline{r}}$.

Denote the locus of those Higgs bundles reducing its structure group to $P_{\overline{r}}$ by

$$\mathrm{Par}_{\overline{r}} = \left\{ (E, \varphi) \in \mathrm{M}_n \;\middle|\; \begin{array}{l} \exists\, \sigma \in H^0(X, E/P_{\overline{r}}), \\ \varphi \in H^0(X, E_\sigma(\mathfrak{p}_{\overline{r}}) \otimes K). \end{array} \right\}.$$

Proceeding as in Proposition 3.2, one can prove that $\mathrm{Par}_{\overline{r}}$ coincides with the preimage of $H_{\overline{r}}$ under the Hitchin map.

Proposition 7.6. *One has the following,*

$$M_n \times_{\mathrm{H}} H_{\overline{r}} = \mathrm{Par}_{\overline{r}}.$$

For $\overline{r} = (r_1, \overset{m_1}{\dots}, r_1, \dots, r_s, \overset{m_s}{\dots}, r_s)$ fixed, we say that J is an ordering of $\overline{r}$ if it is an ordering of the positive integers $\{r_1, \overset{m_1}{\dots}, r_1, \dots, r_s, \overset{m_s}{\dots}, r_s\}$. Let us denote by $\mathrm{Ord}_{\overline{r}}$ the set of orderings of $\overline{r}$. Given $\beta \in V_{\overline{r}}$ one can consider an ordering $J_\beta = \left(\overline{X}_1, \dots, \overline{X}_m\right)$ of the irreducible components of $\overline{X}_\beta$, where the jth element is the irreducible component indexed by $b_{i_j}^{\ell_j}$. Accordingly with J_β denote by π_j the restriction to the irreducible component $\overline{X}_j$ of the projection $\pi : \overline{X}_\beta \to X$ and abbreviate by $r_j := r_{\ell_j}$ the degree of the covering of X associated with $\overline{X}_j \overset{r_j:1}{\to} X$. We say that the ordering J_β respects J if we obtain J out of J_β by setting at the jth position, the rank r_j of the corresponding irreducible component $\overline{X}_j$.

In order to state the equivalent to Proposition 3.7 some extra care is needed, as the fact that the integers r_i are different, breaks the symmetry we have in the case of Borel groups, so that orderings of the indices need to be taken into account.

Proposition 7.7. *Let $\beta \in V_{\overline{r}}$ be associated with a spectral curve $\overline{X}_\beta$ has $m = m_{\overline{r}}$ irreducible components $\overline{X}_{b_1^1}, \dots, \overline{X}_{b_{m_1}^1}, \dots, \overline{X}_{b_1^s}, \dots, \overline{X}_{b_{m_s}^s}$. Let (E, φ) be a Higgs bundle, whose spectral data consist of a line bundle L over $\overline{X}_\beta$. For any ordering of $\overline{r}$, $J \in \mathrm{Ord}_{\overline{r}}$, and any ordering J_β of the irreducible components of $\overline{X}_\beta$ respecting J, one can choose canonically a filtration*

$$(E_{J_\beta})_\bullet : 0 \subsetneq (E_1, \varphi_1) \subsetneq \dots \cdots \subsetneq (E_m, \varphi_m) = (E, \varphi),$$

such that

$$(E_j, \varphi_j)/(E_{j-1}, \varphi_{j-1}) = (\pi_{j,*} L|_{\overline{X}_j} \otimes K^{-R_j^J}, \varphi_j/\varphi_{j-1}),$$

where $R_j^J = \sum_{k \geq j+1} r_k r_j$ depends only on J and φ_j/φ_{j-1} is determined by $\overline{X}_j$ as explained in Equation (2.7). Note that in the expression of R_j^J r_k may be equal to r_j.

Given a line bundle of zero degree $\mathscr{L} \in \mathrm{Jac}^0(X)$ and a point x_0, we define for every r,

$$\hat{\mathscr{L}}_r := \mathscr{L} \otimes \mathcal{O}(x_0)^{(r-1)(g-1)}.$$

Recall from Equation (2.9) the description of the Hitchin section of $h : M_r \to H_r$ associated with a line bundle of degree $(r-1)(g-1)$ over X. Observe that one has

$$\Sigma_{\hat{\mathscr{L}}_r} : H_r \to M_r.$$

For a given $\mathscr{L} \in \mathrm{Jac}^0(X)$, we define the subvariety of $\mathrm{Par}_{\overline{r}}$

$$\mathrm{Uni}_{\overline{r}}(\mathscr{L}) := \left\{ (E, \varphi) \in \mathrm{Par}_{\overline{r}} \;\middle|\; \begin{array}{l} \exists\, \sigma \in H^0(X, E/\mathrm{P}_{\overline{r}}), \text{ and } J \in \mathrm{Ord}_{\overline{r}} : \\ \varphi \in H^0(X, E_\sigma(\mathfrak{p}_{\overline{r}}) \otimes K); \\ (E_\sigma, \varphi)/\mathrm{U}_{\overline{r}} := \Sigma_{\hat{\mathscr{L}}_{r_1}}(\beta) \otimes K^{-R_1^J} \boxplus \cdots \boxplus \Sigma_{\hat{\mathscr{L}}_{r_m}}(\beta) \otimes K^{-R_m^J}. \end{array} \right\}. \tag{7.1}$$

Using Proposition 7.7, we can study the spectral data of the Higgs bundles contained in $\mathrm{Uni}_{\overline{r}}(\mathscr{L})$.

Proposition 7.8. *One has the following,*

1. *The restriction of $\mathrm{Uni}_{\overline{r}}(\mathscr{L})$ to $V_{\overline{r}}^{\mathrm{nod}}$ is surjective.*

2. Let $\beta \in V_{\overline{r}}^{\mathrm{nod}}$, we have that

$$\mathrm{Uni}_{\overline{r}}(\mathscr{L}) \cap h^{-1}(\beta) \cap \mathrm{Jac}(\overline{X}_\beta) = \hat{\nu}^{-1}(\hat{\mathscr{L}}_{r_1}, \overset{m_1}{\dots}, \hat{\mathscr{L}}_{r_1}, \dots, \hat{\mathscr{L}}_{r_s}, \overset{m_s}{\dots}, \hat{\mathscr{L}}_{r_s}).$$

We are now in a position to prove that $\mathrm{Uni}_{\overline{r}}(\mathscr{L})$ is Lagrangian, hence a suitable choice for the support of a (BAA)-brane.

Theorem 7.9. *The subscheme $\mathrm{Uni}_{\overline{r}}(\mathscr{L})$ is Lagrangian.*

Proof. It is enough to prove that the open subset $\mathrm{Uni}_{\overline{r}}(\mathscr{L})^{\mathrm{nod}}$ given by the restriction of $\mathrm{Uni}_{\overline{r}}(\mathscr{L})$ to $V_{\overline{r}}^{\mathrm{nod}}$, is Lagrangian.

Fix $\beta \in V_{\overline{r}}^{\mathrm{nod}}$. By Proposition 7.8 (2) the intersection of $\mathrm{Uni}_{\overline{r}}(\mathscr{L}) \cap h^{-1}(\beta)$ with $\mathrm{Jac}(\overline{X}_\beta)$ is non-empty, so there are Higgs bundles (E, φ) which have a line bundle as spectral data. Those (E, φ) are stable hence are smooth points in $\mathrm{Uni}_{\overline{r}}(\mathscr{L})$. With all this, we prove isotropicity as we did in Proposition 5.3.

By Lemmas 7.4 and Lemma 3.8, there is an exact sequence

$$0 \longrightarrow (\mathbb{C}^\times)^{\delta_{\overline{r}}-s+1} \longrightarrow \mathrm{Jac}(\overline{X}_b) \xrightarrow{\hat{\nu}} \mathrm{Jac}(\widetilde{X}_b) \longrightarrow 0$$

where $\delta_{\overline{r}} = \sum_{1\le i<j\le s} 2r_i r_j(g-1)$. It then follows by Proposition 7.8 (2) that

$$\dim \mathrm{Uni}_{\overline{r}}(\mathscr{L}) \cap h^{-1}(\beta) = \dim \mathrm{Jac}(\overline{X}_\beta) = \delta_{\overline{r}} - s + 1.$$

By Proposition 7.8 (1), one has that

$$V_{\overline{r}}^{\mathrm{nod}} \subset h(\mathrm{Uni}_{\overline{r}}(\mathscr{L})),$$

and recall that $V_{\overline{r}}^{\mathrm{nod}}$ is dense in $H_{\overline{r}}$, so they both have the same dimension. Since there are smooth points in $\mathrm{Uni}_{\overline{r}}(\mathscr{L})$, it follows that the dimension is

$$\begin{aligned}\dim \mathrm{Uni}^{\overline{r}}(\mathscr{L}) &= \delta_{\overline{r}} - s + 1 + \dim H_{\overline{r}} = \delta_{\overline{r}} - s + 1 + \sum_i (r_i^2(g-1)+1) \\ &= n^2(g-1)+1,\end{aligned}$$

which is half of the dimension of M_n. □

Remark 7.10. Propositions 7.5 and 7.8 indicate that a suitable choice of a hyperholomorphic bundle on $M_{\overline{r}}$ and a flat bundle on $\mathrm{Uni}_{\overline{r}}(\mathscr{L})$ would produce a pair of dual (BBB) and (BAA) branes. This would happen similarly to the case of Borel subgroups (i.e., $\overline{r} = (1, \dots, 1)$) in Section 6. The construction involves downward flows to very stable points of higher components of the nilpotent cone [30]. We hope to get back to this in future work.

Now, it is also possible to construct more general unitary Lagragian submanifolds, even in the absence of Hitchin sections. The key is to use very stable bundles to produce Lagrangian multisections of the Hitchin map. Given a vector bundle E we say, after Drinfeld [21, 44], that E is *very stable* if it has no non-zero nilpotent Higgs fields. This implies that E is stable [44, Proposition 3.5] (provided $g \ge 2$). Furthermore, very stable bundles are dense within the moduli space of vector bundles [44, Proposition 3.5]. Gathering the results of Pauly and the second author (see [54, Theorem 1.1 and Corollary 1.2]) with the remark [24, Corollary 7.3], one gets

Theorem 7.11. *Let E be a stable bundle. Then, E is very stable if and only if the Lagrangian subvariety given by the embedding*

$$\begin{array}{ccc} H^0(X, \mathrm{End}(E) \otimes K) & \longrightarrow & M_n \\ \phi & \longmapsto & (E, \phi), \end{array}$$

provides a Lagrangian multisection of the Hitchin fibration (i.e., the restriction of the Hitchin fibration to $H^0(X, \mathrm{End}(E) \otimes K) \hookrightarrow M_n$ is finite and surjective).

Set $m = m_{\overline{r}}$. Given an ordering $J \in \mathrm{Ord}_{\overline{r}}$ consider an m-tuple of very stable vector bundles over X, $\overline{E} = (E_1, \dots, E_m)$, whose ith element has $\mathrm{rk}\, E_i = r_i$ given by the ith position of J. Denote $\deg E_i = e_i$ and

$$f_i^J = e_i + (r_i^2 - r_i)(g-1) + 2R_i^J(g-1),$$

where $R_i^J = \sum_{k \geq i+1} r_{j_i} r_{j_k}$ are defined as in Proposition 7.7. From now on, we shall assume that the choice of J and $\overline{E}$ is done under the following numerical condition on the degrees e_i.

Assumption 7.12. Let $\overline{e} = (e_1, \dots, e_m)$ be an m-tuple of integers and pick $J \in \mathrm{Ord}_{\overline{r}}$. Suppose that, for all subset $I \subset \{1, \dots, m\}$, there are inequalities

$$\sum_{i \in I} f_i^J > (r_I^2 - r_I)(g-1), \tag{7.2}$$

where $r_I = \sum_{i \in I} r_{j_i}$, and when $I = \{1, \dots, m\}$ one has the equality

$$\sum_{i=1}^{m} f_i^J = (n^2 - n)(g-1).$$

Given an m-tuple of very stable bundles $\overline{E}$ whose degrees $\overline{e}$ satisfy Assumption 7.12, we define the following subvariety of $\mathrm{Par}_{\overline{r}}$,

$$\mathrm{Uni}_{\overline{r}}(\overline{E}) := \left\{ (E, \varphi) \;\middle|\; \begin{array}{l} \exists\, \sigma \in H^0(X, E/\mathrm{P}_{\overline{r}}) : \\ \varphi \in H^0(X, E_\sigma(\mathfrak{p}_{\overline{r}}) \otimes K); \\ E_\sigma/\mathrm{U}_{\overline{r}} := E_{\mathrm{L}_{\overline{r}}} \cong \bigoplus_{i=1}^m E_i. \end{array} \right\}. \tag{7.3}$$

In what follows, we prove that $\mathrm{Uni}^{\overline{r}}(\overline{E})$ is a Lagrangian submanifold. As in the case of $\mathrm{Uni}_{\overline{r}}(\mathscr{L})$, this is proven through the study the associated spectral data.

Consider restriction of the Hitchin map h to $\mathrm{Uni}_{\overline{r}}(\overline{E})$. After Proposition 7.6, one has that the image is contained in $H_{\overline{r}}$,

$$h : \mathrm{Uni}_{\overline{r}}(\overline{E}) \longrightarrow H_{\overline{r}}.$$

Before we can give the analogous to Proposition 5.7, we need an intermediate result.

Proposition 7.13. *Let $\beta \in V^{\mathrm{nod}}$. Assume that $\overline{E}$ satisfies Assumption 7.12 and denote by $S_{i,\beta}$ the finite set of Higgs bundles over β admitting E_i as underlying vector bundle. Let $\mathcal{S}_{i,\beta}$ the associated set of spectral data over $\overline{X}_\beta$ associated with each of the Higgs bundles in $S_{i,\beta}$. For each $J \in \mathrm{Ord}_{\overline{r}}$, pick*

$$\hat{\mathscr{L}}^J_{\overline{E},\beta} = (\mathscr{L}_1 \otimes \pi_1^* K^{R_1^J}, \dots, \mathscr{L}_m \otimes \pi_m^* K^{R_m^J}), \tag{7.4}$$

where $\mathscr{L}_i \in \mathcal{S}_{i,\beta}$. Let us denote by $\mathcal{S}^J_\beta$ the set of all tuples of the form (7.4).

Assume that $\overline{E}$ satisfies Assumption 7.12 i). Let $b \in H^{\mathrm{nod}}_{\overline{r}}$, and let $\mathcal{O}rd_s$ denote the set of orderings of $\{1, \dots, s\}$. For each $J \in \mathcal{O}rd_s$, let $\hat{\mathscr{L}}^J$ be as in Equation (7.4). Then, $\mathrm{Uni}_{\overline{r}}(\overline{E}) \cap h^{-1}(b) \cap \mathrm{Jac}^{\overline{d}}(\overline{X}_\beta)$ is either empty or

$$\mathrm{Uni}_{\overline{r}}(\overline{E}) \cap h^{-1}(b) \cap \mathrm{Jac}^{\overline{d}}(\overline{X}_\beta) = \bigcup_{\hat{\mathscr{L}}^J_{\overline{E},\beta} \in \mathcal{S}^J_\beta} \hat{\nu}^{-1}(\hat{\mathscr{L}}^J_{\overline{E},\beta})$$

where we identify $\mathrm{Jac}^{\overline{d}}(\overline{X}_b)$ *with an open subset of* $h^{-1}(b)$ *and define*

$$\hat{\nu} : \mathrm{Jac}^{\overline{d}}(\overline{X}_b) \longrightarrow \mathrm{Jac}^{\overline{d}}(\widetilde{X}_b)$$

to be the pullback map.

Proof. After checking that Equation (7.2) ensures the stability of the points of $\mathrm{Uni}_{\overline{r}}(\overline{E}) \cap h^{-1}(b)$, the proof follows as in Proposition 5.7. □

Continuing the parallelism with $\mathrm{Uni}(\mathscr{L})$, we next prove Lagrangianity of the submanifold $\mathrm{Uni}_{\overline{r}}(\overline{E})$.

Theorem 7.14. *Under Assumption 7.12, the subscheme* $\mathrm{Uni}_{\overline{r}}(\overline{E})$ *is Lagrangian.*

Proof. The proof is analogous to that of Theorem 7.9. □

Remark 7.15. For the sake of clarity, we have chosen to work with the moduli space of degree 0 Higgs bundles. Note however that the subvarieties $\mathrm{M}_{\overline{r}}$ and $\mathrm{Uni}^{\overline{r}}$ make sense in a larger context. Indeed, consider the moduli space of rank n, degree d Higgs bundles $M_X(n,d)$ with $(n,d) \neq 1$. Then, $M_X(n,d)^{sing} \neq \emptyset$, and so there will exist partitions $\overline{r}$ of n for which $\mathrm{M}_{\overline{r}} \neq \emptyset$. Note that in that case the (semi)stability condition for torsion-free sheaves should then be modified accordingly.

ACKNOWLEDGMENTS

We would like to thank P. Gothen, M. Jardim, A. Oliveira, and C. Pauly for their kind support and inspiring conversations. Many thanks to J. Heinloth for reading a preliminary version of this paper and pointing out some mistakes. We are indebted to A. Wienhard, whose support and hospitality made this project possible. E. Franco is currently supported by FCT (Portugal) in the framework of the Investigador FCT program. He has previously been supported by project PTDC/MAT-GEO/2823/2014 funded by FCT with Portuguese national funds and FAPESP postdoctoral grant number 2012/16356-6 and BEPE-2015/06696-2 (Brazil). A. Peón-Nieto is currently supported by the scheme H2020-MSCA-IF-2019, Agreement no. 897722 (GoH). She was formerly funded through a Beatriu de Pinós grant no. 2018 BP 332 (H2020-MSCA-COFUND-2017 Agreement no. 801370), a postdoctoral grant associated with the project FP7-PEOPLE-2013-CIG-GEOMODULI number: 618471, a postdoctoral contract of the Heidelberg Institute for Theoretical Studies, a MATCH postdoctoral fellowship and the European Research Council under ERC-Consolidator grant no.: 614733.

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How to cite this article: E. Franco and A. Peón-Nieto, *Branes on the singular locus of the Hitchin system via Borel and other parabolic subgroups*, Math. Nachr. (2023), 1–39. https://doi.org/10.1002/mana.202000267