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## MINIMAL RAMSEY GRAPHS WITH MANY VERTICES OF SMALL DEGREE\*

SIMONA BOYADZHIYSKA<sup>†</sup>, DENNIS CLEMENS<sup>‡</sup>, AND PRANSHU GUPTA<sup>†</sup>

**Abstract.** Given any graph  $H$ , a graph  $G$  is said to be  $q$ -Ramsey for  $H$  if every coloring of the edges of  $G$  with  $q$  colors yields a monochromatic subgraph isomorphic to  $H$ . Such a graph  $G$  is said to be minimal  $q$ -Ramsey for  $H$  if additionally no proper subgraph  $G'$  of  $G$  is  $q$ -Ramsey for  $H$ . In 1976, Burr, Erdős, and Lovász initiated the study of the parameter  $s_q(H)$ , defined as the smallest minimum degree among all minimal  $q$ -Ramsey graphs for  $H$ . In this paper, we consider the problem of determining how many vertices of degree  $s_q(H)$  a minimal  $q$ -Ramsey graph for  $H$  can contain. Specifically, we seek to identify graphs for which a minimal  $q$ -Ramsey graph can contain arbitrarily many such vertices. We call a graph satisfying this property  $s_q$ -abundant. Among other results, we prove that every cycle is  $s_q$ -abundant for any integer  $q \geq 2$ . We also discuss the cases when  $H$  is a clique or a clique with a pendant edge, extending previous results of Burr and co-authors and Fox and co-authors. To prove our results and construct suitable minimal Ramsey graphs, we use gadget graphs, which we call pattern gadgets and which generalize earlier constructions used in the study of minimal Ramsey graphs. We provide a new, more constructive proof of the existence of these gadgets.

**Key words.** graph theory, Ramsey graphs, minimum degrees

**AMS subject classification.** 05D10

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**1. Introduction.** A classical result of Ramsey from 1930 [21] states that, for every graph  $H$ , there exists an integer  $n$  such that the following property holds: No matter how the edges of  $K_n$  are colored with two colors, there must always exist a *monochromatic* copy of  $H$ , that is, a subgraph of  $K_n$  isomorphic to  $H$  in which all edges have the same color. In fact, the same is true if, instead of two, we use any arbitrary number of colors. Through the last decades, this result has become the starting point of a field of intense studies, giving rise to a branch of combinatorics known as *Ramsey theory*. For an excellent survey on the more recent developments in the field, see [9].

One line of research is concerned with studying properties of (minimal) Ramsey graphs, which is also the focus of this paper. Formally, given any graph  $H$ , a graph  $G$  is said to be  $q$ -Ramsey for  $H$ , denoted by  $G \rightarrow_q H$ , if in every coloring of the edges of  $G$  with  $q$  colors, there exists a monochromatic copy of  $H$ . Ramsey's theorem discussed above then states that, for every graph  $H$ , we have  $K_n \rightarrow_q H$  provided that  $n$  is large enough, thus establishing the existence of a  $q$ -Ramsey graph for  $H$  for any choice of  $H$  and  $q \geq 2$ . We denote the set of all such graphs for  $H$  by  $\mathcal{R}_q(H)$ .

In this language, the well-known  $q$ -color Ramsey number of a graph  $H$ , denoted by  $r_q(H)$ , can be defined as the minimum possible number of vertices in a graph that is  $q$ -Ramsey for  $H$ . Over the years, researchers have worked hard to understand the

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behavior of Ramsey numbers for various classes of graphs, which in some cases has turned out to be notoriously difficult. Perhaps the most natural example here is the clique  $K_t$ . While the determination of  $r_2(K_3)$  is a simple exercise often given in a first course in combinatorics, already for  $t = 5$ , the precise value of  $r_2(K_t)$  is not known. For general  $t$ , Erdős and Szekeres [11] and Erdős [10] showed that  $2^{t/2} \leq r_2(K_t) \leq 2^{2t}$ , establishing that the 2-color Ramsey number of  $K_t$  is exponential in  $t$  but leaving a large gap between the two bounds in the base of the exponent. Now, more than 70 years later, those remain essentially the best known bounds, with improvements only in the lower order terms. The current best known lower bound is due to Spencer [26]; a new upper bound was shown very recently by Sah [24], improving on the previous best known bound due to Conlon [8].

More generally, it is of interest to understand what makes a graph  $q$ -Ramsey for some chosen graph  $H$ , that is, to understand the structural properties of graphs that are  $q$ -Ramsey for  $H$  and, whenever possible, to characterize all such graphs. After considering the number of vertices, it is natural to ask about the behavior of other graph parameters. For example, much work has been done in studying the minimum possible number of edges in a graph that is  $q$ -Ramsey for  $H$ , known as the  $q$ -color size-Ramsey number of  $H$ .

Here, we are interested in questions concerning minimum degrees of graphs that are  $q$ -Ramsey for a graph  $H$ . Note that asking about the smallest possible minimum degree of a graph that is  $q$ -Ramsey for  $H$  is not very interesting, as we can immediately see that the answer is zero. This is because any graph containing a  $q$ -Ramsey graph for  $H$  as a subgraph is itself  $q$ -Ramsey for  $H$ , and we can of course add an isolated vertex to obtain a graph with minimum degree zero. To avoid such trivialities, we restrict our attention to those graphs that are, in some sense, critically  $q$ -Ramsey for  $H$ . This leads to the following natural definition: We say  $G$  is *minimal  $q$ -Ramsey* for  $H$  if  $G \rightarrow_q H$  and, for any proper subgraph  $G' \subsetneq G$ , we have  $G' \not\rightarrow_q H$ , that is,  $G$  loses its Ramsey property whenever we delete any vertex or edge of  $G$ . We denote the set of all minimal  $q$ -Ramsey graphs for  $H$  by  $\mathcal{M}_q(H)$ .

The 1970s saw the beginning of two prominent directions of research concerning  $\mathcal{M}_q(H)$ . One of the questions, first posed in [19] by Nešetřil and Rödl, was whether for a given graph  $H$  the set  $\mathcal{M}_q(H)$  is finite or infinite. We call a graph  $H$   *$q$ -Ramsey finite* (resp., *infinite*) if the set  $\mathcal{M}_q(H)$  is finite (resp., infinite). In a series of paper (see [2, 3, 20, 22]), it was established that a graph  $H$  is 2-Ramsey-finite if and only if it is the disjoint union of an odd star and any number of isolated edges.

Around the same time, Burr, Erdős, and Lovász [4] initiated the general study of graph parameters for graphs in  $\mathcal{M}_q(H)$ . In their seminal paper, they considered the chromatic number, the (vertex) connectivity, and the minimum degree of minimal 2-Ramsey graphs for the clique  $K_t$  when  $t \geq 3$ . In particular, they were interested in how small these parameters can be.

Surprisingly, while the 2-Ramsey number of  $K_t$  is still not known, Burr, Erdős, and Lovász [4] could determine the mentioned values precisely.

Following [13], we set  $s_q(H) = \min\{\delta(G) : G \in \mathcal{M}_q(H)\}$ , where as usual  $\delta(G)$  denotes the minimum degree of  $G$ . One of the results that appeared in [4] establishes that  $s_2(K_t) = (t-1)^2$ , which is perhaps surprising, given that each graph in  $\mathcal{M}_q(K_t)$  has at least exponentially many vertices.

For more colors, Fox et al. [13] established that  $s_q(K_t) \leq 8(t-1)^6 q^3$ , showing that  $s_q(K_t)$  is polynomial in both  $t$  and  $q$ . Recently, this upper bound was improved by Bamberg, Bishnoi, and Lesgourgues [1] to  $C(t-1)^5 q^{5/2}$ . Fox et al. also investigated the growth of  $s_q(K_t)$  as a function of  $q$  (with  $t$  being treated as a constant) and proved

that  $s_q(K_t) = q^2 \text{polylog}(q)$ . However, a logarithmic gap remained between the lower and the upper bound. For the case of the triangle, Guo and Warnke [17] closed this gap, showing that  $s_q(K_3) = \Theta(q^2 \log q)$ . On the other hand, Hàn, Rödl, and Szabó [18] studied the dependence of  $s_q(K_t)$  on the size of the clique with the number of colors kept constant; they showed that  $s_q(K_t) = t^2 \text{polylog}(t)$ .

The parameter  $s_q(H)$  has also been investigated for other choices of the target graph  $H$  when  $q = 2$ . For instance, Szabó, Zumstein, and Zürcher [27] determined  $s_2(H)$  for many interesting classes of bipartite graphs, including trees, even cycles, and biregular bipartite graphs. Later Grinshpun [15] determined  $s_2(H)$  for any 3-connected bipartite graph  $H$ . A rather surprising result in this direction appeared in a paper of Fox et al. [12], who studied  $s_2(K_t \cdot K_2)$ , where  $K_t \cdot K_2$  is the graph obtained from a clique of size  $t$  by adding a new vertex and connecting it to exactly one vertex of the clique (we will call such a graph a *clique with a pendant edge*). The authors proved that  $s_2(K_t \cdot K_2) = t - 1$ , showing that even a single edge can significantly change the value of the parameter  $s_2$ . This result also implies that there exists a 2-Ramsey graph for  $K_t$  that is not 2-Ramsey for  $K_t \cdot K_2$ .

Once we know that a minimal  $q$ -Ramsey graph for a given  $H$  can contain a vertex of small degree, a natural next question is, *how many* vertices of this small degree can a minimal  $q$ -Ramsey graph for  $H$  contain? More specifically, can a minimal  $q$ -Ramsey graph have arbitrarily many vertices of the smallest possible minimum degree? This question motivates the following definition.

**DEFINITION 1.1.** For a given integer  $q \geq 2$ , a graph  $H$  is said to be  $s_q$ -abundant if, for every  $k \geq 1$ , there exists a minimal  $q$ -Ramsey graph for  $H$  with at least  $k$  vertices of degree  $s_q(H)$ .

As it turns out, it is not immediate whether  $s_q$ -abundant graphs exist at all. In [4], Burr, Erdős, and Lovász noted that their construction can be generalized to show that cliques are  $s_2$ -abundant. In this paper, we will give several examples showing that, for all  $q \geq 2$ , there are infinitely many  $s_q$ -abundant graphs.

It is not hard to see that, if a graph is  $q$ -Ramsey finite, then it cannot be  $s_q$ -abundant. This immediately implies that odd stars are not  $s_2$ -abundant. On the other hand, we know that even stars are 2-Ramsey infinite, but as we will see below they are also not  $s_2$ -abundant. This statement follows from the following result.

**THEOREM 1.2 ([4]).** *Let  $m \geq 1$  be an integer. Then a connected graph  $G$  is 2-Ramsey for  $K_{1,m}$  if and only if either  $\Delta(G) \geq 2m - 1$  or  $m$  is even and  $G$  is a  $(2m - 2)$ -regular graph on an odd number of vertices.*

The theorem immediately implies that  $\mathcal{M}_2(K_{1,m}) = \{K_{1,2m-1}\}$  if  $m$  is odd and  $\mathcal{M}_2(K_{1,m}) = \{K_{1,2m-1}\} \cup \{G : G \text{ is connected, } (2m-2)\text{-regular, and } |V(G)| \text{ is odd}\}$  if  $m$  is even. In particular, this implies that no star is  $s_2$ -abundant.

More generally, it turns out that stars are not  $s_q$ -abundant for any  $q \geq 2$ : A simple argument implies that, for any  $m \geq 1$  and  $q \geq 2$ , a minimal  $q$ -Ramsey graph for  $K_{1,m}$  has either zero or  $q(m-1) + 1$  vertices of degree one. Indeed, if  $G$  is a minimal  $q$ -Ramsey graph for  $K_{1,m}$  that is not isomorphic to  $K_{1,q(m-1)+1}$ , then the maximum degree of  $G$  is at most  $q(m-1)$ . Thus, if  $G$  contains a vertex  $v$  of degree one, then the only neighbor  $u$  of  $v$  has at most  $q(m-1) - 1$  other neighbors. By the minimality of  $G$ , the graph  $G - v$  has a  $q$ -coloring  $c$  without a monochromatic copy of  $K_{1,m}$ . Since  $u$  has at most  $q(m-1) - 1$  neighbors in  $G - v$ , there is a color that appears at most  $m - 2$  times on the edges incident to  $u$ . Then this color can be used on the edge  $uv$  to extend  $c$  to a  $q$ -coloring of  $G$  without a monochromatic copy of  $K_{1,m}$ , leading to a contradiction. Hence,  $G$  cannot contain a vertex of degree one.

One of the goals of this paper is to initiate the systematic study of  $s_q$ -abundance. First, we show that all cycles of length at least four are  $s_q$ -abundant. As a byproduct, we determine  $s_q(C_t)$  for all  $q \geq 2$  and  $t \geq 4$ .

**THEOREM 1.3.** *For any given integers  $q \geq 2$ ,  $t \geq 4$ , and  $k \geq 1$ , there exists a minimal  $q$ -Ramsey graph for  $C_t$  that has at least  $k$  vertices of degree  $q + 1$ . In particular,  $s_q(C_t) = q + 1$  and  $C_t$  is  $s_q$ -abundant.*

It turns out that the cycle  $C_3$  behaves differently compared to longer cycles with respect to the value of  $s_q$ . Its behavior is consistent with that of a clique, and we know from an earlier discussion that  $s_2(K_3) = 4$  and  $s_q(K_3) = \Theta(q^2 \log q)$  as a function of  $q$ . While the value of  $s_q$  for  $K_3$ , or for any larger clique, is not known precisely when  $q > 2$ , our theory still allows us to show that any clique  $K_t$  for  $t \geq 3$  is  $s_q$ -abundant for any value of  $q$ . In fact, Theorem 1.4 below is a consequence of a more general result that will be presented in section 3.

**THEOREM 1.4.** *For given integers  $q \geq 2$  and  $t \geq 3$ , the clique  $K_t$  is  $s_q$ -abundant.*

Finally, we show that a clique with a pendant edge is  $s_2$ -abundant. We note that, since  $s_2(K_t) = (t - 1)^2$  and  $s_2(K_t \cdot K_2) = t - 1$  for all  $t \geq 3$ , Theorem 1.5 also yields that there are infinitely many graphs that are minimal 2-Ramsey for  $K_t \cdot K_2$  but not minimal 2-Ramsey for  $K_t$ . One of the main building blocks used in our construction is not known to exist for  $K_t \cdot K_2$  when  $q > 2$ , which is why we focus on the case  $q = 2$ .

**THEOREM 1.5.** *For a given integer  $t \geq 3$ , the graph  $K_t \cdot K_2$  is  $s_2$ -abundant.*

In order to prove the statements above, we will use gadget graphs that we call *pattern gadgets*. These were proven to exist for cycles by Siggers [25] and they generalized other well-known gadgets such as signal senders, originally developed by Burr, Erdős, and Lovász [4] to study  $s_2(K_t)$ . Pattern gadgets help us construct minimal Ramsey graphs with many vertices of small degree. Even though Siggers' proof can be generalized to other graph classes, we will provide a more constructive proof, building and using some intermediate gadgets along the way. The more explicit nature of our approach might make it applicable in other settings, which is why we choose to include it in this paper.

In a nutshell, the main idea behind pattern gadgets is the following: Given some graph  $G$  and some family  $\mathcal{G}$  of colorings of  $E(G)$  with  $q$  colors that do not contain monochromatic copies of  $H$ , we will find some larger graph  $P$  containing  $G$  such that the colorings in  $\mathcal{G}$  are exactly those colorings of  $G$  that can be extended to  $P$  without creating a monochromatic copy of  $H$ . Then, in order to prove each of the above theorems, we will choose  $G$  and  $\mathcal{G}$  in such a way that we can attach  $k$  small-degree vertices to  $G \subseteq P$  so that no coloring in  $\mathcal{G}$  can be extended to the new edges without creating a monochromatic copy of  $H$ , but if we remove any of these new vertices, we can find a coloring in  $\mathcal{G}$  that can be extended in the desired way.

The precise definition of a pattern gadget will be given in section 2. We will show their existence for many target graphs  $H$ , including all 3-connected graphs.

**Organization of the paper.** In section 2, we introduce all necessary auxiliary gadgets and present our proof of the existence of pattern gadgets. Afterward, we continue with the proofs of Theorems 1.3, 1.4, 1.5 in section 3, where we also prove a general statement regarding 3-connected graphs. We end with some concluding remarks and open problems in section 4.

**Notation.** Given an integer  $n \geq 1$ , we write  $[n]$  for the set of the first  $n$  positive integers. For a graph  $G$ , we denote its vertex set by  $V(G)$  and its edge set by  $E(G)$ ,

and we set  $e(G) = |E(G)|$ . For any edge  $\{v, w\} \in E(G)$ , we write  $vw$  for short. We let  $N_G(v) = \{w \in V(G) : vw \in E(G)\}$  denote the neighborhood of  $v$  in  $G$ ,  $d_G(v) = |N_G(v)|$  denote the degree of  $v$  in  $G$ ,  $\delta(G) = \min\{d_G(v) : v \in V(G)\}$ , and  $\Delta(G) = \max\{d_G(v) : v \in V(G)\}$  denote the minimum degree and maximum degree of  $G$ , respectively.

For a graph  $G$  and vertex subsets  $A$  and  $B$  of  $G$ , we denote by  $E_G(A, B)$  the edges in  $G$  with one endpoint in  $A$  and another in  $B$ . Also,  $E_G(A)$  denotes the edges in  $G$  with both endpoints in  $A$ . We sometimes identify a graph  $G$  with its edge set.

Let  $F$  and  $G$  be two graphs. We say that  $F$  and  $G$  are isomorphic, denoted by  $F \cong G$ , if there exists a bijection  $f : V(F) \rightarrow V(G)$  such that  $vw \in E(F)$  if and only if  $f(v)f(w) \in E(G)$ . In this case, we also say that  $F$  forms a copy of  $G$ .

We say that  $F$  is a subgraph of  $G$ , denoted by  $F \subseteq G$ , if there is an injective map  $f : V(F) \rightarrow V(G)$  such that  $f(x)f(y) \in E(G)$  for all  $xy \in E(F)$ ; further,  $F$  is a proper subgraph of  $G$  if  $F \subseteq G$  and  $F \neq G$ . Given any subset  $A \subseteq V$ , the subgraph induced by  $A$ , denoted by  $G[A]$ , is the graph with vertex set  $A$  and edge set  $E_G(A)$ . Moreover, we set  $G - v = G[V(G) \setminus v]$  and  $G - e = (V(G), E(G) \setminus \{e\})$  for any  $v \in V(G)$  and  $e \in E(G)$ . If  $F \cong G[A]$  for some  $A \subseteq V(G)$ , then we say that  $F$  is an induced subgraph of  $G$  and write  $F \subseteq_{ind} G$ .

Given a graph  $G$  and any subsets  $A$  and  $B$  of the vertex set or the edge set of  $G$ , we define the *distance* between  $A$  and  $B$ , denoted by  $\text{dist}_G(A, B)$ , to be the number of edges in a shortest path with one endpoint in (the vertex set of)  $A$  and one endpoint in (the vertex set of)  $B$ . The *girth* of  $G$ , denoted by  $\text{girth}(G)$ , is the length of a shortest cycle in  $G$  (if  $G$  is acyclic, then  $\text{girth}(G)$  is defined to be infinity). A graph  $G$  is said to be  $k$ -connected if it has more than  $k$  vertices and, for any set  $S$  of at most  $k - 1$  vertices, the graph  $G[V(G) \setminus S]$  is connected.

In the rest of the paper, a coloring of some graph  $G$  always refers to a coloring of its edge set. If  $G$  contains no monochromatic subgraph isomorphic to  $H$  under a given coloring, the coloring is said to be  $H$ -free. If a coloring uses at most  $q$  colors, we call it a  $q$ -coloring. Unless otherwise specified, we will assume in this case that our color palette is the set  $[q]$ . If we are only concerned with the case  $q = 2$ , for the sake of convenience we will sometimes call our colors red and blue instead of color 1 and color 2. If  $c$  is a  $q$ -coloring of  $G$  and some subgraph  $F$  is monochromatic in some color  $i$ , we will sometimes write  $c(F) = i$ . Similarly, when defining colorings, we will write, for example,  $c(F) = i$  to indicate that we give color  $i$  to every edge of the subgraph  $F$ .

**2. Construction of pattern gadgets.** Most of our constructions of minimal Ramsey graphs will rely on the existence of certain gadget graphs; these graphs will have the property that, in every coloring not containing a monochromatic copy of our target graph  $H$ , some fixed *color patterns* need to appear on certain sets of edges. Such an approach has already been used in the paper of Burr, Erdős, and Lovász [4] when proving that  $s_2(K_t) = (t - 1)^2$ . In their paper, the authors introduced gadget graphs that are now known as BEL gadgets and are defined as follows: Let  $H$  and  $G$  be fixed graphs such that  $G \not\rightarrow_q H$ , and let  $\varphi$  be an  $H$ -free  $q$ -coloring of  $G$ ; a *BEL gadget* for  $H$  with respect to the pair  $(G, \varphi)$  is a graph  $B$  containing  $G$  as an induced subgraph such that  $B$  is not  $q$ -Ramsey for  $H$  but in every  $H$ -free  $q$ -coloring of the edges of  $B$ , the subgraph  $G$  has the coloring given by  $\varphi$  (up to a permutation of colors). Burr and co-authors showed the existence of BEL gadgets for all cliques on at least three vertices when  $q = 2$  (for any appropriate choice of  $G$  and  $\varphi$ ). Later results imply that BEL gadgets exist for more general graphs and for more colors; we will give an overview of those results in section 2.1.

Suppose we want to construct a minimal  $q$ -Ramsey graph for  $H$  that contains a vertex of degree at most  $d$ . Provided that a BEL gadget with certain properties exists, it suffices to find a graph  $G$  that contains a vertex  $v$  of degree  $d$  and a  $q$ -coloring  $\varphi$  of  $G - v$  that contains no monochromatic copy of  $H$  but cannot be extended to an  $H$ -free coloring of  $G$ . Indeed, we can construct  $\tilde{G}$  by taking a copy  $G'$  of  $G - v$  and a BEL gadget for  $H$  with respect to  $(G', \varphi)$  and adding the vertex  $v$  along with  $d$  edges so that  $V(G') \cup \{v\}$  induces a copy of  $G$ . Now it is not difficult to check that  $\tilde{G} \rightarrow_q H$ , and if  $H$  satisfies certain conditions, then we can also ensure that  $\tilde{G} - v \not\rightarrow_q H$ . This means that any minimal  $q$ -Ramsey subgraph of  $\tilde{G}$  needs to contain  $v$ , that is,  $v$  is *important* for  $\tilde{G}$  to be a  $q$ -Ramsey graph, and  $s_q(H) \leq d_{\tilde{G}}(v)$ .

For our main theorems, we will aim to find graphs  $\tilde{G}$  with many vertices of small degree, each of which is important for  $\tilde{G}$  to be a Ramsey graph for  $H$ . In order to do so, we will use a gadget that allows for more flexibility than a BEL gadget. This gadget again comes with a subgraph  $G$  on which fixed color patterns are forced in any  $H$ -free  $q$ -coloring. However, while for a BEL gadget we fix only a single permissible pattern (up to a permutation of the color classes), our gadget graph allows us to fix a family of permissible color patterns for  $G$  such that each of these patterns, and no other, can be extended to an  $H$ -free coloring of the whole graph.

To make this more precise, let us first define color patterns and an isomorphism relation between them.

**DEFINITION 2.1.** Let  $q \geq 2$  be a given integer and  $H$  and  $G$  be graphs. A  $q$ -color pattern for  $G$  is a partition  $g = \{G_1, G_2, \dots, G_q\}$  of the edges of  $G$ . If  $H \not\subseteq G_i$  for every  $i \in [q]$ , we say that  $g$  is  $H$ -free. Given any subset  $A \subseteq V(G)$ , we call the partition  $g[A] = \{G_1[A], G_2[A], \dots, G_q[A]\}$  the *induced  $q$ -color pattern* on  $A$ .

Let  $G'$  be a copy of  $G$ , and let  $g' = \{G'_1, \dots, G'_q\}$  be a  $q$ -color pattern for  $G'$ . Then we say that  $g$  and  $g'$  are *isomorphic*, denoted by  $g \cong g'$ , if there exists a permutation  $\pi$  of  $[q]$  such that  $G_i \cong G'_{\pi(i)}$  for every  $i \in [q]$ .

Using the above terminology, we can now give a precise definition of the gadget graphs that we are interested in.

**DEFINITION 2.2.** Let  $q \geq 2$  be a given integer and  $H$  and  $G$  be graphs such that  $G \not\rightarrow_q H$ . Also let  $\mathcal{G}$  be a family of  $H$ -free  $q$ -color patterns for  $G$ . Then we call a graph  $P = P(H, G, \mathcal{G}, q)$  a *pattern gadget* if the following properties hold:

- (P1)  $G \subseteq_{\text{ind}} P$ .
- (P2) If  $c : E(P) \rightarrow [q]$  is an  $H$ -free coloring of  $P$ , then  $\{c_{|G}^{-1}(1), \dots, c_{|G}^{-1}(q)\} \in \mathcal{G}$ .
- (P3) For every pattern  $\{G_1, \dots, G_q\} \in \mathcal{G}$ , there exists an  $H$ -free coloring  $c : E(P) \rightarrow [q]$  such that  $\{c_{|G}^{-1}(1), \dots, c_{|G}^{-1}(q)\} = \{G_1, \dots, G_q\}$ .

A variant of these gadgets was defined by Siggers [25], who showed its existence for cycles. The rest of this section is mainly devoted to our proof that pattern gadgets exist for certain choices of the graph  $H$ . In the proof, we will combine various intermediate gadgets and for that to work we will often require them to satisfy an additional property that we refer to as robustness (following Grinshpun [15]). We will also require that our final gadgets satisfy this property, which will be useful in applications.

**DEFINITION 2.3.** Let  $G$  be a graph and  $G_0$  be an induced subgraph of  $G$ . We say that the pair  $(G, G_0)$  is  $H$ -robust if, in any graph obtained from  $G$  by adding any set  $S$  of new vertices and any collection of edges within  $S \cup V(G_0)$ , every copy of  $H$  is entirely contained either in  $G$  or in the subgraph induced by  $S \cup V(G_0)$ .

The main theorem of this section states that if  $H$  is 3-connected or isomorphic to a cycle or  $K_t \cdot K_2$ , then pattern gadgets that satisfy certain robustness properties exist for  $H$ .

**THEOREM 2.4.** *Let  $q \geq 2$  be a given integer, and let  $H$  and  $G$  be graphs with  $G \not\rightarrow_q H$ . Further, let  $\mathcal{G}$  be a family of  $H$ -free  $q$ -color patterns for  $G$ .*

- (a) *If  $H$  is 3-connected or a triangle, then a pattern gadget  $P = P(H, G, \mathcal{G}, q)$  exists.*
- (b) *If  $H$  is a cycle of length at least four, then a pattern gadget  $P = P(H, G, \mathcal{G}, q)$  exists.*
- (c) *If  $H \cong K_t \cdot K_2$  and  $q = 2$  and  $G$  does not contain a copy of  $H$ , then a pattern gadget  $P = P(H, G, \mathcal{G}, q)$  exists. Further, we can ensure that in the 2-colorings in (P3) every monochromatic copy of  $K_t$  using a vertex from  $G$  is fully contained in  $G$ .*

*Further, in parts (a) and (b), the pattern gadget can be taken so that  $(P, G)$  is  $H$ -robust, and in part (c), it can be taken so that  $(P, G)$  is  $K_t$ -robust.*

Before we give the proof of Theorem 2.4 in section 2.3, we need to introduce two different simpler gadgets, known as *signal senders* and *indicators*, in section 2.1, and to construct a generalization of the latter, which we will call *generalized negative indicators*, in section 2.2.

**2.1. Signal senders and indicators.** Signal senders were introduced by Burr, Erdős, and Lovász [4] for the construction of BEL gadgets when  $H \cong K_t$  for  $t \geq 3$  and  $q = 2$ .

**DEFINITION 2.5.** Let  $q \geq 2$  and  $d \geq 1$  be given integers, and let  $H$  be a graph. A *positive signal sender*  $S = S^+(H, e, f, q, d)$  for  $H$  is a graph that contains two distinguished edges  $e, f \in E(S)$ , called the *signal edges* of  $S$ , such that the following properties hold:

- (S1)  $S \not\rightarrow_q H$ .
- (S2) In any  $H$ -free  $q$ -coloring of  $S$ , the edges  $e$  and  $f$  have the same color.
- (S3)  $\text{dist}_S(e, f) \geq d$ .

A *negative signal sender*  $S = S^-(H, e, f, q, d)$  for  $H$  is defined similarly, except that the words “the same color” in (S2) are replaced by “different colors.”

An *interior* vertex of a signal sender is a vertex that is not incident to either of the signal edges. The *interior* of a signal sender is the set of all interior vertices.

Signal senders are known to exist for some important classes of graphs, as given by Theorem 2.6 below. Part (a) is due to Rödl and Siggers [23], generalizing results of Burr, Erdős, and Lovász [4] and Burr, Nešetřil, and Rödl [6], part (b) is due to Siggers [25], and part (c) follows from a result in the PhD thesis of Grinshpun [15, Lemma 2.6.3] combined with the result of Fox et al. [12] concerning  $s_2(K_t \cdot K_2)$ .

**THEOREM 2.6.**

- (a) *For all integers  $q \geq 2$  and  $d \geq 1$  and every graph  $H$  that is 3-connected or isomorphic to  $K_3$ , there exist positive and negative signal senders in which the distance between the signal edges is at least  $d$ .*
- (b) *For all integers  $q \geq 2$ ,  $d \geq 1$ , and  $t \geq 4$ , there exist positive and negative signal senders for  $C_t$  with girth  $t$  and distance at least  $d$  between the signal edges.*
- (c) *For  $q = 2$  and for all integers  $t \geq 3$  and  $d \geq 1$ , there exist positive and negative signal senders for  $K_t \cdot K_2$  in which the distance between the signal*



edges is at least  $d$ . Further, a signal sender  $S$ , positive or negative, with signal edges  $e$  and  $f$  can be chosen so that  $S$  has a  $K_t \cdot K_2$ -free 2-coloring in which all edges incident to  $e$  (resp.,  $f$ ) have a different color from  $e$  (resp.,  $f$ ) and none of the vertices of  $e$  and  $f$  is contained in a monochromatic copy of  $K_t$ .

Before we continue, we make a few remarks about Theorem 2.6. First, in [15], Grinshpun does not explicitly prove that signal senders exist for  $K_3 \cdot K_2$ ; however, his proof easily extends to this case. Further, part (c) is actually a slight strengthening of Grinshpun's result: His result is stated only in terms of negative signal senders and provides a special coloring in which neither signal edge is incident to a monochromatic copy of  $K_t$  but only one of the signal edges, say,  $f$ , is required to have a color different from all edges incident to it. We can derive the version stated above easily. Let  $S'$  be the signal sender constructed by Grinshpun. To construct a positive signal sender  $S^+$  as in Theorem 2.6(c), take two copies of  $S'$  and identify the two copies of  $e$ ; similarly, to construct a negative signal sender as in Theorem 2.6(c), take a copy of  $S^+$  and a copy of  $S'$  and identify the edge  $e$  with one of the signal edges of  $S^+$ . As a final remark, in the original manuscripts where (b) and (c) appear, it is not shown explicitly that the distance between the signal edges can be arbitrarily large. However, it is easy to see that this is indeed the case. Both of the constructions do guarantee that the signal edges are not incident to each other, which means that we can increase the distance between the signal edges by stringing several signal senders together (that is, taking signal senders  $S_1, \dots, S_r$  and, for each  $i \in \{2, \dots, r-1\}$ , identifying one signal edge of  $S_i$  with a signal edge of  $S_{i-1}$  and the other with a signal edge of  $S_{i+1}$ ; if we take  $S_1, \dots, S_{r-1}$  to be positive signal senders, then the resulting signal sender is of the same type (positive or negative) as  $S_r$ ).

Indicators were introduced by Burr, Faudree, and Schelp in [5] for two colors and generalized by Clemens, Liebenau, and Reding in [7] to multiple colors. Together with signal senders, these graphs will serve as basic building blocks for our construction of pattern gadgets. For this, we need to modify slightly the definition appearing in [7], as given below. In addition, we will need both positive and negative indicators.

**DEFINITION 2.7.** Let  $q \geq 2$  and  $d \geq 1$ , and let  $H$  and  $F$  be graphs such that  $H \not\subseteq F$ . A *positive indicator*  $I = I^+(H, F, e, q, d)$  for  $H$  is a graph such that the following properties hold:

- (I1)  $F \subseteq_{ind} I$  and  $e \in E(I)$  with  $\text{dist}_I(F, e) \geq d$ .
- (I2) There exists an  $H$ -free  $q$ -coloring of  $I$  in which  $F$  is monochromatic.
- (I3) For every  $H$ -free  $q$ -coloring  $c$  of  $I$  in which  $F$  is monochromatic, we have  $c(e) = c(F)$ .
- (I4) For any nonconstant coloring  $\varphi_F : E(F) \rightarrow [q]$  and  $k \in [q]$ , there exists an  $H$ -free coloring  $c : E(I) \rightarrow [q]$  such that  $c|_F = \varphi_F$  and  $c(e) = k$ .

If  $I$  is a positive indicator with parameters  $H, F, e, q$ , and  $d$ , we call  $I$  a *positive*  $(H, F, e, q, d)$ -indicator. In this case, we call  $F$  the *indicator subgraph* and  $e$  the *indicator edge* of  $I$ .

A *negative indicator*  $I = I^-(H, F, e, q, d)$  is the same except that in property (I3) we replace “ $c(e) = c(F)$ ” with “ $c(e) \neq c(F)$ .”

An *interior* vertex of an indicator is a vertex that belongs to neither the indicator subgraph nor the indicator edge. The *interior* of an indicator is the set of all interior vertices.

The construction of indicators for the case when  $H$  is 3-connected or isomorphic to  $K_3$  was given in [5] for two colors and in [7] for more than two colors, where (I4)

is replaced with a similar yet slightly weaker property. Essentially the same constructions work for 3-connected graphs as well as cycles and cliques with a pendant edge with this new property (I4). In our constructions, however, we need to ensure that when we put together several gadgets and later on color each of them avoiding a monochromatic copy of our target graph  $H$ , there is still no monochromatic  $H$  in the resulting graph. We do not want to accidentally create monochromatic copies that use vertices from several different pieces of our construction. While we can get this almost immediately for 3-connected graphs, in the latter two cases we need to maintain some extra properties. Despite these additional technicalities and the slight modification in our definition of indicators, our proofs that the constructions given in [5] and [7] indeed give the required positive indicators are very similar to the proofs presented in the original papers. This is why we choose to omit the proof of Theorem 2.8 here; for the convenience of the reader we include it in the appendix of the arXiv version of this paper.

**THEOREM 2.8.** *Let  $q \geq 2$  and  $d \geq 1$  be integers,  $H$  be a graph, and  $F$  be a graph with  $e(F) \geq 2$  such that  $H \not\subseteq F$ .*

- (a) *If  $H$  is a 3-connected graph or  $H \cong K_3$ , then there exist a positive indicator  $I = I^+(H, F, e, q, d)$  and a negative indicator  $I = I^-(H, F, e, q, d)$ .*
- (b) *If  $H \cong C_t$  for  $t \geq 4$  and  $\text{girth}(F) > t$ , then there exist a positive indicator  $I = I^+(H, F, e, q, d)$  and a negative indicator  $I = I^-(H, F, e, q, d)$ , each with girth  $t$ .*
- (c) *If  $H \cong K_t \cdot K_2$  for  $t \geq 3$  and  $q = 2$ , then there exist a positive indicator  $I = I^+(H, F, e, q, d)$  and a negative indicator  $I = I^-(H, F, e, q, d)$ , each satisfying the following additional property: The  $H$ -free 2-colorings in (I2) and (I4) can be chosen so that none of the vertices of  $F$  and  $e$  is a vertex of a monochromatic copy of  $K_t$  and all edges incident to  $e$  have a different color from  $e$ .*

*Further, in parts (a) and (b) the indicators can be taken so that  $(I, F)$  is  $H$ -robust and in part (c) we can ensure that  $(I, F)$  is  $K_t$ -robust.*

Throughout the paper, we will often say that we *join* or *connect* two edges  $e_1, e_2$  of a given graph by a signal sender. What we mean by that is that we create a vertex-disjoint copy of a signal sender  $S$  and identify its signal edges with  $e_1$  and  $e_2$ , that is, the signal sender does not share any vertices or edges with the original graph except for the (vertices of the) signal edges. Similarly, joining or connecting a subgraph  $F$  and an edge  $e$  by an indicator will mean that we create a vertex-disjoint copy of the indicator and identify the indicator subgraph with  $F$  and the indicator edge with  $e$ . We will also use the same terminology in the context of generalized negative indicators, defined in the next section.

**2.2. Generalized negative indicators.** Before we can prove the existence of pattern gadgets as stated in Theorem 2.4, we will first need to construct slightly weaker gadget graphs, which we call generalized negative indicators.

Recall that a negative indicator  $I = I^-(H, F, e, q, d)$  comes with an indicator subgraph  $F$  and an indicator edge  $e$  and has the following property: In any  $H$ -free  $q$ -coloring of  $I$  that colors  $F$  monochromatically,  $e$  needs to get a color different from that of  $F$ ; but once  $F$  is not monochromatic, we can extend the  $q$ -coloring to an  $H$ -free  $q$ -coloring of  $I$ , independently of which color is chosen for  $e$ . That is, in short, when  $F$  is monochromatic we get some information on the color given to  $e$ , while otherwise we do not.

The gadgets  $I^*$  described in the following will generalize this concept by replacing  $e$  with another graph  $G$ . Now, whenever the indicator subgraph  $F$  is monochromatic in an  $H$ -free  $q$ -coloring of  $I^*$ , we again want to get some information on the coloring given to  $G$ , namely, that a certain color pattern is forced on  $G$ . Otherwise, when  $F$  is not monochromatic, we do not get any information on  $G$  in the sense that we can still color this subgraph by any  $H$ -free  $q$ -coloring and then find an  $H$ -free extension to  $I^*$ . We give a precise definition below.

**DEFINITION 2.9.** Let  $q \geq 2$  and  $d \geq 1$  be integers, and let  $H, F$ , and  $G$  be graphs with  $H \not\subseteq F$ . Further, let  $G = G_1 \cup G_2 \cup \dots \cup G_{q-1}$  be a partition with  $H \not\subseteq G_k$  for every  $k \in [q-1]$ . We call a graph  $I^* = I^*(H, F, \{G_k\}_{k \in [q-1]}, q, d)$  a *generalized negative indicator* if the following properties hold:

- (GI1)  $F, G \subseteq_{ind} I^*$  and  $\text{dist}_{I^*}(F, G) \geq d$ .
- (GI2) There exists an  $H$ -free  $q$ -coloring of  $I^*$  such that  $F$  is monochromatic.
- (GI3) In any  $H$ -free coloring  $c : E(I^*) \rightarrow [q]$  in which  $F$  is monochromatic, each of the graphs  $G_i$  needs to be monochromatic so that  $\{c(F), c(G_1), \dots, c(G_{q-1})\} = [q]$ .
- (GI4) Let  $\varphi_F : E(F) \rightarrow [q]$  be any nonconstant coloring, and let  $\varphi_G : E(G) \rightarrow [q]$  be any  $H$ -free coloring. Then there exists an  $H$ -free coloring  $c : E(I^*) \rightarrow [q]$  such that  $c|_F = \varphi_F$  and  $c|_G = \varphi_G$ .

If  $I^*$  is a generalized negative indicator with parameters  $H, F, \{G_k\}_{k \in [q-1]}, q$ , and  $d$ , we call  $I^*$  a *generalized negative  $(H, F, \{G_k\}_{k \in [q-1]}, q, d)$ -indicator*. In this case, we call  $F$  and  $G$  the *indicator subgraphs* of  $I^*$ .

An *interior* vertex of a generalized negative indicator is a vertex that belongs to neither of the indicator subgraphs. The *interior* of a generalized negative indicator is the set of all interior vertices.

The following lemma states that if  $H$  is 3-connected or isomorphic to a cycle or  $K_t \cdot K_2$ , then generalized negative indicators that satisfy additional robustness properties exist for  $H$ .

**LEMMA 2.10.** *Let  $q \geq 2$  and  $d \geq 1$  be integers and  $H, F$ , and  $G$  be graphs with  $H \not\subseteq F$ . Further, let  $G = G_1 \cup \dots \cup G_{q-1}$  be a partition such that  $H \not\subseteq G_k$  for every  $k \in [q-1]$ .*

- (a) *If  $H$  is 3-connected or  $H \cong K_3$ , then a generalized negative indicator  $I^* = I^*(H, F, \{G_k\}_{k \in [q-1]}, q, d)$  exists.*
- (b) *If  $H \cong C_t$  for  $t \geq 4$  and  $\text{girth}(F) > t$ , then a generalized negative indicator  $I^* = I^*(H, F, \{G_k\}_{k \in [q-1]}, q, d)$  exists.*
- (c) *If  $H \cong K_t \cdot K_2$  for  $t \geq 3$  and  $q = 2$ , then a generalized negative indicator  $I^* = I^*(H, F, \{G_k\}_{k \in [q-1]}, q, d)$  with the following additional property exists: The  $H$ -free 2-colorings in (GI2) and (GI4) can be chosen so that every monochromatic copy of  $K_t$  using a vertex from  $F \cup G$  is contained fully in  $F \cup G$ .*

*Further, in parts (a) and (b), the generalized negative indicator can be taken so that  $(I^*, F)$  and  $(I^*, G)$  are  $H$ -robust. In part (c), we can ensure that  $(I^*, F)$  and  $(I^*, G)$  are  $K_t$ -robust.*

*Proof.* Let  $q, d, H, F$ , and  $G$  be as given, and without loss of generality assume that  $d \geq v(H) + 1$ . Let  $M_1, \dots, M_{q-1}$  be matchings of size  $q$ , let  $P_1, \dots, P_{q-1}$  be matchings of size two, and let  $e_k$  be a fixed edge of  $P_k$  for each  $k \in [q-1]$ .

In order to construct  $I^*$ , we take the vertex-disjoint union of  $F, G$  and all of the above matchings and we join them with signal senders and indicators in the following way:

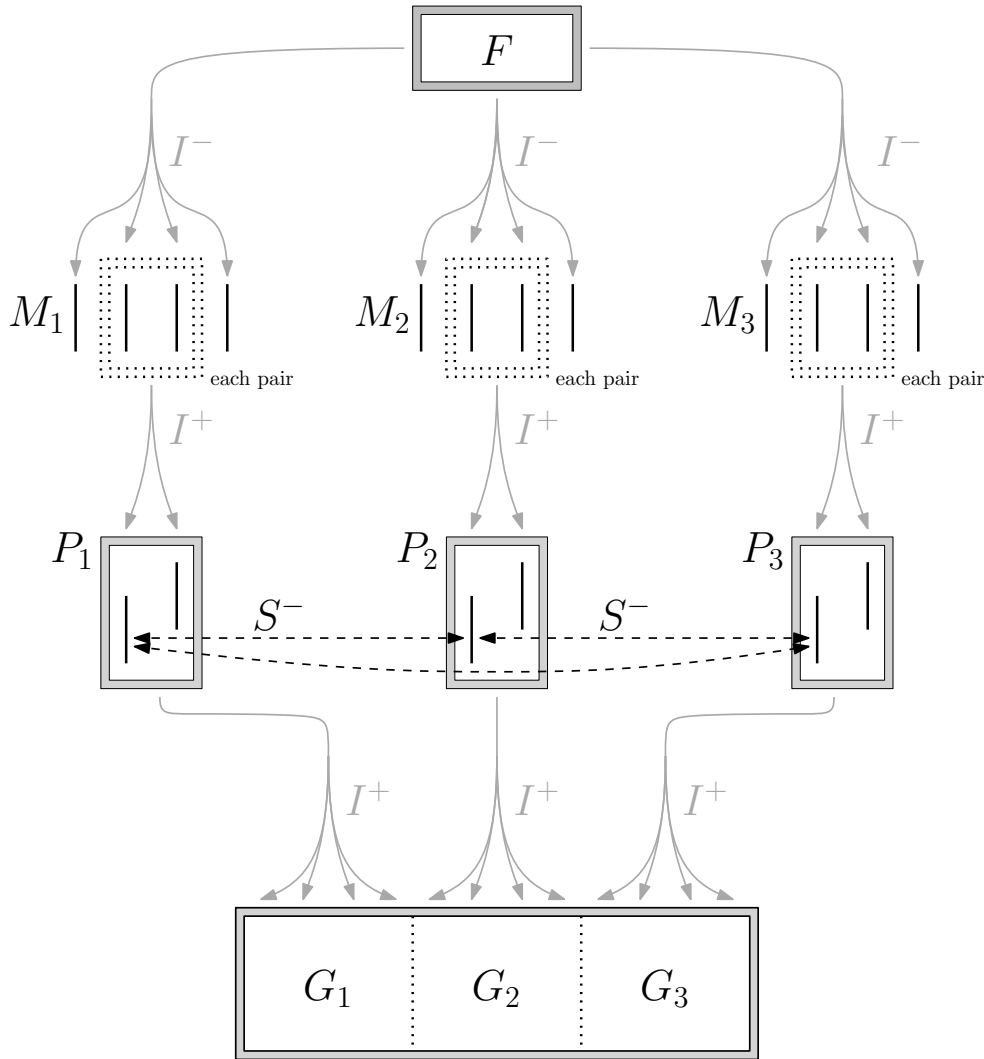


FIG. 1. Generalized negative indicator for  $q = 4$ .

- (i) For every  $k \in [q - 1]$  and every edge  $m \in M_k$ , join  $F$  and  $m$  by a negative  $(H, F, m, q, d)$ -indicator.
- (ii) For every  $k \in [q - 1]$ , every submatching  $S \subseteq M_k$  of size two, and every edge  $p \in P_k$ , join  $S$  and  $p$  by a positive  $(H, S, p, q, d)$ -indicator.
- (iii) For every  $1 \leq k_1 < k_2 \leq q - 1$ , join the distinguished edges  $e_{k_1} \in P_{k_1}$  and  $e_{k_2} \in P_{k_2}$  by a negative signal sender  $S^- = S^-(H, e_{k_1}, e_{k_2}, q, d)$ .
- (iv) For every  $k \in [q - 1]$  and every edge  $g \in E(G_k)$ , join  $P_k$  and  $g$  by a positive  $(H, P_k, g, q, d)$ -indicator.

Moreover, let all the indicators satisfy the robustness property promised by Theorem 2.8, respectively. When  $H$  is a cycle of length  $t \geq 4$ , choose the gadgets in (i)–(iv) so that their girth equals  $t$ . When  $H \cong K_t \cdot K_2$  for some  $t \geq 3$  and  $q = 2$ , choose these gadgets so that they have a  $K_t \cdot K_2$ -special 2-coloring. Note that the existence of all these gadgets and colorings is given by Theorems 2.6 and 2.8. An illustration of the construction for the case  $q = 4$  can be found in Figure 1.

Let  $M_k = \{m_1^k, \dots, m_q^k\}$  for every  $k \in [q-1]$ . Before showing that  $I^*$  satisfies (GI1)–(GI4), we first discuss where copies of  $H$  can be located in the graph  $I^*$ . Note that from the following two observations we immediately obtain the desired robustness properties as stated in Lemma 2.10.

*Observation 2.11.* Let  $H$  be 3-connected or a cycle. Let  $I'$  be a graph obtained from  $I^*$  by adding two new vertex sets  $S_F$  and  $S_G$  and any collection of edges within  $S_F \cup V(F)$  and within  $S_G \cup V(G)$ . Then every copy of  $H$  in  $I'$  is fully contained in one of the indicators from (i), (ii), or (iv), in one of the signal senders from (iii), or in one of the subgraphs induced by  $S_F \cup V(F)$  or  $S_G \cup V(G)$ .

*Proof.* For a contradiction, assume that some copy  $H'$  of  $H$  in  $I'$  forms a counterexample. Consider first the case when  $H'$  uses a vertex  $v \in S_G \cup V(G)$ . Since  $H'$  is a counterexample, we have  $V(H') \not\subseteq S_G \cup V(G)$ . Hence,  $H'$  needs to use an interior vertex of one of the indicators in (iv); without loss of generality, assume it is an indicator  $I_{P_1}^+$  joining  $P_1$  with an edge of  $G_1$ . We then have  $\text{dist}_{I^*}(P_1, G) \geq d > v(H')$  by property (I1) of the indicators in (iv), and thus, since  $H'$  is 3-connected or a triangle or a cycle with  $v(H') = \text{girth}(I_{P_1}^+)$ , it follows that  $H' \subseteq I_{P_1}^+$ , a contradiction. We may therefore assume that  $H'$  is vertex-disjoint from  $S_G \cup V(G)$ .

Consider next the case when  $H'$  uses a vertex  $v \in S_F \cup V(F)$ . As before, we have  $V(H') \not\subseteq S_F \cup V(F)$ . Hence,  $H'$  needs to use an interior vertex of an indicator in (i); without loss of generality, assume it is an indicator  $I_1$  between  $F$  and an edge  $m \in M_1$ . But then, since  $\text{dist}_{I_1}(m, F) \geq d > v(H')$  by property (I1) and since  $(I_1, F)$  is  $H$ -robust by Theorem 2.8, we conclude that  $H' \subseteq I_1$  must hold, contradicting our assumption. Hence, we may also assume that  $H'$  is vertex-disjoint from  $S_F \cup V(F)$ .

Now, if  $H'$  uses an interior vertex of one of the signal senders  $S^-$  in (iii), say, between the edges  $e_{k_1}$  and  $e_{k_2}$ , then again, using that  $\text{dist}_S(e_{k_1}, e_{k_2}) \geq d$  by property (S3) and that  $H'$  is 3-connected or isomorphic to a triangle or  $H'$  is a cycle with  $v(H') = \text{girth}(S)$ , we deduce that  $H'$  must be fully contained in that signal sender.

Next, if  $H'$  uses an interior vertex of one of the indicators in (i), (ii), or (iv), using the same argument and the robustness properties of our indicators, guaranteed by Theorem 2.8 for positive and negative indicators, respectively, we again conclude that  $H'$  must be fully contained in that indicator.

Hence, we are left with the case when  $H'$  uses neither vertices from  $S_F \cup V(F)$ , nor vertices from  $S_G \cup V(G)$ , nor interior vertices from one of the gadgets in (i)–(iv). But then  $H' \subseteq \bigcup_{k \in [q-1]} (M_k \cup P_k)$ , which contradicts the fact that  $H'$  contains at least one cycle.  $\square$

*Observation 2.12.* Let  $H \cong K_t \cdot K_2$ . Let  $I'$  be a graph obtained from  $I^*$  by adding two new vertex sets  $S_F$  and  $S_G$  and any collection of edges within  $S_F \cup V(F)$  and within  $S_G \cup V(G)$ . Then every copy of  $K_t$  in  $I'$  is fully contained in one of the indicators from (i), (ii), or (iv), in one of the signal senders from (iii), or in one of the subgraphs induced by  $S_F \cup V(F)$  or  $S_G \cup V(G)$ .

*Proof.* The proof is analogous to the previous proof, except that we use the robustness properties of all gadget graphs with respect to  $K_t$ , guaranteed by Theorem 2.8 for the indicators in (i), (ii), and (iv).  $\square$

It remains to show that  $I^*$  satisfies (GI1)–(GI4) and to verify the additional property required in case (c) regarding the existence of  $K_t \cdot K_2$ -special 2-colorings for (GI2) and (GI4).

(GI1) The graph  $F$  is an induced subgraph of  $I^*$ , as it is an induced subgraph of each of the negative indicators in (i) by property (I1). Also  $G$  is an induced subgraph

of  $I^*$ , since in the construction of  $I^*$  we attach gadget graphs to single edges of  $G$  without adding any further edges inside  $V(G)$ . Moreover, we have  $\text{dist}_{I^*}(F, G_k) \geq d$ , since, for every  $k \in [q-1]$  and every  $m \in M_k$ , the joining  $(H, F, m, q, d)$ -indicator  $I_F^-$  from (i) satisfies  $\text{dist}_{I_F^-}(F, m) \geq d$  by property (I1).

(GI2) We define a coloring  $c : E(I^*) \rightarrow [q]$  as follows:

- Give color 1 to the edges of  $F$ .
- For every  $k \in [q-1]$ , give color  $k+1$  to the edges  $m_1^k$  and  $m_2^k$ .
- For every  $k \in [q-1]$ , color the edges of  $M_k \setminus \{m_1^k, m_2^k\}$  such that each color from  $[q] \setminus \{1, k+1\}$  is used exactly once.
- For every  $k \in [q-1]$ , give color  $k+1$  to the edges of  $P_k$  and  $G_k$ .
- Finally, extend this coloring to each of the indicators and signal senders in (i)–(iv) so that none of these contains a monochromatic copy of  $H$ . In case (c), choose these colorings to be  $K_t \cdot K_2$ -special.

The extension in the last step of the coloring is possible for the following reason: For the indicators in (i), we can find such an extension by properties (I2) and (I3) for negative indicators and since  $c(F) = 1 \neq c(m)$  for every  $k \in [q-1]$  and  $m \in M_k$ . For the indicators in (ii), consider two cases. If  $S = \{m_1^k, m_2^k\}$ , then we have  $c(S) = c(P_k) = k+1$ , and hence we can color as desired by properties (I2) and (I3). Otherwise, if  $S \in \binom{M_k}{2}$  is different from  $\{m_1^k, m_2^k\}$ , the coloring on  $S$  is not constant and hence we can extend as desired by property (I4). For the signal senders in (iii), the described extension is possible by properties (S1) and (S2) for negative signal senders and since  $c(e_{k_1}) \neq c(e_{k_2})$  for every distinct  $k_1, k_2 \in [q-1]$ . For the indicators in (iv), we again use properties (I2) and (I3) plus the fact that  $c(P_k) = c(G_k)$  for every  $k \in [q-1]$ .

It remains to check that the resulting coloring  $c$  on  $I^*$  is  $H$ -free. Consider first the case when  $H$  is a cycle or 3-connected. By Observation 2.11, we know that each copy of  $H$  must be fully contained in one of the gadgets in (i)–(iv) or in the graph  $G$ . By the choice of the coloring, we know that each of the gadgets is colored without a monochromatic copy of  $H$ . Moreover, the coloring  $c$  splits the graph  $G$  into color classes given by the subgraphs  $G_1, \dots, G_{q-1}$ , none of which contains a copy of  $H$  by the assumption of the lemma. Hence,  $c$  is  $H$ -free in this case.

Next, consider the case when  $H \cong K_t \cdot K_2$ . Assume that there is a monochromatic copy  $H'$  of  $H$ , and let  $K'$  denote the copy of  $K_t$  in  $H'$ . According to Observation 2.12,  $K'$  needs to be fully contained in one of the gadget graphs or in one of the subgraphs  $F$  or  $G$ . If  $H'$  is fully contained in one of these parts, then  $H'$  cannot be monochromatic by the same argument as above. Hence, we may assume that  $K'$  uses a vertex of one of the signal edges, indicator edges, or indicator subgraphs. If  $K'$  is contained in one of the gadget graphs, then by the choice of the  $K_t \cdot K_2$ -special coloring for this gadget graph,  $K'$  cannot be monochromatic, a contradiction.

So assume next that  $K' \subseteq G = G_1$ . We need to check that no edge adjacent to  $K'$  can be of the same color. Indeed, since  $H \not\subseteq G_1$  by the assumption of the lemma, every edge incident to  $K'$  must belong to one of the indicators from (iv) and must be incident to the corresponding indicator edge which is part of  $K'$ . But the 2-coloring of each indicator was chosen to be  $K_t \cdot K_2$ -special, so any such edge has the opposite color, and hence  $H'$  cannot be monochromatic, a contradiction. We are left with the case  $K' \subseteq F$ . As we have  $H \not\subseteq F$  by the assumption of the theorem, we know that any edge adjacent to  $K'$  must be part of one of the indicators from (i). But then  $H'$  is fully contained in such an indicator and hence cannot be monochromatic, as the coloring on every gadget is  $H$ -free, a contradiction.

Note that the last argument also shows half of the additional property in case (c), i.e., that the  $H$ -free 2-colorings in (GI2) can be chosen so that every monochromatic copy of  $K_t$  using a vertex from  $F \cup G$  is contained fully in  $F \cup G$ .

(GI3) Let  $c$  be any  $H$ -free coloring of  $I^*$  such that  $F$  is monochromatic, say,  $c(F) = 1$ . By properties (I2) and (I3) for negative indicators, the indicators in (i) make sure that all edges in the matchings  $M_k$  need to get a color different from 1. Then, by the pigeonhole principle, in each matching  $M_k$  there needs to be at least one color from  $[q] \setminus \{1\}$  that appears at least twice. For each matching  $M_k$ , fix one such color and denote it by  $c_k$ . By symmetry, we assume without loss of generality that  $c(m_1^k) = c(m_2^k) = c_k$ . By property (I3) for the indicators in (ii), we conclude that  $c(P_k) = c(e_k) = c_k$ . Similarly, using property (S2) for the signal senders in (iii), we obtain that all edges in  $\{e_1, \dots, e_{q-1}\}$  need to have distinct colors. Since color 1 is excluded, we may assume by symmetry that  $c_k = c(e_k) = k + 1$  and thus  $c(P_k) = k + 1$ . Then, applying property (I3) for the positive indicators in (iv) yields that  $c(G_k) = k + 1$  and hence  $\{c(F), c(G_1), \dots, c(G_k)\} = [q]$ .

(GI4) Let  $\varphi_F$  and  $\varphi_G$  satisfy the assumption in property (GI4). We define a coloring  $c : E(I^*) \rightarrow [q]$  as follows:

- Color  $F$  according to  $\varphi_F$ .
- Color  $G$  according to  $\varphi_G$ .
- For every  $k \in [q - 1]$  and  $\ell \in [q]$ , give color  $\ell$  to  $m_\ell^k$ .
- For every  $k \in [q - 1]$ , give color  $k$  to  $e_k$  and give color  $k + 1$  to the edge in  $P_k - e_k$ .
- Finally, extend this coloring to each of the indicators and signal senders in (i)–(iv) so that none of these contains a monochromatic copy of  $H$ . In case (c), choose these colorings to be  $K_t \cdot K_2$ -special.

The extension in the last step of the coloring is possible for the following reason: For the indicators in (i), we can find such an extension by property (I4) for negative indicators and since  $\varphi_F$  is not constant by assumption. For the indicators in (ii), such an extension exists by property (I4) and since no subgraph  $S \subseteq M_k$  of size two is colored monochromatically. For the signal senders in (iii), this extension is possible by properties (S1) and (S2) and since  $c(e_{k_1}) \neq c(e_{k_2})$  for every distinct  $k_1, k_2 \in [q - 1]$ . For the indicators in (iv), we again use property (I4) plus the fact that  $P_k$  is not monochromatic.

Finally, as in the discussion of (GI2), it follows that  $c$  must be  $H$ -free. Moreover, if  $H \cong K_t \cdot K_2$  and  $q = 2$ , then, taking a  $K_t \cdot K_2$ -special 2-coloring for each of the gadget graphs, we deduce that every monochromatic copy of  $K_t$  that uses a vertex from  $F \cup G$  is fully contained in  $F \cup G$ . That is, we obtain the second half of the additional property required in case (c). □

**2.3. Existence of pattern gadgets.** We now prove Theorem 2.4.

*Proof.* Set  $t = |\mathcal{G}|$ . For every  $g = \{G_1, \dots, G_q\} \in \mathcal{G}$ , fix an ordered color pattern  $\vec{g} = (G_1, \dots, G_q)$  with an arbitrary ordering of the subgraphs  $G_i \in g$ , and denote the  $j$ th component of  $\vec{g}$  by  $\vec{g}_j$ . Further, let  $\vec{\mathcal{G}} = \{\vec{g} : g \in \mathcal{G}\}$ . Choose  $r \in \mathbb{Z}_{\geq 1}$  such that

$$\binom{(r - 1)q + 1}{r} \geq t.$$

Fix a matching  $M$  of size  $(r - 1)q + 1$  and a surjection  $s : \binom{M}{r} \rightarrow \vec{\mathcal{G}}$ , which exists by the choice of  $r$ . We construct a pattern gadget  $P = P(H, G, \mathcal{G}, q)$  as follows. Take  $G$

together with the given family  $\mathcal{G}$  of  $H$ -free  $q$ -color patterns for  $G$ . Further, take the matching  $M$  to be vertex-disjoint from  $G$  and join submatchings of  $M$  and edges of  $G$  by generalized negative indicators and positive indicators as described below. For this, choose an integer  $d$  such that  $d > v(H)$ .

- (i) For every  $A \in \binom{M}{r}$  and every edge  $e \in E(s(A)_q)$ , join the submatching  $A$  and the edge  $e$  by a positive  $(H, A, e, q, d)$ -indicator.
- (ii) For every  $A \in \binom{M}{r}$ , join the submatching  $A$  and the graph  $G - (s(A))_q$  by a generalized negative  $(H, A, \{s(A)_k\}_{k \in [q-1]}, q, d)$ -indicator.

The existence of the indicators needed in (i) and (ii) is given by Theorem 2.8 and Lemma 2.10.

In the case when  $H \cong K_t \cdot K_2$  and  $q = 2$ , we additionally choose all gadgets so that they have  $K_t \cdot K_2$ -special 2-colorings as described in Theorem 2.8 and Lemma 2.10(c), respectively. Moreover, we choose all the indicators so that they satisfy the robustness properties described in Theorem 2.8 and Lemma 2.10. Then, analogously to Observations 2.11 and 2.12, we can prove the following.

*Observation 2.13.* Let  $P'$  be a graph obtained from  $P$  by adding a vertex set  $S$  and any collection of edges within  $S \cup V(G)$ . If  $H$  is 3-connected or a cycle, then every copy of  $H$  in  $P'$  is fully contained in one of the indicators from (i) or (ii) or in the subgraph induced by  $S \cup V(G)$ . If  $H \cong K_t \cdot K_2$ , then every copy of  $K_t$  in  $P'$  is fully contained in one of the indicators from (i) or (ii) or in the subgraph induced by  $S \cup V(G)$ .

Given this observation, it follows immediately that  $(P, G)$  is  $H$ -robust if  $H$  is 3-connected or a cycle and that  $(P, G)$  is  $K_t$ -robust if  $H \cong K_t \cdot K_2$ . Hence, it remains to verify that  $P$  satisfies (P1)–(P3) and that in the case when  $H \cong K_t \cdot K_2$  and  $q = 2$  we can find 2-colorings for (P3) as described in part (c) of Theorem 2.4.

(P1) Since  $P$  is constructed by attaching different gadgets to  $G$  without adding edges inside  $V(G)$ , we have  $G \subseteq_{ind} P$ .

(P2) Let  $c : E(P) \rightarrow [q]$  be any  $H$ -free coloring of  $P$ . By the pigeonhole principle, at least one color is used at least  $r$  times on the matching  $M$ . Without loss of generality, say  $c(A) = q$  for some  $A \in \binom{M}{r}$ . Consider the pattern  $g = \{s(A)_k\}_{k \in [q]}$ . By property (I3) of the indicators in (i), we deduce that every edge in  $E(s(A)_q)$  also needs to have color  $q$ . Moreover, by property (GI3) of the generalized negative indicators in (ii), each of the subgraphs  $s(A)_k$  with  $k \neq q$  is forced to be monochromatic, and all colors except for  $c(A) = q$  get used among these subgraphs. Hence,  $\{c_{|G}^{-1}(1), \dots, c_{|G}^{-1}(q)\} = g \in \mathcal{G}$ .

(P3) Let  $g = \{G_1, \dots, G_q\} \in \mathcal{G}$  be given. Fix an arbitrary set  $A_0 \in \binom{M}{r}$  such that  $s(A_0) = \vec{g}$ . Without loss of generality, assume that  $s(A_0)_k = G_k$  for every  $k \in [q]$ ; otherwise relabel the subgraphs in  $g$ . We define a coloring  $c : E(P) \rightarrow [q]$  as follows:

- Give color  $q$  to each edge in  $A_0$ .
- Color  $M \setminus A_0$  so that each color from  $[q - 1]$  appears exactly  $r - 1$  times.
- For every  $k \in [q]$ , give color  $k$  to the edges of  $G_k$ .
- Finally, extend this coloring to each of the gadgets in (i) and (ii) so that none of these contains a monochromatic copy of  $H$ . In case (c), choose these colorings to be  $K_t \cdot K_2$ -special.

We claim that the extension in the last step of the coloring is indeed possible. Recall that each gadget from (i) and (ii) is associated to a submatching  $A \in \binom{M}{r}$ . Suppose first that  $A = A_0$ . Then we have  $c(A) = q = c(s(A)_q)$ , and by properties (I2)



and (I3), we find an extension as desired for the corresponding positive indicators in (i). Moreover, we have  $c(s(A)_k) = k \neq q = c(A)$  for every color  $k \in [q - 1]$ . Hence, by properties (GI2) and (GI3), we find extensions as desired for the corresponding generalized negative indicators in (ii). Consider next the case when  $A \neq A_0$ . Then  $A$  is not monochromatic, since  $A_0$  is the only monochromatic subset of  $M$  of size  $r$ . Now, let  $I$  be any positive indicator between  $A$  and any edge  $e \in E(s(A)_q)$  as described in (i). Then, by property (I4), we find an extension for  $I$  as desired. Finally, let  $I$  be the generalized negative indicator from (ii) for the set  $A$ . Then, using property (GI4), we conclude analogously that an extension for  $I$  can be found.

Finally, we have  $\{G_1, \dots, G_q\} = \{c_G^{-1}(1), \dots, c_G^{-1}(q)\}$ . Since  $g = \{G_1, \dots, G_q\}$  is an  $H$ -free  $q$ -color pattern by the assumption of the theorem, we know that  $c_G$  is  $H$ -free. Now, if  $H$  is 3-connected or a cycle, then every copy of  $H$  in  $P$  that is not contained in  $G$  must be a subgraph of some indicator from (i) or (ii), according to Observation 2.13. But we already know that the coloring  $c$  is  $H$ -free on every indicator, and hence it is  $H$ -free on the whole graph  $P$ .

It remains to consider the case when  $H \cong K_t \cdot K_2$  and  $q = 2$ . Assume that there is a monochromatic copy  $H'$  of  $H$ , and let  $K'$  denote its copy of  $K_t$ . As above, if  $H'$  is fully contained in one of the indicators, then it cannot be monochromatic. Hence, we may assume that  $K'$  intersects the vertex set of an indicator edge or an indicator subgraph. Then, by the  $K_t \cdot K_2$ -special 2-colorings for the indicators, we know that  $K'$  needs to be a subgraph of  $G$ . Without loss of generality, let  $K' \subseteq G_1$ . Since  $G$  does not contain a copy of  $K_t \cdot K_2$  by assumption, we know that  $E_G(V(K'), V(G_2)) = \emptyset$ . Hence, the pendant edge  $f$  of  $H'$  needs to belong either to a positive indicator between some  $A \in \binom{M}{r}$  and some  $e \in E(K')$  or to a generalized negative indicator between some  $A \in \binom{M}{r}$  and the graph  $G_1 \supseteq K'$ . In the former case, the edge  $f$  needs to be incident to the indicator edge  $e$  and hence  $c(e) \neq c(f)$  by the  $K_t \cdot K_2$ -special 2-coloring of the corresponding positive indicator. In the latter case, we have  $c(f) \neq c(K')$  as the coloring of the generalized negative indicator was chosen to be  $H$ -free. Hence, in both cases  $H'$  cannot be monochromatic, a contradiction.  $\square$

**3. Applications of pattern gadgets.** In this section, we present several applications of the pattern gadgets constructed in the previous section. We first prove Theorems 1.3 and 1.5 directly. The proof of Theorem 1.4 is given later in the section as a consequence of a more general result about 3-connected graphs (Theorem 3.1).

*Proof of Theorem 1.3.* Let  $H \cong C_t$  and  $t \geq 4$  and  $q \geq 2$  be fixed. We first note that  $s_q(H) \geq q + 1$ . Indeed, suppose there is a minimal  $q$ -Ramsey graph  $G$  for  $C_t$  with a vertex  $v$  of degree at most  $q$ ; by the minimality of  $G$ , there exists a  $C_t$ -free  $q$ -coloring of  $G - v$ . Now, coloring the edges incident to  $v$  so that no two of them share a color gives a  $q$ -coloring of  $G$  with no monochromatic  $C_t$ , a contradiction.

We now turn our attention to showing that there can be arbitrarily many vertices of degree  $q + 1$ , also implying that  $s_q(C_t) = q + 1$ . Let  $k \geq 1$ . We now construct a minimal  $q$ -Ramsey graph for  $H$  with at least  $k$  vertices of degree  $q + 1$ . Our graph will be constructed in several steps. We refer the reader to Figure 2 for an illustration of our construction in the case when  $q = 2$ ,  $t = 6$ , and  $k = 3$ .

To begin with, let  $W$  be a set of  $q + 1$  vertices. For every  $u, w \in W$  and  $u \neq w$ , add  $q$  internally vertex-disjoint paths of length  $t - 2$  with  $u$  and  $w$  as endpoints. Call the resulting graph  $F$ . Let  $c_1 : E(F) \rightarrow [q]$  be a coloring of the edges of  $F$  such that, for every distinct  $u, w \in W$ , every path between  $u$  and  $w$  is monochromatic but no two such paths are monochromatic in the same color. Let  $c_2 : E(F) \rightarrow [q]$  be another coloring of the edges of  $F$  such that, for every distinct  $u, w \in W$ , no path between  $u$

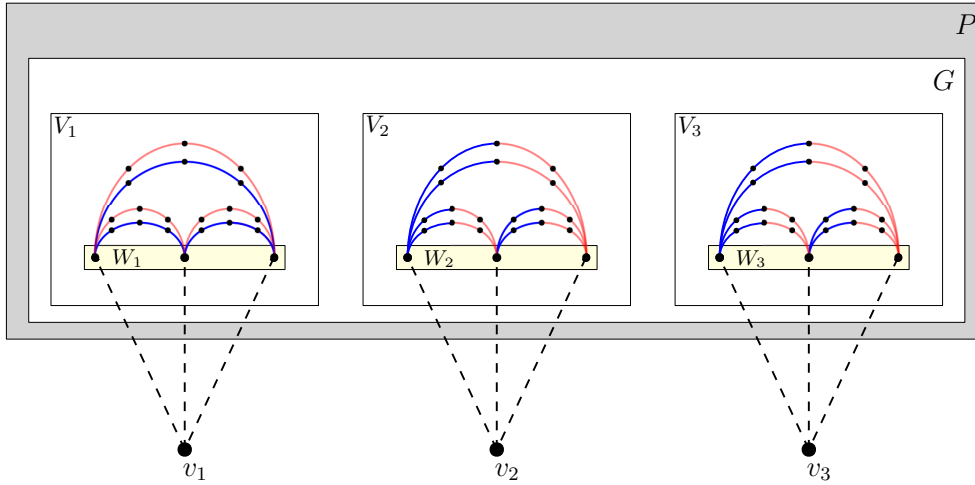


FIG. 2. Graph  $\tilde{G}$  for  $q = 2$ ,  $t = 6$ , and  $k = 3$ . Light solid lines correspond to color 1 and dark solid lines correspond to color 2. (Color available online.)

and  $w$  is monochromatic. We define  $f_1$  and  $f_2$ , two  $q$ -color patterns for  $F$ , by setting  $f_1 = \{c_1^{-1}(i)\}_{i \in [q]}$  and  $f_2 = \{c_2^{-1}(i)\}_{i \in [q]}$ . Note that  $f_1$  and  $f_2$  are  $H$ -free.

We now take  $k$  vertex-disjoint copies  $F_1, \dots, F_k$  of  $F$ , where  $F_i = (V_i, E_i)$  for all  $1 \leq i \leq k$ , and denote by  $W_i$  the subset of  $V_i$  corresponding to  $W$  in  $V(F)$ . Call this graph  $G$ , and define  $V = \cup_{i=1}^k V_i$ . Note that  $G \not\rightarrow_q H$ , since  $F \not\rightarrow_q H$ . Let  $\mathcal{G}$  be a family of  $q$ -color patterns for  $G$  such that  $g \in \mathcal{G}$  if and only if there exists an  $i \in [k]$  such that  $g[V_i] \cong f_1$  and  $g[V_j] \cong f_2$  for all  $j \neq i$ . Note that  $\mathcal{G}$  is a family of  $H$ -free  $q$ -color patterns for  $G$ .

By Theorem 2.4, we know that there exists a pattern gadget  $P = P(H, G, \mathcal{G}, q)$ . Moreover, we can choose the pattern gadget  $P$  in such a way that the pair  $(P, G)$  is  $H$ -robust. We add  $k$  additional vertices  $v_1, \dots, v_k$  to  $P$ , and for all  $i \in [k]$ , we add edges from  $v_i$  to all vertices in  $W_i$ . We call the resulting graph  $\tilde{G}$ .

We now show that  $\tilde{G} \rightarrow_q H$  and that each of the new vertices  $v_i$  is important for  $\tilde{G}$  to have this property, that is,  $\tilde{G} - v_i \not\rightarrow_q H$  for every  $i \in [k]$ . This then implies the existence of a minimal  $q$ -Ramsey graph for  $H$  with the desired properties. Indeed, consider any minimal  $q$ -Ramsey graph  $\tilde{G}' \subseteq \tilde{G}$ . Since  $\tilde{G} - v_i \not\rightarrow_q H$ , we know that  $v_i \in V(\tilde{G}')$  for every  $i \in [k]$ . Also  $q + 1 \leq s_q(C_t) \leq d_{\tilde{G}'}(v_i) \leq q + 1$ , which means that  $d_{\tilde{G}'}(v_i) = s_q(C_t) = q + 1$ .

First, we show that  $\tilde{G} \rightarrow_q H$ . Let  $c : E(\tilde{G}) \rightarrow [q]$  be a  $q$ -coloring of the edges of  $\tilde{G}$ , and assume that  $c$  is  $H$ -free. For each  $i \in [q]$ , define  $c_i = c^{-1}(i)$  to be the  $i$ th color class with respect to  $c$ . By property (P2) of the pattern gadget  $P$ , we know that  $g = \{c_1[V], \dots, c_q[V]\} \in \mathcal{G}$ ; by the definition of  $\mathcal{G}$ , there exists an  $i \in [k]$  such that  $\{c_1[V_i], \dots, c_q[V_i]\} \cong f_1$ . Without loss of generality, we may assume that  $i = 1$ . Consider the edges from  $v_1$  to the vertices of  $W_1$ . There are  $q + 1$  such edges and they are colored in  $q$  colors, so by the pigeonhole principle there are two vertices in  $W_1$ , say,  $u$  and  $w$ , such that  $c(v_1u) = c(v_1w)$ . Again without loss of generality, we may assume that  $c(v_1u) = 1$ . By our choice of  $f_1$ , we know that there is a monochromatic path of length  $t - 2$  in color 1 between the vertices  $u$  and  $w$ . This monochromatic path along with the edges  $v_1u$  and  $v_1w$  gives a monochromatic cycle of length  $t$ , contradicting our assumption.

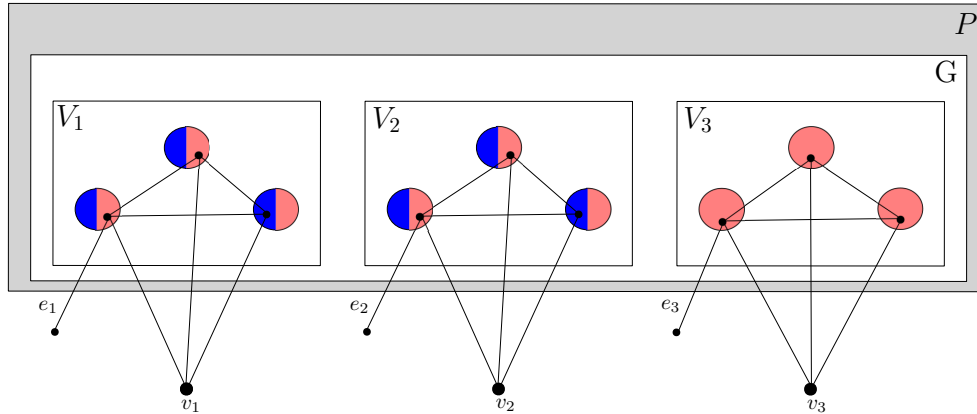


FIG. 3. Graph  $\tilde{G}$  for  $t = 4$  and  $k = 3$ . Light shading of discs corresponds to red, and dark shading corresponds to blue. (Color available online.)

Next, we show that  $\tilde{G} - v_i \not\rightarrow_q H$  for every  $i \in [k]$ . By symmetry, it is enough to show this for  $i = 1$ . Partition the vertices in  $V$  in the following way: For every  $\ell \in [k]$ , write  $G[V_\ell] = G_{\ell,1} \cup \dots \cup G_{\ell,q}$  so that  $\{G_{1,j}\}_{j \leq q} \cong f_1$  and  $\{G_{\ell,j}\}_{j \leq q} \cong f_2$  for  $\ell \neq 1$ . We define a coloring  $c : E(G) \rightarrow [q]$  by setting  $c(G_{\ell,j}) = j$  for every  $\ell \in [k]$  and  $j \in [q]$ . The  $q$ -color pattern on  $V$  defined by  $c$ , namely,  $\{c \mid G^{-1}(1), \dots, c \mid G^{-1}(q)\}$ , is in  $\mathcal{G}$ , and by property (P3), we can extend  $c$  to an  $H$ -free coloring of  $P$ . We then color the remaining edges in  $\tilde{G} - v_1$  arbitrarily and denote the resulting  $q$ -coloring of  $\tilde{G} - v_1$  by  $\tilde{c}$ . Since  $\tilde{c}|_P$  is  $H$ -free, any monochromatic copy of  $H$  in  $\tilde{G} - v_1$  needs to contain a vertex  $v_\ell$  for some  $\ell \geq 2$ . Now, due to the  $C_t$ -robustness of the pair  $(P, G)$ , any possible monochromatic copy of  $H$  must be contained in some  $V_\ell \cup \{v_\ell\}$ . Such a copy then needs to contain two vertices of  $W_\ell$  and a path of length  $t - 2$  between them. But we know that  $\{c|_{G[V_\ell]}(j)\}_{j \in [q]} \cong f_2$ , and by the definition of  $f_2$ , no such path is monochromatic. Hence, no monochromatic copy of  $H$  exists.  $\square$

*Proof of Theorem 1.5.* It was shown by Fox et al. [12] that  $s_2(K_t \cdot K_2) = t - 1$  for every  $t \geq 3$ . We now show that a minimal 2-Ramsey graph for  $K_t \cdot K_2$  can contain arbitrarily many vertices of this minimum degree.

Let  $H \cong K_t \cdot K_2$  for some  $t \geq 3$ , and let  $k \geq 1$  be fixed. Our construction of a minimal 2-Ramsey graph for  $H$  containing at least  $k$  vertices of degree  $t - 1$  will combine ideas similar to those in the proof of Theorem 1.3 with ideas from the construction given by Fox et al. [12]. We again refer the reader to Figure 3 for an illustration of the case  $t = 4$  and  $k = 3$ .

We begin by defining  $F$  to be the vertex disjoint union of  $t - 1$  copies of  $K_t$ . For every copy of  $K_t$ , we fix an arbitrary vertex and call the set of all these vertices  $W$ . Let  $c_1 : E(F) \rightarrow \{\text{red}, \text{blue}\}$  be a 2-coloring that colors every edge of  $F$  red. Let  $c_2 : E(F) \rightarrow \{\text{red}, \text{blue}\}$  be another 2-coloring of the edges of  $F$  such that no copy of  $K_t$  is monochromatic (in either color). We define two color patterns  $f_1$  and  $f_2$  for  $F$  by setting  $f_1 = \{c_1^{-1}(\text{red}), c_1^{-1}(\text{blue})\}$  and  $f_2 = \{c_2^{-1}(\text{red}), c_2^{-1}(\text{blue})\}$ . Note that  $f_1$  and  $f_2$  are  $H$ -free.

Now take  $k$  vertex-disjoint copies  $F_1, \dots, F_k$  of  $F$ , where  $F_i = (V_i, E_i)$  for  $i \in [k]$ , and let  $W_i$  be the subset of  $V_i$  corresponding to the set  $W$  in  $V(F)$ . Call this graph  $G$ , and define  $V = \cup_{i=1}^k V_i$ . Note that  $G$  does not contain any copies of  $H$ . Let  $\mathcal{G}$  be a family of 2-color patterns for  $G$  such that  $g \in \mathcal{G}$  if and only if there exists an  $i \in [k]$  such that  $g[V_i] \cong f_1$  and  $g[V_j] \cong f_2$  for all  $j \neq i$ . Note that  $\mathcal{G}$  is a family of  $H$ -free 2-color patterns for  $G$ .

By Theorem 2.4, we deduce that there exists a pattern gadget  $P = P(H, G, \mathcal{G}, 2)$ . Moreover, we can choose the pattern gadget  $P$  in such a way that the pair  $(P, G)$  is  $K_t$ -robust and that for property (P3) there is always an  $H$ -free 2-coloring such that if a monochromatic copy of  $K_t$  uses a vertex from  $G$ , then it lies entirely in  $G$ . We add  $k$  additional vertices  $v_1, \dots, v_k$  to  $P$  with edges from  $v_i$  to all vertices of  $W_i$  for all  $i \in [k]$ ; also, for all  $i \in [k]$ , we add an edge between each pair of distinct vertices in  $W_i$ . Last we choose an arbitrary vertex in  $W_i$  and add a pendant edge  $e_i$  incident to that vertex. We call the resulting graph  $\tilde{G}$ .

We now show that  $\tilde{G} \rightarrow_2 H$  and that  $\tilde{G} - v_i \not\rightarrow_2 H$  for every  $i \in [k]$ . This, as argued in the proof of Theorem 1.3, implies the existence of a minimal 2-Ramsey graph with the desired properties.

First we show that  $\tilde{G} \rightarrow_2 H$ . Let  $c : E(\tilde{G}) \rightarrow \{red, blue\}$  be a 2-coloring of the edges of  $\tilde{G}$ ; assume that  $c$  is  $H$ -free. Define  $c_{red} = c^{-1}(red)$  and  $c_{blue} = c^{-1}(blue)$  to be the two color classes with respect to  $c$ . By property (P2) of the pattern gadget  $P$ , we know that  $g = \{c_{red}[V], c_{blue}[V]\} \in \mathcal{G}$ , and by the definition of  $\mathcal{G}$ , there exists an  $i \in [k]$  such that  $\{c_{red}[V_i], c_{blue}[V_i]\} \cong f_1$ . Without loss of generality, we may assume that  $i = 1$  and every edge inside  $V_i$  is red. Consider the edges with endpoints in the set  $W' = W_1 \cup \{v_1\}$ . Since  $c$  is an  $H$ -free coloring of  $\tilde{G}$  and each such edge  $e$  has at least one endpoint in  $W_1$  (and is hence incident to an all-red copy of  $K_t$ ), we obtain that  $c(e) = blue$ . As a result, the graph induced by  $W'$  is a monochromatic blue copy of  $K_t$ . Now, the pendant edge  $e_1$  is incident to monochromatic copies of  $K_t$  in both colors and thus creates a monochromatic copy of  $H$  irrespective of its color. This contradicts our assumption.

Next, we show that, for every  $i \in [k]$ , we have  $\tilde{G} - v_i \not\rightarrow_2 H$ . By symmetry, it suffices to show this for  $i = 1$ . For every  $\ell \in [k]$ , take a partition  $G[V_\ell] = G_{\ell,red} \cup G_{\ell,blue}$  such that  $\{G_{1,red}, G_{1,blue}\} \cong f_1$  and  $\{G_{\ell,red}, G_{\ell,blue}\} \cong f_2$  for  $\ell \neq 1$ . We define a coloring  $c : E(\tilde{G}) \rightarrow \{red, blue\}$  by first setting  $c(G_{\ell,j}) = j$  for every  $\ell \in [k]$  and  $j \in \{red, blue\}$ . The color pattern defined on  $G$  by  $c$  is in  $\mathcal{G}$ , and by property (P3) of  $P$ , we can extend this to all of  $P$  so that the coloring  $c|_P$  is  $H$ -free and has the following additional property:

- (P) If a monochromatic copy of  $K_t$  in the coloring  $c|_P$  uses a vertex from  $G$ , then it lies entirely in  $G$ .

Now, for every  $\ell \geq 2$ , color one edge between  $v_\ell$  and  $W_\ell$  red and color the remaining edges in  $E_{\tilde{G}}(W_\ell \cup \{v_\ell\}) \cup \{e_\ell\}$  blue. Further, color all edges in  $E_{\tilde{G}}(W_1) \cup \{e_1\}$  with the color not used on  $G[V_1]$  (recall that  $G[V_1]$  was colored monochromatically as  $\{G_{1,red}, G_{1,blue}\} \cong f_1$ ).

We claim that this coloring is  $H$ -free. For a contradiction, assume that there is a monochromatic copy  $H'$  of  $H$  produced by the coloring  $c$ . Since  $c|_P$  is  $H$ -free,  $H'$  needs to use at least one edge  $e_0$  from  $E_{\tilde{G}}(W_1) \cup \{e_1\}$  or from  $E_{\tilde{G}}(W_\ell \cup \{v_\ell\}) \cup \{e_\ell\}$  for some  $\ell \geq 2$ .

Consider first the case when  $e_0 \in E_{\tilde{G}}(W_1) \cup \{e_1\}$ . We know that  $G[V_1]$  is monochromatic and that  $e_0$  has the opposite color, say,  $G[V_1]$  is red and  $e_0$  is blue. Then, by property (P) and the fact that  $|W_1| = t - 1$ , there can be no blue copy of  $K_t$  in the subgraph induced by the set  $V_1 \supseteq W_1$ . But this means that  $e_0$  cannot be part of a blue copy of  $K_t \cdot K_2$ , a contradiction.

Consider now the case when  $e_0 \in E_{\tilde{G}}(W_\ell \cup \{v_\ell\}) \cup \{e_\ell\}$  for some  $\ell \geq 2$ , and assume without loss of generality that  $\ell = 2$ . By the  $K_t$ -robustness of the pair  $(P, G)$ , the copy  $H'$  of  $H$  must be contained within  $E_{\tilde{G}}(V_2 \cup \{v_2\}) \cup \{e_2\}$ . Since  $c|_{G[V_2]} \cong f_2$ , i.e., the copies of  $K_t$  in  $F_2$  are not monochromatic, and  $c$  satisfies property (P), we

obtain that  $G[V_2]$  does not contain a monochromatic copy of  $K_t$ . From this and the fact that  $e_2$  is a pendant edge it follows that the vertices of the copy of  $K_t$  in  $H'$  must be contained entirely in  $W_2 \cup \{v_2\}$ . But this set contains precisely  $t$  vertices that do not form a monochromatic copy of  $K_t$ , again giving a contradiction.  $\square$

Before turning to the proof of Theorem 1.4, we state and prove a more general statement concerning 3-connected graphs. Roughly speaking, it reduces the problem of showing  $s_q$ -abundance to that of finding a suitable minimal  $q$ -Ramsey graph containing at least one vertex of the desired small degree. In fact, we can even relax the condition that the  $q$ -Ramsey graph be minimal and that the desired small degree be precisely  $s_q(H)$  for the given graph  $H$ .

**THEOREM 3.1.** *Let  $H$  be 3-connected or a triangle and assume there exists a graph  $F$  together with a vertex  $v \in V(F)$  and an edge  $e \in E(F)$  satisfying the following properties:*

- (F1)  $F \rightarrow_q H$ .
- (F2)  $v$  and  $e$  do not share a copy of  $H$  in  $F$ .
- (F3)  $F - e \not\rightarrow_q H$ .
- (F4)  $F - g \not\rightarrow_q H$  for every  $g \in E(F)$  which is incident to  $v$ .

*Then, for any  $k \in \mathbb{Z}_{\geq 1}$ , there exists a minimal  $q$ -Ramsey graph for  $H$  that has  $k$  vertices of degree  $d_F(v)$ .*

*Proof.* Given a graph  $F$  with the required properties, denote the edges incident to  $v$  in  $F$  by  $g_1, \dots, g_{d_F(v)}$ . Let  $F' = F - v - e$ . In order to define  $q$ -color patterns for an application of Theorem 2.4, we first observe the existence of two types of  $H$ -free  $q$ -colorings on  $F'$ .

**CLAIM 3.2.** *For every  $j \in [d_F(v)]$ , there exists an  $H$ -free  $q$ -coloring  $c_{1,j}$  of  $F'$  such that*

- $c_{1,j}$  can be extended to an  $H$ -free  $q$ -coloring of  $F - \{e, g_j\}$ , and
- $c_{1,j}$  cannot be extended to an  $H$ -free  $q$ -coloring of  $F - e$ .

*Proof.* By property (F4), there exists an  $H$ -free  $q$ -coloring  $\varphi$  of  $F - g_j$ . We set  $c_{1,j} := \varphi|_{F'}$ . One observes easily that this is an  $H$ -free  $q$ -coloring of  $F'$  and that  $\varphi|_{F - \{e, g_j\}}$  is an extension to  $F - \{e, g_j\}$  that is  $H$ -free. Hence, it remains to check that there is no  $H$ -free extension to the graph  $F - e$ .

For a contradiction, assume that there is some  $H$ -free coloring  $\psi : E(F - e) \rightarrow [q]$  extending  $c_{1,j}$ . The  $q$ -coloring  $\tilde{\psi} : E(F) \rightarrow [q]$  defined by

$$\tilde{\psi}(f) = \begin{cases} \psi(f) & \text{if } f \neq e, \\ \varphi(e) & \text{if } f = e \end{cases}$$

cannot be  $H$ -free by property (F1). Thus, there must be a copy  $H'$  of  $H$  that is monochromatic under  $\tilde{\psi}$ ; moreover,  $H'$  needs to use the edge  $e$  as  $\tilde{\psi}|_{F - e} = \psi$  is  $H$ -free. By property (F2), we have  $v \notin V(H')$ , that is,  $H'$  lies entirely in the graph  $F - v$ . However,  $\tilde{\psi}|_{F - v} = \varphi|_{F - v}$ , since  $\tilde{\psi}|_{F'} = \psi|_{F'} = c_{1,j} = \varphi|_{F'}$  and  $\tilde{\psi}(e) = \varphi(e)$ . Hence, since  $\varphi$  is  $H$ -free,  $H'$  cannot be monochromatic, a contradiction.  $\square$

**CLAIM 3.3.** *There exists an  $H$ -free  $q$ -coloring  $c_2$  of  $F'$  that can be extended to an  $H$ -free  $q$ -coloring of  $F - e$ .*

*Proof.* By property (F3) there exists an  $H$ -free  $q$  coloring  $\varphi$  of  $F - e$ . We set  $c_2 := \varphi|_{F'}$ .  $\square$

Given the colorings of our previous claims, we next define  $H$ -free  $q$ -color patterns  $f_{1,j}$ , with  $j \in [d_F(v)]$ , and  $f_2$  for  $F'$  by partitioning  $F'$  into its color classes with respect to  $c_{1,j}$  and  $c_2$ , respectively. More precisely, we set

$$f_{1,j} = \{c_{1,j}^{-1}(i)\}_{i \in [q]} \quad \text{and} \quad f_2 = \{c_2^{-1}(i)\}_{i \in [q]}.$$

Now let  $k \geq 1$  be an integer. We proceed similarly as in the proof of Theorem 1.3 and construct a graph  $\tilde{G}$  that will be a  $q$ -Ramsey graph for  $H$  with the additional property that there are at least  $k$  vertices of degree  $d_F(v)$ , each of which is important for  $G$  to be  $q$ -Ramsey for  $H$ .

First, let  $F_1, \dots, F_q$  be  $k$  vertex-disjoint copies of  $F - e$ . For each  $i \in [k]$ , let  $v_i \in V(F_i)$  represent the vertex  $v \in V(F - e)$  and let  $g_1^i, \dots, g_{d_F(v)}^i \in E(F_i)$  be the edges representing  $g_1, \dots, g_{d_F(v)}$ . Moreover, for every  $i \in [k]$ , let  $F'_i = F_i - v_i$  and  $W_i := N_{F_i}(v_i)$ .

We fix  $G = (V, E)$  to be the vertex-disjoint union of the graphs  $F'_i = (V'_i, E'_i)$ , i.e., we set  $V = \cup_{i=1}^k V'_i$  and  $E = \cup_{i=1}^k E'_i$ . Then we fix a family  $\mathcal{G}$  of  $q$ -color patterns for  $G$  such that  $g \in \mathcal{G}$  if and only if there exist  $i \in [k]$  and  $j \in [d_F(v)]$  such that  $g[V'_i] \cong f_{1,j}$  and such that  $g[V'_\ell] \cong f_2$  for all  $\ell \neq i$ .

By the definition of the patterns  $f_{1,j}$  and  $f_2$ , and since the vertex sets  $V'_i$  for  $i \in [k]$  are pairwise disjoint, we know that  $\mathcal{G}$  is a family of  $H$ -free  $q$ -color patterns for  $G$ . Hence, applying Theorem 2.4, we can find a pattern gadget  $P = P(H, G, \mathcal{G}, q)$  such that  $(P, G)$  is  $H$ -robust. Finally, we obtain  $\tilde{G}$  from  $P$  by adding the vertices  $v_1, \dots, v_k$  and by connecting  $v_i$  to all vertices in  $W_i$  via the edges  $g_1^i, \dots, g_{d_F(v)}^i$  for all  $i \in [k]$ .

Analogously to the proof of Theorem 1.3, we now show that  $\tilde{G} \rightarrow_q H$  and that each of the edges  $g_j^i$  for  $i \in [k]$  and  $j \in [d_F(v)]$  is important for  $\tilde{G}$  to be Ramsey in the sense that  $\tilde{G} - g_j^i \not\rightarrow_q H$ . This then implies the existence of a minimal  $q$ -Ramsey graph as claimed by the theorem. Indeed, assuming these properties, let  $\tilde{G}' \subseteq \tilde{G}$  be minimal  $q$ -Ramsey for  $H$ . Since  $\tilde{G} - g_j^i \not\rightarrow_q H$ , we can conclude that  $g_j^i \in E(\tilde{G}')$  for every  $i \in [k]$  and  $j \in [d_F(v)]$ . This then implies that  $d_{\tilde{G}'}(v_i) = d_F(v)$ . Hence,  $\tilde{G}'$  is a minimal  $q$ -Ramsey graph for  $H$  with at least  $k$  vertices of degree  $d_F(v)$ .

Let us show first that  $\tilde{G} \rightarrow_q H$ . For a contradiction, suppose we can find an  $H$ -free  $q$ -coloring  $c : E(\tilde{G}) \rightarrow [q]$ . For each  $i \in [q]$ , define  $c_i = c^{-1}(i)$  to be the  $i$ th color class with respect to  $c$ . By property (P2) of the pattern gadget  $P$ , we know that  $g := \{c_{\tilde{G}}^{-1}(1), \dots, c_{\tilde{G}}^{-1}(q)\} \in \mathcal{G}$ . Hence, by the definition of  $\mathcal{G}$ , there exist  $i \in [k]$  and  $j \in [d_F(v)]$  such that  $g[V'_i] \cong f_{1,j}$ . But then, by the choice of  $f_{1,j}$  and the properties of  $c_{1,j}$ , we deduce that  $c_{\tilde{G}[V'_i]}$  cannot be extended to an  $H$ -free  $q$ -coloring of  $\tilde{G}[V'_i \cup \{v_i\}]$ . This is a contradiction, since  $c_{\tilde{G}[V'_i \cup \{v_i\}]}$  is already such an  $H$ -free extension by the assumption on  $c$ .

Next, we show that  $\tilde{G} - g_j^i \not\rightarrow_q H$  for every  $i \in [k]$  and  $j \in [d_F(v)]$ . By symmetry, we may only consider the case when  $i = j = 1$ . We first partition  $G$  in the following way: For every  $\ell \in [k]$ , we fix a partition  $G[V'_\ell] = G_{\ell,1} \cup \dots \cup G_{\ell,q}$  such that  $\{G_{1,r}\}_{r \leq q} \cong f_{1,1}$  and  $\{G_{\ell,r}\}_{r \leq q} \cong f_2$  for  $\ell \neq 1$ . By the choice of  $f_{1,1}$  and  $f_2$ , we know that the coloring  $c : E(G) \rightarrow [q]$  defined by  $c(G_{\ell,r}) = r$  for every  $\ell \in [k]$  and  $r \in [q]$  is  $H$ -free. Moreover,  $\{c^{-1}(1), \dots, c^{-1}(q)\} \in \mathcal{G}$  and therefore, by property (P3), we can extend  $c$  to an  $H$ -free  $q$ -coloring  $\varphi_P$  of  $P$ . By the definition of  $f_{1,1}$  and the properties of  $c_{1,1}$ , we know that the coloring  $c|_{G[V_1]}$  can be extended to an  $H$ -free  $q$ -coloring  $\varphi_1$  of  $\tilde{G}[V_1 \cup \{v_1\}] - g_1^1$ . By the definition of  $f_2$  and the properties

of  $c_2$  we know that, for each  $\ell \neq 1$ , the coloring  $c|_{G[V_\ell]}$  can be extended to an  $H$ -free  $q$ -coloring  $\varphi_\ell$  of  $\tilde{G}[V_\ell \cup \{v_\ell\}]$ . We now put all these colorings together to form the coloring  $\varphi : E(\tilde{G} - g_1^1) \rightarrow [q]$  given by

$$\varphi(f) := \begin{cases} \varphi_P(f) & \text{if } f \in E(P), \\ \varphi_\ell(f) & \text{if } v_\ell \in f \text{ for some } \ell \in [k]. \end{cases}$$

We claim that this coloring is  $H$ -free.

Assume for a contradiction that there is a monochromatic copy  $H'$  of  $H$  in the coloring  $\varphi$ . Then, since  $(P, G)$  is  $H$ -robust, we know that either  $H' \subseteq P$  or  $H' \subseteq \tilde{G}[V \cup \{v_\ell\}_{\ell \in [k]}] - g_1^1$ . Since the coloring  $\varphi_P$  on  $P$  is  $H$ -free, we can assume that  $H' \subseteq \tilde{G}[V \cup \{v_\ell\}_{\ell \in [k]}] - g_1^1$ . But since  $H'$  is connected, we have  $H' \subseteq \tilde{G}[V_\ell \cup \{v_\ell\}]$  for some  $\ell \neq 1$  or  $H' \subseteq \tilde{G}[V_1 \cup \{v_1\}] - g_1^1$ . In both cases we know that  $H'$  cannot be monochromatic, since the colorings  $\varphi_1, \dots, \varphi_k$  are  $H$ -free. This is a contradiction.  $\square$

Finally, we illustrate how to apply Theorem 3.1 by deriving Theorem 1.4 as a consequence of it.

*Proof of Theorem 1.4.* In order to show that  $K_t$  is  $s_q$ -abundant, it will be enough to prove the existence of a graph  $F$  with a vertex  $v \in V(F)$  and an edge  $e \in E(F)$  satisfying (F1)–(F4) with  $d_F(v) = s_q(K_t)$ . Implicitly, such a graph is given in an argument of Fox et al. [13], which was a first step for finding an upper bound on  $s_q(K_t)$ . In the following, we will briefly sketch their argument and then conclude the existence of a graph  $F$  as desired.

Let  $P_q(t - 1)$  be the smallest integer  $n$  such that the following holds: There exist a graph  $G$  on  $n$  vertices and a  $K_t$ -free  $q$ -color pattern  $\{G_1, \dots, G_q\}$  for  $G$  such that, for every partition  $V(G) = \cup_{j \in [q]} V_j$ , there exists a copy  $H$  of  $K_{t-1}$  and an integer  $i \in [q]$  such that  $H \subseteq G_i[V_i]$ . Fox et al. proved that  $s_q(K_t) = P_q(t - 1)$  (Theorem 1.5 in [13]). For a proof of the inequality  $s_q(K_t) \leq P_q(t - 1)$  (Theorem 2.3 in [13]), they gave the following construction of a graph  $\tilde{G}$ .

Fix a graph  $G$  on  $P_q(t - 1)$  vertices with a  $K_t$ -free  $q$ -color pattern  $\{G_1, \dots, G_q\}$  as described above. We take the given graph  $G$ , an isolated vertex  $v$ , and a matching  $M = \{e_1, \dots, e_q\}$  that is vertex-disjoint from  $G$  and  $v$ ; next, we take a negative signal sender  $S^- := S^-(K_t, e, f, q, d)$  and a positive signal sender  $S^+ := S^+(K_t, e, f, q, d)$  with  $d > t$ , the existence of which is guaranteed by Theorem 2.6. We then obtain  $\tilde{G}$  as follows:

- (i) For every distinct  $i, j \in [q]$ , join  $e_i$  and  $e_j$  by a copy of  $S^-$ .
- (ii) For every  $i \in [q]$  and every  $f \in E(G_i)$ , join  $e_i$  and  $f$  by a copy of  $S^+$ .
- (iii) Connect  $v$  to all vertices in  $V(G)$  by an edge.

We will see in the following that  $\tilde{G} \rightarrow_q K_t$ ,  $\tilde{G} - v \not\rightarrow_q K_t$ , and  $\tilde{G} - M \not\rightarrow_q K_t$ . From this, we can then conclude the existence of a graph  $F \in \mathcal{M}_q(K_t)$  satisfying the hypothesis of Theorem 3.1. Indeed, consider any minimal  $q$ -Ramsey graph  $F$  for  $K_t$  contained in  $\tilde{G}$ . Since  $\tilde{G} - v \not\rightarrow_q K_t$ , we conclude that  $F$  must contain the vertex  $v$ ; moreover, we have  $s_q(K_t) \leq d_F(v) \leq d_{\tilde{G}}(v) = P_q(t - 1) = s_q(K_t)$ , so  $d_F(v) = s_q(K_t)$ . Further, using that  $\tilde{G} - M \not\rightarrow_q K_t$ , we also deduce that  $\tilde{G}$  must contain at least one edge  $e \in M$ . Since  $\text{dist}_F(v, e) \geq \text{dist}_{\tilde{G}}(v, e) \geq d > v(K_t)$ ,  $v$  and  $e$  cannot share a copy of  $K_t$ , implying that property (F2) holds. By the minimality of  $F$ , properties (F1), (F3), and (F4) are immediate. We split the remainder of the proof into three claims.

CLAIM 3.4. *We have  $\tilde{G} \rightarrow_q K_t$  and  $\tilde{G} - v \not\rightarrow_q K_t$ .*

*Proof.* Both statements were already proven in [13]. We include the argument here for completeness.

We begin by showing that  $\tilde{G} \rightarrow_q K_t$ . For a contradiction, assume that there exists a  $K_t$ -free coloring  $c : E(\tilde{G}) \rightarrow [q]$ . The signal senders in (i) then ensure that the edges of  $M$  must receive distinct colors, say, without loss of generality that  $c(e_i) = i$  for every  $i \in [q]$ . The signal senders in (ii) ensure that  $c(G_i) = c(e_i) = i$  for every  $i \in [q]$ . Now, consider the partition  $V(G) = \cup_{j \in [q]} V_j$ , where, for every  $j \in [q]$ , we have  $w \in V_j$  if and only if  $c(vw) = j$ . Then, by the choice of  $G$  and the definition of  $P_q(t-1)$ , there exists a graph  $H \cong K_{t-1}$  and an integer  $i \in [q]$  such that  $H \subseteq G_i[V_i]$ . Hence, the edges in  $E(H) \cup \{vw : w \in V(H)\}$  all have color  $i$  and thus induce a monochromatic copy of  $K_t$ . This is a contradiction.

Next, let us show that  $\tilde{G} - v \not\rightarrow_q K_t$ . In order to do so, we define a  $q$ -coloring  $c$  of  $\tilde{G} - v$ . We first set  $c(G_i) = c(e_i) = i$  for every  $i \in [q]$ ; afterward we extend the coloring  $c$  to  $\tilde{G} - v$  in such a way that  $c$  is  $K_t$ -free on each signal sender from (i) and (ii). Note that the latter is possible by property (S1) and (S2). Analogously to previous proofs, each copy of  $K_t$  is fully contained either in a signal sender or in the graph  $G$ . Since the coloring restricted to any signal sender is  $K_t$ -free and since  $\{G_1, \dots, G_q\}$  is a  $K_t$ -free  $q$ -color pattern, it follows that  $c$  is  $K_t$ -free.  $\square$

The next two claims were not shown in [13].

CLAIM 3.5. *If  $s_q(K_t) \leq r_q(K_t) - 2$ , then  $\tilde{G} - M \not\rightarrow_q K_t$ .*

*Proof.* In order to see this claim, we define a  $q$ -coloring  $c$  of  $\tilde{G} - M$  as follows: We first fix a  $K_t$ -free  $q$ -coloring of  $\tilde{G}[N_{\tilde{G}}(v) \cup \{v\}] = \tilde{G}[V(G) \cup \{v\}]$ , which is possible since by assumption we have

$$|N_{\tilde{G}}(v) \cup \{v\}| = s_q(K_t) + 1 \leq r_q(K_t) - 1 .$$

Afterward we extend the coloring to every signal sender so that it is  $K_t$ -free. The latter is possible since every signal sender is missing at least one signal edge in the graph  $\tilde{G} - M$  (and hence we can always pretend that the missing signal edge has a color that fits property (S2)). Now, each copy of  $K_t$  is fully contained either in a signal sender or in the graph  $G$ , and hence the resulting coloring of  $\tilde{G} - M$  is  $K_t$ -free.  $\square$

CLAIM 3.6. *For all  $q \geq 2$  and  $t \geq 3$ , we have  $s_q(K_t) \leq r_q(K_t) - 2$ .*

*Proof.* Let  $t \geq 3$  be fixed. For all  $q \geq 2$ , define  $N_q = (t-1)^q$ . To show the claim it suffices to prove that  $K_{N_q}$  satisfies the following properties:

- (i) There is a  $K_t$ -free  $q$ -coloring  $\varphi_q$  of  $K_{N_q}$  that cannot be extended to a  $K_t$ -free coloring of  $K_{N_q+1}$ .
- (ii) There exists a  $K_t$ -free coloring  $\psi_q$  of  $K_{N_q+1}$ .

Note that, by an argument similar to that given in Claim 3.4, property (i) implies that  $P_q(t-1) \leq N_q$ . Property (ii) implies that  $N_q + 1 < r_q(K_t)$ . These two inequalities together with the fact that  $s_q(K_t) = P_q(t-1)$  imply the claim.

We now proceed by induction on  $q$  and show properties (i) and (ii). First consider the case  $q = 2$ . We can use the idea of Burr, Erdős, and Lovász [4]. Partition the vertices of the graph  $K_{(t-1)^2}$  into  $t-1$  equally sized sets  $Q_1, \dots, Q_{t-1}$ . Consider the coloring  $\varphi_2$  of  $K_{(t-1)^2}$  in which the edges lying within a single  $Q_i$  are colored red and the edges with endpoints in two different  $Q_i$  are colored blue. It is not difficult to check that this coloring is  $K_t$ -free but there is no way to extend it to  $K_{(t-1)^2+1}$



without creating a monochromatic  $K_t$ , establishing property (i). On the other hand, we can define a  $K_t$ -free 2-coloring  $\psi_2$  of  $K_{(t-1)^2+1}$  as follows. Let  $Q_1, \dots, Q_{t-1}$  be as before; fix an arbitrary vertex  $v_i \in Q_i$  for every  $i \in [t-1]$ . Color all edges of  $K_{(t-1)^2}$  as before except for the edge  $v_1v_2$ , which we now color red. Let  $v$  be a new vertex connected to all vertices of  $K_{(t-1)^2}$ . Color  $vv_i$  blue for all  $i \in [t-1]$ , and color all other edges incident to  $v$  red. It is not difficult to check that this coloring is  $K_t$ -free.

Assume that (i) and (ii) hold for some  $q \geq 2$ . Consider the graph  $K_{N_{q+1}}$ . Partition its vertex set into  $t-1$  equally sized sets  $Q_1, \dots, Q_{t-1}$ . Let  $\varphi_{q+1}$  be the coloring in which the edges inside each  $Q_i$  are colored according to  $\varphi_q$  and the edges between two different  $Q_i$  are given color  $q+1$ . Again, it is easily seen that this coloring is  $K_t$ -free. Now, let  $v$  be a vertex connected to all vertices of  $K_{N_{q+1}}$ , and consider any coloring of  $K_{N_{q+1}+1}$  extending  $\varphi_{q+1}$ . If all edges from  $v$  to some  $Q_i$  have colors in  $[q]$ , then by induction the graph induced by  $Q_i \cup \{v\}$  contains a monochromatic copy of  $K_t$ . So we may assume that, for all  $i \in [t-1]$ , there is a vertex  $v_i \in Q_i$  such that the edge  $vv_i$  has color  $q+1$ . But then the vertices  $v_1, \dots, v_{t-1}, v$  induce a monochromatic copy of  $K_t$ . Hence property (i) is satisfied. For property (ii), notice that if  $Q_1, \dots, Q_{t-1}$  are as above and  $v$  is a new vertex connected to all vertices of  $K_{N_{q+1}}$ , then coloring the graph induced by  $Q_i \cup \{v\}$  according to  $\psi_q$  for all  $i \in [t-1]$  and giving all edges with endpoints in different  $Q_i$  color  $q+1$  gives the required  $K_t$ -free coloring  $\psi_{q+1}$  of  $K_{N_{q+1}+1}$ .  $\square$

Putting Claims 3.4–3.6 together, we obtain the theorem.  $\square$

**4. Concluding remarks and open problems.** In the present paper, we formalized a new tool for studying (minimal) Ramsey graphs and showed some applications to questions concerning minimum degrees. In particular, we used pattern gadgets to find examples of graphs  $H$  such that a minimal  $q$ -Ramsey graph for  $H$  can contain arbitrarily many vertices of degree  $s_q(H)$ , that is,  $s_q$ -abundant graphs. A number of interesting problems remain open.

Questions concerning minimum degrees of minimal Ramsey graphs are particularly interesting for the class of so-called  *$q$ -Ramsey-simple graphs*. Observe that  $s_q(H) \geq q(\delta(H) - 1) + 1$  for any graph  $H$  and integer  $q \geq 2$ . This was shown by Fox and Lin [14] for two colors and generalizes easily to any number of colors. Indeed, assume there exists  $G \in \mathcal{M}_q(H)$  with a vertex  $v \in V(G)$  such that  $d_G(v) \leq q(\delta(H) - 1)$ . Since  $G$  is minimal  $q$ -Ramsey for  $H$ , we can color the graph  $G - v$  with  $q$  colors without a monochromatic copy of  $H$ . Then we can extend this coloring to all of  $G$  by coloring at most  $\delta(H) - 1$  of the edges incident to  $v$  in any given color. It is not difficult to check that this is an  $H$ -free coloring of  $G$ , a contradiction. Now, a graph  $H$  without isolated vertices is said to be  *$q$ -Ramsey-simple* if  $s_q(H) = q(\delta(H) - 1) + 1$ . In [27], Szabó, Zumstein, and Zürcher found many classes of 2-Ramsey-simple bipartite graphs; in particular, all trees were shown to be 2-Ramsey simple. Later Grinshpun [15, Theorems 2.1.2 and 2.1.3] gave further examples of Ramsey-simple graphs, showing in particular that all 3-connected bipartite graphs are 2-Ramsey-simple. Despite this progress, the following question, posed by Szabó and co-authors, remains open.

QUESTION 4.1 ([27, Problem 2]). *Is every bipartite graph with no isolated vertices 2-Ramsey-simple?*

In fact, Grinshpun made the following bolder conjecture.

CONJECTURE 4.2 ([15, Conjecture 2.8.2]). *Every connected triangle-free graph is 2-Ramsey-simple.*

Some evidence in favor of this conjecture was given in [16], where the authors showed that the statement is true for regular 3-connected triangle-free graphs satisfying one extra technical condition. It is of course natural to ask the same questions for larger values of  $q$ .

In view of the results presented in this paper, it is also interesting to investigate which bipartite (or triangle-free) graphs are  $s_q$ -abundant. This question is particularly interesting for trees. As discussed above, Szabó, Zumstein, and Zürcher [27] showed that, for all trees  $T$ , we have  $s_2(T) = 1$ . This result might appear surprising at first, since we might not expect a degree-one vertex to be essential for the Ramsey properties of a graph. Having established that a degree-one vertex can indeed play a significant role in a minimal Ramsey graph for a tree  $T$ , we might wonder whether we can find many such vertices in a minimal Ramsey graph for  $T$ .

It is simple to show that the path  $P_4$  with three edges is  $s_2$ -abundant. Indeed, let  $k \geq 3$  be an odd integer and  $G$  be the graph obtained from the cycle  $C_k$  by adding a distinct pendant edge to each vertex of the cycle. Using the fact that in every 2-coloring of  $C_k$  there must be two consecutive edges of the same color, it is not difficult to check that  $G$  is a minimal 2-Ramsey graph for  $P_4$ . Further,  $G$  has  $k$  vertices of degree one, establishing the claim.

Thus, we have seen that stars are not  $s_2$ -abundant but  $P_4$  is. For all other trees  $T$ , the question of whether  $T$  is  $s_2$ -abundant (or, more generally,  $s_q$ -abundant) remains open. This leads us to propose the following problem.

QUESTION 4.3. *Is every tree that is not a star  $s_q$ -abundant for  $q \geq 2$ ?*

As explained above, a positive answer to this question would be rather surprising.

More generally, we would like to understand better which graphs are  $s_q$ -abundant. In particular, besides stars, we do not have any examples of graphs that are not  $s_q$ -abundant; we propose the following question.

QUESTION 4.4. *Let  $q \geq 2$  be an integer. Does there exist a graph  $H$  that is not  $s_q$ -abundant but has infinitely many minimal  $q$ -Ramsey graphs of minimum degree  $s_q(H)$ ?*

We saw in Theorem 1.4 that we can sometimes show  $s_q$ -abundance without knowing the precise value of  $s_q$ . Further, we established a sufficient condition for a given 3-connected graph to be  $s_q$ -abundant in Theorem 3.1. Given the tools developed in this paper, we believe that all 3-connected graphs should be  $s_q$ -abundant and propose Conjecture 4.5 below.

CONJECTURE 4.5. *Every 3-connected graph  $H$  is  $s_q$ -abundant for any integer  $q \geq 2$ .*

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