

# Error estimation and adaptivity for stochastic collocation finite elements part II

Bespalov, Alex; Silvester, DJ

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# ERROR ESTIMATION AND ADAPTIVITY FOR STOCHASTIC COLLOCATION FINITE ELEMENTS PART II: MULTILEVEL APPROXIMATION

ALEX BESPALOV AND DAVID J. SILVESTER

ABSTRACT. A multilevel adaptive refinement strategy for solving linear elliptic partial differential equations with random data is recalled in this work. The strategy extends the a posteriori error estimation framework introduced by Guignard & Nobile in 2018 (*SIAM J. Numer. Anal.*, **56**, 3121–3143) to cover problems with a *nonaffine* parametric coefficient dependence. A suboptimal, but nonetheless reliable and convenient implementation of the strategy involves approximation of the decoupled PDE problems with a common finite element approximation space. Computational results obtained using such a *single-level* strategy are presented in part I of this work (Bespalov, Silvester & Xu, *SIAM J. Sci. Comp.*, **44** (2022), A3393–A3412). Results obtained using a potentially more efficient *multilevel* approximation strategy, where meshes are individually tailored, are discussed herein. The results demonstrate that the optimal convergence rates can be achieved, but only when solving specific types of problems. The codes used to generate the numerical results are available online.

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## 1. INTRODUCTION

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Partial differential equations (PDEs) with uncertain inputs have provided engineers and scientists with enhanced fidelity in the modelling of real-life phenomena, especially within the last two decades. Sparse grid stochastic collocation representations of parametric uncertainty, in combination with finite element discretization of physical space, have emerged as an efficient alternative to Monte-Carlo strategies over this period, especially in the context of nonlinear PDE models or linear PDE problems that are nonlinear in the parameterization of the uncertainty.

While the combination of sparse grid interpolation with *hierarchies* of spatial approximations has given rise to effective multilevel and multi-index stochastic collocation approaches in [6, 18, 14], enabling sample-dependent adaptivity in this context is a relatively new development, see, for example [15, 17]. In our precursor paper [5] (part I), we proposed a novel error estimation strategy and the associated adaptive framework for stochastic collocation finite element method (SC-FEM) and presented a critical comparison of alternative strategies in the context of solving a model problem that combines strong anisotropy in the parametric dependence with singular behavior in the physical space. The *hierarchical* a posteriori error estimates and indicators proposed in [5] and utilized in the present work require additional PDE solves and thus incur additional computational cost compared to the *residual-based* error estimates proposed in [13] and used in [11, 9].

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24 However, unlike the error estimates in [13, 11, 9], our hierarchical error estimation frame-  
 25 work is not restricted to PDEs with affine-parametric representation of the coefficient in  
 26 combination with a deterministic right-hand side function.

27 The numerical results presented in [5] demonstrate the effectivity and robustness of  
 28 our error estimation strategy as well as the utility of the error indicators guiding the  
 29 adaptive refinement process. The results in [5] also showed that optimality of convergence  
 30 is difficult to achieve using a simple single-level approach where a single finite element  
 31 space is associated with all active collocation points. The main aim of this contribution is  
 32 to see if optimal convergence rates can be recovered by computing results using a multilevel  
 33 implementation of the algorithm outlined in [5]. Here, the optimal rate is understood as  
 34 the best possible algebraic rate that can be achieved for parametric solutions from a  
 35 given approximation class; for instance, for problems with sufficiently smooth parametric  
 36 inputs, this is the rate of the chosen finite element approximations for the corresponding  
 37 parameter-free problem; see, for example, [7, 12, 3] in the context of stochastic Galerkin  
 38 finite element method (SGFEM).

39 While the convergence of a modified version of the adaptive algorithm in [13] has been  
 40 established by Eigel et al. [9] and independently by Feischl & Scaglioni [11], our focus in the  
 41 present contribution is different. In particular, we address “the interplay of parametric  
 42 refinement and finite element refinement”, which is identified by the authors of [11] as  
 43 playing a critical role in establishing the convergence of adaptive SC-FEM algorithms.

44 The model problems that are of interest are stated in section 2. The only difference  
 45 from the problem statement in [5] is that we also cover the case where the right-hand  
 46 side function has a parametric dependence. The adaptive solution algorithm from [5] is  
 47 extended to cover the case of a non-deterministic right-hand side function in section 3. The  
 48 novel contribution of this work primarily lies in section 4, where we compare numerical  
 49 results obtained with our multilevel algorithm with those generated using a single-level  
 50 strategy and with those computed using a multilevel SGFEM code.

## 51 2. PARAMETRIC MODEL PROBLEMS

52 Let  $D \subset \mathbb{R}^2$  be a bounded Lipschitz domain with polygonal boundary  $\partial D$ . Let  $\Gamma :=$   
 53  $\Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_M$  denote the parameter domain in  $\mathbb{R}^M$ , where  $M \in \mathbb{N}$  and each  $\Gamma_m$   
 54 ( $m = 1, \dots, M$ ) is a bounded interval in  $\mathbb{R}$ . We introduce a probability measure  $\pi(\mathbf{y}) :=$   
 55  $\prod_{m=1}^M \pi_m(y_m)$  on  $(\Gamma, \mathcal{B}(\Gamma))$ ; here,  $\pi_m$  denotes a Borel probability measure on  $\Gamma_m$  ( $m =$   
 56  $1, \dots, M$ ) and  $\mathcal{B}(\Gamma)$  is the Borel  $\sigma$ -algebra on  $\Gamma$ .

57 The first model problem is the parametric elliptic problem analyzed in [5]: we seek  
 58  $u: \overline{D} \times \Gamma \rightarrow \mathbb{R}$  satisfying

$$59 \quad \begin{aligned} -\nabla \cdot (a(\cdot, \mathbf{y}) \nabla u(\cdot, \mathbf{y})) &= f \quad \text{in } D, \\ u(\cdot, \mathbf{y}) &= 0 \quad \text{on } \partial D, \end{aligned} \tag{1a}$$

61  $\pi$ -almost everywhere on  $\Gamma$ . The second model problem is to find  $u: \overline{D} \times \Gamma \rightarrow \mathbb{R}$  satisfying

$$62 \quad \begin{aligned} -\nabla^2 u(\cdot, \mathbf{y}) &= f(\cdot, \mathbf{y}) \quad \text{in } D, \\ u(\cdot, \mathbf{y}) &= 0 \quad \text{on } \partial D, \end{aligned} \tag{1b}$$

64  $\pi$ -almost everywhere on  $\Gamma$ .

65 In the first model problem, the deterministic right-hand side function  $f \in L^2(D)$  and  
 66 the coefficient  $a$  is a random field on  $(\Gamma, \mathcal{B}(\Gamma), \pi)$  over  $L^\infty(D)$ . In this case we will assume  
 67 that there exist constants  $a_{\min}, a_{\max}$  such that

$$68 \quad 0 < a_{\min} \leq \operatorname{ess\,inf}_{x \in D} a(x, \mathbf{y}) \leq \operatorname{ess\,sup}_{x \in D} a(x, \mathbf{y}) \leq a_{\max} < \infty \quad \pi\text{-a.e. on } \Gamma. \quad (2)$$

69 This assumption implies the following norm equivalence: for any  $v \in \mathbb{X} := H_0^1(D)$  there  
 70 holds

$$71 \quad a_{\min}^{1/2} \|\nabla v\|_{L^2(D)} \leq \|a^{1/2}(\cdot, \mathbf{y}) \nabla v\|_{L^2(D)} \leq a_{\max}^{1/2} \|\nabla v\|_{L^2(D)} \quad \pi\text{-a.e. on } \Gamma. \quad (3)$$

72 The parametric problem (1a) is understood in the weak sense: given  $f \in L^2(D)$ , find  
 73  $u : \Gamma \rightarrow \mathbb{X}$  such that

$$74 \quad \int_D a(x, \mathbf{y}) \nabla u(x, \mathbf{y}) \cdot \nabla v(x) \, dx = \int_D f(x) v(x) \, dx \quad \forall v \in \mathbb{X}, \quad \pi\text{-a.e. on } \Gamma. \quad (4)$$

76 The above assumptions on  $a$  and  $f$  guarantee that the parametric problem (1a) admits  
 77 a unique weak solution  $u$  in the Bochner space  $\mathbb{V} := L_\pi^p(\Gamma; \mathbb{X})$  for any  $p \in [1, \infty]$ ; see [1,  
 78 Lemma 1.1] for details. In the sequel, we restrict attention to  $p = 2$  and denote by  $\|\cdot\|$   
 79 the norm in  $\mathbb{V} = L_\pi^2(\Gamma; \mathbb{X})$ ; we also define  $\|\cdot\|_{\mathbb{X}} := \|\nabla \cdot\|_{L^2(D)}$ .

80 The second parametric elliptic problem (1b) combines uncertainty in the source term  
 81 with an isotropic diffusion coefficient field. In this case the right-hand side function  $f$   
 82 simply needs to be a random field that is smooth enough to ensure that (1b) also admits  
 83 a unique weak solution  $u$  in the Bochner space  $\mathbb{V}$ .

### 84 3. MULTILEVEL STOCHASTIC COLLOCATION FINITE ELEMENT METHOD

85 Full details of the construction of a multilevel stochastic collocation finite element  
 86 approximation of the first parametric elliptic problem can be found in [5]. The parametric  
 87 approximation is associated with a monotone (or, downward-closed) finite set  $\Lambda_\bullet \subset \mathbb{N}^M$   
 88 of multi-indices, where  $\Lambda_\bullet = \{\boldsymbol{\nu} = (\nu_1, \dots, \nu_M) : \nu_m \in \mathbb{N}, \forall m = 1, \dots, M\}$  is such that  
 89  $\#\Lambda_\bullet < \infty$ <sup>1</sup>. Each component  $\nu_m$  ( $m = 1, \dots, M$ ) of the multi-index  $\boldsymbol{\nu} \in \Lambda_\bullet$  corresponds  
 90 to a set of  $\kappa(\nu_m)$  points along the  $m$ th coordinate axis in  $\mathbb{R}^M$ , and the associated *sparse*  
 91 *grid*  $\mathcal{Y}_\bullet = \mathcal{Y}_{\Lambda_\bullet}$  of collocation points on  $\Gamma$  is given by<sup>2</sup>

$$92 \quad \mathcal{Y}_{\Lambda_\bullet} := \bigcup_{\boldsymbol{\nu} \in \Lambda_\bullet} \mathcal{Y}^{(\boldsymbol{\nu})} = \bigcup_{\boldsymbol{\nu} \in \Lambda_\bullet} \mathcal{Y}_1^{\kappa(\nu_1)} \times \mathcal{Y}_2^{\kappa(\nu_2)} \times \dots \times \mathcal{Y}_M^{\kappa(\nu_M)}.$$

93 Each collocation point  $\mathbf{z} \in \mathcal{Y}_{\Lambda_\bullet} \subset \Gamma$  is associated with a piecewise linear finite element  
 94 approximation space  $\mathbb{X}_{\bullet, \mathbf{z}} = \mathcal{S}_0^1(\mathcal{T}_{\bullet, \mathbf{z}})$  defined on a mesh  $\mathcal{T}_{\bullet, \mathbf{z}}$  and an enhanced space  $\widehat{\mathbb{X}}_{\bullet, \mathbf{z}}$   
 95 defined on the mesh  $\widehat{\mathcal{T}}_{\bullet, \mathbf{z}}$  obtained by *uniform refinement* of  $\mathcal{T}_{\bullet, \mathbf{z}}$ . The spatial detail space  
 96  $\mathbb{Y}_{\bullet, \mathbf{z}}$  is the approximation space associated with the newly introduced (mid-edge) nodes,  
 97 i.e.,  $\widehat{\mathbb{X}}_{\bullet, \mathbf{z}} = \mathbb{X}_{\bullet, \mathbf{z}} \oplus \mathbb{Y}_{\bullet, \mathbf{z}}$ . We assume that any finite element mesh employed for the spa-  
 98 tial discretization is obtained by (uniform or local) refinement of a given (coarse) initial  
 99 mesh  $\mathcal{T}_0$ .

<sup>1</sup>Here and throughout the paper, we use  $\bullet$  as a placeholder for the iteration counter, see, for example,  $\Lambda_\ell$  in Algorithm 1. The notation is identical to that used in [5].

<sup>2</sup>In particular, the definition of the sparse grid  $\mathcal{Y}_{\Lambda_\bullet}$  hinges on the nestedness property of the underlying 1D nodes.

100 The SC-FEM approximation of the solution  $u$  to either of the parametric problems (1a)  
 101 or (1b) is given by

$$102 \quad u_{\bullet}^{\text{SC}}(x, \mathbf{y}) := \sum_{\mathbf{z} \in \mathcal{Y}_{\bullet}} u_{\bullet\mathbf{z}}(x) L_{\bullet\mathbf{z}}(\mathbf{y}), \quad (5)$$

103 where  $u_{\bullet\mathbf{z}} \in \mathbb{X}_{\bullet\mathbf{z}}$  are Galerkin approximations satisfying (6a) or (6b) for  $\mathbf{z} \in \mathcal{Y}_{\bullet}$ , and  
 104  $\{L_{\bullet\mathbf{z}}(\mathbf{y}) = L_{\mathbf{z}}^{\mathcal{Y}_{\bullet}}(\mathbf{y}) : \mathbf{z} \in \mathcal{Y}_{\bullet}\}$  is a set of polynomial basis functions associated with  $\mathcal{Y}_{\bullet}$   
 105 and satisfying  $L_{\bullet\mathbf{z}}(\mathbf{z}') = \delta_{\mathbf{z}\mathbf{z}'}$  for any  $\mathbf{z}, \mathbf{z}' \in \mathcal{Y}_{\bullet}$ .<sup>3</sup> The enhancement of the parametric  
 106 component of the SC-FEM approximation (5) is done by enriching the index set  $\Lambda_{\bullet}$  with  
 107 multi-indices selected from the *reduced margin* set  $\mathbb{R}_{\bullet} = \mathbb{R}(\Lambda_{\bullet})$ ; this corresponds to adding  
 108 some collocation points from the set  $\widehat{\mathcal{Y}}_{\bullet} \setminus \mathcal{Y}_{\bullet}$ , where  $\widehat{\mathcal{Y}}_{\bullet} := \mathcal{Y}_{\Lambda_{\bullet} \cup \mathbb{R}(\Lambda_{\bullet})}$ .

109 To keep the discussion concise we simply summarize the components of the adaptive  
 110 refinement strategy. The three components are:

- 111 • solution of a deterministic finite element problem at each sparse grid collocation  
 112 point. That is, the computation of  $u_{\bullet\mathbf{z}} \in \mathbb{X}_{\bullet\mathbf{z}}$  satisfying either

$$113 \quad \int_D a(x, \mathbf{z}) \nabla u_{\bullet\mathbf{z}}(x) \cdot \nabla v(x) \, dx = \int_D f(x) v(x) \, dx \quad \forall v \in \mathbb{X}_{\bullet\mathbf{z}} \quad (6a)$$

114  
 115 in the case of the first parametric problem (1a), or

$$116 \quad \int_D \nabla u_{\bullet\mathbf{z}}(x) \cdot \nabla v(x) \, dx = \int_D f(x, \mathbf{z}) v(x) \, dx \quad \forall v \in \mathbb{X}_{\bullet\mathbf{z}} \quad (6b)$$

117  
 118 in the case of the second parametric problem (1b). The *enhanced* Galerkin solution  
 119 satisfying (6a) or (6b) for all  $v \in \widehat{\mathbb{X}}_{\bullet\mathbf{z}}$  is denoted by  $\widehat{u}_{\bullet\mathbf{z}} \in \widehat{\mathbb{X}}_{\bullet\mathbf{z}}$ .

- 120 • computation of the spatial hierarchical error indicators. For each  $\mathbf{z} \in \mathcal{Y}_{\bullet}$ , we define  
 121  $\mu_{\bullet\mathbf{z}} := \|e_{\bullet\mathbf{z}}\|_{\mathbb{X}}$ , where  $e_{\bullet\mathbf{z}} \in \mathbb{Y}_{\bullet\mathbf{z}}$  satisfies

$$122 \quad \int_D \nabla e_{\bullet\mathbf{z}}(x) \cdot \nabla v(x) \, dx = \int_D f(x) v(x) \, dx$$

$$123 \quad - \int_D a(x, \mathbf{z}) \nabla u_{\bullet\mathbf{z}}(x) \cdot \nabla v(x) \, dx \quad \forall v \in \mathbb{Y}_{\bullet\mathbf{z}} \quad (7a)$$

124 in the case of the first parametric problem (1a), or satisfies

$$125 \quad \int_D \nabla e_{\bullet\mathbf{z}}(x) \cdot \nabla v(x) \, dx = \int_D f(x, \mathbf{z}) v(x) \, dx$$

$$126 \quad - \int_D \nabla u_{\bullet\mathbf{z}}(x) \cdot \nabla v(x) \, dx \quad \forall v \in \mathbb{Y}_{\bullet\mathbf{z}} \quad (7b)$$

127 in the case of the second parametric problem (1b); the corresponding *local* error  
 128 indicators  $\mu_{\bullet\mathbf{z}}(\xi)$  associated with interior edge midpoints  $\xi \in \mathcal{N}_{\bullet\mathbf{z}}^+$  are given by  
 129 components of the solution vector to the linear system stemming from the discrete  
 130 formulation (7a) or (7b).

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<sup>3</sup> An efficient implementation of the representation given in (5) can be effected using the standard combination technique (see equation (10) in [5]).

- computation of the parametric error indicators<sup>4</sup>

$$\tilde{\tau}_{\bullet\nu} = \sum_{\mathbf{z}' \in \tilde{\mathcal{Y}}_{\bullet\nu}} \left\| u_{0\mathbf{z}'} - \sum_{\mathbf{z} \in \mathcal{Y}_{\bullet}} u_{0\mathbf{z}} L_{\bullet\mathbf{z}}(\mathbf{z}') \right\|_{\mathbb{X}} \|\hat{L}_{\bullet\mathbf{z}'}\|_{L^2_{\pi}(\Gamma)} \quad \forall \nu \in \mathbb{R}(\Lambda_{\bullet}), \quad (8)$$

where  $\tilde{\mathcal{Y}}_{\bullet\nu} := \mathcal{Y}_{\Lambda_{\bullet} \cup \{\nu\}} \setminus \mathcal{Y}_{\Lambda_{\bullet}} \subset \hat{\mathcal{Y}}_{\bullet} \setminus \mathcal{Y}_{\bullet}$  are the collocation points ‘generated’ by the multi-index  $\nu \in \mathbb{R}(\Lambda_{\bullet})$ , the functions  $u_{0\mathbf{z}'} \in \mathbb{X}_{0\mathbf{z}'}$  for  $\mathbf{z}' \in \tilde{\mathcal{Y}}_{\bullet\nu}$  and  $u_{0\mathbf{z}} \in \mathbb{X}_{0\mathbf{z}}$  for  $\mathbf{z} \in \mathcal{Y}_{\bullet}$  are Galerkin approximations on some meshes  $\mathcal{T}_{0\mathbf{z}'}$  and  $\mathcal{T}_{0\mathbf{z}}$ , respectively, that are to be specified (e.g.,  $u_{0\mathbf{z}}$  satisfies (6a) or (6b) with  $\mathbb{X}_{\bullet\mathbf{z}}$  replaced by  $\mathbb{X}_{0\mathbf{z}}$ ), and  $\hat{L}_{\bullet\mathbf{z}'}(\mathbf{y}) = L_{\mathbf{z}'}^{\hat{\mathcal{Y}}_{\bullet}}(\mathbf{y})$  denotes the Lagrange polynomial basis function associated with the point  $\mathbf{z}' \in \hat{\mathcal{Y}}_{\bullet}$  satisfying  $\hat{L}_{\bullet\mathbf{z}'}(\mathbf{z}'') = \delta_{\mathbf{z}'\mathbf{z}''}$  for any  $\mathbf{z}', \mathbf{z}'' \in \hat{\mathcal{Y}}_{\bullet}$ . The error indicator  $\tilde{\tau}_{\bullet\nu}$  given by (8) provides an upper bound for the norm of the hierarchical surplus associated with the parametric enhancement of the current SC-FEM approximation (effected by adding  $\nu \in \mathbb{R}(\Lambda_{\bullet})$  to  $\Lambda_{\bullet}$ ); cf. [5, Remarks 1, 3 and 4].

We emphasize that the computation of parametric error indicators according to (8) is in line with the hierarchical a posteriori error estimation strategy developed in [5] (see section 4 therein). In the standard *single-level* SC-FEM setting discussed in [5, section 5], the meshes  $\mathcal{T}_{0\mathbf{z}'}$  and  $\mathcal{T}_{0\mathbf{z}}$  underlying the Galerkin approximations  $u_{0\mathbf{z}'}$  and  $u_{0\mathbf{z}}$  in (8) are all selected to be identical to the (single) finite element mesh  $\mathcal{T}_{\bullet\mathbf{z}} = \mathcal{T}_{\bullet}$  that underlies the current SC-FEM solution  $u_{\bullet}^{\text{SC}}$  in (5). In this case, the indicators in (8) are written as

$$\tilde{\tau}_{\bullet\nu} = \sum_{\mathbf{z}' \in \tilde{\mathcal{Y}}_{\bullet\nu}} \|u_{\bullet\mathbf{z}'} - u_{\bullet}^{\text{SC}}(\cdot, \mathbf{z}')\|_{\mathbb{X}} \|\hat{L}_{\bullet\mathbf{z}'}\|_{L^2_{\pi}(\Gamma)} \quad \forall \nu \in \mathbb{R}(\Lambda_{\bullet}),$$

where  $u_{\bullet\mathbf{z}'} \in \mathbb{X}_{\bullet\mathbf{z}'} = \mathcal{S}_0^1(\mathcal{T}_{\bullet})$  for all  $\mathbf{z}' \in \tilde{\mathcal{Y}}_{\bullet\nu}$  and for all  $\nu \in \mathbb{R}(\Lambda_{\bullet})$ .

In the multilevel SC-FEM setting presented in the adaptive algorithm below, the meshes underlying Galerkin approximations for different collocation points might be different. In this case, when computing the parametric error indicators in (8), the meshes  $\mathcal{T}_{0\mathbf{z}'}$  ( $\mathbf{z}' \in \tilde{\mathcal{Y}}_{\bullet\nu}$ ) and  $\mathcal{T}_{0\mathbf{z}}$  ( $\mathbf{z} \in \mathcal{Y}_{\bullet}$ ) are all selected to be identical to the *coarsest* finite element mesh  $\mathcal{T}_0$ .

With the above ingredients in place, the solution to the problems in section 2 can be generated using the iterative strategy described in Algorithm 1 together with the marking strategy in Algorithm 2.

**Algorithm 1. Input:**  $\Lambda_0 = \{\mathbf{1}\}$ ;  $\mathcal{T}_{0\mathbf{z}} := \mathcal{T}_0$  for all  $\mathbf{z} \in \hat{\mathcal{Y}}_0 = \mathcal{Y}_{\Lambda_0 \cup \mathbb{R}(\Lambda_0)}$ ; *marking criterion.* Set the iteration counter  $\ell := 0$ , the output counter  $k$  and the tolerance.

- (i) Compute Galerkin approximations  $\{u_{\ell\mathbf{z}} \in \mathbb{X}_{\ell\mathbf{z}} : \mathbf{z} \in \hat{\mathcal{Y}}_{\ell}\}$  by solving (6a) or (6b).
- (ii) Compute spatial error indicators  $\{\mu_{\ell\mathbf{z}} = \|e_{\ell\mathbf{z}}\|_{\mathbb{X}} : \mathbf{z} \in \mathcal{Y}_{\ell}\}$  by solving (7a) or (7b).
- (iii) Compute the parametric error indicators  $\{\tilde{\tau}_{\ell\nu} : \nu \in \mathbb{R}(\Lambda_{\ell})\}$  given by (8).
- (iv) Use a marking criterion (e.g., Algorithm 2) to determine  $\mathcal{M}_{\ell\mathbf{z}} \subseteq \mathcal{N}_{\ell\mathbf{z}}^+$  for all  $\mathbf{z} \in \mathcal{Y}_{\ell}$  and  $\Upsilon_{\ell} \subseteq \mathbb{R}(\Lambda_{\ell})$ .
- (v) For all  $\mathbf{z} \in \mathcal{Y}_{\ell}$ , set  $\mathcal{T}_{(\ell+1)\mathbf{z}} := \text{refine}(\mathcal{T}_{\ell\mathbf{z}}, \mathcal{M}_{\ell\mathbf{z}})$ <sup>5</sup>.

<sup>4</sup>This construction assumes that the enriched index set  $\hat{\Lambda}_{\bullet}$  is obtained using the reduced margin of  $\Lambda_{\bullet}$ , see Remark 3 in [5].

<sup>5</sup>Hereafter,  $\mathcal{T}_{\circ} := \text{refine}(\mathcal{T}_{\bullet}, \mathcal{M}_{\bullet})$  means that  $\mathcal{T}_{\circ}$  is the coarsest newest vertex bisection refinement of  $\mathcal{T}_{\bullet}$  such that all marked edge midpoints in  $\mathcal{M}_{\bullet}$  are vertices of  $\mathcal{T}_{\circ}$ .

- 167 (vi) Set  $\Lambda_{\ell+1} := \Lambda_\ell \cup \Upsilon_\ell$ , run Algorithm 3 for each  $\mathbf{z}' \in \bigcup_{\nu \in \Upsilon_\ell} \tilde{\mathcal{Y}}_{\ell\nu}$  to construct meshes  
 168  $\mathcal{T}_{(\ell+1)\mathbf{z}'}$  and initialize  $\mathcal{T}_{(\ell+1)\mathbf{z}} := \mathcal{T}_{0\mathbf{z}} = \mathcal{T}_0$  for all  $\mathbf{z} \in \hat{\mathcal{Y}}_{\ell+1} \setminus \mathcal{Y}_{\ell+1}$ .  
 169 (vii) If  $\ell = jk$ ,  $j \in \mathbb{N}$ , compute the spatial and parametric error estimates  $\mu_\ell$  and  $\tau_\ell$   
 170 given by (11) and (12), respectively, and exit if  $\mu_\ell + \tau_\ell < \text{errortolerance}$ .  
 171 (viii) Increase the counter  $\ell \mapsto \ell + 1$  and goto (i).

172 **Output:** For some specific  $\ell_* = jk \in \mathbb{N}$ , the algorithm returns the multilevel SC-FEM  
 173 approximation  $u_{\ell_*}^{\text{SC}}$  computed via (5) from Galerkin approximations  $\{u_{\ell_*\mathbf{z}} \in \mathbb{X}_{\ell_*\mathbf{z}} : \mathbf{z} \in \mathcal{Y}_{\ell_*}\}$   
 174 together with a corresponding error estimate  $\mu_{\ell_*} + \tau_{\ell_*}$ .

175 A general marking strategy for step (iv) of Algorithm 1 is specified next. We will adopt  
 176 this strategy in the numerical experiments discussed in the next section.

177 **Algorithm 2. Input:** error indicators  $\{\mu_{\ell\mathbf{z}} : \mathbf{z} \in \mathcal{Y}_\ell\}$ ,  $\{\mu_{\ell\mathbf{z}}(\xi) : \mathbf{z} \in \mathcal{Y}_\ell, \xi \in \mathcal{N}_{\ell\mathbf{z}}^+\}$ , and  
 178  $\{\tilde{\tau}_{\ell\nu} : \nu \in \mathbb{R}(\Lambda_\ell)\}$ ; marking parameters  $0 < \theta_{\mathbb{X}}, \theta_{\mathbb{Y}} \leq 1$  and  $\vartheta > 0$ .

- 179 • If  $\sum_{\mathbf{z} \in \mathcal{Y}_\ell} \mu_{\ell\mathbf{z}} \|L_{\ell\mathbf{z}}\|_{L_\pi^2(\Gamma)} \geq \vartheta \sum_{\nu \in \mathbb{R}(\Lambda_\ell)} \tilde{\tau}_{\ell\nu}$ , then proceed as follows:  
 180 ◦ set  $\Upsilon_\ell := \emptyset$   
 181 ◦ for each  $\mathbf{z} \in \mathcal{Y}_\ell$ , determine  $\mathcal{M}_{\ell\mathbf{z}} \subseteq \mathcal{N}_{\ell\mathbf{z}}^+$  such that

$$182 \theta_{\mathbb{X}} \sum_{\mathbf{z} \in \mathcal{Y}_\ell} \sum_{\xi \in \mathcal{N}_{\ell\mathbf{z}}^+} \mu_{\ell\mathbf{z}}(\xi) \|L_{\ell\mathbf{z}}\|_{L_\pi^2(\Gamma)} \leq \sum_{\mathbf{z} \in \mathcal{Y}_\ell} \sum_{\xi \in \mathcal{M}_{\ell\mathbf{z}}} \mu_{\ell\mathbf{z}}(\xi) \|L_{\ell\mathbf{z}}\|_{L_\pi^2(\Gamma)} \quad (9)$$

183 with a cumulative cardinality  $\sum_{\mathbf{z} \in \mathcal{Y}_\ell} \#\mathcal{M}_{\ell\mathbf{z}}$  that is minimized over all the sets  
 184 that satisfy (9).

- 185 • Otherwise, i.e., if  $\sum_{\mathbf{z} \in \mathcal{Y}_\ell} \mu_{\ell\mathbf{z}} \|L_{\ell\mathbf{z}}\|_{L_\pi^2(\Gamma)} < \vartheta \sum_{\nu \in \mathbb{R}(\Lambda_\ell)} \tilde{\tau}_{\ell\nu}$ , proceed as follows:  
 186 ◦ set  $\mathcal{M}_{\ell\mathbf{z}} := \emptyset$  for all  $\mathbf{z} \in \mathcal{Y}_\ell$   
 187 ◦ determine  $\Upsilon_\ell \subseteq \mathbb{R}(\Lambda_\ell)$  of minimal cardinality such that

$$188 \theta_{\mathbb{Y}} \sum_{\nu \in \mathbb{R}(\Lambda_\ell)} \tilde{\tau}_{\ell\nu} \leq \sum_{\nu \in \Upsilon_\ell} \tilde{\tau}_{\ell\nu}. \quad (10)$$

189 **Output:**  $\mathcal{M}_{\ell\mathbf{z}} \subseteq \mathcal{N}_{\ell\mathbf{z}}^+$  for all  $\mathbf{z} \in \mathcal{Y}_\ell$  and  $\Upsilon_\ell \subseteq \mathbb{R}(\Lambda_\ell)$ .

190 The purpose of the marking strategy is twofold. First, the (global) error estimates  
 191  $\bar{\mu}_\ell := \left\| (\mu_{\ell\mathbf{z}} \|L_{\ell\mathbf{z}}\|_{L_\pi^2(\Gamma)})_{\mathbf{z} \in \mathcal{Y}_\ell} \right\|_{\ell_1}$  and  $\bar{\tau}_\ell := \left\| (\tilde{\tau}_{\ell\nu})_{\nu \in \mathbb{R}(\Lambda_\ell)} \right\|_{\ell_1}$  stemming from the corresponding  
 192 error indicators (cf. (13)) are used to identify the refinement type (spatial vs. paramet-  
 193 ric). Specifically, if the spatial estimate  $\bar{\mu}_\ell$  dominates the parametric estimate  $\bar{\tau}_\ell$  then a  
 194 spatial refinement is enforced (by keeping the same set of collocation points but enhancing  
 195 finite element spaces associated with these points); otherwise, a parametric refinement is  
 196 effected (by keeping the finite element spaces for existing collocation points unchanged  
 197 but augmenting the index set and thus adding new collocation points). The role of the  
 198 parameter  $\vartheta$  in Algorithm 2 is to prioritize one of these refinement types; e.g., choosing  
 199  $\vartheta > 1$  prioritizes parametric refinement. The second purpose of the marking strategy is  
 200 to actually generate the marking sets (of interior edge midpoints of the current spatial  
 201 meshes or multi-indices from the current index set) that will feed into the refinement  
 202 process in steps (v) and (vi) of Algorithm 1. To that end, we use Dörfler marking on the  
 203 corresponding set of error indicators (see (9), (10)). We note that Algorithm 2 performs  
 204 the marking of spatial degrees of freedom (i.e., the interior edge midpoints of finite ele-  
 205 ment meshes) across all current collocation points (see (9)). Empirically we have found

206 that the multilevel SC-FEM with this ‘combined’ spatial marking performs better than  
 207 the multilevel SC-FEM with a ‘separate’ marking of spatial degrees of freedom for in-  
 208 dividual current collocation points that was proposed in the first part of this work; cf.  
 209 equation (36) in [5].

210 We point out that if the computational mesh  $\mathcal{T}_{\bullet\mathbf{z}}$  for a collocation point  $\mathbf{z} \in \mathcal{Y}_{\bullet}$  does  
 211 not change from one iteration to another, then the corresponding Galerkin approximation  
 212  $u_{\bullet\mathbf{z}} \in \mathbb{X}_{\bullet\mathbf{z}}$  and the associated spatial error indicator  $\mu_{\bullet\mathbf{z}}$  do not need to be recomputed  
 213 at the new iteration. In particular, at the iterations that follow parametric refinement,  
 214 Galerkin approximations and the associated spatial error indicators need to be computed  
 215 only for the newly added collocation points.

216 As discussed in section 4 of [5], the computation of the *error estimates* in step (vii) of  
 217 Algorithm 1 is best done periodically because of the significant computational overhead.  
 218 Specifically, recalling the notation  $\|\cdot\| := \|\cdot\|_{L^2_{\pi}(\Gamma;\mathbb{X})}$ , the spatial error estimate is given by

$$219 \quad \mu_{\bullet} := \left\| \sum_{\mathbf{z} \in \mathcal{Y}_{\bullet}} (\hat{u}_{\bullet\mathbf{z}} - u_{\bullet\mathbf{z}}) L_{\bullet\mathbf{z}} \right\|. \quad (11)$$

221 It requires computation of the enhanced Galerkin approximation  $\hat{u}_{\bullet\mathbf{z}} \in \hat{\mathbb{X}}_{\bullet\mathbf{z}}$  and thus  
 222 requires the solution of the PDE on the mesh  $\hat{\mathcal{T}}_{\bullet\mathbf{z}}$ —a uniform refinement of  $\mathcal{T}_{\bullet\mathbf{z}}$ —for each  
 223 collocation point generated by the current index set. Recalling the earlier discussion of  
 224 the error indicators (8) (see also Remarks 1 and 3 in [5]), we see that the parametric error  
 225 estimate

$$226 \quad \tau_{\bullet} := \left\| \sum_{\mathbf{z}' \in \hat{\mathcal{Y}}_{\bullet} \setminus \mathcal{Y}_{\bullet}} \left( u_{0\mathbf{z}'} - \sum_{\mathbf{z} \in \mathcal{Y}_{\bullet}} u_{0\mathbf{z}} L_{\bullet\mathbf{z}}(\mathbf{z}') \right) \hat{L}_{\bullet\mathbf{z}'} \right\| \quad (12)$$

228 requires additional PDE solves on the coarsest mesh  $\mathcal{T}_{0\mathbf{z}'} := \mathcal{T}_0$  for all margin collocation  
 229 points  $\mathbf{z}' \in \hat{\mathcal{Y}}_{\bullet} \setminus \mathcal{Y}_{\bullet}$ . (The coarsest-mesh Galerkin approximations  $u_{0\mathbf{z}}$  in (12) for the current  
 230 collocation points  $\mathbf{z} \in \mathcal{Y}_{\bullet}$  will have been computed in preceding iterations and, thus, can  
 231 be reused). The key point here is that computation of the error estimates is only needed  
 232 to give reliable termination of the adaptive process (and to provide reassurance that the  
 233 SC-FEM error is decreasing at an acceptable rate). On the other hand, the error estimates  
 234  $\mu_{\bullet}$  and  $\tau_{\bullet}$  satisfy the following inequalities (see equation (27) and Remarks 1, 3, 4 in [5],  
 235 respectively)

$$236 \quad \mu_{\bullet} \lesssim \sum_{\mathbf{z} \in \mathcal{Y}_{\bullet}} \mu_{\bullet\mathbf{z}} \|L_{\bullet\mathbf{z}}\|_{L^2_{\pi}(\Gamma)} \quad \text{and} \quad \tau_{\bullet} \leq \sum_{\nu \in \mathbb{R}(\Lambda_{\bullet})} \tilde{\tau}_{\bullet\nu} \quad (13)$$

237 that motivate the use of the spatial and parametric error indicators in the marking strategy  
 238 within the adaptive algorithm.

239 Regarding the implementation aspects of computing the above error estimates, we note  
 240 that the sum in (11) involves Galerkin approximations over different finite element meshes.  
 241 In our implementation, the computation of this sum is effected by interpolating piecewise  
 242 linear functions  $u_{\bullet\mathbf{z}}$  and  $\hat{u}_{\bullet\mathbf{z}}$  at the nodes of the mesh  $\bigoplus_{\mathbf{z} \in \mathcal{Y}_{\bullet}} \hat{\mathcal{T}}_{\bullet\mathbf{z}}$ —the overlay (or, the  
 243 coarsest common refinement) of the meshes  $\hat{\mathcal{T}}_{\bullet\mathbf{z}}$ ,  $\mathbf{z} \in \mathcal{Y}_{\bullet}$ —and by subtracting/summing  
 244 the obtained coefficient vectors representing these piecewise linear functions over the same  
 245 mesh  $\bigoplus_{\mathbf{z} \in \mathcal{Y}_{\bullet}} \hat{\mathcal{T}}_{\bullet\mathbf{z}}$ . In this respect, the implementation of the parametric error estimate  
 246 in (12) is rather straightforward, as the involved Galerkin approximations  $u_{0\mathbf{z}}$  and  $u_{0\mathbf{z}'}$   
 247 are all computed on the same coarsest finite element mesh  $\mathcal{T}_0$ .



248 The other detail that is missing in the statement of Algorithm 1 is the identification of a  
 249 strategy for defining suitable meshes  $\mathcal{T}_{(\ell+1)\mathbf{z}'}$  corresponding to the newly added collocation  
 250 points in step (vi). This specification of sample-specific *initial meshes* turns out to be  
 251 crucial if optimal rates of convergence are to be realized in practice. If an initial mesh  
 252 associated with a collocation point is too coarse, then adding this collocation point will  
 253 introduce a large spatial error at the next iteration step. Conversely, if the initial mesh is  
 254 too fine, as in the case of a single-level implementation of the algorithm, then the growth  
 255 in the number of degrees of freedom is not matched by the resulting error reduction.  
 256 Indeed, the conclusion reached in [11] on this point is that “while the theoretical results  
 257 are strongest for the fully adaptive algorithm ... the single mesh algorithm seems to be  
 258 more efficient”. A mesh initialization strategy that attempts to balance the conflicting  
 259 requirements is given in Algorithm 3. Specifically, for a given (newly added) collocation  
 260 point  $\mathbf{z}' \notin \mathcal{Y}_\bullet$ , we start with the coarsest mesh  $\mathcal{T}_0$  and iterate the standard SOLVE  $\rightarrow$   
 261 ESTIMATE  $\rightarrow$  MARK  $\rightarrow$  REFINES loop until the resolution of the mesh is such that  
 262 the estimated error in the corresponding Galerkin solution  $u_{\bullet\mathbf{z}'}$  is on par with the error  
 263 estimates for Galerkin solutions associated with other collocation points  $\mathbf{z} \in \mathcal{Y}_\bullet$  (that  
 264 are ‘rolled over’ from the previous iteration). This is ensured by the choice of stopping  
 265 tolerance  $\text{tol}$  in Algorithm 3. We note that in the multilevel SGFEM, such a mesh  
 266 initialization procedure is not needed. Instead, for every newly ‘activated’ multi-index, the  
 267 associated finite element mesh is set to the coarsest mesh  $\mathcal{T}_0$ ; see [2]. Due to the inherent  
 268 orthogonality of the parametric components of SGFEM approximations associated with  
 269 different multi-indices, this initialization by the coarsest mesh does not affect optimal  
 270 convergence properties of the multilevel SGFEM; see [3].

271 **Algorithm 3. Input:** spatial error indicators  $\{\mu_{\ell\mathbf{z}} : \mathbf{z} \in \mathcal{Y}_\ell\}$ ; the set of collocation points  
 272  $\mathcal{Y}_{\ell+1} = \mathcal{Y}_{\Lambda_{\ell+1}}$ ; the collocation point  $\mathbf{z}' \in \mathcal{Y}_{\ell+1} \setminus \mathcal{Y}_\ell$ ; marking parameter  $\theta$ .  
 273 Set the tolerance  $\text{tol} := (\#\mathcal{Y}_\ell)^{-1} \sum_{\mathbf{z} \in \mathcal{Y}_\ell} \mu_{\ell\mathbf{z}} \|L_{(\ell+1)\mathbf{z}}\|_{L^2_\pi(\Gamma)}$  and the iteration counter  $n := 0$ ;  
 274 initialize the mesh  $\mathcal{T}_{0\mathbf{z}'} := \mathcal{T}_0$ .

- 275 (i) Compute the Galerkin approximation  $u_{n\mathbf{z}'} \in \mathbb{X}_{n\mathbf{z}'}$  by solving (6a) or (6b).
- 276 (ii) Compute the error estimate  $\mu_{n\mathbf{z}'} = \|e_{n\mathbf{z}'}\|_{\mathbb{X}}$  by solving (7a) or (7b) and compute  
 277 the corresponding local error indicators  $\{\mu_{n\mathbf{z}'}(\xi) : \xi \in \mathcal{N}_{n\mathbf{z}'}^+\}$ .
- 278 (iii) If  $\mu_{n\mathbf{z}'} \|L_{(\ell+1)\mathbf{z}'}\|_{L^2_\pi(\Gamma)} < \text{tol}$ , set  $\mathcal{T}_{(\ell+1)\mathbf{z}'} := \mathcal{T}_{n\mathbf{z}'}$  and exit.
- 279 (iv) Determine  $\mathcal{M}_{n\mathbf{z}'} \subseteq \mathcal{N}_{n\mathbf{z}'}^+$  of minimal cardinality such that

$$280 \quad \theta \sum_{\xi \in \mathcal{N}_{n\mathbf{z}'}^+} \mu_{n\mathbf{z}'}(\xi)^2 \leq \sum_{\xi \in \mathcal{M}_{n\mathbf{z}'}} \mu_{n\mathbf{z}'}(\xi)^2.$$

- 281 (v) Set  $\mathcal{T}_{(n+1)\mathbf{z}'} := \text{refine}(\mathcal{T}_{n\mathbf{z}'}, \mathcal{M}_{n\mathbf{z}'})$ .
- 282 (vi) Increase the counter  $n \mapsto n + 1$  and goto (i).

283 **Output:** The mesh  $\mathcal{T}_{(\ell+1)\mathbf{z}'}$  associated with the collocation point  $\mathbf{z}'$ .

284 Results presented in the next section will show that a well-designed multilevel strat-  
 285 egy can give significant efficiency gains compared to a single-level SC-FEM algorithm  
 286 if the parameterized problem has local features that vary in spatial location across the  
 287 parameter space.

Results for three test cases are discussed in this section of the paper. The performance of our adaptive SC multilevel algorithm will be directly compared with that of the single-level algorithm discussed in [5] to see if any gains in efficiency can be realized. The output counter  $k$  is set to 1 in all the experiments to facilitate a comparison of the error estimates and indicators. The first two test cases are identical to those discussed in §5 of [5]. The third test case is a refinement of the *one-peak* test problem that was introduced by Kornhuber & Youett [16] in order to assess the efficiency of adaptive Monte Carlo methods. In all computations we employ sparse grids based on Clenshaw–Curtis points with the standard doubling rule.

The single-level refinement strategy that is the basis for comparison is the obvious and natural simplification of the multilevel strategy described in §3. In particular, to effect a spatial refinement in the single-level case, we use a Dörfler-type marking with threshold  $\theta_{\mathbf{x}}$  to produce sets of marked interior edge midpoints from the (single) grid  $\mathcal{T}_\ell$ . A refined triangulation  $\mathcal{T}_{\ell+1}$  can then be constructed using the *union* of these individual marking sets, i.e.,  $\mathcal{T}_{\ell+1} := \text{refine}(\mathcal{T}_\ell, \bigcup_{\mathbf{z} \in \mathcal{Y}_\ell} \mathcal{M}_{\ell\mathbf{z}})$ .

**4.1. Test case I: affine coefficient data.** We set  $f = 1$  and look to solve the first model problem on the square-shaped domain  $D = (0, 1)^2$  with random field coefficient given by

$$a(x, \mathbf{y}) = a_0(x) + \sum_{m=1}^M a_m(x) y_m, \quad x \in D, \mathbf{y} \in \Gamma. \quad (14)$$

The specific problem we consider is taken from [4]. The parameters  $y_m$  in (14) are the images of uniformly distributed independent mean-zero random variables, so that  $\pi_m = \pi_m(y_m)$  is the associated probability measure on  $\Gamma_m = [-1, 1]$ . The expansion coefficients  $a_m$ ,  $m \in \mathbb{N}_0$  are chosen to represent planar Fourier modes of increasing total order. Thus, we fix  $a_0(x) := 1$  and set

$$a_m(x) := \alpha_m \cos(2\pi\beta_1(m)x_1) \cos(2\pi\beta_2(m)x_2), \quad x = (x_1, x_2) \in (0, 1) \times (0, 1). \quad (15)$$

The modes are ordered so that for any  $m \in \mathbb{N}$ ,

$$\beta_1(m) = m - k(m)(k(m) + 1)/2 \quad \text{and} \quad \beta_2(m) = k(m) - \beta_1(m) \quad (16)$$

with  $k(m) = \lfloor -1/2 + \sqrt{1/4 + 2m} \rfloor$  and the amplitude coefficients are constructed so that  $\alpha_m = \bar{\alpha}m^{-2}$  with  $\bar{\alpha} = 0.547$ . This is referred to as the *slow decay case* in [4].

A reference solution to this problem with  $M$  set to 4 is illustrated in Fig. 1 in [5]. This solution was generated by running the *single-level* algorithm with the `errortolerance` set to `6e-3`, starting from a uniform initial mesh with 81 vertices and a sparse grid consisting of a single collocation point. The threshold parameter  $\vartheta$  was set to 1, the marking parameters  $\theta_{\mathbf{x}}$  and  $\theta_{\mathbf{y}}$  were set to 0.3. The error tolerance was satisfied after 25 iterations comprising 20 spatial refinement steps and 5 parametric refinement steps. There were 13 Clenshaw–Curtis sparse grid collocation points when the iteration terminated. These points are visualized in Fig. 1. The associated sparse grid indices are listed in Table 1 in [5]. The final spatial mesh is shown in Fig. 2 in [5]. The number of vertices in this mesh is 16,473 so the total number of degrees of freedom when the error tolerance was satisfied when running the single-level algorithm was 214,149.

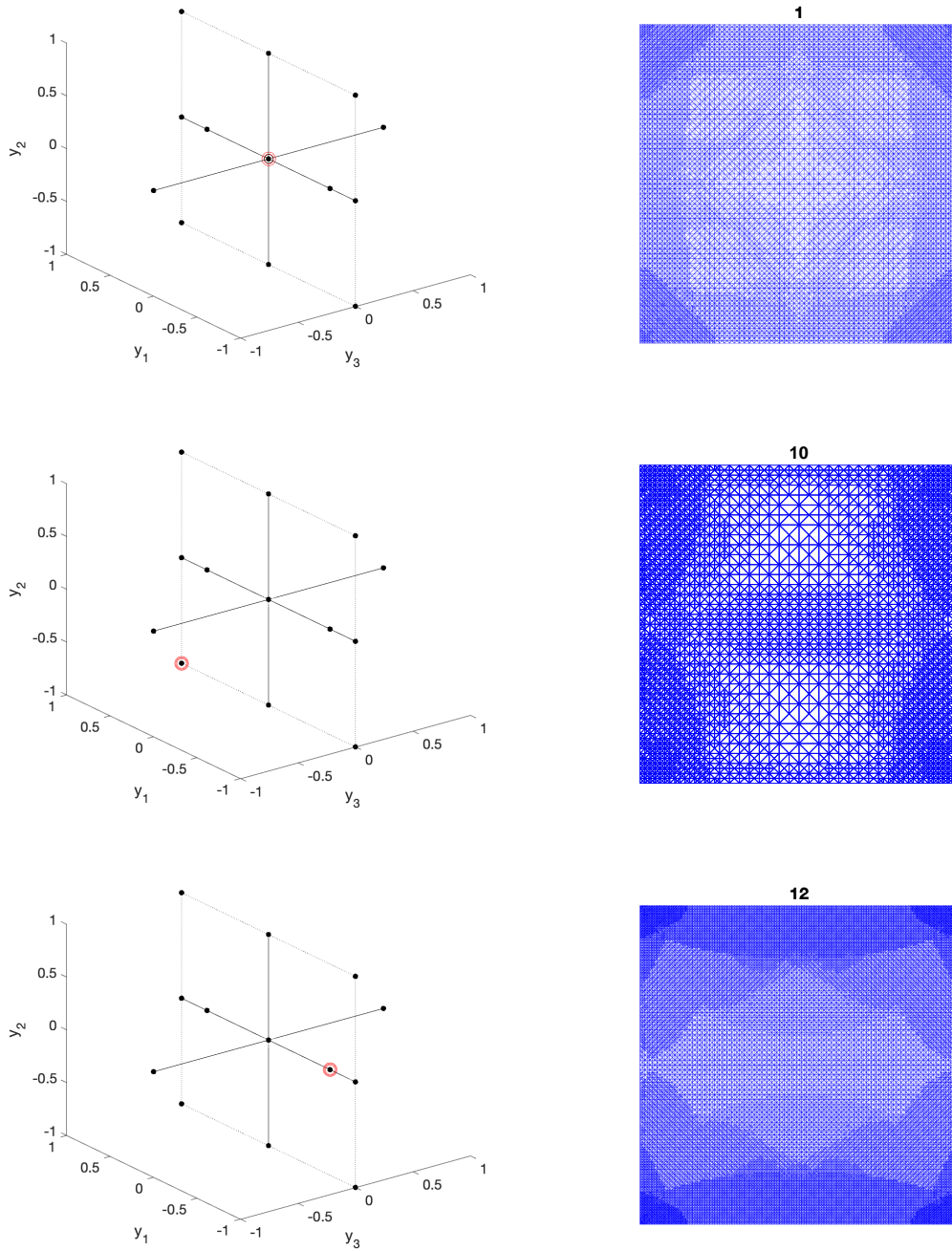


FIGURE 1. Selected collocation point (left) and corresponding spatial mesh (right) that is generated by the multilevel adaptive strategy for test case I.

330 The first test of the *multilevel* algorithm is to repeat the above experiment; that is,  
 331 starting from the same point with identical marking parameters  $\vartheta = 1$ ,  $\theta_x = \theta_y = 0.3$   
 332 and the same initial coarse mesh  $\mathcal{T}_0$  (we also set the marking parameter  $\theta$  in Algorithm 3  
 333 to the same value as  $\theta_x$  in all our experiments). Specifying the same error tolerance  $6e-3$   
 334 led to the same sparse grid of 13 collocation points, in this case after 26 rather than

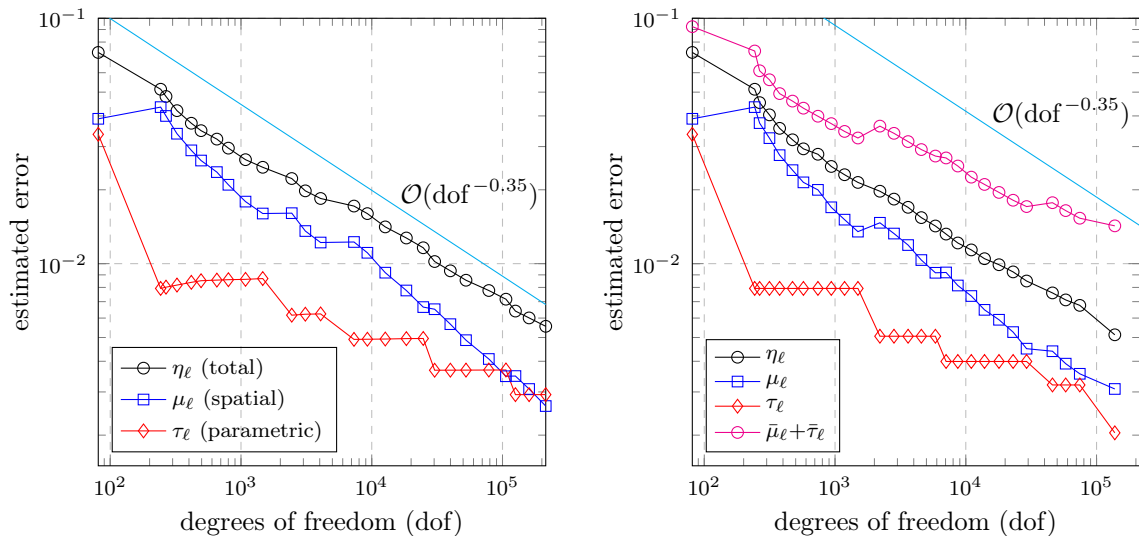


FIGURE 2. Evolution of the single-level error estimates (left) and the multilevel error estimates and sum of the error indicators (right) for test case I with error tolerance set to  $6\mathbf{e-3}$ . The axes limits are identical in the left and right plots.

25 iterations. A comparison of the single-level and multilevel error estimates is given in Fig. 2. While the final number of degrees of freedom is reduced from 214,149 to 137,943 in the multilevel case, the *rate of convergence* is still far from optimal (close to  $O(\text{dof}^{-1/3})$ ).

The degree of refinement of the final meshes associated with some specific collocation points is illustrated in Fig. 1. The two finest meshes had over 32,000 vertices and are associated with the pair of collocation points that are activated by the multi-index  $(3, 1, 1, 1)$  that is introduced at the final iteration (one of these collocation points and the corresponding mesh are shown in the bottom plot). The two coarsest meshes had close to 3,600 vertices; one of these is shown in the middle plot. The mesh that is associated with the mean field  $a_0 = 1$  has 11,157 vertices and is shown in the topmost plot. As might be anticipated, the level of refinement of this mesh is less than that of the final mesh that is generated by the single-level strategy.

It is worth pointing out that in our extensive experimentations with other choices of marking parameters the adaptive multilevel SC-FEM algorithm did not exhibit a faster convergence rate compared to that of the single-level algorithm for the respective choice of marking parameters. This is in contrast to SGFEM, where multilevel adaptivity always results in a faster convergence rate than that of the single-level counterpart for problems with affine-parametric coefficients including the test case considered here; see [10, 8, 2, 3]. Furthermore, for this class of problems, the analysis in [3] has shown that, under an appropriate saturation assumption, the adaptive multilevel SGFEM algorithm driven by a two-level a posteriori error estimator and employing a Dörfler-type marking on the joint set of spatial and parametric indicators yields optimal convergence rates with respect to the number of degrees of freedom in the underlying multilevel approximation space.

**4.2. Test case II: nonaffine coefficient data.** In this case, we set  $f = 1$  and look to solve the first model problem on the L-shaped domain  $D = (-1, 1)^2 \setminus (-1, 0]^2$  with coefficient  $a(x, \mathbf{y}) = \exp(h(x, \mathbf{y}))$ , where the exponent field  $h(x, \mathbf{y})$  has affine dependence

361 on parameters  $y_m \in [-1, 1]$  that are images of uniformly distributed independent mean-  
 362 zero random variables,

$$363 \quad h(x, \mathbf{y}) = h_0(x) + \sum_{m=1}^4 h_m(x) y_m, \quad x \in D, \mathbf{y} \in \Gamma. \quad (17)$$

364  
 365 We further specify  $h_0(x) = 1$  and  $h_m(x) = \sqrt{\lambda_m} \varphi_m(x)$  ( $m = 1, \dots, 4$ ). Here  $\{(\lambda_m, \varphi_m)\}_{m=1}^{\infty}$   
 366 are the eigenpairs of the integral operator  $\int_{D \cup (-1, 0]^2} \text{Cov}[h](x, x') \varphi(x') dx'$  with a synthetic  
 367 covariance function given by

$$368 \quad \text{Cov}[h](x, x') = \sigma^2 \exp(-|x_1 - x'_1| - |x_2 - x'_2|). \quad (18)$$

370 The standard deviation  $\sigma$  is set to 1.5 in order to mirror the most challenging test case  
 371 in §5.2 of [5]. The convergence of the multilevel adaptive algorithm, starting with one  
 372 collocation point and with the initial grid shown in Fig. 7 of [5] is compared with the  
 373 single-level result in Fig. 3. The multilevel algorithm is again run using the marking  
 374 parameters  $\theta_x = \theta_y = 0.3$  specified in [5] and the same error tolerance, that is  $6e-3$ .

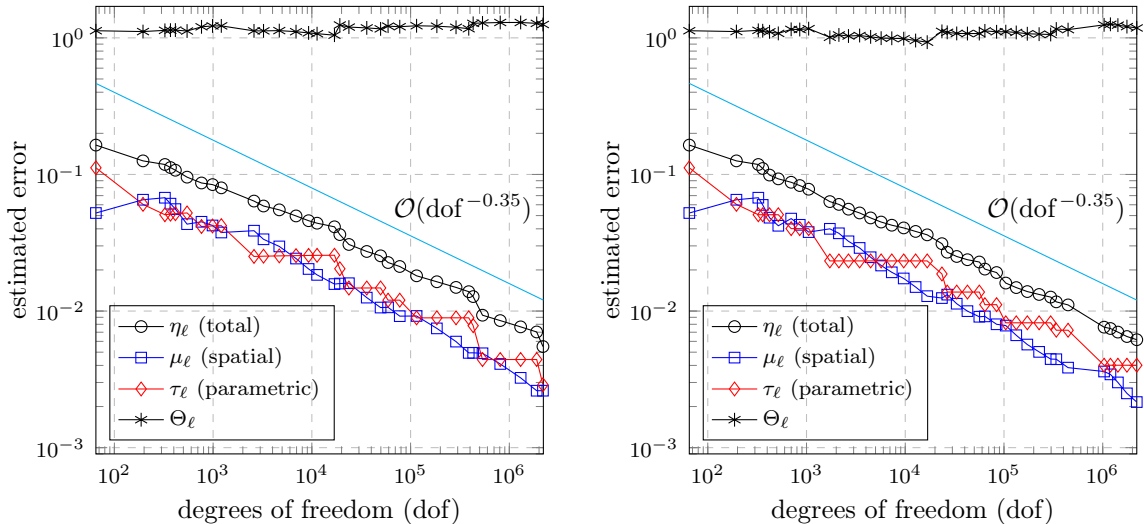


FIGURE 3. Evolution of the single-level error estimates (left) and the multilevel error estimates (right) for test case II with error tolerance set to  $6e-3$ . The axes limits are identical in the left and right plots.

375 These results reinforce the view that performance gains from the multilevel strategy  
 376 are difficult to realize. While the number of active collocation points is smaller in the  
 377 multilevel case (51 vs 57; the multi-index  $(2, 1, 2, 2)$  added at the final single-level iteration  
 378 is not included), the total number of degrees of freedom when the tolerance is reached  
 379 is almost identical (2,212,393 vs 2,190,847). The issue here is that meshes associated  
 380 with mixed indices with multiple active dimensions have multiple features that require  
 381 resolution. Thus, the most refined grid associated with the index that is introduced in  
 382 the final parametric enhancement has 428,972 vertices. This is significantly more refined  
 383 than the final grid that is generated in the single-level implementation, which had 37,133  
 384 vertices. This fact, together with the increase in the number of adaptive steps taken  
 385 (37 vs 31) means that the overall computation time is significantly increased when the

386 multilevel strategy is adopted. In addition, since the generated locally refined meshes  
 387 are not necessarily nested, the need to store all the meshes imposes significant memory  
 388 requirements.

389 The plots in Fig. 3 also show that the use of the *coarsest-mesh* approximations for  
 390 computing the parametric error estimates  $\tau_\ell$  in (12) does not affect the overall effectivity  
 391 of the error estimation in the multilevel algorithm. Indeed, in the single-level algorithm  
 392 (where parametric error estimates employ the (single) *refined mesh* underlying the cur-  
 393 rent SC-FEM solution  $u_\ell^{\text{SC}}$ ), the effectivity indices  $\Theta_\ell$  computed<sup>6</sup> at each iteration range  
 394 between 1.047 and 1.296, whereas for the multilevel algorithm they stay between 0.930  
 395 and 1.257.

396 **4.3. Test case III: one-peak problem.** We are looking to solve the Poisson equation  
 397  $-\nabla^2 u = f$  in a unit square domain  $D = (-4, 4) \times (-4, 4)$  with Dirichlet boundary data  
 398  $u = g$ . The source term  $f$  and boundary data are *uncertain* and are parameterized  
 399 by  $\mathbf{y} = (y_1, y_2)$ , representing the image of a pair of independent random variables with  
 400  $y_j \sim U[-1, 1]$ . In the vanilla case discussed in [16], the same test problem is posed on the  
 401 unit domain  $I = (-1, 1) \times (-1, 1)$  with  $y_j \sim U[-1/4, 1/4]$ . The source term  $f$  and the  
 402 boundary data  $g$  are chosen so that the problem has a specific pathwise solution given by

$$403 \quad u(x, \mathbf{y}) = \exp(-\beta\{(x_1 - y_1)^2 + (x_2 - y_2)^2\}),$$

404 where a scaling factor  $\beta = 50$  is chosen to generate a highly localized Gaussian profile  
 405 centered at the uncertain spatial location  $(y_1, y_2)$ .  
 406

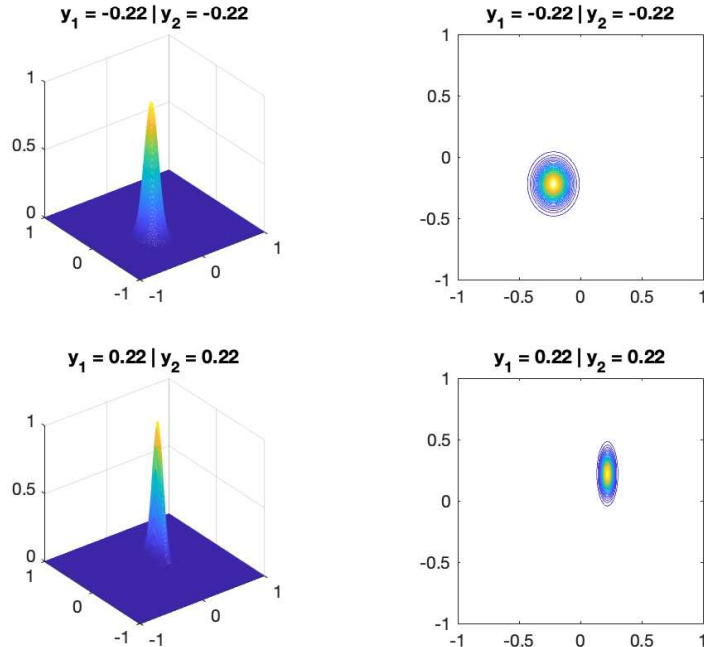


FIGURE 4. One-peak problem solutions on the unit domain:  $\alpha = 1.54$  (top),  $\alpha = 9.46$  (bottom).

<sup>6</sup>The effectivity indices are computed using a reference solution as explained in [5], see equation (43) therein.

407 In the paper [17], the one-peak test problem defined on the unit domain is made  
 408 *anisotropic* by scaling the solution in the first coordinate direction by a linear function  
 409  $\alpha(y_1) = 18y_1 + 11/2$  so that  $\alpha$  takes values in the interval  $[1, 10]$ . The corresponding  
 410 pathwise solution is then given by

$$411 \quad u(x, \mathbf{y}) = \exp(-\beta\{\alpha(y_1)(x_1 - y_1)^2 + (x_2 - y_2)^2\}). \quad (19)$$

413 The solution (19) is generated by specifying an uncertain forcing function

$$414 \quad f(x, \mathbf{y}) = d(x_1, x_2, y_1, y_2) \cdot \exp(-\beta\{\alpha(y_1)(x_1 - y_1)^2 + (x_2 - y_2)^2\}) \quad (20a)$$

416 with

$$417 \quad d(x_1, x_2, y_1, y_2) = -4\beta^2 \{\alpha^2(y_1)(x_1 - y_1)^2 + (x_2 - y_2)^2\} + 2\beta(\alpha(y_1) + 1). \quad (20b)$$

419 Realisations of the exact solution (19) with  $\beta = 50$  are shown at two distinct sample  
 420 points in Fig. 4. The anisotropy introduced by the scaling with  $\alpha$  is a clear feature.

421 Our specific goal is to compute the following quantity of interest (QoI)

$$422 \quad \mathbb{E}[\phi_I(u)] = \int_{[-\frac{1}{4}, \frac{1}{4}]^2} \int_I u^2(x, \mathbf{y}) \, dx \, d\pi(\mathbf{y}), \quad (21)$$

424 where  $\phi_I(u) = \int_I u^2(x, \cdot) \, dx$ . The choice  $\beta = 50$  is then helpful for two reasons:

- 425 • The Dirichlet boundary condition ( $u$  satisfying (19) on  $\partial I$ ) may be replaced with-
- 426 out significant loss of accuracy by the numerical approximation  $u_{\bullet \mathbf{z}} = 0$  on  $\partial I$ .
- 427 • A reference value (accurate to more than 10 digits)

$$428 \quad \mathbb{E}[\phi_I(u)] \approx Q := \frac{1}{9} \cdot (\sqrt{10} - 1) \cdot \frac{\pi}{\beta} = 0.015095545 \dots \quad (22)$$

430 may be readily computed; see [17, Appendix] for details.

431 We compute estimates of the QoI by solving the problem (1b) using the coordinate trans-  
 432 formations  $x_j \leftarrow 4x_j$  and  $y_j \leftarrow 4y_j$  ( $j = 1, 2$ ). In this case, the pathwise solution on  
 433 the scaled domain  $D \times \Gamma$  is given by (19) with  $\beta = 50/16$  and  $\alpha(y_1) = (9y_1 + 11)/2$ .  
 434 Moreover, the QoI in (21) (and its reference value given in (22)) can be estimated within  
 435 Algorithm 1 by computing the following quantity:

$$436 \quad \frac{1}{16} \mathbb{E}[\phi_D(u_\ell^{\text{SC}})] = \frac{1}{16} \int_\Gamma \int_D (u_\ell^{\text{SC}}(x, \mathbf{y}))^2 \, dx \, d\pi(\mathbf{y}).$$

438 Statistics of a solution to the scaled problem are illustrated in Fig. 5.

439 A comparison of the single-level and multilevel SC-FEM algorithms with default mark-  
 440 ing parameters, when applied to the one-peak test problem, is given by the evolution of  
 441 error estimates and errors in the QoI shown in Fig. 6. The single-level algorithm reached  
 442 the tolerance in 37 steps with 169 active collocation points and the final approximation  
 443 had 42,961,659 degrees of freedom. The multilevel algorithm proved to be much more  
 444 efficient. The same tolerance was reached in 34 steps with 153 collocation points in the  
 445 final approximation space. Crucially, each collocation point is associated with a mesh  
 446 that is locally refined in the vicinity of the respective point in  $D$  (as illustrated in Fig. 7).  
 447 In contrast, the final mesh generated by the adaptive single-level SC-FEM has refinement  
 448 everywhere in a larger region corresponding to the union of supports of all sampled solu-  
 449 tions. When the error tolerance was reached, both algorithms gave estimates of the QoI

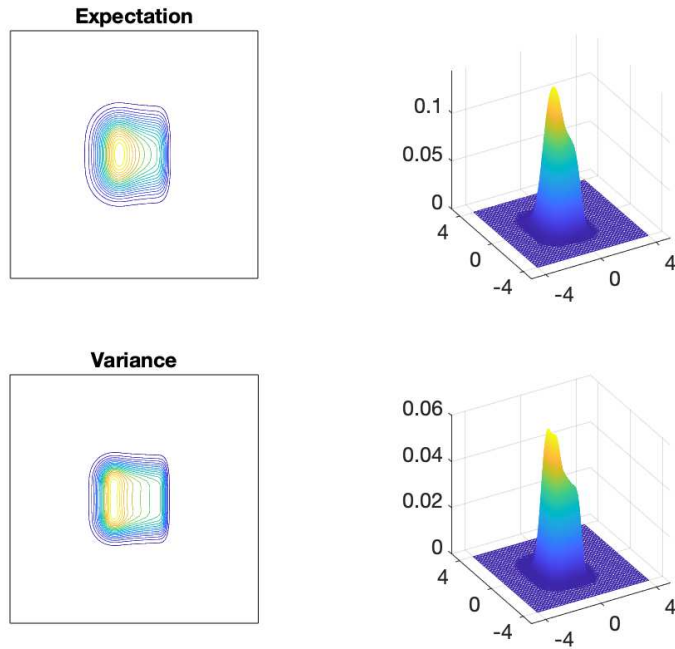


FIGURE 5. The expectation and the variance of the solution for test case III.

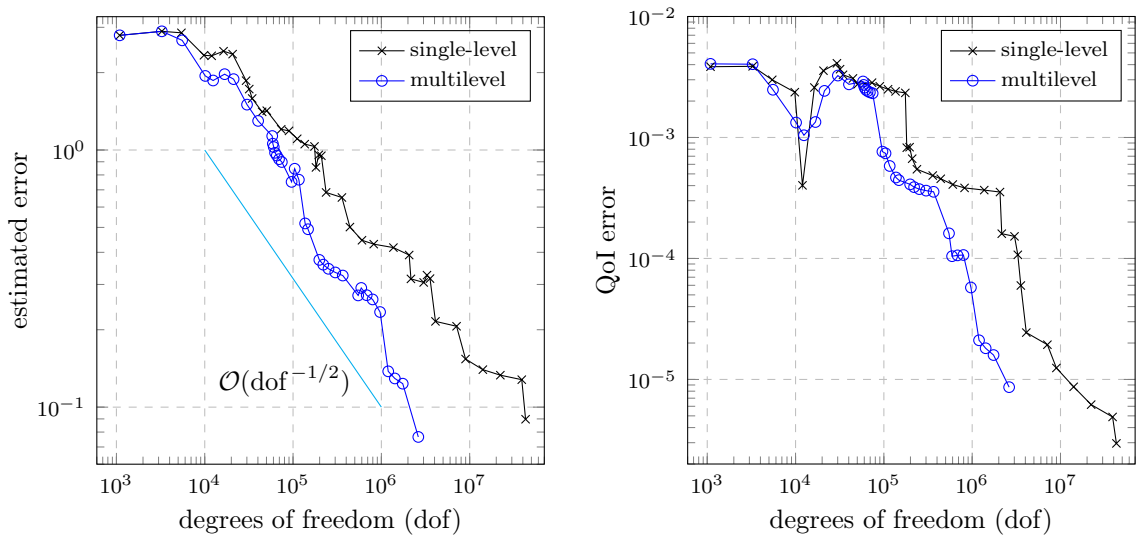


FIGURE 6. Evolution of the single-level and multilevel error estimates (left) and the corresponding errors in the QoI (right) for the one-peak test problem with error tolerance set to  $1e-1$ .

450 that agreed with the reference value to five decimal places (0.015092 for the single-level  
 451 case vs 0.015087 for the multilevel case).

452 The upshot of the effective use of tailored refinement is an order of magnitude decrease  
 453 in the overall computation time. The total number of degrees of freedom in the multilevel  
 454 case was 2,620,343—a factor of 16 reduction overall. Looking at the associated rates of



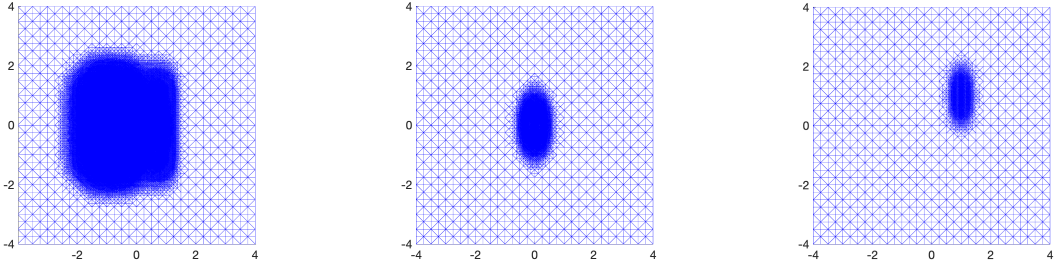


FIGURE 7. Single-level mesh (left) and meshes associated with the central collocation point (middle) and top right corner point (right) when the tolerance is reached for test case III.

455 convergence we see that the optimal rate  $O(\text{dof}^{-1/2})$  is recovered in the multilevel case.  
 456 We anticipate that similar performance gains will be realized whenever a problem has  
 457 local features that can be effectively resolved using sample-dependent meshes.

458 We have also solved the one-peak test problem using an efficient adaptive stochastic  
 459 Galerkin approximation strategy. While the linear algebra associated with the Galerkin  
 460 formulation is decoupled in this case, the computational overhead of evaluating the right-  
 461 hand side vector is a significant limiting factor in terms of the relative efficiency. The  
 462 overall CPU time taken to compute 4 digits in the QoI using adaptive stochastic Galerkin  
 463 FEM is comparable to the CPU time taken to compute 5 digits using the multilevel  
 464 SC-FEM strategy.

## 465 5. CONCLUSIONS

466 Adaptive methods hold the key to efficient approximation of solutions to linear elliptic  
 467 partial differential equations with random data. The numerical results presented in this  
 468 series of two papers demonstrate the effectiveness and the robustness of our novel SC-  
 469 FEM error estimation strategy, as well as the utility of the error indicators guiding the  
 470 adaptive refinement process. Furthermore, the proposed error estimation strategy, error  
 471 indicators, and adaptive algorithms can be easily extended to other parametric PDE  
 472 problems with either affine or nonaffine parametric dependence of inputs. Our results  
 473 also suggest that optimal rates of convergence are more difficult to achieve in a sparse  
 474 grid collocation framework than in a multilevel stochastic Galerkin framework. It is  
 475 demonstrated herein that the overhead of generating specially tailored sample-dependent  
 476 meshes can be worthwhile and optimal convergence rates can be recovered when the  
 477 solutions to the sampled problems have local features in space. The single-level strategy  
 478 discussed in part I of this work is, however, likely to be more efficient (certainly in terms  
 479 of overall CPU time and memory requirements) when a single adaptively refined grid can  
 480 adequately resolve spatial features associated with solutions to a range of individually  
 481 sampled problems.

482 An efficient implementation of the multilevel SC-FEM would benefit from dedicated  
 483 memory optimization algorithms for storing computational meshes as well as from accel-  
 484 eration methods for computing sampled Galerkin solutions (e.g., by an iterative process  
 485 initialized at the previously computed Galerkin approximation on a coarser mesh). These  
 486 implementation aspects require further investigation.

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530 SCHOOL OF MATHEMATICS, UNIVERSITY OF BIRMINGHAM, EDGBASTON, BIRMINGHAM B15 2TT,  
 531 UK

532 *E-mail address:* a.bespalov@bham.ac.uk

533 DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANCHESTER, OXFORD ROAD, MANCHESTER  
 534 M13 9PL, UK

535 *E-mail address:* d.silvester@manchester.ac.uk