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RAMSEY NUMBERS WITH PRESCRIBED RATE OF GROWTH

MATÍAS PAVEZ-SIGNÉ, SIMÓN PIGA, AND NICOLÁS SANHUEZA-MATAMALA

ABSTRACT. Let $R(G)$ be the two-colour Ramsey number of a graph G . In this note, we prove that for any non-decreasing function $n \leq f(n) \leq R(K_n)$, there exists a sequence of connected graphs $(G_n)_{n \in \mathbb{N}}$, with $|V(G_n)| = n$ for all $n \geq 1$, such that $R(G_n) = \Theta(f(n))$. In contrast, we also show that an analogous statement does not hold for hypergraphs of uniformity at least 5.

We also use our techniques to answer a question posed by DeBiasio about the existence of sequences of graphs whose 2-colour Ramsey number is linear whereas their 3-colour Ramsey number has superlinear growth.

1. INTRODUCTION

For a graph G and $r \geq 2$, the r -colour Ramsey number $R_r(G)$ of G is the smallest number n such that every r -edge-colouring of the edges of the complete graph K_n contains a monochromatic copy of G , that is, a copy of G with all its edges in the same colour. For $r = 2$ we will simply write $R_2(G) = R(G)$ and refer to this as *the* Ramsey number of G . The most notorious open problem here is to determine the Ramsey number of cliques. The classical bounds on $R(K_n)$ by Erdős [Erd47] and Erdős and Szekeres [ES35] imply that $\sqrt{2}^n \leq R(K_n) \leq 4^n$, so $R(K_n)$ is exponential in n , but despite tremendous efforts its exact behaviour remains unknown.

In general, if a graph G on n vertices has m edges, then $2^{\Omega(m/n)} \leq R(G) \leq 2^{O(\sqrt{m})}$, where the lower bound follows from a probabilistic construction and the upper bound was shown by Sudakov [Sud11]. Given additional structure on G , there are many cases where we can even obtain $R(H) = O(n)$. This holds, for instance, for graphs with bounded maximum degree [Chv+83], bounded arrangeability [CS93], and bounded degeneracy [Lee17]. We recommend [CFS15] for a survey in the area.

As we have seen, the Ramsey number of an n -vertex graph can vary between linear and exponential in n . A natural question is thus to ask which values (between n and $R(K_n)$) can be attained as the Ramsey number of some n -vertex graph. The aim of this note is to study this question, and, in particular, to determine which functions $f : \mathbb{N} \rightarrow \mathbb{N}$, with $n \leq f(n) \leq R(K_n)$ for all $n \in \mathbb{N}$, are the rate of growth of the Ramsey numbers of some sequence of n -vertex graphs.

It is natural here to restrict our analysis to connected graphs. Note that after adding $n - r$ isolated vertices to an r -vertex graph H , we end with an n -vertex graph H' satisfying $R(H') = \max\{n, R(H)\}$. This means that we can obtain values for the Ramsey number of n -vertex graphs which in essence correspond to the Ramsey number of r -vertex graphs; restricting to connected graphs rules out such constructions. Our first result is that every function can be attained as the rate of growth of some sequence of graphs, up to a multiplicative factor.

Theorem 1. *There exist positive constants C and n_0 such that for every non-decreasing function $f : \mathbb{N} \rightarrow \mathbb{N}$, with $n \leq f(n) \leq R(K_n)$, there exists a sequence of connected graphs $(G_n)_{n \in \mathbb{N}}$ such that for all $n \geq n_0$, $|V(G_n)| = n$ and $f(n) \leq R(G_n) \leq Cf(n)$.*

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In other words, we have $R(G_n) = \Theta(f(n))$, where the implicit constants do not depend on the function f . We remark that by a result of Burr and Erdős [BE76] on the Ramsey number of trees, it is known that every n -vertex connected graph G satisfies $R(G) \geq \lceil \frac{4}{3}n \rceil - 1$; thus taking the function $f(n) = \alpha n$ for any $1 \leq \alpha < 4/3$ shows that the conclusion of Theorem 1 cannot hold with $R(G_n) = (1 + o(1))f(n)$ instead. We discuss the structure of these ‘gaps’ further in Section 5.

Our second result concerns k -uniform hypergraphs. A k -graph H is a pair $H = (V, E)$ where V is the set of vertices of H and every edge $e \in E$ is a k -element subset of V . For $n \in \mathbb{N}$, the k -uniform clique on n vertices $K_n^{(k)}$ is the hypergraph consisting of n vertices such that every k -element subset of vertices is an edge. Given a k -graph H , the Ramsey number $R(H)$ of H is the smallest number n such that every red-blue colouring of the edges of $K_n^{(k)}$ yields a monochromatic copy of H .

We prove that an analogous of Theorem 1 fails for k -graphs if $k \geq 5$ (even without any kind of connectivity restrictions).

Theorem 2. *Let $k \geq 5$. There exists a non-decreasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ with $n \leq f(n) \leq R(K_n^{(k)})$, such that for all $c, C > 0$ and any n_0 , there is an $n > n_0$ such that*

$$R(H) \leq cf(n) \quad \text{or} \quad R(H) \geq Cf(n)$$

for every n -vertex k -graph H .

Using our techniques we can also answer a question posed by DeBiasio [DeB], who asked about the existence of a sequence G_n of graphs where $R_2(G_n)$ is linear whilst $R_3(G_n)$ is superlinear. Similar differences in behaviour depending on the number of colours have been observed before in infinite graphs (see [CDM20, Section 10.1]) and in k -graphs with $k \geq 3$ (see [CFR17]). We answer DeBiasio’s question in the affirmative.

Theorem 3. *There exists a sequence $(G_n)_{n \in \mathbb{N}}$ of graphs such that $|V(G_n)| = n$, $R_2(G_n) = O(n)$ and $R_3(G_n) = \Omega(n \log n)$.*

The graphs we construct for Theorem 3 have isolated vertices. If we insist on sequences of connected graphs, we can get the following.

Theorem 4. *There is a sequence $(G_n)_{n \in \mathbb{N}}$ of connected graphs such that $|V(G_n)| = n$, $R_2(G_n) = O(n \log n)$ and $R_3(G_n) = \Omega(n \log^2 n)$.*

2. PROOF OF THEOREM 1

Conlon, Fox and Sudakov [CFS20] proved that the Ramsey number of a dense graph cannot decrease by much under the deletion of one vertex. A graph on n vertices has density d if it has $d \binom{n}{2}$ edges.

Lemma 5 ([CFS20]). *There exists a constant $c > 0$ such that for every graph H of density at least d and any graph H' obtained by deleting a single vertex from H , we have $R(H) \leq (c \log(1/d)/d)R(H')$.*

The following is an immediate corollary.

Lemma 6. *There exist $c_1, c_2 > 0$ so that for any $n \geq 1$,*

- (i) $R(K_{n+1}) \leq c_1 R(K_n)$,
- (ii) $R(K_{n+1, n+1}) \leq c_2 R(K_{n, n})$.

We also need bounds on the Ramsey number of a path P_n with n edges, which is due to Gerencsér and Gyarfás [GG67].

Lemma 7. For every $n \geq 1$, $R(P_n) = \lceil (3n + 1)/2 \rceil$.

We shall also use a lower bound on the Ramsey number of complete bipartite graphs, which follows from a standard probabilistic construction.

Lemma 8. For $t \geq 1$, $R(K_{t,t}) \geq 2^{t/2}$.

We are now ready for the proof of our first result.

Proof of Theorem 1. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing function such that $n \leq f(n) \leq R(K_n)$ for all $n \in \mathbb{N}$. Fixing a large $n \in \mathbb{N}$, we will construct an appropriate n -vertex graph G_n . We will split the proof into two cases, depending on how large $f(n)$ is. In fact, the two ranges we consider are not disjoint, but they are enough to cover all possibilities between n and $R(K_n)$.

Case 1: $n \leq f(n) \leq 2^{n/8}$. We construct the graph G_n as follows. Let t be the minimal number such that $R(K_{t,t}) > f(n)$. We note that by the choice of t , we have $R(K_{t-1,t-1}) \leq f(n) < R(K_{t,t})$. By Lemma 8, we have $2^{(t-1)/2} \leq f(n)$ and thus $t \leq 2 \log_2(f(n)) + 1$. Since $f(n) \leq 2^{n/8}$, we certainly have $2t \leq n$. Construct G_n by taking $K_{t,t}$ and joining it to a path on $n - 2t$ new vertices, so $|V(G_n)| = n$.

The lower bound on $R(G_n)$ follows from the fact that $K_{t,t} \subseteq G_n$ and the definition of t , as

$$R(G_n) \geq R(K_{t,t}) > f(n).$$

Now we show that $R(G_n) \leq 6c_2f(n)$, where c_2 is the constant appearing in Lemma 6. Let $N = 6c_2f(n)$ and consider any red-blue colouring of the edges of K_N . By Lemma 7, K_N must contain a monochromatic path P on at least $2N/3 \geq 4c_2f(n)$ vertices, which we may assume is red. Let $P' \subseteq P$ be obtained by removing n vertices in one of its extremes, so that P' is a red path on at least $4c_2f(n) - n \geq 3c_2f(n)$ vertices.

Let $S = V(P')$. Note that if S contains a red monochromatic copy of $K_{t,t}$, then we surely obtain a red copy of G_n . Hence, we may assume that S contains no red copy of $K_{t,t}$. Note that by Lemma 6, we have

$$R(K_{t,t}) \leq c_2R(K_{t-1,t-1}) \leq c_2f(n).$$

Thus we can greedily find vertex-disjoint blue monochromatic copies K^1, \dots, K^s of $K_{t,t}$ in S until less than $c_2f(n)$ vertices remain uncovered. Note that these copies cover together at least $|S| - c_2f(n) \geq 3c_2f(n) - c_2f(n) \geq f(n) \geq n$ vertices.

For all $1 \leq i \leq s$, let A_i, B_i be the two classes of K^i . Given $1 \leq i < s$, note that not all edges between B_i and A_{i+1} can be red, as that would yield a red monochromatic copy of $K_{t,t}$ in S . Therefore, there are blue edges e_1, \dots, e_{s-1} where each e_i has one endpoint $b_i \in B_i$ and other endpoint $a_{i+1} \in A_{i+1}$. Let $a_1 \in A_1$ be arbitrary. For all $1 \leq i < s$, take a blue path $P_i \subseteq K^i$ which spans $V(K^i)$ and has endpoints a_i and b_i . Thus, the concatenation $P_1 + e_1 + \dots + P_{s-1} + e_{s-1}$, together with K^s , forms a blue copy of G_n , as required.

Case 2: $2n \log_2 n \leq f(n) \leq R(K_n)$. Take t minimal subject to $R(K_t) \geq f(n)$. Clearly, such t always exists and is at most n . Thus, we have $R(K_{t-1}) < f(n) \leq R(K_t)$. Moreover, since $R(K_r) \geq 2^{r/2}$ holds for all r , we know that $t \leq \min\{n, 2 \log_2 f(n)\}$. By Lemma 6, we also know that $R(K_t) \leq c_1R(K_{t-1}) \leq c_1f(n)$.

Let G_n be the graph consisting of K_t joined to a path on $n - t$ new vertices, so that G_n is a connected graph on n vertices. We know that $R(G_n) \geq R(K_t) \geq f(n)$.

Let $N = 6c_1f(n)$ where c_1 comes from Lemma 6. We want to show that $R(G_n) \leq N$. Suppose there is no monochromatic copy of G_n in K_N . As before, we may find a monochromatic path on at least $2N/3 = 4c_1f(n)$ vertices, and suppose it is red. We remove n vertices from the beginning of the path to obtain a set S on at least $3c_1f(n)$

vertices. Since there is no red copy of K_t in S , all monochromatic copies of K_t in S are blue. As $R(K_t) \leq c_1 f(n)$, we can find vertex-disjoint blue copies of K_t in S until at most $c_1 f(n)$ vertices remain. These copies together cover at least $2c_1 f(n)$ vertices of S . Let Q^1, \dots, Q^r be the r copies of K_t found in this way.

Define a *clique-path* P to be a sequence of vertex-disjoint blue cliques Q^1, \dots, Q^l such that for each $1 \leq i < l$ there is a blue edge e_i between Q^i and Q^{i+1} , and the edges e_i are vertex-disjoint for all $1 \leq i < l$.

Claim 9. *There is a set of at most $t - 1$ clique-paths that together cover all cliques Q^1, \dots, Q^r exactly once.*

Proof. Suppose otherwise and let P_1, \dots, P_{t-1} be $t - 1$ clique-paths which use pairwise-disjoint sets of cliques and together use the maximum possible number of cliques. Let $P_0 = Q^0$ be a clique-path consisting of any clique not used by any P_i , and for each $1 \leq i \leq t - 1$, let Q^i be an ‘‘end-clique’’ of each P_i . In each Q^0, \dots, Q^{t-1} , we select a vertex q_i which is not in any of the edges of the clique-paths. Since S contains no red K_t , there must be a blue edge between some pair $q_i q_j$. But then we can merge P_i and P_j into a longer clique-path using the blue edge $q_i q_j$, and thus we have found a set of $t - 1$ clique-paths covering one more clique, a contradiction. \square

Therefore, there is a clique-path which uses at least $r/(t - 1) \geq r/t$ cliques. In such a clique-path we can easily find a blue clique K_t together with a blue path which together use at least $t \cdot (r/t) = r$ vertices. Thus we are done if $r \geq n$. Indeed, since the cliques use $2c_1 f(n)$ vertices in total and each clique has t vertices, we have at least $r \geq 2c_1 f(n)/t \geq f(n)/\min\{n, \log_2 f(n)\}$, where in the last inequality we used $c_1 \geq 1$ and $t \leq \min\{n, \log_2 f(n)\}$.

If $f(n) \leq 2^n$ then $\min\{n, \log_2 f(n)\} = \log_2 f(n)$, and therefore the bound in the previous paragraph becomes $r \geq f(n)/\log_2 f(n) \geq 2n \log_2 n / \log_2(2n \log_2 n) \geq n$, as required. Otherwise, if $f(n) > 2^n$, then $\min\{n, \log_2 f(n)\} = n$, in which case we obtain $r \geq f(n)/n \geq n$. \square

3. PROOF OF THEOREM 2

For k -graphs, the so-called ‘stepping-up lemma’ by Erdős, Hajnal, and Rado [EHR65] allows us to deduce a tower-type lower bound for the Ramsey number $R(K_n^{(k)})$ for every $k \geq 3$, namely

$$2^{an^2} \leq \log^{(k-2)}(R(K_n^{(k)})), \quad (1)$$

where $a > 0$ is a constant depending only on k and $\log^{(i)}(\cdot)$ denotes the i th iterated logarithm.

Proof of Theorem 2. Let $k \geq 5$. We find a function $g: \mathbb{N} \rightarrow \mathbb{N}$, with $n \leq g(n) \leq R(K_n^{(k)})$ as follows. For every $n \in \mathbb{N}$, let $I_n = [\log n, \log R(K_n^{(k)})]$ be an interval in \mathbb{R} . Note that, since $k \geq 5$, inequality (1) implies that

$$\log R(K_n^{(k)}) - \log n \geq 2^{2^{an}} - \log n.$$

Since the number of k -graphs on n vertices is at most 2^{n^k} , by averaging we find a sub-interval $I'_n \subseteq I_n$ which does not contain $\log R(H)$ for any n -vertex k -graph H , and such that I'_n has length at least

$$\frac{2^{2^{an}} - \log n}{2^{n^k}} \geq 2n,$$

where we used that n is sufficiently large. Let $m_n \in I'_n$ be the middle point of I'_n . Then, for large n and every n -vertex k -graph H , we have

$$\log R(H) \leq m_n - n \quad \text{or} \quad \log R(H) \geq m_n + n. \quad (2)$$

Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined by $g(n) = 2^{m_n}$. Since $m_n \in I_n$ for large n , we have $n \leq g(n) \leq R(K_n^{(k)})$. Then, due to (2) we deduce that for every n and every n -vertex k -graph H ,

$$R(H) \leq 2^{-n}g(n) \quad \text{or} \quad R(H) \geq 2^n g(n).$$

In particular, for every two positive constants $c, C > 0$ and for every a sufficiently large n , we have $R(H) < cg(n)$ or $R(H) > Cg(n)$, as required.

Note that g might decrease. To overcome this, we define $f : \mathbb{N} \rightarrow \mathbb{N}$ by setting $f(1) = g(1)$ and, for $n \geq 2$,

$$f(n) = \begin{cases} g(n) & \text{if } g(n) \geq f(n-1), \\ f(n-1) & \text{if } g(n) < f(n-1). \end{cases}$$

Thus, it is straightforward to check that f is non-decreasing and satisfies the desired conditions. \square

Notice that the proof of Theorem 2 relies on the fact that $\log R(K_n^k) = \omega(2^{n^k})$ for every $k \geq 5$. Erdős, Hajnal, and Rado [EHR65] conjectured that the lower bound in inequality (1) can be improved to $\log^{(k-1)}(R(K_n^{(k)}))$ for every $k \geq 3$, in which case our proof of Theorem 2 works for 4-uniform hypergraphs as well. The situation for 3-uniform hypergraphs is not clear, even if this conjecture were true.

4. PROOF OF THEOREMS 3 AND 4

We shall use the following simple lemma.

Lemma 10. *For every graph G and connected $H \subseteq G$, we have*

$$R_3(G) \geq (\chi(H) - 1)(R_2(H) - 1) + 1.$$

Proof. Let $N = (\chi(H) - 1)(R_2(H) - 1)$. We construct a red-blue-green colouring of K_N as follows: partition $V(K_N)$ into $\chi(H) - 1$ sets $V_1, \dots, V_{\chi(H)-1}$ of size $R_2(H) - 1$ each. Inside each V_i use colours red and blue in such a way that the colouring does not contain a red-blue copy of H ; and colour every other edge green.

This colouring does not contain a monochromatic copy of G . Indeed, an hypothetical such copy cannot be red or blue, as otherwise there must exist a red or blue copy of H . Since H is connected, such a copy of H must lie inside one of the sets V_i , but we have chosen the red-blue edges so that this does not happen. Also, there are no green copies of G_n , since the graph formed by the green edges is $(t-1)$ -partite but $\chi(G) \geq \chi(H) \geq t$. We conclude that $R_3(G) > N$. \square

Proof of Theorem 3. Given n , let t be the least integer such that $n \leq R_2(K_t)$. By choice, we have $R_2(K_{t-1}) < n$ and, by Lemma 6, we have $R_2(K_t) \leq c_1 R_2(K_{t-1}) < c_1 n$. Let G_n be the graph obtained from K_t by adding $n - t$ isolated vertices. Therefore, $|V(G_n)| = n$ and $R_2(G_n) = \max\{n, R_2(K_t)\} = R_2(K_t) < c_1 n = O(n)$. On the other hand, since $n \leq R_2(K_t) \leq 4^t$; by the choice of t we know that $t \geq \frac{1}{2} \log_2 n$ and therefore by Lemma 10 we have $R_3(G_n) = \Omega(n \log n)$. \square

Proof of Theorem 4. As in Case 2 of the proof of Theorem 1, by taking $f(n) = 2n \log_2 n$ we construct a sequence of connected graphs G_n with $|V(G_n)| = n$ and $R_2(G_n) = O(f(n)) = O(n \log n)$.

Note that in Case 2 the graphs G_n are formed by attaching a path to a clique. More precisely, they contain a clique on t vertices with $t \geq \frac{1}{2} \log_2 n$ and a path of length $n - t$. Thus, we have $\chi(G_n) = \Omega(\log n)$ and therefore, by Lemma 10, we have $R_3(G_n) > (\chi(G_n) - 1)(R_2(G_n) - 1) = \Omega(n \log^2 n)$, as required. \square

5. CONCLUDING REMARKS

For $n \in \mathbb{N}$, let us consider the sets

$$\mathcal{R}_n = \{R(G) : |V(G)| = n\},$$

$$\mathcal{R}_n^\circ = \{R(G) : G \text{ does not contain isolated vertices and } |V(G)| = n\}, \quad \text{and}$$

$$\mathcal{R}_n^c = \{R(G) : G \text{ is connected and } |V(G)| = n\}.$$

It is clear that $\mathcal{R}_n^c \subseteq \mathcal{R}_n^\circ \subseteq \mathcal{R}_n \subseteq [n, R(K_n)]$. Observe that $n \in \mathcal{R}_n$ since $R(\overline{K_n}) = n$, where $\overline{K_n}$ corresponds to an independent set on n vertices. Furthermore, consider a disjoint union of two stars $\Sigma_{a,b} = K_{1,a} \cup K_{1,b}$. A result due to Grossman [Gro79] implies that $R(\Sigma_{a,a-i}) = 3a - 2i$ for $i \in \{0, 1, 2\}$. Thus, by adding $n - (2a - i + 2)$ extra isolated vertices to $\Sigma_{a,a-i}$ and letting the value of a vary from $\lfloor n/3 \rfloor$ to $\lfloor (n-2)/2 \rfloor$, we can deduce that $[n, \lfloor \frac{3(n-2)}{2} \rfloor - 3] \subseteq \mathcal{R}_n$. Other families of sparse graphs can also be used to show other inclusions of this kind.

As mentioned in the introduction, $R(G) \geq \lceil \frac{4}{3}n \rceil - 1$ holds for every connected graph G on n vertices, and this bound is tight. In particular, it implies that

$$\mathcal{R}_n^c \subseteq \left[\lceil \frac{4}{3}n \rceil - 1, R(K_n) \right].$$

In a similar fashion, Burr and Erdős [BE76] showed that $R(G) \geq n + \log n - O(\log \log n)$ holds for every $G \in \mathcal{R}_n^\circ$, which is almost tight as shown by Csákány and Komlós [CK99]. It would be interesting to get a better understanding of the structures of \mathcal{R}_n , \mathcal{R}_n° , and \mathcal{R}_n^c .

Given a constant $c > 1$, we say that $a \in [n, R(K_n)]$ is a c -gap for \mathcal{R}_n^c if $[a, ca] \cap \mathcal{R}_n^c = \emptyset$. It is not difficult to see that Theorem 1 is equivalent to the existence of a constant $c \geq 1$ for which \mathcal{R}_n^c has no c -gaps for every sufficiently large n . In this direction a proper (but non-empty) subset of the authors of this paper believe that the answer to the following question should be affirmative.

Question 11. Does there exist an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$

$$\mathcal{R}_n = [n, R(K_n)] \quad \text{and} \quad \mathcal{R}_n^c = \left[\lceil \frac{4}{3}n \rceil - 1, R(K_n) \right] ?$$

Observe that the first equality would imply that for every function $f: \mathbb{N} \rightarrow \mathbb{N}$ with $n \leq f(n) \leq R(K_n)$ there is a sequence of graphs $(G_n)_{n \in \mathbb{N}}$ such that $f(n) = R(G_n)$. An analogous statement would hold for connected graphs if the second identity was true.

Finally, observe that the proof of the first case of Theorem 1 can be modified by replacing the rôle of $K_{t,t}$ with a complete k -partite graph $K_{t,\dots,t}$. In this way, we may ensure that every graph in the sequence $(G_n)_{n \in \mathbb{N}}$ has an arbitrarily large chromatic number.

Theorem 12. For every $k \geq 2$, there are positive constants c, C , and $n_0 \in \mathbb{N}$ such that for every non-decreasing function $f: \mathbb{N} \rightarrow \mathbb{N}$, with $n \leq f(n) \leq R(K_n)$, there is a sequence of connected graphs $(G_n)_{n \in \mathbb{N}}$ with $|G_n| = n$ such that $cf(n) \leq R(G_n) \leq Cf(n)$ for all $n \in \mathbb{N}$. Moreover, $\chi(G_n) \geq k$ for every $n \geq n_0$.

It would be interesting to ensure other properties for the graphs in this sequence. In particular, we believe the graphs can also be taken to have large connectivity.

Conjecture 13. For every $k \geq 2$ and for every non-decreasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ with $n \leq f(n) \leq R(K_n)$ there is a sequence of graphs $(G_n)_{n \in \mathbb{N}}$ with $|G_n| = n$ such that $R(G_n) = \Theta(f(n))$, and G_n is k -connected for all n sufficiently large.

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REFERENCES

- [BE76] S. A. Burr and P. Erdős. “Extremal Ramsey theory for graphs”. *Utilitas Math.* 9 (1976), 247–258. URL: https://www.renyi.hu/~p_erdos/1976-13.pdf.
- [CDM20] Jan Corsten, Louis DeBiasio, and Paul McKenney. “Density of monochromatic infinite subgraphs II”. Preprint (2020). DOI: [10.48550/ARXIV.2007.14277](https://doi.org/10.48550/ARXIV.2007.14277).
- [CFR17] David Conlon, Jacob Fox, and Vojtěch Rödl. “Hedgehogs are not colour blind”. *J. Comb.* 8.3 (2017), 475–485. DOI: [10.4310/JOC.2017.v8.n3.a4](https://doi.org/10.4310/JOC.2017.v8.n3.a4).
- [CFS15] David Conlon, Jacob Fox, and Benny Sudakov. “Recent developments in graph Ramsey theory”. *Surveys in Combinatorics 2015*. Vol. 424. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2015, 49–118. URL: <https://www.cambridge.org/core/books/abs/surveys-in-combinatorics-2015>.
- [CFS20] David Conlon, Jacob Fox, and Benny Sudakov. “Short proofs of some extremal results III”. *Random Structures Algorithms* 57.4 (2020), 958–982. DOI: [10.1002/rsa.20953](https://doi.org/10.1002/rsa.20953).
- [Chv+83] C. Chvátal, V. Rödl, E. Szemerédi, and W.T. Trotter. “The Ramsey number of a graph with bounded maximum degree”. *Journal of Combinatorial Theory, Series B* 34.3 (1983), 239–243. DOI: [https://doi.org/10.1016/0095-8956\(83\)90037-0](https://doi.org/10.1016/0095-8956(83)90037-0).
- [CK99] Rita Csákány and János Komlós. “The smallest Ramsey numbers”. *Discrete Math.* 199.1-3 (1999), 193–199. DOI: [10.1016/S0012-365X\(98\)00300-8](https://doi.org/10.1016/S0012-365X(98)00300-8).
- [CS93] Guantao Chen and R. H. Schelp. “Graphs with linearly bounded Ramsey numbers”. *J. Combin. Theory Ser. B* 57.1 (1993), 138–149. DOI: [10.1006/jctb.1993.1012](https://doi.org/10.1006/jctb.1993.1012).
- [DeB] Louis DeBiasio. “Graphs with linear Ramsey number for two colors, but super-linear Ramsey number for three colors?” MathOverflow. URL: <https://mathoverflow.net/q/385498>.
- [EHR65] P. Erdős, A. Hajnal, and R. Rado. “Partition relations for cardinal numbers”. *Acta Math. Acad. Sci. Hungar.* 16 (1965), 93–196. DOI: [10.1007/BF01886396](https://doi.org/10.1007/BF01886396).
- [Erd47] P. Erdős. “Some remarks on the theory of graphs”. *Bull. Amer. Math. Soc.* 53 (1947), 292–294. DOI: [10.1090/S0002-9904-1947-08785-1](https://doi.org/10.1090/S0002-9904-1947-08785-1).
- [ES35] P. Erdős and G. Szekeres. “A combinatorial problem in geometry”. *Compositio Math.* 2 (1935), 463–470. URL: http://www.numdam.org/item?id=CM_1935__2__463_0.
- [GG67] L. Gerencsér and A. Gyárfás. “On Ramsey-type problems”. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* 10 (1967), 167–170. URL: https://users.renyi.hu/~gyarfas/Cikkek/01_GerencserGyarfas_1967.pdf.
- [Gro79] Jerrold W. Grossman. “The Ramsey numbers of the union of two stars”. *Utilitas Math.* 16 (1979), 271–279.
- [Lee17] Choongbum Lee. “Ramsey numbers of degenerate graphs”. *Ann. of Math. (2)* 185.3 (2017), 791–829. DOI: [10.4007/annals.2017.185.3.2](https://doi.org/10.4007/annals.2017.185.3.2).
- [Sud11] Benny Sudakov. “A conjecture of Erdős on graph Ramsey numbers”. *Adv. Math.* 227.1 (2011), 601–609. DOI: [10.1016/j.aim.2011.02.004](https://doi.org/10.1016/j.aim.2011.02.004).

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