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# Saturated fusion systems on p-groups of maximal class 

Parker, Chris; Grazian, Valentina

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# Saturated Fusion Systems on $p$-groups of Maximal Class 

Valentina Grazian<br>Chris Parker

Author address:<br>Department of Mathematics and Applications, University of Milano - Bicocca, Via Roberto Cozzi 55, 20125 Milano, Italy<br>Email address: valentina.grazian@unimib.it<br>School of Mathematics, University of Birmingham, Birmingham B15 2TT, United Kingdom<br>Email address: c.w.parker@bham.ac.uk

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#### Abstract

For a prime number $p$, a finite $p$-group of order $p^{n}$ has maximal class if and only if it has nilpotency class $n-1$. Here we examine saturated fusion systems $\mathcal{F}$ on maximal class $p$-groups $S$ of order at least $p^{4}$. The Alperin-Goldschmidt Theorem for saturated fusion systems yields that $\mathcal{F}$ is entirely determined by the $\mathcal{F}$-automorphisms of its $\mathcal{F}$-essential subgroups and of $S$ itself. If an $\mathcal{F}$-essential subgroup either has order $p^{2}$ or is non-abelian of order $p^{3}$, then it is called an $\mathcal{F}$-pearl. The facilitating and technical theorem in this work shows that an $\mathcal{F}$ essential subgroup is either an $\mathcal{F}$-pearl, or one of two explicitly determined maximal subgroups of $S$. This result is easy to prove if $S$ is a 2 -group and can be read from the work of Díaz, Ruiz, and Viruel together with that of Parker and Semeraro when $p=3$. The main contribution is for $p \geq 5$ as in this case there is no classification of the maximal class $p$-groups. The main Theorem describes all the reduced saturated fusion systems on a maximal class $p$-group of order at least $p^{4}$ and follows from two more extensive theorems. These two theorems describe all saturated fusion systems, not restricting to the reduced ones for example, on exceptional and non-exceptional maximal class $p$-groups respectively. As a corollary we have the easy to remember result that states that, if $O_{p}(\mathcal{F})=1$, then either $\mathcal{F}$ has $\mathcal{F}$-pearls or $S$ is isomorphic to a Sylow $p$-subgroup of $\mathrm{G}_{2}(p)$ with $p \geq 5$ and the fusion systems are explicitly described.


[^0]
## 1. Introduction and main results

Let $p$ be a prime and $S$ be a finite $p$-group. A fusion system on $S$ is a category which has objects the subgroups of $S$ and morphisms which are injective group homomorphisms between the subgroups of $S$. A fusion system is saturated if it satisfies certain technical axioms which are described in some detail in Section 5 , where standard terminology used for saturated fusion systems can also be found. Of special importance is the Alperin-Goldschmidt Theorem, Theorem 5.9, which says that a saturated fusion system $\mathcal{F}$ on $S$ is entirely determined by the $\mathcal{F}$ automorphisms of $S$, and the $\mathcal{F}$-automorphism groups of the so-called $\mathcal{F}$-essential subgroups. All finite groups $G$ determine a saturated fusion system $\mathcal{F}_{S}(G)$ on a fixed Sylow $p$-subgroup $S$ of $G$. In this case the morphisms between subgroups of $S$ are exactly the restrictions of conjugation maps induced by elements of $G$. When a saturated fusion system can be constructed from a group in this way, we say the system is realizable. A saturated fusion system which is not realizable is called exotic.

A $p$-group has maximal class if it has order $p^{n}$ and nilpotency class $n-1$. The intention of this work is to study the structure of saturated fusion systems $\mathcal{F}$ on maximal class $p$-groups of order at least $p^{4}$. Our main theorems precisely describe the $\mathcal{F}$-essential subgroups of $\mathcal{F}$ and their $\mathcal{F}$-automorphism groups. In contrast with other works on fusion systems, we will consider all saturated fusion systems on the selected class of $p$-groups, not limiting our investigation, for instance, to reduced saturated fusion systems. Aside from proving a broader result, our decision to do this will become evident when we discuss the inductive approach to the proof of our theorems.
1.1. Historical context. The categories we call saturated fusion systems were first introduced by Puig in the early 1990s and recorded in handwritten notes which developed much of the fundamental theory. Puig did not formally publish his discoveries until 2006 [52] and in this work, for a $p$-group $S$, a Frobenius $S$ category is exactly what we call a saturated fusion system on $S$. Puig's motivation was to gain a deeper understanding of the local to global conjectures which are of fundamental importance in modular representation theory. In particular, saturated fusion systems can be constructed on the defect group of a $p$-block in much the same way as they are constructed on Sylow $p$-subgroups. This connection is described by Kessar in [5, Part IV]. Recent developments in this direction include work of Kessar, Linckelmann, Lynd and Semeraro [35] in which certain numerical conjectures in representation theory are formulated for fusion systems; in particular, for exotic fusion systems. This is exciting as, to-date, no saturated fusion system for an odd prime $p$ has been shown to be exotic without using the classification of finite simple groups and the research in [35] offers potential for achieving this without the classification. Finding an alternative way to show that exotic systems are exotic is listed as [6, Question 7.7] in the survey by Aschbacher and Oliver.

The christening of Frobenius $S$-categories as saturated fusion systems is traced back to [10] where in 2003 Broto, Levi and Oliver defined a centric linking system related to a saturated fusion system. Later in 2013, Chermak [14] demonstrated that each saturated fusion system determines a unique centric linking system. From the linking system of a saturated fusion system Broto, Levi and Oliver were able to construct the $p$-completion of its geometric realization and this is the classifying
space associated with the saturated fusion system. The homotopy properties of these spaces share properties of $p$-completed classifying spaces of finite groups. Readers are directed to [ $\mathbf{5}$, Part III] for details and motivation for this study.

The theory of fusion systems provides an idealized environment in which to study the $p$-local structure of a finite group $G$ and, in particular, the structure of normalizers of non-trivial $p$-subgroups of $G$. This formalism has been exploited most notably and energetically by Aschbacher in a deep series of papers aimed at simplifying some parts of the classification of finite simple groups. This approach is described in the surveys $[\mathbf{4 - 6}]$. Other notable contributions to this goal are [43] in which Oliver determines reduced fusion systems on 2-groups in which every subgroup is generated by at most 4 elements, Andersen, Oliver and Ventura [3] which determines, using computational assistance, all reduced fusion systems on 2-groups of order at most $2^{9}$ and Henke and Lynd [31] where they consider saturated fusion systems with components related to the Solomon fusion systems. Using specially developed computational methods, Parker and Semeraro [50, Main Result] have explicitly enumerated all saturated fusion systems with $O^{p}(\mathcal{F})=\mathcal{F}$ and $O_{p}(\mathcal{F})=1$ on $p$-groups of order $p^{n}$ with $p^{n} \in\left\{3^{4}, 3^{5}, 3^{6}, 3^{7}, 5^{4}, 5^{5}, 5^{6}, 7^{4}, 7^{5}\right\}$. Their Magma code for computing with fusion systems is available publicly [49] and is used for some of the computations in this article (see Appendix C).

The Solomon fusion systems [5, III.6.7], which are supported on a Sylow 2subgroup of $\operatorname{Spin}_{7}(q), q$ odd, are exotic. These are the only known exotic fusion systems on 2-groups and are still a fascinating subject of research [31]. In contrast, there is a large variety of exotic fusion systems on $p$-groups with $p$ odd and the reason for this is still not transparent. Ruiz [53] and more recently Oliver and Ruiz [44] have considered non-abelian simple groups and determined instances where the fusion system $\mathcal{F}=\mathcal{F}_{S}(G)$ has $O^{p^{\prime}}(\mathcal{F})$ exotic. These fusion systems, from their very construction are closely related to finite simple groups. Other exotic systems are constructed as fusion systems of free amalgamated products. Examples of these can be found in $[\mathbf{1 0}, \mathbf{1 5}, \mathbf{5 1}]$ and these are often far away from being realized by finite groups in that the $p$-groups are usually not closely related to the Sylow $p$ subgroup of a finite simple group. Many of these fusion systems are defined on $p$-groups of maximal class.

In the early 2000's Ruiz and Viruel [54] famously handled the case when $\mathcal{F}$ is defined on a non-abelian $p$-group of order $p^{3}$. These are maximal class $p$-groups. The shocking outcome was the discovery of three exotic fusion systems when $p=7$. In their work on rank 2 groups (groups which have no elementary abelian subgroups of order $p^{3}$ and are not cyclic or quaternion), Díaz, Ruiz and Viruel [20] examine saturated fusion systems on the infinite families as classified by Blackburn [8]. The outcome of their research was the discovery of several infinite families of exotic fusion systems on certain of the maximal class 3 -groups. In fact, every maximal class 3 -group except the Sylow 3 -subgroup of the simple group Alt(9) has rank 2. In all of the examples they discovered, the 3-group has an abelian subgroup of index 3 [8]. In 2019, Parker and Semeraro, using their computational approach to saturated fusion systems [50] uncovered a saturated fusion system on a rank 2 group of order $3^{6}$ which has maximal class and no abelian subgroups of index 3 . This gave rise to the article [48]. A description of all the saturated fusion systems on maximal class 3-groups is provided in Appendix B. From a different direction, Clelland and Parker [15] constructed families of saturated fusion systems on a Sylow
$p$-subgroup $T$ of groups of shape $q^{a}: \mathrm{SL}_{2}(q)$ where $2 \leq a \leq p$ and $q=p^{b}$. When $a=2, T$ is a Sylow $p$-subgroup of $\mathrm{PSL}_{3}(q)$ and, for $a=3, T$ is a Sylow $p$-subgroup of $\operatorname{PSp}_{4}(q)$. For $a>3$ and for $p \geq 5$, the fusion systems $\mathcal{F}$ with $O_{p}(\mathcal{F})=1$ discovered in [15] are typically exotic. If $q=p$, then $T$ has maximal class. This construction therefore yields infinite families of exotic saturated fusion systems on such maximal class $p$-groups. In each case, the underlying $p$-group has an abelian subgroup of index $p$. In a remarkable series of articles, Oliver takes this property as his starting point and in $[\mathbf{1 9}, \mathbf{4 2}, \mathbf{4 5}]$ he, and his co-authors, determine the reduced saturated fusion systems on $p$-groups with an abelian subgroup of index $p$. A compilation of their results when applied to $p$-groups of maximal class is provided in Appendix A. It turns out that these saturated fusion systems are overwhelmingly exotic. This perhaps leaves the impression that for odd primes exotic fusion systems are not exotic at all. This is possibly an illusion created by considering groups which are in some way small. Evidence that exotic fusion systems may be exotic after all comes, for example, from [46] where it is shown that, in certain good situations, a saturated fusion system determines a locally finite classical Tits chamber system and so is not exotic. Work of van Beek generalizing the classification of groups with a weak $B N$-pair has also not revealed any surprises $[\mathbf{5 7}]$.

Grazian has classified the saturated fusion systems on $p$-groups of rank 3 for $p \geq 5$ [29]. Unlike in the earlier work of Díaz, Ruiz and Viruel for rank 2 groups, there is no list of groups to examine such as those given in [8] and her methods invoke deep results from group theory developed for the classification of the finite simple groups. The resulting theorem reveals that saturated fusion systems on such groups are realizable with just one isolated exotic example on a maximal class 7 -group of order $7^{5}$.

Moragues Moncho [41] classified all saturated fusion systems $\mathcal{F}$ on $p$-groups $S$ with an extraspecial subgroup of index $p$ and $O_{p}(\mathcal{F})=1$. The resulting theorem, which is required for the proof of our Theorem B, is much more uniform than the result concerning $p$-groups with an abelian subgroup of index $p$. In particular, he shows that if the $p$-group considered has order at least $p^{7}$ then it has maximal class and it is a uniquely determined group of order $p^{p-1}$. The fusion systems uncovered are closely related to those found by Parker and Stroth $[\mathbf{5 1}]$ and are all exotic.
1.2. The main theorems. The first step towards a classification of the saturated fusion systems $\mathcal{F}$ on a class of $p$-groups is the study of the so-called $\mathcal{F}$-essential subgroups (see Definition 5.8). The set of all $\mathcal{F}$-essential subgroups is denoted by $\mathcal{E}_{\mathcal{F}}$. In Grazian's research, a certain special type of $\mathcal{F}$-essential subgroups played an important role. She named these subgroups $\mathcal{F}$-pearls. They are defined as follows:

Definition 1.1. Let $p$ be a prime and $\mathcal{F}$ be a saturated fusion system on a p-group $P$. An $\mathcal{F}$-essential subgroup $E$ of $P$ is called an $\mathcal{F}$-pearl if it is isomorphic to either the elementary abelian group of order $p^{2}$, or the extraspecial group of exponent $p$ and order $p^{3}$ if $p$ is odd or the quaternion of order 8 if $p=2$. We denote the set of abelian $\mathcal{F}$-pearls by $\mathcal{P}_{a}(\mathcal{F})$, the set of extraspecial $\mathcal{F}$-pearls by $\mathcal{P}_{e}(\mathcal{F})$ and we write $\mathcal{P}(\mathcal{F})=\mathcal{P}_{a}(\mathcal{F}) \cup \mathcal{P}_{e}(\mathcal{F})$.

In [28], Grazian develops many of the fundamental properties of saturated fusion systems $\mathcal{F}$ which have $\mathcal{F}$-pearls, one of the most basic being that the $p$-group on which $\mathcal{F}$ is defined must have maximal class. She also presents a fascinating lemma [28, Lemma 3.7] (included here as Lemma 5.22), that constructs proper
saturated subfusion systems with $\mathcal{F}$-pearls of any saturated fusion system that has $\mathcal{F}$-pearls. These subsystems are frequently exotic and if the $\mathcal{F}$-pearls are abelian the systems $O^{p}(\mathcal{F})$ are simple [5, Definition I.6.1] (see also Theorem C). The exotic simple saturated fusion system on the group of order $7^{5}$ discovered in [29] can be constructed using Lemma 5.22 from the saturated fusion system of the monster sporadic simple group at the prime 7 .

It is important to clarify that not all saturated fusion systems on a maximal class $p$-group contain $\mathcal{F}$-pearls. The determination of saturated fusion systems $\mathcal{F}$ on $S$ with $O_{p}(\mathcal{F})=1$ and without $\mathcal{F}$-pearls is a consequence of the main results of this work and is recorded as Corollary 1.3.

Before we can state our main theorems, we require some further notation related to maximal class groups. Assume that $S$ has maximal class and order at least $p^{4}$. Set $\gamma_{2}(S)=S^{\prime}=[S, S]$ and define $\gamma_{j}(S)=\left[\gamma_{j-1}(S), S\right]$ for $j \geq 3$. We set

$$
\gamma_{1}(S)=C_{S}\left(\gamma_{2}(S) / \gamma_{4}(S)\right)
$$

As $S$ has maximal class, $\gamma_{2}(S) / \gamma_{4}(S)$ has order $p^{2}$ and so $\left|S: \gamma_{1}(S)\right|=p$ which means that $\gamma_{1}(S)$ is a maximal subgroup of $S$. We also have $\gamma_{n-2}(S)$ is the second centre $Z_{2}(S)$ of $S$ and $\left|Z_{2}(S)\right|=p^{2}$. Hence $C_{S}\left(Z_{2}(S)\right)$, just like $\gamma_{1}(S)$, is a maximal subgroup of $S$. These subgroups are examples of 2 -step centralizers. If $S$ has more than one 2-step centralizer, $S$ is called exceptional. It is a fact that $S$ is exceptional if and only if $\gamma_{1}(S) \neq C_{S}\left(Z_{2}(S)\right)$.

An example of an exceptional group of maximal class is a Sylow $p$-subgroup of the simple group of Lie type $\mathrm{G}_{2}(p)$ for $p \geq 5$. The saturated fusion systems $\mathcal{F}$ on a Sylow $p$-subgroup of the simple group $\mathrm{G}_{2}(p)$ satisfying $O_{p}(\mathcal{F})=1$ have been classified by Parker and Semeraro in [47]. Their work uncovers 27 exotic fusion systems when $p=7$.

The work of Oliver and his co-workers Craven, Ruiz and Semeraro [19, 42, 45] described earlier can be applied to maximal class $p$-groups whenever $\gamma_{1}(S)$ is abelian. This straightforward application is presented for completeness as Theorem A. 1 in Appendix A. Hence throughout the main body of this work we may and do assume that $\gamma_{1}(S)$ is not abelian.

A consequence of the main results of our research can be presented as follows. The notation used for sporadic simple groups is consistent with [26].

Theorem A. Suppose that $\mathcal{F}$ is a reduced saturated fusion system on a p-group $S$ of maximal class of order at least $p^{4}$. Then one of the following statements holds.
(i) $\gamma_{1}(S)$ is non-abelian, and $S$ is not exceptional, $\mathcal{E}_{\mathcal{F}}=\mathcal{P}_{a}(\mathcal{F}), \mathcal{F}$ is simple and exotic.
(ii) $\gamma_{1}(S)$ is non-abelian, $S$ is exceptional and either
(a) $p \geq 5$ and $\mathcal{F}=\mathcal{F}_{S}\left(\mathrm{G}_{2}(p)\right)$;
(b) $p=5, S$ is isomorphic to a Sylow 5 -subgroup of $\mathrm{G}_{2}(5)$ and $\mathcal{F}=$ $\mathcal{F}_{S}(G)$ where $G$ is one of the sporadic simple groups $\mathrm{Ly}, \mathrm{HN}$ or B ;
(c) $p=7, S$ is isomorphic to a Sylow 7 -subgroup of $\mathrm{G}_{2}(7)$ and either $\mathcal{F}$ is exotic (20 examples) or $\mathcal{F}=\mathcal{F}_{S}(\mathrm{M})$ where M denotes the monster; or
(d) $p \geq 11, S$ is uniquely determined of order $p^{p-1}, \mathcal{P}(\mathcal{F})=\mathcal{P}_{a}(\mathcal{F}) \neq \emptyset$ and, if $\gamma_{1}(S)$ is $\mathcal{F}$-essential, then $\operatorname{Out}_{\mathcal{F}}(S) \cong \operatorname{GF}(p)^{\times} \times \operatorname{GF}(p)^{\times}$, $O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{2}(S)\right)\right) \cong \mathrm{SL}_{2}(p)$ and $\gamma_{1}(S) / Z\left(\gamma_{1}(S)\right)$ is the $(p-3)$ dimensional irreducible $\mathrm{GF}(p) \mathrm{SL}_{2}(p)$-module.
(iii) $\gamma_{1}(S)$ is abelian and $\mathcal{F}$ is described by Theorem A.1.

Theorem A is proved by extracting special cases from Theorems B and C below. We remark here that our proofs require the Classification Theorem of the nonabelian simple groups. This is used to provide the names of groups with a strongly $p$-embedded subgroup which contain an elementary abelian subgroup of order $p^{2}$ (Proposition 2.12) and also to understand so-called quadratic pairs [13]. Of course, it is also used when we assert that a given fusion system is exotic for otherwise we would have answered [6, Question 7.7].

The maximal class 2 -groups are either dihedral, quaternion or semidihedral [37, Corollary 3.3.4(iii)] and the fusion systems on such groups are known; in particular it is easy to demonstrate that the $\mathcal{F}$-essential subgroups are all $\mathcal{F}$-pearls (see Lemma 6.1). As for $p=3$, the saturated fusion systems on maximal class 3 -groups are all known due to the work of Díaz, Ruiz, Viruel and Parker and Semeraro (see Lemma 6.2). Therefore for our main theorems we focus our attention on the primes $p \geq 5$.

To state our next theorem we require some additional terminology. Let $S(p)$ be the unique split extension of an extraspecial group of exponent $p$ and order $p^{p-2}$ by a cyclic group of order $p$ which has maximal class [41, Proposition 8.1]. When $p=7$, we have $S(7)$ is isomorphic to the Sylow 7 -subgroup of $\mathrm{G}_{2}(7)$. Also, the group denoted by $\operatorname{SmallGroup}\left(5^{6}, 661\right)$ is group number 661 in the MAGMA [9] small group library of groups of order $5^{6}$. In Lemma 3.3 (v) we see that the maximal class $p$-groups which are exceptional have order at least $p^{6}$ and at most $p^{p+1}$. In particular, there are no exceptional maximal class 3 -groups. This explains our assumption that $p \geq 5$ and $|S| \geq p^{6}$ in the next theorem.

Theorem B. Suppose that $p \geq 5, S$ is an exceptional maximal class $p$-group of order at least $p^{6}$ and $\mathcal{F}$ is a saturated fusion system on $S$. Assume that $\mathcal{F} \neq N_{\mathcal{F}}(S)$. Then one of the following holds.
(i) $\gamma_{1}(S)$ is extraspecial, and, if $\mathcal{F} \neq N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$, then one of the following holds:
(a) $S$ is isomorphic to a Sylow p-subgroup of $\mathrm{G}_{2}(p)$ and either
( $\alpha$ ) $\mathcal{F}=N_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right), O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right)\right) \cong \mathrm{SL}_{2}(p)$;
( $\beta$ ) $p=5,1 \neq O_{p}(\mathcal{F}) \leq \gamma_{2}(S), \mathcal{F} \cong \mathcal{F}_{S}\left(5^{3} \cdot \mathrm{SL}_{3}(5)\right)$;
$(\gamma) p \geq 5$ and $\mathcal{F}=\mathcal{F}_{S}\left(\mathrm{G}_{2}(p)\right)$;
( $\delta) ~ p=5$ and $\mathcal{F}=\mathcal{F}_{S}(G)$ where $G=\mathrm{Ly}, \mathrm{HN}, \operatorname{Aut}(\mathrm{HN})$ or B ; or
(ع) $p=7$ and either $\mathcal{F}$ is exotic (27 examples) or $\mathcal{F}=\mathcal{F}_{S}(\mathrm{M})$.
(b) $p \geq 11, S \cong S(p), \mathcal{P}(\mathcal{F})=\mathcal{P}_{a}(\mathcal{F}) \neq \emptyset$ and, if $\gamma_{1}(S)$ is $\mathcal{F}$-essential, then $\operatorname{Out}_{\mathcal{F}}(S) \cong \operatorname{GF}(p)^{\times} \times \operatorname{GF}(p)^{\times}, O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{2}(S)\right)\right) \cong \mathrm{SL}_{2}(p)$ and $\gamma_{1}(S) / Z\left(\gamma_{1}(S)\right)$ is the unique $(p-3)$-dimensional irreducible $\mathrm{GF}(p) \mathrm{SL}_{2}(p)$-module.
(ii) $p=5, S=\operatorname{SmallGroup}\left(5^{6}, 661\right), O_{5}(\mathcal{F})=C_{S}\left(Z_{2}(S)\right)$ is the unique $\mathcal{F}$ essential subgroup, Out $\mathcal{F}(S)$ is cyclic of order 4 , $\operatorname{Out}_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right) \cong$ $\mathrm{SL}_{2}(5)$ and $\mathcal{F}$ is unique.
In particular, if $\mathcal{F} \neq N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$, then $\mathcal{F}=O^{p}(\mathcal{F})$ and, in addition, $O_{p}(\mathcal{F})=1$ in all cases other than (i)(a)( $\alpha$ ), (i)(a)( $\beta$ ) and (ii).

Theorem B does not describe $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ when $\mathcal{F}=N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$. This additional detail can be determined as follows. First observe that $|S| \leq p^{p+1}$ as $S$ is exceptional. Therefore $\left|\gamma_{1}(S)\right|=p^{1+2 a}$ where $2 \leq a \leq(p-1) / 2$. Now $\operatorname{Aut}_{S}\left(\gamma_{1}(S)\right)$
acts on $\gamma_{1}(S) / Z(S)$ with a single Jordan block and so we can apply [18] to determine the candidates for $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$. For non-exceptional $p$-groups, we prove the following theorem.

Theorem C. Suppose that $p$ is an odd prime, $S$ is a maximal class p-group of order at least $p^{4}$ and $\mathcal{F}$ is a saturated fusion system on $S$. Assume that $S$ is not exceptional, $\gamma_{1}(S)$ is not abelian and $\mathcal{F} \neq N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$. Then one of the following holds:
(i) $\mathcal{E}_{\mathcal{F}}=\mathcal{P}_{a}(\mathcal{F}),|S: \operatorname{hyp}(\mathcal{F})| \leq p$ with $|S: \operatorname{hyp}(\mathcal{F})|=p$ if and only if $|S|=p^{j(p-1)+1}$ for some $j \geq 2$. Furthermore, either $O^{p}(\mathcal{F})$ is simple and exotic or $p=3$ and $O^{3}(\mathcal{F})$ is realized by $\operatorname{PSL}_{3}(q)$ for suitable prime powers $q$.
(ii) $p \geq 5, \mathcal{E}_{\mathcal{F}}=\mathcal{P}_{e}(\mathcal{F}), O_{p}(\mathcal{F})=Z(S),|S: \operatorname{hyp}(\mathcal{F})| \leq p$ with $\mid S:$ $\operatorname{hyp}(\mathcal{F}) \mid=p$ if and only if $|S|=p^{j(p-1)+2}$ for some $j \geq 2$. Furthermore, $O^{p}(\mathcal{F} / Z(S))$ is simple and exotic.
(iii) $p \geq 5, \mathcal{E}_{\mathcal{F}}=\mathcal{P}_{a}(\mathcal{F}) \cup\left\{\gamma_{1}(S)\right\}, O_{p}(\mathcal{F})=1, \mathcal{F} \neq O^{p}(\mathcal{F})$ and
(a) $\mathcal{P}_{a}(\mathcal{F})$ is a single $\mathcal{F}$-class, $|S|=p^{j(p-1)+1}$ for some $j \geq 2$ and $S$ has sectional rank $p-1$;
(b) $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right) \cong \operatorname{Sym}(p)$ or $\mathrm{PGL}_{2}(p)$;
(c) $Z\left(\gamma_{1}(S)\right)=\mho^{1}\left(\gamma_{1}(S)\right)$ has index $p^{p-1}$ in $\gamma_{1}(S), \gamma_{1}(S)^{\prime}<\Omega_{1}\left(\gamma_{1}(S)\right)$ has order $p^{p-2}$ and $\gamma_{2}(S)$ is abelian but not elementary abelian;
(d) every composition factor of $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ on $\gamma_{1}(S)$ has order $p$ or $p^{p-2}$ and the composition factors of order $p$ are centralized by the automorphism group $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$;
(e) for $P \in \mathcal{P}_{a}(\mathcal{F})$, $\operatorname{hyp}(\mathcal{F})=P \gamma_{2}(S), O^{p}(\mathcal{F})$ is a saturated fusion system on $P \gamma_{2}(S)$, and $\operatorname{Aut}_{O^{p}(\mathcal{F})}\left(\gamma_{2}(S)\right) \cong \operatorname{Sym}(p)$ or $\mathrm{PGL}_{2}(p)$.
Furthermore, in all cases $\operatorname{Out}_{\mathcal{F}}(S)$ is a Hall $p^{\prime}$-subgroup of $\operatorname{Out}(S)$ and is cyclic of order $p-1$ and, if $|S|=p^{n}$, and $P \in \mathcal{P}(\mathcal{F})$, then either $\mathcal{P}(\mathcal{F})=P^{S}$ or $\mathcal{E}_{\mathcal{F}}=\mathcal{P}(\mathcal{F})$ and $n \equiv \epsilon(\bmod p-1)$ where $\epsilon=0$ if $P \in \mathcal{P}_{a}(\mathcal{F})$ and $\epsilon=1$ if $P \in \mathcal{P}_{e}(\mathcal{F})$.

Suppose that $p$ odd, $r$ a prime with $r^{a}-1 \equiv 0(\bmod p)^{k}$. In Section 15 , we show that an automorphism group $G$ of $G_{0}=\operatorname{PSL}_{p}\left(r^{p}\right)$ which projects diagonally into a Sylow $p$-subgroups of $\operatorname{Out}\left(G_{0}\right)$ between the image of $\mathrm{PGL}_{p}(r)$ and the image of a field automorphism of order $p$ provides realizable examples of Theorem C (iii) with, for $S \in \operatorname{Syl}_{p}(G)$, $\operatorname{Out}_{\mathcal{F}_{S}(G)}\left(\gamma_{1}(S)\right) \cong \operatorname{Sym}(p)$. By [50, Theorem 6.2], the subfusion systems generated by the $\mathcal{F}_{S}(G)$-pearls gives an example of (i) in the case that $\gamma_{2}(S)$ is abelian. In Theorem C (iii) the fusion systems $O^{p}(\mathcal{F})$ can be found in Table 2 in Appendix A and are listed in Lines (3) and (4) in the case that $\gamma_{2}(S)$ is elementary abelian and otherwise in Lines (29) and (33). If $\mathcal{F}$ is a saturated fusion system with $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right) \cong \operatorname{PGL}_{2}(p)$ with $p \geq 7$ and $\gamma_{1}(S)$ is non-abelian, then, as $\gamma_{1}(\operatorname{hyp}(\mathcal{F}))=\gamma_{2}(S)$ is abelian, we may use [45, Theorem 4.5] to see that $O^{p}(\mathcal{F})$ is exotic.

Theorem C does not specify the structure of a saturated fusion system $\mathcal{F}=$ $N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ and, as we don't know the structure of $\gamma_{1}(S)$, it could be more difficult to determine the precise structure than in the exceptional case. This leads to some complications in our inductive arguments. Theorems C leaves open the question of what more we can say about saturated fusion systems which only have $\mathcal{F}$-pearls. For example, the isomorphism type of $S$ is not determined in Theorem C (i) and (ii). As an indicator that the structure of $S$ can be more complicated than the
structures described in Theorem C (iii) we have the following example which was obtained by computer using the procedures developed in [50] and implemented in Magma [9]. In Example 1.2, $|S|=5^{7}$ and so $S$ is not exceptional.

Example 1.2. Suppose that $S$ is a maximal class 5 -group of order $5^{7}$ and suppose that $\gamma_{1}(S)$ is not abelian. If $\mathcal{F}$ is a saturated fusion system on $S$ and $\mathcal{P}(\mathcal{F})$ is non-empty, then $S$ is one of the seven groups

$$
\begin{aligned}
& \operatorname{SmallGroup}\left(5^{7}, 1297\right), \operatorname{SmallGroup}\left(5^{7}, 1308\right), \\
& \operatorname{SmallGroup}\left(5^{7}, 1321\right), \operatorname{SmallGroup}\left(5^{7}, 1360\right), \\
& \operatorname{SmallGroup}\left(5^{7}, 1363\right), \operatorname{SmallGroup}\left(5^{7}, 1374\right), \\
& \operatorname{SmallGroup}\left(5^{7}, 1384\right) .
\end{aligned}
$$

Furthermore, each possibility for $S$ supports a unique (up to isomorphism) saturated fusion system $\mathcal{F}$ with $\mathcal{F}$-pearls and $\gamma_{1}(S)$ is not $\mathcal{F}$-essential. For $S=$ SmallGroup $\left(5^{7}, 1308\right), \mathcal{F}$ has a unique $\mathcal{F}$-class of extraspecial $\mathcal{F}$-pearls and the saturated fusion systems on the remaining groups have a single $\mathcal{F}$-class of abelian $\mathcal{F}$-pearls. For $S$ any of the groups $\operatorname{SmallGroup}\left(5^{7}, 1360\right)$, SmallGroup $\left(5^{7}, 1363\right)$, SmallGroup $\left(5^{7}, 1374\right)$, or SmallGroup $\left(5^{7}, 1384\right), \gamma_{1}(S)$ has nilpotency class 3; otherwise $\gamma_{1}(S)$ has nilpotency class 2.

All the fusion systems in Example 1.2 with abelian pearls are simple and exotic by Theorem 5.25 . To provide some perspective to this calculation, we remark that there are 99 maximal class groups of order $5^{7}$. Three of them have an abelian subgroup of index 5 . Suppose that $S$ is a maximal class 5 -group with $\gamma_{1}(S)$ nonabelian and let $\mathcal{F}$ be saturated fusion system on $S$. Assume $\mathcal{F}$ has an $\mathcal{F}$-pearl $P$. Then Theorem C says that $\operatorname{Aut}(S)$ has a Hall $5^{\prime}$-subgroup of order 4. Of the maximal class 5 -groups of order $5^{7}$, with non-abelian 2 -step centralizer just 12 of them have a Hall $5^{\prime}$-subgroup of $\operatorname{Aut}(S)$ of order 4. All of these have an elementary abelian subgroup of order 25 not contained in the 2-step centralizer and so have candidates for pearls. We also remark that the fusion system $\mathcal{F}$ on $S=\operatorname{SmallGroup}\left(5^{7}, 1308\right)$ has a unique class of extraspecial $\mathcal{F}$-pearls and so this shows that there are examples in Theorem C (ii) when $p=5$. The Magma routines for this computation can be found in Subsection C.1. The code exploits an idea we are just about to explain.

The study of maximal class $p$-groups with large automorphism group as in [21] requires more research to make substantial headway on the problem of determining all fusion systems which have every essential subgroup a pearl. To make this statement clearer, let us fix $B=\left\{\left.\left(\begin{array}{ccc}a & 0 & 0 \\ b & a^{-1} & 0 \\ c & d & 1\end{array}\right) \right\rvert\, a, b, c, d \in \operatorname{GF}(p), a \neq 0\right\}$. Let $S$ be maximal class $p$-group of order at least $p^{5}$, and suppose that $x \in S$ has order $p$ and $P=C_{S}(x)=\langle x, Z(S)\rangle$. Then $N_{S}(P)$ is extraspecial of order $p^{3}$. If $S$ has an automorphism $\phi$ of order $p-1$ which normalizes $p$ and satisfies $N_{S}(P) \rtimes\langle\phi\rangle \cong B$, then there exists a saturated fusion system $\mathcal{F}$ on $S$ with $P$ an abelian $\mathcal{F}$-pearl just as in [42, Theorem 2.8]. The existence of an automorphism of order $p-1$ is precisely the condition that Dietrich and Eick use in their work [21].

We now state the corollary as advertised above.
Corollary 1.3. Let $p$ be a prime, $S$ be a p-group of maximal class and let $\mathcal{F}$ be a saturated fusion system on $S$ with $O_{p}(\mathcal{F})=1$. If $\mathcal{P}(\mathcal{F})$ is empty, then $S$ is isomorphic to a Sylow p-subgroup of $\mathrm{G}_{2}(p)$ and either
(i) $\mathcal{F}=\mathcal{F}_{S}\left(\mathrm{G}_{2}(p)\right)$;
(ii) $p=5$ and $\mathcal{F}=\mathcal{F}_{S}(G)$ where $G=\mathrm{Ly}, \mathrm{HN}, \operatorname{Aut}(\mathrm{HN})$ or B ;
(iii) $p=7, \mathcal{F}$ is exotic and the $\mathcal{F}$-essential subgroups are $C_{S}\left(Z_{2}(S)\right.$ ) and $\gamma_{1}(S)$, with $\operatorname{Out}_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right) \cong \operatorname{GL}_{2}(7)$, $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right) \cong 3 \times 2 \cdot \operatorname{Sym}(7)$, and $\operatorname{Out}_{\mathcal{F}}(S) \cong \mathrm{GF}(7)^{\times} \times \mathrm{GF}(7)^{\times}$.
We remark that the exotic saturated fusion system in part (iii) of the corollary is obtained from the saturated fusion system in the monster $M$ by pruning the pearl [50, Lemma 6.5].
1.3. An overview of the paper. The article develops as follows. We start in Section 2 with some background group theoretical results. Especially, we explain what it means for a group to have a strongly $p$-embedded subgroup and present some elementary facts about such groups.

Section 3 commences with two lemmas which detail properties of maximal class $p$-groups, most of which are drawn from [37]. Of particular importance are the facts that $\gamma_{1}(S)$ is a regular $p$-group and $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right| \leq p^{p}$, with equality if and only if $|S|=p^{p+1}$. In the second part of Section 3, we start to study the automorphism group of $S$. As we are interested in the structure of $\operatorname{Out}_{\mathcal{F}}(S)$, we specifically study $p^{\prime}$-automorphisms. Two very important results for our work, which were known to Juhász [34], are Lemma 3.10 which describes the action of a single $p^{\prime}$ automorphism of $S$ on $\gamma_{k}(S) / \gamma_{k+1}(S)$ for $k \geq 1$, and Lemma 3.11 which says that if some $p^{\prime}$-automorphism of $S$ centralizes $S / \gamma_{1}(S)$, then $\gamma_{1}(S)$ is either abelian or $S$ is exceptional and $\gamma_{1}(S)$ is extraspecial. In our work this lemma has the consequence that most of the time we can assume that $\operatorname{Out}_{\mathcal{F}}(S)$ is cyclic of order dividing $p-1$. But perhaps more important than both these results is Lemma 3.14 which applies when $S$ is not exceptional and, for example, controls the size of $\left|C_{\gamma_{1}(S)}(\psi)\right|$ for $\psi$ a $p^{\prime}$-automorphism of $S$. A particular consequence of Lemma 3.14 is that if $\tau$ is an automorphism of $S$ and $\tau$ centralizes $\gamma_{k}(S) / \gamma_{k+2}(S)$ then $\tau$ is a $p$-element. A final important result is Theorem 3.15 which is [34, Theorem 6.2]. This has the consequence that if $\gamma_{w}(S)$ is elementary abelian and $w \geq 3$, then $\gamma_{w-1}(S)$ has nilpotency class at most 2 . For completeness, we present a modestly simplified version of Juhász's proof.

In Section 4, we gather a collection of results about representations of groups with cyclic Sylow $p$-subgroups of order $p$. These results are applied later in the paper in the case when $\gamma_{1}(S)$ is known to be an $\mathcal{F}$-essential subgroup, in order to obtain the structure of Out $\mathcal{F}\left(\gamma_{1}(S)\right)$. Of particular significance is Feit's Theorem (Theorem 4.2) which says that if a group with a cyclic Sylow $p$-subgroup is not closely related to $\mathrm{PSL}_{2}(p)$, then any faithful representation is relatively large. Section 4 also contains a description of the irreducible $\mathrm{GF}(p) \mathrm{SL}_{2}(p)$-modules, and results which help decompose tensor products of such modules (see Proposition 4.6). These results are exploited throughout the proof of our main theorems.

In Section 5 we recall basic definitions and known facts about fusion systems, referring mostly to [5] (especially for the terminology). We also present a number of results about saturated fusion systems with $\mathcal{F}$-pearls: Lemmas 5.18, 5.19, 5.20 and 5.22 as well as Theorems 5.21 and 5.25 which are mostly taken from [28]. The last result of Section 5 is Proposition 5.27 which allows us in Sections 13 to construct a saturated subfusion system on a $p$-group with a maximal abelian subgroup.

Because of the Alperin-Goldschmidt Theorem, Theorem 5.9, our first significant objective is to determine all the candidates for $\mathcal{F}$-essential subgroups in a maximal
class $p$-group. This is the foundation for our proof of Theorems A, B and C, for without knowing the $\mathcal{F}$-essential subgroups nothing more can be said.

Theorem D. Suppose that $p$ is a prime, $S$ is a p-group of maximal class and order at least $p^{4}$ and $\mathcal{F}$ is a saturated fusion system on $S$. If $E$ is an $\mathcal{F}$-essential subgroup, then either $E$ is an $\mathcal{F}$-pearl, $E=\gamma_{1}(S)$ or $E=C_{S}\left(Z_{2}(S)\right)$. Furthermore, if $S$ is exceptional, then $\mathcal{P}(\mathcal{F})=\mathcal{P}_{a}(\mathcal{F})$.

The proof of Theorem D spans Sections 6 to 11. This means that Section 6 is where the real work begins. We start by analyzing the properties of $\mathcal{F}$-essential subgroups of maximal class $p$-groups. The most relevant results are Lemma 6.4, that says that an $\mathcal{F}$-essential subgroup $E$ of $S$ is not an $\mathcal{F}$-pearl if and only if it is contained in either $\gamma_{1}(S)$ or $C_{S}\left(Z_{2}(S)\right.$ ), and Lemma 6.7, in which we prove that every normal $\mathcal{F}$-essential subgroup of $S$ is a maximal subgroup of $S$.

In Section 7 we focus our attention on the $\mathcal{F}$-essential subgroups of $S$ in the case that $S$ is an exceptional group. The goal of this section is to prove Proposition 7.2, which implies Theorem D in the case in which $\gamma_{1}(S)$ is extraspecial and gives the first ingredients toward our proof of Theorem B.

Section 8 considers the case when $S$ is not exceptional and $\gamma_{1}(S)$ is $\mathcal{F}$-essential. Of particular interest is Lemma 8.5 in which we prove that $\Omega_{1}\left(\gamma_{1}(S)\right)$ has nilpotency class at most 2 and that, if the class is exactly 2 , then $O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right) \cong \operatorname{PSL}_{2}(p)$. This lemma is crucial for the proof of Theorem D.

In Section 9, in the case that $S$ is exceptional, Proposition 9.1 states that if $\gamma_{1}(S)$ is $\mathcal{F}$-essential, then $\gamma_{1}(S)$ is extraspecial. Thus, in combination with Proposition 7.2 , we may assume that, if $S$ is exceptional, then $\gamma_{1}(S)$ is not $\mathcal{F}$-essential in our minimal counterexample. The proof of Proposition 9.1 invokes Lemma 8.5, Proposition 4.6 and a detailed commutator calculation which in the end reaches contradiction by using a 1902 result due to Burnside (Theorem 2.4).

Sections 10 and 11 contain the series of results that leads to the proof of Theorem D. Our proof is achieved by contradiction, considering a minimal counterexample $\mathcal{F}$ to Theorem D (first with respect to the size of $S$ and then to the number of morphisms). In other words, we consider a minimal fusion system $\mathcal{F}$ containing an $\mathcal{F}$-essential subgroup $E$ that is not an $\mathcal{F}$-pearl and is not equal to $\gamma_{1}(S)$ or $C_{S}\left(Z_{2}(S)\right)$ (we say that $E$ is a witness). So assume that $\mathcal{F}$ is such a fusion system. The most important result of Section 10 is Proposition 10.6 which asserts $O_{p}(\mathcal{F})=1$. The proof of Proposition 10.6 uses the fact that we know those non-abelian simple groups which have a $\operatorname{GF}(p) G$-module on which some non-trivial $p$-elements act with minimal polynomial of degree 2 . The fact that we require originally goes back to Thompson in unpublished work and we cite Chermak [14] for the proof. We also show that for $\mathcal{F}$ a minimal counterexample we must have $p \geq 7$, $p^{7} \leq|S|<p^{2 p-4}$ and $\Omega_{1}\left(\gamma_{1}(S)\right)$ non-abelian (Lemmas 10.9 and 10.10). With the scene set, in Section 11 we choose a subgroup $T$ of $\gamma_{1}(S)$ whose automorphism group is not formed by restrictions of $\mathcal{F}$-automorphisms of $\gamma_{1}(S)$ and that is maximal first with respect to the order of its normalizer in $S$ and second with respect to its own order. Note that a witness $E$ is a candidate for $T$, but the key idea is that $T$ is not necessarily $\mathcal{F}$-essential. The study of the subgroup $T$ is divided in two major cases: the case in which $T$ is $\mathcal{F}$-centric and the case in which $T$ is not $\mathcal{F}$-centric. We show that both cases are impossible and so $T$ cannot exist, proving Theorem D.

With Theorem D proved, Section 12 contains the proof of Theorem B. By this stage, this is relatively straightforward because of [41], where Moragues Moncho classified all saturated fusion systems $\mathcal{F}$ on $p$-groups with an extraspecial subgroup of index $p$ and $O_{p}(\mathcal{F})=1$.

In Section 13 we prepare for the proof of Theorem C. We are interested in the case in which $S$ is not exceptional. Because of Theorem D, once we assume that $\mathcal{F} \neq N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$, we know that $\mathcal{P}(\mathcal{F})$ is non-empty. Thus we choose an $\mathcal{F}$-pearl $P$. If $\gamma_{1}(S)$ is not $\mathcal{F}$-essential, then we readily obtain (i) and (ii) of Theorem C. So in Hypothesis 13.5 we assume that $\gamma_{1}(S)$ is $\mathcal{F}$-essential and non-abelian. In this case we let $V=\Omega_{1}\left(Z\left(\gamma_{1}(S)\right)\right)$ and $S_{1}=V P$. Then Proposition 5.27 can be applied to produce a saturated reduced subfusion system of $S_{1}$ in which $V$ is $\mathcal{F}$ essential. The application of $[\mathbf{1 9}]$ (see Appendix A for the main result of $[\mathbf{1 9}, \mathbf{4 5}]$ applied to maximal class $p$-groups) gives us that $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ is either $\mathrm{PGL}_{2}(p)$ or $\operatorname{Sym}(p)$ and it also dictates the isomorphism type of $V$ as a $\operatorname{GF}(p) \operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ module. We then determine in Lemma 13.11 detailed information about the action of $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ on $\gamma_{1}(S)$. In particular, this shows that the chief factors alternate between having order $p$ and being central and having order $p^{p-2}$. After a few more observations, in Lemma 13.14 we show that $\mathcal{F} \neq O^{p}(\mathcal{F})$; the proof of this uses detailed knowledge about the submodule structure of exterior squares of the $(p-2)$-dimensional modules for $\operatorname{PSL}_{2}(p)$ and $\operatorname{Alt}(p)$. In Lemma 13.15, we show that $P$ must be abelian and so there are no extraspecial $\mathcal{F}$-pearls and $O_{p}(\mathcal{F})=1$. We also determine the structure of $\gamma_{1}(S)$. With this information available, the proof of Theorem C is quickly brought together in Section 14. In Section 14 we also prove Theorem A and Corollary 1.3.

The final section of the paper presents a group which realizes examples of Theorem C and in Appendix A we give a description of the reduced fusion systems on maximal class $p$-groups with $p$ odd and having an abelian subgroup of index $p$ taken from $[\mathbf{1 9}, \mathbf{4 5}]$. In Appendix B we present the classification of saturated fusion systems on maximal class 3 -groups due to $[\mathbf{2 0}, \mathbf{4 8}]$. Finally Appendix C lists the Magma code used in various examples and results of the paper.
1.4. Non-standard notation. We follow one of $[\mathbf{2 4}, \mathbf{2 5}, \mathbf{3 2}]$ for group theoretic notation and we follow Leedham-Green and McKay [37] for notation and facts surrounding $p$-groups of maximal class. In particular, for a maximal class $p$-group $R$, we mention that $R^{\prime}=\Phi(R)$ is denoted by $\gamma_{2}(R)$. We apply almost all maps on the right. If $G$ is a group and $g \in G$ then $c_{g}$ is the conjugation map $c_{g}: G \rightarrow G$ defined by $x c_{g}=x^{g}=g^{-1} x g$ for $x \in G$. If $X, Y \leq G$ are groups and $n \in \mathbb{N}$, then $[X, Y ; n]$ is defined recursively by $[X, Y ; 1]=[X, Y]$ and $[X, Y ; n]=[[X, Y ; n-1], Y]$ for $n \geq 1$. It is also convenient to define $[X, Y ; 0]=X$. For $x, y, z \in G$, we write $[x, y, z]$ for $[[x, y], z]$.

Our nomenclature for specific groups is for the most part standard or selfexplanatory, for example, we use $\operatorname{Alt}(n)$ and $\operatorname{Sym}(n), n \geq 3$ to denote the alternating and symmetric groups of degree $n$ respectively. Similarly, we use Frob $(n)$ to denote a Frobenius group of order $n$, whenever such a group is uniquely defined. For $p$ odd, the extraspecial groups of order $p^{3}$ are denoted by $p_{+}^{1+2}$ and $p_{-}^{1+2}$ where the first group has exponent $p$ and the second exponent $p^{2}$. For a field $\mathbb{K}, \mathbb{K}^{\times}$ denotes its multiplicative group.

## 2. General group theoretical results

We use the commutator formulae as in [24, Theorem 2.2.1 and Lemma 2.2.2] regularly and without reference. We also often refer to [24, Sections 5.2 and 5.3] for results concerning coprime action. Here we catalogue less familiar results.

Let $p$ be a prime. A $p$-group $T$ is regular if, and only if, for all $x, y \in T$, there exist $g_{1}, \ldots, g_{t} \in\langle x, y\rangle^{\prime}$ such that

$$
(x y)^{p}=x^{p} y^{p} g_{1}^{p} \ldots g_{t}^{p}
$$

We will need a handful of properties of regular $p$-groups. These attributes say that regular $p$-groups have similar properties to abelian groups with respect to taking powers. Recall that, for a $p$-group $X, \Omega_{1}(X)$ is the subgroup of $X$ generated by elements of order $p$ and $\mho^{1}(X)=\left\langle x^{p} \mid x \in X\right\rangle$.

Lemma 2.1. Suppose that $P$ is a regular $p$-group and assume that $Q \leq P$. Then
(i) $Q$ is regular;
(ii) $\Omega_{1}(P)$ has exponent $p$;
(iii) $\Omega_{1}(Q)=Q \cap \Omega_{1}(P)$; and
(iv) $\left|P / \Omega_{1}(P)\right|=\left|\mho^{1}(P)\right|$.

Proof. Part (i) is obvious from the definition of a regular p-group. For (ii) see [32, Haupsatz III.10.5]. Part (iii) follows from (ii). Part (iv) comes from [32, Satz III.10.7].

Lemma 2.2. Assume that $P$ is a regular $p$-group and $T \leq P$. If $T \geq P^{\prime}$ and $P=T \Omega_{1}(P)$, then $\mho^{1}(P)=\mho^{1}(T)$.

Proof. Let $x \in P$. Then there is $t \in T$ and $w \in \Omega_{1}(P)$ such that $x=t w$. By Lemma 2.1 (ii) we have $w^{p}=1$. Thus [37, Lemma 1.2.10 (iii)] together with $T \geq P^{\prime} \geq\langle t, w\rangle^{\prime}$, imply there exists $s \in\langle t, w\rangle^{\prime}$ such that

$$
x^{p}=(t w)^{p}=t^{p} w^{p} s^{p}=t^{p} s^{p} \in \mho^{1}(T) .
$$

Hence $\mho^{1}(P) \leq \mho^{1}(T) \leq \mho^{1}(P)$ and this gives the result.
For a group $P$, we define

$$
E_{2}(P)=\{x \in P \mid[x, y, y]=1 \text { for all } y \in P\}
$$

Lemma 2.3. Suppose that $p$ is a prime and $P$ is a p-group of nilpotency class 3. Then $E_{2}(P)$ is a characteristic subgroup of $P$ which contains $Z_{2}(P)$.

Proof. Obviously $Z_{2}(P) \leq E_{2}(P)$. Assume that $a, b \in E_{2}(P)$ and let $y \in P$. Then we calculate

$$
\left[a b^{-1}, y\right]=[a, y]^{b^{-1}}\left[b^{-1}, y\right]=[a, y]\left[a, y, b^{-1}\right]\left[b^{-1}, y\right] \in C_{P}(y) Z(P) C_{P}(y)=C_{P}(y)
$$

Hence $E_{2}(P)$ is a subgroup of $P$. Since $E_{2}(P)$ is a characteristic subset of $P, E_{2}(P)$ is a characteristic subgroup of $P$.

THEOREM 2.4 (Burnside). Suppose that $p \neq 3$ is a prime and $P$ is a finite p-group such that $P=E_{2}(P)$. Then $P$ has nilpotency class 2 .

Proof. Notice that $[x, y, y]=1$ if and only if $\left[\left(y^{-1}\right)^{x}, y\right]=1$ if and only if $\left\langle y^{x}\right\rangle$ and $\langle y\rangle$ commute. Therefore this theorem dates back to 1902 [12].

Proposition 2.5. Suppose that $p$ is a prime, $L$ is a group and $P$ is a Sylow p-subgroup of $L$. Let $V$ be a faithful, irreducible $\mathrm{GF}(p) L$-module. If $\left[O_{p^{\prime}}(L), P\right] \neq 1$, then $\operatorname{dim} V \geq p-1$ and $[V, P ; p-2] \neq 0$.

Proof. Set $H=O_{p^{\prime}}(L)\langle x\rangle$ where $x \in P$ does not centralize $O_{p^{\prime}}(H)$. Let $U$ be a non-trivial composition factor for $H$ in $V$ which is not centralized by $\left[O_{p^{\prime}}(H), x\right]$. Then $H$ is $p$-soluble, $O_{p}\left(H / C_{H}(U)\right)=1$ and we may apply the Hall-Higman Theorem [24, Theorem 11.1.1] to $H / C_{H}(U)$ to obtain the minimal polynomial for $x$ acting on $U$ is $(X-1)^{r}$ with $r \in\{p, p-1\}$. In particular, $\operatorname{dim} V \geq r \geq p-1$ and $[V, P ; p-2] \geq[V, x ; p-2]=V(x-1)^{(p-2)} \neq 0$.

Lemma 2.6. Suppose that $p$ is a prime, $S$ is a p-group, $E, K \leq S$ with $E K a$ subgroup of $S$. If $N_{K}(E) \leq E$, then $K \leq E$.

Proof. Let $t \in N_{E K}(E)$. Then $t=e k$ for some $e \in E$ and $k \in K$. Thus

$$
E=E^{t}=E^{e k}=E^{k}
$$

and so $k \in N_{K}(E) \leq E$. Hence $N_{E K}(E) \leq E$ and this means that $E=E K \geq$ $K$.

We require the following cohomological type result which is a consequence of a theorem of Gaschütz.

Lemma 2.7. Suppose that $p$ is a prime, $G$ a group and $V$ a $\operatorname{GF}(p) G$-module. Let $W=\left[V, O^{p}(G)\right]$ and $T \in \operatorname{Syl}_{p}(G)$. Then $W+C_{V}(T)=W+C_{V}(G)$. In particular, if $C_{V}(G)=0$, then $C_{V}(T) \leq W$.

Proof. See [40, Lemma C.17].
2.1. Groups with a strongly $p$-embedded subgroup. In this subsection we collect together results about groups with a strongly $p$-embedded subgroup.

Definition 2.8. Suppose that $p$ is a prime, $H$ is a group and $M$ is a proper subgroup of $H$ of order divisible by $p$. Then $M$ is strongly $p$-embedded in $H$ if and only if $M \cap M^{h}$ has order coprime to $p$ for all $h \in H \backslash M$.

It is easy to establish, see [25, Definition 17.11, Proposition 17.11], that $M$ is strongly $p$-embedded in $H$ if and only if $M$ contains a Sylow $p$-subgroup $T$ of $H$ and $N_{H}(R) \leq M$ for all $1 \neq R \leq T$. In particular, if $H$ has a strongly $p$-embedded subgroup, then $O_{p}(H)=1$.

Lemma 2.9. Suppose that $p$ is a prime, $H$ is a group and $M$ is strongly $p$ embedded in $H$. If $K \leq M$ is a subnormal subgroup of $H$, then $K$ is a $p^{\prime}$-subgroup.

Proof. We may suppose that $N_{H}(K) \not \leq M$. Let $R \in \operatorname{Syl}_{p}(K)$. Then, by the Frattini Argument, $N_{H}(K)=N_{N_{H}(K)}(R) K$ and so $N_{H}(R) \not \subset M$. As $M$ is strongly $p$-embedded in $M, R=1$. Hence $K$ is a $p^{\prime}$-subgroup.

Lemma 2.10. Suppose that $p$ is a prime and $H$ is a group with a strongly pembedded subgroup $M$. If $M$ contains an elementary abelian subgroup of order $p^{2}$, then $O_{p^{\prime}}(H) \leq M, M / O_{p^{\prime}}(H)$ is strongly p-embedded in $H / O_{p^{\prime}}(H)$ and $H / O_{p^{\prime}}(H)$ is an almost simple group.

Proof. See [50, Lemma 4.3].

Lemma 2.11. Suppose that $p$ is a prime, $H$ is a group with a strongly pembedded subgroup and that $K$ is a normal subgroup of $H$ which commutes with an element of order $p$. Then $H / K$ has a strongly p-embedded subgroup.

Proof. Assume that $M$ is strongly $p$-embedded in $H$. Then $M$ contains a Sylow $p$-subgroup $T$ of $H$ and $N_{H}(R) \leq M$ for all $1 \neq R \leq T$. Since $K$ centralizes a $p$-element in $H$ and $K$ is normal in $H$ by assumption, $K \leq M$. Hence $K \leq$ $O_{p^{\prime}}(H)$ by Lemma 2.9. Set $\bar{H}=H / K$. Then $\bar{H}$ and $\bar{M}$ have order divisible by $p$. Let $\bar{R}$ be a non-trivial $p$-subgroup of $\bar{T}$ with $R \leq T$. Then, by coprime action, $N_{\bar{H}}(\bar{R})=\overline{N_{H}(R)} \leq \bar{M}$ and so we conclude $\bar{H}$ has a strongly $p$-embedded subgroup as claimed.

For a group $X, F^{*}(X)$ denotes the generalized Fitting subgroup of $X$. This is the subgroup of $X$ generated by the subnormal nilpotent subgroups and subnormal quasisimple subgroups of $X$. See [25, Definition 3.4].

Proposition 2.12. Suppose that $p$ is a prime, $X$ is a group, $K=F^{*}(X)$ and $T \in \operatorname{Syl}_{p}(X)$. Assume that $O_{p^{\prime}}(X)=1$ and that $M$ is a strongly p-embedded subgroup of $X$ containing $T$. Then $O_{p}(X)=1, K$ is a non-abelian simple group and $M \cap K$ is strongly p-embedded in $K$, and $p$ and $K$ are as follows:
(i) $p$ is any prime, $a \geq 1$ and $K \cong \operatorname{PSL}_{2}\left(p^{a+1}\right), \operatorname{PSU}_{3}\left(p^{a}\right)\left(p^{a} \neq 2\right)$, ${ }^{2} \mathrm{~B}_{2}\left(2^{2 a+1}\right)(p=2)$ or ${ }^{2} \mathrm{G}_{2}\left(3^{2 a+1}\right)(p=3)$ and $X / K$ is a $p^{\prime}$-group.
(ii) $p>3, K \cong \operatorname{Alt}(2 p),|X / K| \leq 2$ and $T$ is elementary abelian of order $p^{2}$.
(iii) $p=3, K \cong \operatorname{PSL}_{2}(8), X \cong \operatorname{Aut}\left(\operatorname{PSL}_{2}(8)\right) \cong{ }^{2} \mathrm{G}_{2}(3) \cong \mathrm{PSL}_{2}(8): 3, T \cong$ $3_{-}^{1+2}$ and $T \cap K$ is cyclic of order 9 .
(iv) $p=3, K \cong \operatorname{PSL}_{3}(4), X / K$ is a 2-group and $T$ is elementary abelian of order $3^{2}$.
(v) $p=3, X=K \cong \mathrm{M}_{11}$ and $T$ is elementary abelian of order $3^{2}$.
(vi) $p=5, K \cong{ }^{2} \mathrm{~B}_{2}(32), X \cong \operatorname{Aut}\left({ }^{2} \mathrm{~B}_{2}(32)\right) \cong{ }^{2} \mathrm{~B}_{2}(32): 5, T \cong 5_{-}^{1+2}$ and $T \cap K$ is cyclic of order 25 .
(vii) $p=5, K \cong{ }^{2} \mathrm{~F}_{4}(2)^{\prime},|X / K| \leq 2$ and $T$ is elementary abelian of order $5^{2}$.
(viii) $p=5, K \cong \mathrm{McL},|X / K| \leq 2$ and $T \cong 5_{+}^{1+2}$.
(ix) $p=5, K \cong \mathrm{Fi}_{22},|X / K| \leq 2$ and $T$ is elementary abelian of order $5^{2}$.
(x) $p=11, X=K \cong \mathrm{~J}_{4}$ and $T \cong 11_{+}^{1+2}$.
(xi) $p$ is odd and $T=T \cap K$ is cyclic.

Proof. This is mainly [27, Chapter 4, Lemma 10.3]. The formulation presented here comes from [50, Proposition 4.5].

## 3. Maximal class $p$-groups

Throughout this section, $S$ represents a $p$-group of maximal class of order $p^{n}$ with $n \geq 3$. This means that $S$ has nilpotency class $n-1$. The maximal class 2 -groups are the dihedral groups, generalized quaternion groups and semidihedral groups and so everything about these groups is easy to calculate. The maximal class $p$-groups of order $p^{3}$ are extraspecial and we assume that the reader is familiar with their structure. Thus for this section we concentrate on the case when $p$ is odd, even though many of the results are obviously true when $p=2$, and we also assume that $n \geq 4$. We set $Z_{1}(S)=Z(S)$ and, for $2 \leq j \leq n-1, Z_{j}(S)$ is the complete preimage of $Z\left(S / Z_{j-1}(S)\right)$. So the $Z_{j}(S)$ are the terms of the upper central series of $S$. Similarly, we put $\gamma_{2}(S)=S^{\prime}=[S, S]$ and define $\gamma_{j}(S)=\left[\gamma_{j-1}(S), S\right]$ for $j \geq 3$. These are the members of the lower central series of $S$. Notice that $\gamma_{j}(S)=1$ for $j \geq n$. As $S$ has maximal class, $Z_{j}(S)=\gamma_{n-j}(S)$ for $1 \leq j \leq n-2$ and this subgroup has order $p^{j}$ and index $p^{n-j}$ in $S$. The 2-step centralizers in $S$ are the subgroups

$$
C_{S}\left(\gamma_{j}(S) / \gamma_{j+2}(S)\right)
$$

where $2 \leq j \leq n-2$. As in the introduction, we define

$$
\gamma_{1}(S)=C_{S}\left(\gamma_{2}(S) / \gamma_{4}(S)\right)
$$

Notice that, as $|S| \geq p^{4}$, then $\left|S: \gamma_{1}(S)\right|=\left|\gamma_{1}(S): \gamma_{2}(S)\right|=p$ and all the 2-step centralizers in $S$ are maximal subgroups of $S$.

Definition 3.1. A maximal class group with more than one 2-step centralizer is called an exceptional group.

By [37, Lemma 1.1.25 (i)], we always have $\left[\gamma_{i}(S), \gamma_{j}(S)\right] \leq \gamma_{i+j}(S)$. Hence $\gamma_{t}(S)$ is abelian for all $t \geq n / 2$. The degree of commutativity of $S$, is the greatest integer $c$ such that $\left[\gamma_{i}(S), \gamma_{j}(S)\right] \leq \gamma_{i+j+c}(S)$ for all $1 \leq i \leq j \leq n$. We say that $S$ has a positive degree of commutativity if and only if $c$ is positive.

Lemma 3.2. Suppose that $S$ is a maximal class p-group of order $p^{n}$ with $n \geq 4$.
(i) If $N$ is a proper normal subgroup of $S$, then either $N=\gamma_{j}(S)$ for some $j \geq 2$ or $N$ is a maximal subgroup of $S$. Furthermore, $S / \gamma_{j}(S)$ has maximal class for all $j \geq 2$.
(ii) $\gamma_{1}(S)$ is a regular $p$-group.
(iii) If $n>p+1$, then $\Omega_{1}\left(\gamma_{1}(S)\right)=\gamma_{n-(p-1)}(S)$ has exponent $p$ and order $p^{p-1}$ and $\mho^{1}\left(\gamma_{i}(S)\right)=\gamma_{i+(p-1)}(S)$ for all $1 \leq i \leq n-(p-1)$.
(iv) $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right| \leq p^{p}$ and if equality holds then $\Omega_{1}\left(\gamma_{1}(S)\right)=\gamma_{1}(S)$ and $|S|=$ $p^{p+1}$.
(v) If $n>p+1, A \leq \gamma_{1}(S)$ and $B$ is normal in $A$ with $A / B$ elementary abelian, then $|A / B| \leq p^{p-1}$.
(vi) If $n \leq p+1$, then $\gamma_{2}(S)$ and $S / Z(S)$ have exponent $p$.
(vii) There exists a unique maximal abelian normal subgroup $\gamma_{w}(S)$ of $S$ and either $\Omega_{1}\left(\gamma_{1}(S)\right) \leq \gamma_{w}(S)$ or $\gamma_{w}(S)$ is elementary abelian.

Proof. The first part of (i) is [37, Proposition 3.1.2] while the second is obvious by definition.

For part (ii), if $n \leq p+1$, then $\left|\gamma_{1}(S)\right| \leq p^{p}$ and [37, Lemma 1.2.11] yields $\gamma_{1}(S)$ is regular. If $n>p+1,[\mathbf{3 7}$, Corollary 3.3.4 (i)] gives the same result.

For statement (iii) we first note that $\gamma_{1}(S)$ is regular by (ii) and so $\Omega_{1}\left(\gamma_{1}(S)\right.$ has exponent $p$ by Lemma 2.1 (ii). The result now follows from [37, Corollary 3.3.6(i)].

Part (iv) is a consequence of (iii).
Part (v) is included in [37, Exercise 3.3 (3)] and part (vi) is [37, Proposition 3.3.2].

Part (vii) follows from (i) as $n \geq 4$.

Lemma 3.3. Suppose that $S$ is a maximal class $p$-group of order $p^{n}$ with $n \geq 4$ and $M$ is a maximal subgroup of $S$.
(i) If $M \notin\left\{\gamma_{1}(S), C_{S}\left(Z_{2}(S)\right)\right\}$, then $M$ has maximal class and $\gamma_{i}(M)=$ $\gamma_{i+1}(S)$ for $i=2, \ldots, n-1$ and $\gamma_{1}(M)=\gamma_{2}(S)$ whenever $n \geq 5$.
(ii) $M$ is a 2-step centralizer in $S$ if and only if $M \in\left\{\gamma_{1}(S), C_{S}\left(Z_{2}(S)\right)\right\}$.
(iii) $S$ is exceptional if and only if $\gamma_{1}(S) \neq C_{S}\left(Z_{2}(S)\right)$.
(iv) The degree of commutativity of $S$ is positive if and only if $S$ is not exceptional.
(v) If $n$ is odd, or $n=4$ or $n>p+1$, then $S$ is not exceptional.
(vi) Assume that $|S| \geq p^{5}$. Then $S / Z(S)$ is not exceptional.
(vii) $\gamma_{1}(S)=C_{S}\left(\gamma_{j}(S) / \gamma_{j+2}(S)\right)$ for all $2 \leq j \leq n-3$.

Proof. For statement (i) see [36, Lemma (1.2)].
For (ii), we know that $\gamma_{1}(S)$ and $C_{S}\left(Z_{2}(S)\right)$ are 2-step centralizers by definition. That they are the only 2 -step centralizers follows from (i).

Part (iii) follows from (i) and the definition of an exceptional group.
Part (iv) is [ $\mathbf{3 7}$, Corollary 3.2.7].
Part (v) follows from [37, Theorems 3.2 .11 and 3.3.5] for $|S| \geq p^{5}$. If $|S|=p^{4}$ then $\gamma_{1}(S)=C_{S}\left(Z_{2}(S)\right)$ is abelian and is the unique 2-step centralizer.

For part (vi). If $n$ is even, then $|S / Z(S)|=p^{n-1} \geq p^{5}$ is not exceptional by (v). Whereas, if $n$ is odd, then $S$ is not exceptional by (v) and $\gamma_{1}(S)$ is the unique 2-step centralizer in $S$ by (ii). Since the preimage of a 2 -step centralizer in $S / Z(S)$, is a 2 -step centralizer in $S$, we deduce that $S / Z(S)$ is not exceptional.

Finally (vii) follows from (vi).

## Lemma 3.4. The following hold:

(i) Suppose that $t \in S$ and $t \notin \gamma_{1}(S) \cup C_{S}\left(Z_{2}(S)\right)$. Then $C_{S}(t)=\langle t, Z(S)\rangle$ has order $p^{2}$ and $C_{S}(t) \cap \gamma_{1}(S)=C_{S}(t) \cap C_{S}\left(Z_{2}(S)\right)=Z(S)$.
(ii) If $T$ is a $p$-group and there exists $t \in T$ with $\left|C_{T}(t)\right|=p^{2}$, then $T$ has maximal class.

Proof. Suppose that $t \notin \gamma_{1}(S) \cup C_{S}\left(Z_{2}(S)\right)$. Then $t$ is not contained in any 2 -step centralizer by Lemma 3.3 (ii).

Assume for a contradiction that $\left|C_{S}(t)\right| \geq p^{3}$. Then there exists $y \in\left(C_{S}(t) \cap\right.$ $\left.\gamma_{1}(S)\right) \backslash Z(S)$ so that $y \in \gamma_{j}(S) \backslash \gamma_{j+1}(S)$ with $\gamma_{j+1}(S) \geq Z(S)$. Now $t$ centralizes

$$
\gamma_{j}(S) / \gamma_{j+2}(S)=\left\langle\gamma_{j+1}(S) / \gamma_{j+2}(S), y \gamma_{j+2}(S)\right\rangle
$$

contrary to $t$ not being in a 2 -step centralizer. Since $\gamma_{1}(S)$ and $C_{S}\left(Z_{2}(S)\right)$ have index $p$ in $S$ and $t \notin \gamma_{1}(S) \cup C_{S}\left(Z_{2}(S)\right), C_{S}(t) \cap \gamma_{1}(S)=C_{S}(t) \cap C_{S}\left(Z_{2}(S)\right)=Z(S)$. This proves (i).

Part (ii) is [32, Satz III.14.23].

The next lemma provides a weak upper bound for the number of commutators by $\gamma_{1}(S)$ required to annihilate $\Omega_{1}\left(\gamma_{1}(S)\right)$.

Lemma 3.5. Suppose that $S$ is a maximal class p-group of order at least $p^{4}$. Then $\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S) ; \frac{p+1}{2}\right]=1$. Furthermore, if $\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S) ; \frac{p-1}{2}\right] \neq 1$, then $S$ is exceptional.

Proof. If $|S|=p^{4}$, then $\gamma_{1}(S)$ is abelian and so the statement holds in this case. Suppose $|S| \geq p^{5}$. Lemmas 3.2 (iv) and 3.3 (vi) give that $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right| \leq p^{p}$ and $S / Z(S)$ is not exceptional. Hence, the fact that $\gamma_{1}(S) / Z(S)$ is the unique 2step centralizer of $S / Z(S)$ implies $\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S) ; \frac{p-1}{2}\right] \leq Z(S)$. This proves the first bound.

Suppose now that $\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S) ; \frac{p-1}{2}\right] \neq 1$. We first show that

$$
\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S) ; \frac{p-1}{2}-1\right] \leq Z_{2}(S)
$$

This is clear if $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right| \leq p^{p-1}$, since $\gamma_{1}(S)$ is the unique 2-step centralizer of $S / Z(S)$. Assume $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right|=p^{p}$. Then $\Omega_{1}\left(\gamma_{1}(S)\right)=\gamma_{1}(S)$ by Lemma 3.2(iv) and we know that $\gamma_{1}(S) / \gamma_{4}(S)$ is abelian. Hence, in this case we also have

$$
\left[\Omega\left(\gamma_{1}(S)\right), \gamma_{1}(S) ; \frac{p-1}{2}-1\right] \leq Z_{2}(S)
$$

Therefore we get $1 \neq\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S) ; \frac{p-1}{2}\right] \leq\left[Z_{2}(S), \gamma_{1}(S)\right]$. This means that $Z_{2}(S) \not \leq Z\left(\gamma_{1}(S)\right)$ and so $S$ is exceptional by Lemma 3.3(iii).

Lemma 3.6. Suppose that $S$ is a maximal class p-group of order at least $p^{4}$. If $T \leq S$ and $T \nsubseteq \gamma_{1}(S) \cup C_{S}\left(Z_{2}(S)\right)$, then $T$ has maximal class.

Proof. Suppose that $S$ is a counterexample of minimal order and let $T \leq S$ with $T \nsubseteq \gamma_{1}(S) \cup C_{S}\left(Z_{2}(S)\right)$ be a subgroup that does not have maximal class. In particular, $T$ is not abelian of order $p$ or $p^{2}$, and $T \neq S$. So we have $|T| \geq$ $p^{3}$. Let $M$ be a maximal subgroup of $S$ which contains $T$. Then $M \nsubseteq \gamma_{1}(S) \cup$ $C_{S}\left(Z_{2}(S)\right)$ and so $M$ has maximal class by Lemma 3.3 (i). Therefore $T<M$ and $|M| \geq p^{4}$. By Lemma $3.3(\mathrm{i}), \gamma_{1}(M)=\gamma_{2}(S)$ and $Z_{2}(M)=Z_{2}(S)$. In particular, $\gamma_{1}(M)$ centralizes $Z_{2}(S)$ and so $\gamma_{1}(M)=C_{M}\left(Z_{2}(M)\right)$ and $T \nsubseteq \gamma_{1}(M) \cup$ $C_{M}\left(Z_{2}(M)\right)$. Using the minimality of $S$, we conclude that $T$ has maximal class, a contradiction.

Lemma 3.7. Suppose that $S$ is a maximal class p-group of order at least $p^{4}$. If $T \subseteq \gamma_{1}(S) \cup C_{S}\left(Z_{2}(S)\right)$ and $N_{S}(T) \nsubseteq \gamma_{1}(S) \cup C_{S}\left(Z_{2}(S)\right)$, then $T$ is normal in $S$. Furthermore, if $Z(S) \leq W \subseteq \gamma_{1}(S) \cup C_{S}\left(Z_{2}(S)\right)$ and $N_{S}(W) \not \pm \gamma_{1}(S)$, then $W$ is normal in $S$.

Proof. By Lemma 3.6, $N_{S}(T)$ has maximal class. Assume that $T$ is not normal in $S$. Then $T \neq C_{S}\left(Z_{2}(S)\right)$ and $T \neq \gamma_{1}(S)$. Hence, as $N_{S}(T) \not \leq \gamma_{1}(S)$ or $C_{S}\left(Z_{2}(S)\right.$ ), we have $\left|N_{S}(T): T\right| \geq p^{2}$. Since $N_{S}(T)$ has maximal class and normalizes $T, T=\gamma_{i}\left(N_{S}(T)\right)$ for some $i \geq 2$. In particular, $T$ is characteristic in $N_{S}(T)$ and this means $T$ is normal in $S$, a contradiction.

Now suppose that $Z(S) \leq W$. If $S$ is not exceptional, then we have the $N_{S}(W) \not \leq \gamma_{1}(S)=\gamma_{1}(S) \cup C_{S}\left(Z_{2}(S)\right)$ and so $W$ is normal in $S$. If $S$ is exceptional,
then $|S| \geq p^{6}$ and $S / Z(S)$ is not exceptional. Now the result follows by applying our primary statement to $W / Z(S)$ in $S / Z(S)$.

We close this subsection by determining some of the finite simple groups which have a maximal class Sylow $p$-subgroup.

Lemma 3.8. Suppose that $p \geq 5$ is a prime, $G$ is a finite simple group and $S \in \operatorname{Syl}_{p}(G)$. Assume that $S$ has maximal class and has no abelian subgroup of index $p$. Then $S$ is isomorphic to a Sylow p-subgroup of $\mathrm{G}_{2}(p), \gamma_{1}(S)$ is extraspecial and either $G \cong \mathrm{G}_{2}(p)$ or $p=5$ and $G \cong \mathrm{Ly}, \mathrm{HN}$ or B or $p=7$ and $G \cong \mathrm{M}$.

Proof. We may assume that $|S| \geq p^{4}$. We consider each of the types of nonabelian simple groups. If $G$ is an alternating group of degree $d$, then, as $S$ has no abelian subgroup of index $p, d \geq p^{3}$. But then $S$ contains $p^{p}$ commuting $p$-cycles and has order greater than $p^{p+2}$. This contradicts Lemma 3.2 (iv).

Suppose $G$ is a Lie type group in characteristic $p$. We use the fact that $\left|S / \gamma_{2}(S)\right|=p^{2}$. Using [26, Theorem 3.3.1] we see that $G$ has untwisted Lie-rank at most 2 and that $G$ is defined over $\operatorname{GF}(p)$ if it has rank 2 . Since $S$ has no abelian subgroup of index $p, G$ is not $\mathrm{PSL}_{2}\left(p^{2}\right), \mathrm{PSU}_{3}(p), \mathrm{PSL}_{3}(p), \mathrm{PSp}_{4}(p)$. This leaves $\mathrm{G}_{2}(p)$. Hence the result holds for Lie type groups in characteristic $p$.

Suppose that $G$ is a Lie type group in characteristic $r$ with $r \neq p$. We use [26, Theorem 4.10.2]. Thus we can write $S=P_{T} P_{W}$ where $P_{T}$ is an abelian normal subgroup of $S$ and $P_{W}$ is a complement to $P_{T}$. Since

$$
\gamma_{2}(S)=[S, S]=\left[P_{T} P_{W}, P_{T} P_{W}\right]=\left[P_{T}, P_{W}\right] P_{W}^{\prime}
$$

$\left|S / \gamma_{2}(S)\right|=p^{2}$ implies that $P_{W} / P_{W}^{\prime}$ has order $p$. But then, $P_{W}$ has order $p$ and $S$ has an abelian subgroup of index $p$, a contradiction.

Finally, suppose that $G$ is a sporadic simple group. We deploy the tables in [26, Tables 5.3]. The sporadic groups listed in the conclusion of our lemma have Sylow $p$-subgroups isomorphic to those of $\mathrm{G}_{2}(5)$ in the first three cases and to those of $\mathrm{G}_{2}(7)$ in the last case. Since $p \geq 5$, we only need to think about the Sylow 5 -subgroup of the monster M. This group has order $5^{9}$. Using [26, Table 5.3 z ], the monster has a 5 -local subgroup $5^{1+6} \cdot\left(\left(4 \circ 2 \cdot \mathrm{~J}_{2}\right) \cdot 2\right)$. Thus $S / Z(S)$ is of maximal class and has a normal subgroup of 5 -rank 6 , as $|S / Z(S)|=5^{8}>5^{7}$, this contradicts Lemma 3.2(v).

We give a list of the realizable fusion systems on maximal class $p$-groups which have an abelian subgroup of index $p$ in Appendix A, Table 3. This table is extracted from [19, Table 2.2].
3.1. Automorphisms of $p^{\prime}$-order. We now present some conclusions about the automorphism group of $p$-groups $S$ of maximal class of order $p^{n}$ with $n \geq 4$. Our first result is well-known.

Theorem 3.9. Suppose that $S$ is a p-group of maximal class and order at least $p^{4}$. Then $\operatorname{Aut}(S) / O_{p}(\operatorname{Aut}(S))$ is isomorphic to a subgroup of the diagonal matrices in $\mathrm{GL}_{2}(p)$. In particular, if $H$ is a subgroup of $\operatorname{Aut}(S)$ and $|H|$ is coprime to $p$, then $H$ is abelian and is isomorphic to a subgroup of the diagonal matrices in $\mathrm{GL}_{2}(p)$.

Proof. We know that $\left|S / \gamma_{2}(S)\right|=p^{2}$ and $S / \gamma_{2}(S)$ is elementary abelian. Hence $\operatorname{Aut}(S) / C_{\operatorname{Aut}(S)}\left(S / \gamma_{2}(S)\right)$ is isomorphic to a subgroup of $\mathrm{GL}_{2}(p)$. Since $\gamma_{1}(S)$
is a characteristic subgroup of $S$ and $\left|\gamma_{1}(S) / \gamma_{2}(S)\right|=p$, Aut $(S) / C_{\text {Aut }(S)}\left(S / \gamma_{2}(S)\right)$ is isomorphic to a subgroup of the lower triangular matrices of $\mathrm{GL}_{2}(p)$. By Burnside's Theorem $[\mathbf{2 4}, 5.1 .4], C_{\operatorname{Aut}(S)}\left(S / \gamma_{2}(S)\right)$ is a $p$-group. The result now follows.

One consequence of Theorem 3.9 is that if $\varphi \in \operatorname{Aut}(S)$ has order coprime to $p$, then by Maschke's Theorem there exists a maximal subgroup $M$ of $S$ with $M \neq \gamma_{1}(S)$ such that $M \varphi=M$. Choose $x \in M \backslash \gamma_{1}(S)$ and $s_{1} \in \gamma_{1}(S) \backslash \gamma_{2}(S)$ and define

$$
\begin{gathered}
s_{i}=\left[x, s_{i-1}\right] \text { for every } 2 \leq i \leq n-2 \text { and } \\
s_{n-1}= \begin{cases}{\left[x, s_{n-2}\right]} & \text { if } M=C_{S}\left(Z_{2}(S)\right) \\
{\left[s_{1}, s_{n-2}\right]} & \text { otherwise. }\end{cases}
\end{gathered}
$$

The choice of $x$ and $s_{1}$ is plainly not unique, and it is even possible that $\left[s_{1}, s_{2}\right] \in \gamma_{j}(S)$ and for a different choice of $x$ and $s_{1}$ we have $\left[s_{1}, s_{2}\right] \in \gamma_{k}(S)$ with $k<j$. In particular, there are examples of maximal class groups $S$ and a choice of $x$ and $s_{1}$ such that $\left[s_{1}, s_{2}\right]=1$ and $\gamma_{1}(S)$ is non-abelian.

The next lemma is extracted from part of the proof of [28, Theorem 2.19].
Lemma 3.10. Suppose that $S$ is a maximal class $p$-group of order $p^{n}$ with $n \geq 4$. Let $\varphi \in \operatorname{Aut}(S)$ be an automorphism of order coprime to $p$ and let $x \in S \backslash \gamma_{1}(S)$ be such that $\varphi$ leaves $\langle x\rangle \gamma_{2}(S)$ invariant. If $a, b \in \mathrm{GF}(p)$ are such that $x \varphi \equiv x^{a}$ $\bmod \gamma_{2}(S)$ and $s_{1} \varphi \equiv s_{1}^{b} \bmod \gamma_{2}(S)$ then

$$
\begin{aligned}
s_{i} \varphi & \equiv s_{i}^{a^{i-1} b} \bmod \gamma_{i+1}(S) \text { for every } 1 \leq i \leq n-2 \text { and } \\
s_{n-1} \varphi & = \begin{cases}s_{n-1}^{a^{n-2} b} & \text { if } S \text { is not exceptional } \\
s_{n-1}^{a^{n-3} b^{2}} & \text { if } S \text { is exceptional. }\end{cases}
\end{aligned}
$$

Proof. We demonstrate the result by induction on $i$. If $i=1$, then this is just the definition of $b$. Assume $1<i<n-2$. Then by the inductive hypothesis there exists $u \in \gamma_{2}(S), v \in \gamma_{i}(S)$ such that

$$
s_{i} \varphi=\left[x, s_{i-1}\right] \varphi=\left[x^{a} u, s_{i-1}^{a^{i-2} b} v\right]
$$

Thus

$$
s_{i} \varphi \equiv\left[x^{a}, s_{i-1}^{a^{i-2} b}\right] \quad \bmod \gamma_{i+1}(S) \equiv s_{i}^{a^{i-1} b} \quad \bmod \gamma_{i+1}(S)
$$

The same argument works for $i=n-1$ when $S$ is not exceptional and this yields the result in this case.

If $S$ is exceptional and $i=n-1$, then we have

$$
s_{n-1} \varphi=\left[s_{1}, s_{n-2}\right] \varphi=\left[s_{1}^{b} u, s_{n-2}^{a^{n-3} b} v\right]=s_{n-1}^{a^{n-3} b^{2}}
$$

for some $u \in \gamma_{2}(S)$ and $v \in Z(S)$ and this yields the result.

Lemma 3.11. Suppose that $S$ is a p-group of maximal class and order at least $p^{4}$. Assume that $\gamma_{1}(S)$ is not abelian or extraspecial. Then every non-trivial automorphism of $S$ of order coprime to $p$ acts faithfully on $S / \gamma_{1}(S)$. In particular, $C_{\operatorname{Aut}(S)}\left(S / \gamma_{1}(S)\right)$ is a p-group and $\operatorname{Aut}(S) / C_{\operatorname{Aut}(S)}\left(S / \gamma_{1}(S)\right)$ has order dividing $p-1$.

Proof. Suppose $\alpha \in \operatorname{Aut}(S)$ has order coprime to $p$ and centralizes $S / \gamma_{1}(S)$. We show that $\alpha$ is the trivial automorphism. Aiming for a contradiction, suppose that $\alpha$ is non-trivial. Let $b \in \operatorname{GF}(p)$ be such that $s_{1} \alpha \equiv s_{1}^{b} \bmod \gamma_{2}(S)$. Because $\alpha$ is non-trivial, $b \neq 1$ as $\alpha$ acts faithfully on $S / \gamma_{2}(S)$ by [24, Theorem 5.1.4]. Employing Lemma 3.10 with $a=1$, we get

$$
\begin{aligned}
s_{i} \alpha & \equiv s_{i}^{b} \bmod \gamma_{i+1}(S) \text { for every } 1 \leq i \leq n-2 \text { and } \\
s_{n-1} \alpha & = \begin{cases}s_{n-1}^{b} & \text { if } S \text { is not exceptional } \\
s_{n-1}^{b^{2}} & \text { if } S \text { is exceptional }\end{cases}
\end{aligned}
$$

Since $\gamma_{1}(S)$ is not abelian, we have $Z\left(\gamma_{1}(S)\right)=\gamma_{k+1}(S)$ for some $2 \leq k \leq n-2$. Then $\gamma_{k}(S)$ is abelian and is non-central. Choose $j$ maximal such that $\left[s_{k}, s_{j}\right] \neq 1$. As $\left[s_{j}, s_{k}\right] \neq 1,\left[s_{j}, s_{k}\right] \in \gamma_{i}(S) \backslash \gamma_{i+1}(S)$ for some $k+1 \leq i \leq n-1$. Note that $\left[s_{j}, s_{k}\right] \in Z\left(\gamma_{1}(S)\right)$ commutes with both $s_{j}$ and $s_{k}$. Also, $s_{j} \alpha=s_{j}^{b} z$ for some $z \in \gamma_{j+1}(S)$, that commutes with $s_{k}$ by the maximal choice of $j$, and $s_{k} \alpha=s_{k}^{b} u$, for some $u \in \gamma_{k+1}(S)=Z\left(\gamma_{1}(S)\right)$. Therefore using the commutator formulae we get

$$
\left[s_{j}, s_{k}\right] \alpha=\left[s_{j} \alpha, s_{k} \alpha\right]=\left[s_{j}^{b} z, s_{k}^{b} u\right]=\left[s_{j}^{b}, s_{k}^{b}\right]=\left[s_{j}, s_{k}\right]^{b^{2}} .
$$

Suppose $S$ is not exceptional. Then

$$
\left[s_{j}, s_{k}\right]^{b^{2}} \equiv\left[s_{j}, s_{k}\right]^{b} \quad \bmod \gamma_{i+1}(S)
$$

and so $b=1$, which is a contradiction.
Suppose $S$ is exceptional. Then $\bar{S}=S / Z(S)$ is not exceptional by Lemma 3.3 (vi) and $\alpha$ is a non-trivial automorphism of $\bar{S}$ of order coprime to $p$ centralizing $\bar{S} / \gamma_{1}(\bar{S})$. Since $S / Z(S)$ is not exceptional, using what we proved above we obtain $\gamma_{1}(\bar{S})=\gamma_{1}(S) / Z(S)$ is abelian. Hence $\gamma_{1}(S)$ is extraspecial, contradicting the hypothesis.

Corollary 3.12. Suppose that $S$ is a p-group of maximal class and order at least $p^{4}$. Assume that $\gamma_{1}(S)$ is not abelian or extraspecial. If $H$ is a subgroup of $\operatorname{Aut}(S)$ of order coprime to $p$, then $H$ is cyclic of order $m$ dividing $(p-1)$ and $H$ acts faithfully on $S / \gamma_{1}(S)$. Furthermore, $|\operatorname{Aut}(S)|=p^{a} m$ for some natural number $a$.

Proof. We know that $\gamma_{1}(S)$ is a characteristic subgroup of $S$. Hence there is a homomorphism $\theta: \operatorname{Aut}(S) \rightarrow \operatorname{Aut}\left(S / \gamma_{1}(S)\right)$ and the latter group is cyclic of order $p-1$. By Lemma 3.11, $\operatorname{ker} \theta$ is a $p$-group. This proves the claim.

Lemma 3.13. Suppose that $S$ is a p-group of maximal class and order at least $p^{4}$. If $S$ is exceptional and $\alpha \in \operatorname{Aut}(S)$ is an involution which inverts $S / \gamma_{1}(S)$, then $\alpha$ inverts $Z(S)$.

Proof. Because $S$ is exceptional, $n$ is even by Lemma 3.3 (v). Since $\alpha$ is an involution which inverts $S / \gamma_{1}(S),\left(x \gamma_{2}(S)\right) \alpha=x^{-1} \gamma_{2}(S)$ and $\left(s_{1} \gamma_{2}(S)\right) \alpha=s_{1}^{b} \gamma_{2}(S)$ where $b= \pm 1$. By Lemma 3.10, $\alpha$ acts on $Z(S)$ by raising elements to the power

$$
(-1)^{n-3} b^{2}=(-1)^{n-3}=-1
$$

Hence $\alpha$ inverts $Z(S)$.

Lemma 3.14. Suppose that $S$ is a p-group of maximal class and order at least $p^{4}$ that is not exceptional. Let $\alpha$ be an automorphism of $S$ of order $m \neq 1$ in its action on $S / \gamma_{1}(S)$. Assume that there exists $c \in \mathrm{GF}(p)$ such that $s_{j} \alpha \equiv s_{j}^{c} \bmod \gamma_{j+1}(S)$ and $s_{k} \alpha \equiv s_{k}^{c} \bmod \gamma_{k+1}(S)$ for some $j, k \geq 1$. Then $j \equiv k(\bmod m)$.

Proof. As $m$ and $p$ are coprime, there is $x \in S \backslash \gamma_{1}(S)$ such that $\alpha$ leaves invariant $\langle x\rangle \gamma_{2}(S)$. Let $a, b \in \operatorname{GF}(p)$ be such that $x \alpha \equiv x^{a} \bmod \gamma_{2}(S)$ and $s_{1} \alpha \equiv$ $s_{1}^{b} \bmod \gamma_{2}(S)$. By assumption there is $c \in \operatorname{GF}(p)$ and $j, k \geq 1$ such that

$$
s_{j} \alpha \equiv s_{j}^{c} \quad \bmod \gamma_{j+1}(S) \text { and } s_{k} \alpha \equiv s_{k}^{c} \quad \bmod \gamma_{k+1}(S)
$$

Since $S$ is not exceptional, by Lemma 3.10 we have

$$
s_{j} \alpha \equiv s_{j}^{a^{j-1} b} \quad \bmod \gamma_{j+1}(S) \text { and } s_{k} \alpha \equiv s_{k}^{a_{k-1}^{k-1}} \quad \bmod \gamma_{k+1}(S)
$$

Therefore

$$
a^{j-1} b \equiv c \equiv a^{k-1} b \quad(\bmod p)
$$

Thus $a^{j-k} \equiv 1(\bmod p)$, that is $j-k \equiv 0(\bmod m)$ and so

$$
j \equiv k \quad(\bmod m)
$$

3.2. A theorem of Juhász. For completeness, we now present a proof of $[\mathbf{3 4}$, Theorem 6.2] slightly modified for our application.

Theorem 3.15 (Juhász). Assume that $S$ has maximal class and order at least $p^{4}$. If $\gamma_{3}(S)$ is abelian, then either $\gamma_{2}(S)$ has nilpotency class at most 2 or $|S| \geq$ $p^{2 p+4}$. Furthermore, if $\gamma_{j}(S)>1$ is elementary abelian for some $j \geq 3$, then $\gamma_{j-1}(S)$ has nilpotency class at most 2.

Proof. Let $k, \ell \geq 1$ be such that

$$
\gamma_{k}(S)=\left[\gamma_{3}(S), \gamma_{2}(S)\right]=\gamma_{2}(S)^{\prime} \text { and } \gamma_{\ell}(S)=\left[\gamma_{3}(S), \gamma_{2}(S), \gamma_{2}(S)\right]
$$

Assume that $\gamma_{2}(S)$ does not have nilpotency class 2. Then $\gamma_{\ell}(S) \neq 1$. We shall show that $|S| \geq p^{2 p+4}$.

Observe that for $j \geq 3$,

$$
\left[\gamma_{j}(S), \gamma_{2}(S)\right]=\gamma_{j+k-3}(S)
$$

To see this, we note the case $j=3$ is just the definition of $k$. Assume that the statement holds for $j \geq 3$. Then $\left[S, \gamma_{2}(S), \gamma_{j}(S)\right]=\left[\gamma_{3}(S), \gamma_{j}(S)\right]=1$ because $\gamma_{j}(S) \leq \gamma_{3}(S)$ and $\gamma_{3}(S)$ is abelian. Hence the Three Subgroup Lemma yields the second equality of

$$
\begin{aligned}
{\left[\gamma_{j+1}(S), \gamma_{2}(S)\right] } & =\left[\gamma_{j}(S), S, \gamma_{2}(S)\right]=\left[\gamma_{2}(S), \gamma_{j}(S), S\right] \\
& =\left[\gamma_{j+k-3}(S), S\right]=\gamma_{(j+1)+k-3}(S)
\end{aligned}
$$

and this verifies the observation. In particular, $\gamma_{\ell}(S)=\left[\gamma_{k}(S), \gamma_{2}(S)\right]=\gamma_{2 k-3}(S)$ and so

$$
\ell=2 k-3 \text { and }|S|=p^{n} \geq p^{2 k-2} .
$$

Recall the definition of $x$ and $s_{i}, 1 \leq i \leq n-1$ from Subsection 3.1 and for $i \geq n$ we set $s_{i}=1$.

Define $u_{3}=\left[s_{2}, s_{1}\right] \in \gamma_{3}(S)$ and $u_{i}=\left[x, u_{i-1}\right] \in \gamma_{i}(S)$ for $i \geq 4$. Since $\gamma_{2}(S) / \gamma_{k}(S)$ is abelian and $\left[x, s_{1}\right] \in \gamma_{2}(S)$, using the variant of the Hall-Witt identity

$$
\left[w, y, z^{w}\right]\left[y, z, w^{y}\right]\left[z, w, y^{z}\right]=1
$$

and using $\left[s_{2}^{-1}, s_{j}\right] \in Z\left(S / \gamma_{j+k-2}(S)\right)$ we calculate

$$
\begin{aligned}
\gamma_{j+k-2}(S) & =\left[x, s_{j}, s_{1}^{x}\right]\left[s_{j}, s_{1}, s^{s_{j}}\right]\left[s_{1}, x, s_{j}^{s_{1}}\right] \gamma_{j+k-2}(S) \\
& =\left[s_{j+1}, s_{1}^{x}\right]\left[s_{j}, s_{1}, x^{s_{j}}\right]\left[s_{2}^{-1}, s_{j}^{s_{1}}\right] \gamma_{j+k-2}(S) \\
& =\left[s_{j+1}, s_{1}\left[s_{1}, x\right]\right]\left[\left[s_{j}, s_{1}\right]^{s_{j}^{-1}}, x\right]^{s_{j}}\left[s_{2}^{-1}, s_{j}\right] \gamma_{j+k-2}(S) \\
& =\left[s_{j+1}, s_{1} s_{j}^{-1}\right]\left[s_{j}, s_{1}, x\right]\left[s_{j}, s_{2}\right] \gamma_{j+k-2}(S) \\
& =\left[s_{j+1}, s_{1}\right]\left[s_{j}, s_{1}, x\right]\left[s_{j}, s_{2}\right] \gamma_{j+k-2}(S) .
\end{aligned}
$$

Hence

$$
\begin{align*}
{\left[s_{j+1}, s_{1}\right]\left[s_{j}, s_{2}\right] \gamma_{j+k-2}(S) } & =\left[x,\left[s_{j}, s_{1}\right]\right] \gamma_{j+k-2}(S) ; \text { and } \\
{\left[s_{3}, s_{1}\right] \gamma_{k}(S) } & =u_{4} \gamma_{k}(S) \tag{1}
\end{align*}
$$

Using $\gamma_{2}(S) / \gamma_{\ell}(S)$ has class 2 and $\gamma_{3}(S)$ is abelian together with the Hall-Witt identity and we have, for $t \geq 3$,

$$
\begin{aligned}
\gamma_{\ell}(S) & =\left[x, s_{t}, s_{2}\right]^{s_{t}^{-1}}\left[s_{t}^{-1}, s_{2}^{-1}, x\right]^{s_{2}}\left[s_{2}, x^{-1}, s_{t}^{-1}\right]^{x} \gamma_{\ell}(S) \\
& =\left[x, s_{t}, s_{2}\right]\left[s_{t}^{-1}, s_{2}^{-1}, x\right] \gamma_{\ell}(S) \\
& =\left[x, s_{t}, s_{2}\right]\left[s_{t}, s_{2}, x\right] \gamma_{\ell}(S)
\end{aligned}
$$

So we know for $t \geq 3$

$$
\begin{equation*}
\left[s_{t+1}, s_{2}\right] \gamma_{\ell}(S)=\left[s_{t}, s_{2}, x\right]^{-1} \gamma_{\ell}(S)=\left[x,\left[s_{t}, s_{2}\right]\right] \gamma_{\ell}(S) \tag{2}
\end{equation*}
$$

We claim that, for $j \geq 3$,

$$
\begin{equation*}
\left[s_{j}, s_{1}\right]\left[s_{j-1}, s_{2}\right]^{j-3} \gamma_{j+k-3}(S) \gamma_{\ell}(S)=u_{j+1} \gamma_{j+k-3}(S) \gamma_{\ell}(S) \tag{3}
\end{equation*}
$$

This is valid for $j=3$ by (1). We prove the claim by induction. We have

$$
\begin{aligned}
{\left[x,\left[s_{j}, s_{1}\right]\left[s_{j-1}, s_{2}\right]^{j-3}\right] } & \in\left[x, u_{j+1}\right] \gamma_{j+k-2}(S) \gamma_{\ell}(S)=u_{j+2} \gamma_{j+k-2}(S) \gamma_{\ell}(S) . \\
{\left[x,\left[s_{j}, s_{1}\right]\left[s_{j-1}, s_{2}\right]^{j-3}\right] } & \in\left[x,\left[s_{j-1}, s_{2}\right]^{j-3}\right]\left[x,\left[s_{j}, s_{1}\right]\right]^{\left[s_{j}, s_{1}\right]^{j-3}} \gamma_{j+k-2}(S) \gamma_{\ell}(S) \\
& =\left[x,\left[s_{j-1}, s_{2}\right]\right]^{j-3}\left[x,\left[s_{j}, s_{1}\right]\right] \gamma_{j+k-2}(S) \gamma_{\ell}(S) \\
& =\left[s_{j}, s_{2}\right]^{j-2}\left[s_{j+1}, s_{1}\right] \gamma_{j+k-2}(S) \gamma_{\ell}(S)
\end{aligned}
$$

where we have used $\gamma_{3}(S)$ is abelian for the second equality and (1) and (2) for the third. Thus (3) follows by induction.

Write $\left[s_{3}, s_{2}\right]=\prod_{j=k}^{n-1} s_{j}^{a_{j}}$ where $0 \leq a_{j} \leq p-1$ and $a_{k} \neq 0$. As $\gamma_{3}(S)$ is abelian, we obtain

$$
\left[s_{3}, s_{2}, s_{1}\right]=\left[\prod_{j=k}^{n-1} s_{j}^{a_{j}}, s_{1}\right]=\prod_{j=k}^{n-1}\left[s_{j}, s_{1}\right]^{a_{j}}
$$

Hence, using equation (3), taking suitable $g_{j+k-3} \in \gamma_{j+k-3}(S)$, observing that $\left[s_{2}, s_{j}\right] \in \gamma_{j+k-3}(S) \leq \gamma_{\ell}(S)$ for $j \geq k$ and remembering $\gamma_{3}(S)$ is abelian, we
calculate

$$
\begin{align*}
& {\left[s_{3}, s_{2}, s_{1}\right] \gamma_{\ell}(S)=\prod_{j=k}^{n-1}\left[s_{j}, s_{1}\right]^{a_{j}} \gamma_{\ell}(S)=\prod_{j=k}^{n-1}\left(u_{j+1}\left[s_{2}, s_{j-1}\right]^{j-3} g_{j+k-3}\right)^{a_{j}} \gamma_{\ell}(S)} \\
& =\left[s_{2}, s_{k-1}\right]^{a_{k}(k-3)} \prod_{j=k}^{n-1} u_{j+1}^{a_{j}} g_{j+k-3}^{a_{j}} \gamma_{\ell}(S)=\left[s_{2}, s_{k-1}\right]^{a_{k}(k-3)} \prod_{j=k}^{n-1} u_{j+1}^{a_{j}} \gamma_{\ell}(S) \tag{4}
\end{align*}
$$

where $g_{j+k-3} \in \gamma_{j+k-3}(S) \leq \gamma_{\ell}(S)$ as, for $j \geq k, j+k-3 \geq 2 k-3 \geq \ell$. We next determine $\prod_{j=k}^{n-1} u_{j+1}^{a_{j}} \gamma_{\ell}(S)$. So decompose

$$
u_{3}=\left[s_{2}, s_{1}\right]=\prod_{t=3}^{n} s_{t}^{b_{t}} .
$$

Then, since $\gamma_{3}(S)$ is abelian,

$$
u_{4}=\left[x, u_{3}\right]=\left[x, \prod_{t=3}^{n} s_{t}^{b^{t}}\right]=\prod_{t=3}^{n}\left[x, s_{t}\right]^{b_{t}}=\prod_{t=3}^{n} s_{t+1}^{b_{t}} .
$$

and by induction, for $r \geq 3, u_{r}=\prod_{t=3}^{n} s_{t+r-3}^{b_{t}}$.
Using induction and (2), for $t \geq 3,\left[x,\left[s_{t}, s_{2}\right]\right] \gamma_{\ell}(S)=\prod_{j=k}^{n} s_{j+t-2}^{a_{j}} \gamma_{\ell}(S)$ and so

$$
\begin{aligned}
{\left[u_{4}, s_{2}\right] \gamma_{\ell}(S) } & =\prod_{t=3}^{n}\left[s_{t+1}, s_{2}\right]^{b_{t}} \gamma_{\ell}(S)=\prod_{t=3}^{n}\left(\left[x,\left[s_{t}, s_{2}\right]\right)^{b_{t}} \gamma_{\ell}(S)\right. \\
& =\prod_{t=3}^{n}\left(\prod_{j=k}^{n} s_{j+t-2}^{a_{j}}\right)^{b_{t}} \gamma_{\ell}(S) \\
& =\prod_{j=k}^{n}\left(\prod_{t=3}^{n} s_{j+t-2}^{b_{t}}\right)^{a_{j}} \gamma_{\ell}(S)=\prod_{j=k}^{n} u_{j+1}^{a_{j}} \gamma_{\ell}(S) .
\end{aligned}
$$

In combination with equation (4) this provides

$$
\begin{equation*}
\left[s_{3}, s_{2}, s_{1}\right] \gamma_{\ell}(S)=\left[s_{2}, s_{k-1}\right]^{a_{k}(k-3)}\left[u_{4}, s_{2}\right] \gamma_{\ell}(S) \tag{5}
\end{equation*}
$$

On the other hand, commutating Equation (1) on the right with $s_{2}$ yields

$$
\begin{equation*}
\left[s_{3}, s_{1}, s_{2}\right] \gamma_{\ell}(S)=\left[u_{4} g_{k}, s_{2}\right] \gamma_{\ell}(S)=\left[u_{4}, s_{2}\right] \gamma_{\ell}(S) \tag{6}
\end{equation*}
$$

Since $\left[s_{3}, s_{2}, s_{1}\right] \gamma_{\ell}(S)=\left[s_{3}, s_{1}, s_{2}\right] \gamma_{\ell}(S)$, we deduce that $\left[s_{2}, s_{k-1}\right]^{a_{k}(k-3)} \in \gamma_{\ell}(S)$. However,

$$
\left\langle\left[s_{k-1}, s_{2}\right]\right\rangle \gamma_{\ell}(S)=\left[\gamma_{k-1}(S), \gamma_{2}(S)\right]=\gamma_{2 k-4}(S)>\gamma_{\ell}(S)=\gamma_{2 k-3}(S)
$$

and therefore $\left[s_{k-1}, s_{2}\right] \notin \gamma_{\ell}(S)$ and so $a_{k}(k-3) \equiv 0(\bmod p)$. Since $0<a_{k} \leq$ $p-1, k-3 \equiv 0(\bmod p)$ and, as $k>3, k-3=m p$ for some $m \geq 1$. Hence $n \geq 2 k-2 \geq 2 p+4$. We conclude, $|S| \geq p^{2 p+4}$. This proves the main statement.

Assume that $\gamma_{j}(S)$ is non-trivial and elementary abelian for some $j \geq 3$. Let $x \in S \backslash\left(\gamma_{1}(S) \cup C_{S}\left(Z_{2}(S)\right)\right)$. Then $x^{p} \in C_{S}(x) \cap \gamma_{1}(S)=Z(S) \leq \gamma_{2}(S)$ by Lemma 3.4. Hence $T=\langle x\rangle \gamma_{j-2}(S)$ has order $p^{n-j+2} \geq p^{4}$. By Lemma 3.6, $T$ has maximal class and $\gamma_{j}(S)=\gamma_{3}(T)$. Hence $\gamma_{3}(T)$ is elementary abelian. Lemma 3.2 (iv) implies $\left|\gamma_{3}(T)\right| \leq p^{p-1}$ and thus $|T| \leq p^{p+2}$. Since $p+2<2 p+4$, the main claim yields $\gamma_{2}(T)=\gamma_{j-1}(S)$ has nilpotency class at most 2. This completes the proof.

## 4. Representations of groups with a cyclic Sylow p-subgroup

In this section, for $p$ an odd prime, we gather together various facts about representations of groups with cyclic Sylow $p$-subgroups.

Definition 4.1. A group $X$ is said to be of $\mathrm{L}_{2}(p)$-type provided each composition factor of $X$ is either a $p$-group, a $p^{\prime}$-group or is isomorphic to $\operatorname{PSL}_{2}(p)$.

We will require the following result due to Feit.
ThEOREM 4.2 (Feit). Suppose that $p$ is a prime, $\mathbb{K}$ is a field of characteristic $p, L$ is a finite group with a cyclic Sylow p-subgroup $P$ and $V$ is a faithful indecomposable $\mathbb{K} L$-module with $d=\operatorname{dim} V \leq p$. Assume that $L$ is not of $\mathrm{L}_{2}(p)$-type. Then $p$ is odd, $|P|=p,\left.V\right|_{P}$ is indecomposable, $C_{L}(P)=P \times Z(L)$ and $d \geq \frac{2}{3}(p-1)$.

Proof. This is [23, Theorem 1].
Theorem 4.2 illuminates the importance of representations of $\mathrm{SL}_{2}(p)$. Some of the results in this section hold for an arbitrary field $\mathbb{K}$ but our applications will only be for $\mathbb{K}=\operatorname{GF}(p)$. We follow the standard construction of certain irreducible modules for $\mathrm{GL}_{2}(\mathbb{K})$. Let $\mathbb{K}[x, y]$ be the polynomial ring in two commuting variables $x$ and $y$ and coefficients in $\mathbb{K}$. Then, for $\left(\begin{array}{c}\alpha \\ \gamma \\ \gamma\end{array}\right) \in \mathrm{GL}_{2}(\mathbb{K})$ and $a, b \geq 0$ natural numbers, the extension linearly of

$$
x^{a} y^{b} \cdot\left(\begin{array}{cc}
\alpha & \beta \\
\gamma & \delta
\end{array}\right)=(\alpha x+\beta y)^{a}(\gamma x+\delta y)^{b}
$$

makes $\mathbb{K}[x, y]$ into a $\mathbb{K} \mathrm{GL}_{2}(\mathbb{K})$-module. The subspaces of $\mathbb{K}[x, y]$ consisting of homogenous polynomials of fixed degree at most $p-1$ provide us with an explicit construction of the basic irreducible ${\mathbb{K} \mathrm{SL}_{2}}^{(p)}$-modules by restriction.

Notation 4.3. For $0 \leq e \leq p-1, \mathbf{V}_{e}$ represents the $(e+1)$-dimensional $\mathrm{GL}_{2}(\mathbb{K})$-submodule of $\mathbb{K}[x, y]$ consisting of degree $e$ homogeneous polynomials. We use the same notation for $\mathbf{V}_{e}$ when we consider $\mathbf{V}_{e}$ as a module for certain subgroups of $\mathrm{GL}_{2}(\mathbb{K})$, for example, when considered as a $\mathbb{K} \mathrm{SL}_{2}(p)$-module. We often call $\mathbf{V}_{1}$ the natural $\mathbb{K} \mathrm{SL}_{2}(p)$-module.

For $\mathrm{SL}_{2}(p)$, every irreducible $\mathbb{K} \mathrm{SL}_{2}(p)$-module is basic and can be realized over $\mathrm{GF}(p)$ (see [1, page 15] for example). In particular, in this case there are $p$ irreducible modules and they have dimensions $1,2, \ldots, p$. The faithful modules are the ones of even-dimension and the odd-dimensional modules are representations of $\mathrm{PSL}_{2}(p)$.

The next eight results provide the facts that we shall require about these representations. The first result is used silently in the text.

Lemma 4.4. Suppose that $L \cong \mathrm{SL}_{2}(p), T \in \operatorname{Syl}_{p}(L)$ and $V$ is an irreducible $\mathrm{GF}(p) \mathrm{SL}_{2}(p)$-module. Then $V$ is indecomposable as a $\mathrm{GF}(p) T$-module. In particular, for all $0 \leq k \leq \operatorname{dim} V-1$, $\operatorname{dim}[V, T ; k] /[V, T ; k+1]=1$ and, if $\operatorname{dim}[V, T]=1$, then $V \cong \mathbf{V}_{1}$ has dimension 2 .

Proof. This is calculated using the description of the modules above.
Obviously, if $p$ is odd and $V$ is a faithful $\mathrm{SL}_{2}(p)$-module, then the centre of $\mathrm{SL}_{2}(p)$ negates $V$ and so a complement to a Sylow $p$-subgroup of $\mathrm{SL}_{2}(p)$ acts fixed-point-freely on any faithful module. The same is not true for the irreducible $\mathrm{PSL}_{2}(p)$-modules.

Lemma 4.5. Suppose that $L \cong \operatorname{SL}_{2}(p), T \in \operatorname{Syl}_{p}(L)$ and $H$ is a complement to $T$ in $N_{L}(T)$. Assume that $V$ is an irreducible d-dimensional $\operatorname{GF}(p) L$-module. If $C_{V}(H) \neq 0$, then $d$ is odd and either
(i) $d \leq p-2$, $\operatorname{dim} C_{V}(H)=1$ and $[V, T ;(d-1) / 2] /[V, T ;(d+1) / 2]$ is centralized by $H$; or
(ii) $d=p, \operatorname{dim} C_{V}(H)=3$ and $V /[V, T],[V, T ;(p-1) / 2] /[V, T ;(p+1) / 2]$ and $C_{V}(T)$ are centralized by $H$.
In particular, if $C_{V}(T)$ is centralized by $H$, then either $V$ is the trivial module or $\operatorname{dim} V=p$ and $\operatorname{dim} C_{V}(H)=3$.

Proof. Let $e=d-1$ and remember that $V=\mathbf{V}_{e}$. Take $\tau=\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ to be a generator of $T$ and $\delta=\left(\begin{array}{cc}\lambda & 0 \\ 0 & \lambda^{-1}\end{array}\right)$ where $\lambda \in \mathrm{GF}(p)$ has order $p-1$ to be a generator of $H$. Then we calculate

$$
[V, T ; k]=\left\langle x^{j} y^{e-j} \mid 0 \leq j \leq e-k\right\rangle
$$

We also calculate that $\delta$ acts as the scalar $\lambda^{e-k} \lambda^{-k}=\lambda^{e-2 k}$ on the quotient

$$
[V, T ; k] /[V, T ; k+1]=\left\langle x^{e-k} y^{k}+[V, T ; k+1]\right\rangle
$$

Hence $\tau$ centralizes $[V, T ; k] /[V, T ; k+1]$ if and only if either $k=e / 2$ or $e=p-1$ and $k=0$ or $p-1$. In particular, $e=d-1$ is even and this gives the result.

Assume that $\mathbb{K}$ has characteristic $p \geq 0$ (allowing $p=0$ for a moment) and that $\mathbf{V}_{d}$ is the $(d+1)$-dimensional $\mathbb{K} \mathrm{GL}_{2}(\mathbb{K})$-module of homogeneous polynomials of degree $d$. Define

$$
\begin{aligned}
\Omega: \mathbb{K}[x, y] \otimes \mathbb{K}[x, y] & \rightarrow \mathbb{K}[x, y] \otimes \mathbb{K}[x, y] \\
(f \otimes g) & \mapsto \frac{\partial f}{\partial x} \otimes \frac{\partial g}{\partial y}-\frac{\partial f}{\partial y} \otimes \frac{\partial g}{\partial x}
\end{aligned}
$$

Then $\Omega$ is $\mathbb{K}$-linear. Let $A=\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ and $B_{\theta}=\left(\begin{array}{cc}0 & -\theta \\ 1 & 0\end{array}\right)$ with $\theta \in \mathbb{K}^{\times}$be elements of $\mathrm{GL}_{2}(\mathbb{K})$. Then $\mathrm{GL}_{2}(\mathbb{K})=\left\langle A, B_{\theta} \mid \theta \in \mathbb{K}^{\times}\right\rangle$. We calculate

$$
\begin{aligned}
\left(x^{a} y^{b} \otimes x^{c} y^{d}\right) A \Omega= & \left((x+y)^{a} y^{b} \otimes(x+y)^{c} y^{d}\right) \Omega \\
= & a(x+y)^{a-1} y^{b} \otimes\left(c(x+y)^{c-1} y^{d}+d(x+y)^{c} y^{d-1}\right) \\
& -\left(a(x+y)^{a-1} y^{b}+b(x+y)^{a} y^{b-1}\right) \otimes c(x+y)^{c-1} y^{d} \\
= & a d(x+y)^{a-1} y^{b} \otimes(x+y)^{c} y^{d-1} \\
& -b c(x+y)^{a} y^{b-1} \otimes(x+y)^{c-1} y^{d} \\
= & \left(a d x^{a-1} y^{b} \otimes x^{c} y^{d-1}-b c x^{a} y^{b-1} \otimes x^{c-1} y^{d}\right) A \\
= & \left(x^{a} y^{b} \otimes x^{c} y^{d}\right) \Omega A .
\end{aligned}
$$

Hence $\Omega A=A \Omega$. Similarly, we calculate that $B_{\theta} \Omega=\theta \Omega B_{\theta}$ and so $C \Omega=(\operatorname{det} C) \Omega C$ for all $C \in \mathrm{GL}_{2}(\mathbb{K})$. In particular, $\Omega$ is a $\mathbb{K} \mathrm{SL}_{2}(\mathbb{K})$-module homomorphism. The multiplication map $\mu: \mathbb{K}[x, y] \otimes \mathbb{K}[x, y] \rightarrow \mathbb{K}[x, y]$ defined by $(f \otimes g) \mu \mapsto f g$ is also a $\mathbb{K S L}_{2}(\mathbb{K})$-module homomorphism. It follows that, for $r \geq 0$, the $r$-transvectant

$$
\Theta_{r}=\Omega^{r} \mu: \mathbb{K}[x, y] \otimes \mathbb{K}[x, y] \rightarrow \mathbb{K}[x, y]
$$

is also a $\mathbb{K} \mathrm{SL}_{2}(\mathbb{K})$-module homomorphism.
Suppose that $d$ and $e$ are natural numbers and observe that by restriction

$$
\Theta_{r}: \mathbf{V}_{d} \otimes \mathbf{V}_{e} \rightarrow \mathbf{V}_{d+e-2 r}
$$

Assume now that $\mathbb{K}$ has characteristic $p>0$. Then, for $\ell \leq p-1, \mathbf{V}_{\ell}$ is an irreducible $\mathbb{K} \mathrm{SL}_{2}(\mathbb{K})$-module. Therefore, if $d+e-2 r \leq p-1$, the restriction of $\Theta_{r}$ is either the zero map or is a surjection. In particular, if $r \leq e \leq d$ and $d+e \leq p-1$, then, as $\left(x^{d} \otimes y^{e}\right) \Theta_{r} \neq 0, \Theta_{r}$ restricts to a surjection. By counting dimensions, we conclude

$$
\mathbf{V}_{d} \otimes \mathbf{V}_{e} \cong \mathbf{V}_{d+e} \oplus \mathbf{V}_{d+e-2} \oplus \cdots \oplus \mathbf{V}_{d-e}
$$

which in characteristic 0 is known as the Clebsch-Gordan decomposition. Let $\iota$ be the $\mathbb{K} \mathrm{SL}_{2}(\mathbb{K})$-module endomorphism of $\mathbb{K}[x, y]$ which maps $f \otimes g$ to $g \otimes f$. Then $\Theta_{r} \iota=(-1)^{r} \Theta_{r}$ and, for a fixed natural number $d, \Lambda^{2}\left(\mathbf{V}_{d}\right)$ is the submodule of $\mathbf{V}_{d} \otimes \mathbf{V}_{d}$ negated by $\iota$ and $S^{2}\left(\mathbf{V}_{d}\right)$ is the submodule centralized by $\iota$. We have recreated the following

Proposition 4.6. Suppose that $d, e \in \mathbb{N}$ with $d \geq e$ and $d+e \leq p-1$. Then, as $\mathbb{K} \mathrm{SL}_{2}(\mathbb{K})$-modules,

$$
\mathbf{V}_{d} \otimes \mathbf{V}_{e} \cong \mathbf{V}_{d+e} \oplus \mathbf{V}_{d+e-2} \oplus \cdots \oplus \mathbf{V}_{d-e}
$$

Furthermore, if $2 d \leq p-1$,

$$
S^{2}\left(\mathbf{V}_{d}\right) \cong \mathbf{V}_{2 d} \oplus \mathbf{V}_{2 d-4} \oplus \cdots \oplus \mathbf{V}_{a}
$$

and

$$
\Lambda^{2}\left(\mathbf{V}_{d}\right)=\mathbf{V}_{2 d-2} \oplus \mathbf{V}_{2 d-6} \oplus \cdots \oplus \mathbf{V}_{b}
$$

where $a=2 d(\bmod 4)$ and $b=2 d-2(\bmod 4)$ with $a, b \leq 2$.
Notice that, if we take $d+1=(p-1) / 2$, then $\mathbf{V}_{d}$ is involved in $\Lambda^{2}\left(\mathbf{V}_{d}\right)$ if and only if $d \equiv 2(\bmod 4)$.

Lemma 4.7. We have

$$
\mathbf{V}_{p-3} \otimes \mathbf{V}_{p-3}=\mathbf{V}_{0} \oplus \mathbf{V}_{2} \oplus \mathbf{V}_{p-1} \oplus P(p-3) \oplus \cdots \oplus P(4)
$$

where, for $1 \leq j \leq p-1, P(j)$ is the projective cover of $\mathbf{V}_{j}$. In particular, $\left(\mathbf{V}_{p-3} \otimes\right.$ $\left.\mathbf{V}_{p-3}\right) / \operatorname{Rad}\left(\mathbf{V}_{p-3} \otimes \mathbf{V}_{p-3}\right)$ is the direct sum of all the irreducible $\operatorname{GF}(p) \operatorname{PSL}_{2}(p)$ modules.

Proof. It suffices to work over an algebraically closed field. From [17, Lemma 3.1 (ii)], we have

$$
\mathbf{V}_{p-3} \otimes \mathbf{V}_{p-3}=\mathbf{V}_{0} \oplus \mathbf{V}_{2} \oplus \mathbf{V}_{p-1} \oplus T(p+1) \oplus \cdots \oplus T(2 p-6)
$$

where, for an integer $j, T(j)$ is the tilting module associated to $j$. We know $\mathbf{V}_{p-1}$ has dimension $p$ and is a projective module (see [1]). Using [17, Lemma 3.1 (iii)]

$$
\mathbf{V}_{p-3} \otimes \mathbf{V}_{p-1}=\mathbf{V}_{p-1} \oplus T(p+1) \oplus \cdots \oplus T(2 p-4)
$$

Since $\mathbf{V}_{p-1}$ is projective, so is $\mathbf{V}_{p-3} \otimes \mathbf{V}_{p-1}$ and every direct summand of $\mathbf{V}_{p-3} \otimes$ $\mathbf{V}_{p-1}$ by [7, Lemma 1.5.2]. Hence the tilting modules $T(p+1), \ldots, T(2 p-6)$ are projective $\operatorname{GF}(p) \mathrm{SL}_{2}(p)$-modules. Now, the discussion before [17, Lemma 3.1] reveals that for $p+1 \leq j \leq 2 p-2, T(j)$ has a quotient $\mathbf{V}_{2(p-1)-j}$ and dimension $2 p$. We conclude that $T(j)$ is the projective cover $P(2(p-1)-j)$ of $\mathbf{V}_{2(p-1)-j}$. With this notation, we have

$$
\mathbf{V}_{p-3} \otimes \mathbf{V}_{p-3}=\mathbf{V}_{0} \oplus \mathbf{V}_{2} \oplus \mathbf{V}_{p-1} \oplus P(p-3) \oplus \cdots \oplus P(4)
$$

In particular, we note that every irreducible $\mathrm{GF}(p) \mathrm{PSL}_{2}(p)$-module appears exactly once as a quotient of $\left(\mathbf{V}_{p-3} \otimes \mathbf{V}_{p-3}\right) / \operatorname{Rad}\left(\mathbf{V}_{p-3} \otimes \mathbf{V}_{p-3}\right)$. This proves the claim.

We recall from [1, pages 15 and 48] that $P(0)$ has dimension $p$ and is a uniserial $\mathrm{GF}(p) \mathrm{PSL}_{2}(p)$-module with composition factors $\mathbf{V}_{0}, \mathbf{V}_{p-3}$ and $\mathbf{V}_{0}$. In particular, there is a unique indecomposable $\mathrm{GF}(p) \mathrm{PSL}_{2}(p)$-module with socle of dimension 1 and quotient $\mathbf{V}_{p-3}$.

LEMMA 4.8. Suppose that $L \cong \operatorname{PSL}_{2}(p)$ and $W$ is the unique indecomposable $\mathrm{GF}(p) L$-module with socle of dimension 1 and quotient $\mathbf{V}_{p-3}$. Then $\Lambda^{2}(W)$ has a submodule $U \leq \operatorname{Rad}\left(\Lambda^{2}(W)\right)$ with $U \cong \mathbf{V}_{p-3}$ and $\Lambda^{2}(W) / U \cong \Lambda^{2}\left(\mathbf{V}_{p-3}\right)$.

Proof. Let $R$ be the socle of $W$. Then $W \otimes W$ has a submodule

$$
U^{*}=\langle r \otimes w, w \otimes r \mid r \in R, w \in W\rangle
$$

Plainly $W / U^{*} \cong \mathbf{V}_{p-3} \otimes \mathbf{V}_{p-3}$. Set $U=\langle w \otimes r-r \otimes w \mid r \in R, w \in W\rangle$. Then $U \cong W / R=\mathbf{V}_{p-3}$ and $\Lambda^{2}(W) / U \cong \Lambda^{2}\left(\mathbf{V}_{p-3}\right)$. Hence we only need to show that $U \leq \operatorname{Rad}\left(\Lambda^{2}(W)\right)$. Since $\Lambda^{2}(W)$ is a direct summand of $W \otimes W$, we have $\operatorname{Rad}\left(\Lambda^{2}(W)\right)=\operatorname{Rad}(W \otimes W) \cap \Lambda^{2}(W)$. In particular, if $U \not \leq \operatorname{Rad}\left(\Lambda^{2}(W)\right)$, then $U$ is a direct summand of $W \otimes W$. So suppose that this is the case. Then $W \otimes W \cong \mathbf{V}_{p-3} \oplus\left(\mathbf{V}_{p-3} \otimes \mathbf{V}_{p-3}\right)$. Let $T \in \operatorname{Syl}_{p}(L)$. Then using Lemma 4.7, as a $\mathrm{GF}(p) T$-module $\mathbf{V}_{p-3} \oplus\left(\mathbf{V}_{p-3} \otimes \mathbf{V}_{p-3}\right)$ is a sum of indecomposable modules $\mathbf{V}_{0}$ of dimension 1, $\mathbf{V}_{2}$ of dimension $3, \mathbf{V}_{p-3}$ of dimension $p-2$ and a number of free modules of dimension $p$. On the other hand, as $W$ is indecomposable with socle $\mathbf{V}_{0}$ and quotient $\mathbf{V}_{p-3}$, using Lemma 4.4 and [ $\mathbf{7}$, Corollary 3.6.10] we have that $W$ restricted to $T$ is indecomposable of dimension $p-1$. It follows that, as a GF $(p) T$ module, $W \otimes W$ is a direct sum of a trivial module and a free module (for example use [17, Lemma 3.1 (ii)] to write down $\mathbf{V}_{p-2} \otimes \mathbf{V}_{p-2}$ and then restrict to $T$ ). Since the two structures are incompatible, we conclude that $U \leq \operatorname{Rad}\left(\Lambda^{2}(W)\right)$.

We can now establish the technical point that we require.
Lemma 4.9. Suppose that $L \cong \operatorname{PSL}_{2}(p), T \in \operatorname{Syl}_{p}(L), H \leq N_{L}(T)$ is a complement to $T$ and $W$ is the unique indecomposable $\mathrm{GF}(p) L$-module with socle of dimension 1 and quotient $\mathbf{V}_{p-3}$. Assume that $\theta: W \times W \rightarrow \mathbf{V}_{p-3}$ is a surjective alternating L-invariant bilinear map. Let $u \in W \backslash[W, T]$ and $w \in[W, T ; p-3] \backslash C_{W}(T)$ be such that $\langle w\rangle$ and $\langle u\rangle$ are $H$-invariant. Then $(u, w) \theta \neq 0$.

Proof. Since $\theta: W \times W \rightarrow \mathbf{V}_{p-3}$ is a surjective $L$-invariant bilinear map, there is a unique surjective $\operatorname{GF}(p) L$-module homomorphism $\widetilde{\theta}$ from $\Lambda^{2}(W)$ to $\mathbf{V}_{p-3}$. Using Lemma 4.8, we have that $\widetilde{\theta}$ determines a surjective homomorphisms $\theta^{*}$ from $\Lambda^{2}\left(\mathbf{V}_{p-3}\right)$ to $\mathbf{V}_{p-3}$. By Lemma 4.7, $\Lambda^{2}\left(\mathbf{V}_{p-3}\right)$ either has no quotient isomorphic to $\mathbf{V}_{p-3}$, which is against our assumption that $\theta$ is surjective, or $\operatorname{ker} \theta^{*}$ is the unique maximal submodule of $\Lambda^{2}\left(\mathbf{V}_{p-3}\right)$ which has quotient $\mathbf{V}_{p-3}$. Now the $(p-$ 3)/2-transvectant $\Theta_{(p-3) / 2}$ restricted to $\Lambda^{2}\left(\mathbf{V}_{p-3}\right)$ also has image in $\mathbf{V}_{p-3}$. Hence $\operatorname{ker} \theta^{*}=\operatorname{ker} \Theta_{(p-3) / 2}$.

We may take $u=x^{p-3}$ and $w=y^{p-3}$ in $\mathbf{V}_{p-3}$. We calculate $(u \otimes w) \Theta_{(p-3) / 2}=$ $\binom{(p-3)}{(p-3) / 2}^{2}(x y)^{(p-3) / 2} \neq 0$. Therefore $(u, w) \notin \operatorname{ker} \Theta_{(p-3) / 2}=\operatorname{ker} \theta^{*}$. This shows that $(u, w) \theta \neq 0$.

In the next lemma we are interested in modules for $L=\operatorname{Sym}(p)$ defined in characteristic $p$. The notation $S^{\lambda}$ denotes the Specht module for $L$ corresponding to the partition $\lambda$ of $p$. The module $D^{\lambda}$ is the unique irreducible quotient of $S^{\lambda}$ (see [33]). Thus $S^{p-1,1}$ is a characteristic $p$ representation of $L$ of dimension $p-1$ and
can be identified with the submodule of the natural $\operatorname{GF}(p) L$-permutation module $\left\langle v_{i} \mid 1 \leq i \leq p\right\rangle$ which is the kernel of the augmentation map $\sum_{i=1}^{p} \lambda_{i} v_{i} \mapsto \sum_{i=1}^{p} \lambda_{i}$. In this case, $D^{p-1,1}=S^{p-1,1} /\left\langle\sum_{i=1}^{p} v_{i}\right\rangle$ has dimension $p-2$. The result we shall need is as follows.

Lemma 4.10. Suppose that $p \geq 5$ is a prime, $L \cong \operatorname{Sym}(p)$ and $V=S^{p-1,1}$ considered as a $\mathrm{GF}(p) L$-module. Then $\Lambda^{2}(V)=S^{p-2,1^{2}}$ has irreducible composition factors $D^{p-1,1}$ and $D^{p-2,1^{2}}$ both with multiplicity 1 and, furthermore, $\Lambda^{2}(V)$ has no quotient of dimension 1 or isomorphic to $D^{p-1,1}$.

Proof. For prime $p \geq 11$, the isomorphism type and composition factors of $\Lambda^{2} V$ are explicitly given in $[\mathbf{3 8}$, Section 2$]$. The fact that $D^{p-1,1}$ is not a quotient of $\Lambda^{2}(V)$ follows from [33, Corollary 12.2]. For $p=5$ and $p=7$, we have checked the assertion by computer (see Subsection C.2).

Lemma 4.11. Assume that $p \geq 5$ is a prime, $X=\operatorname{Sym}(p), Y=X^{\prime}=$ $\operatorname{Alt}(p)$ and $V=D^{p-1,1}$ is the $p-2$-dimensional module. Let $W=\left.V\right|_{Y}$. Then $\operatorname{dim} \mathrm{H}^{1}(Y, W)=1$.

Proof. Let $\tau=(1,2,3)$ and $\sigma=(3, \ldots, p)$. Then $H=\langle\tau, \sigma\rangle$ acts transitively on $\Omega=\{1, \ldots, p\}$ and, as $p$ is a prime, it is primitive. Since $H$ contains a 3 -cycle, $Y=H$ by Jordan's Theorem [32, II.4.5]. If $U$ is a $\operatorname{GF}(p) Y$-module with $W$ of codimension 2 and $[U, Y]=W$, then $\operatorname{dim} C_{U}(\tau)=2+\operatorname{dim} C_{W}(\tau)=2+p-4=p-2$ and $\operatorname{dim} C_{U}(\sigma)=2+\operatorname{dim} C_{W}(\sigma)=2+1=3$. Hence $\operatorname{dim} C_{U}(H) \geq 1$ and this proves $\operatorname{dim} \mathrm{H}^{1}(Y, W) \leq 1$. Since the natural $p$-point permutation module has a quotient $T$ with $[T, Y]=W$ and $C_{T}(Y)=0$, we have $\operatorname{dim} \mathrm{H}^{1}(Y, W)=1$.

## 5. A primer on fusion systems

We assume some basic familiarity with fusion systems and recommend the references $[\mathbf{5}, \mathbf{1 6}]$ as introductory texts. We follow the notation from these sources. We start by flying over the standard definitions and at the same time introduce some of the standard terminology from $[\mathbf{5}, \mathbf{1 6}]$.

Definition 5.1. For a finite group $G$ and subgroups $H, K \leq G$, define
$\operatorname{Hom}_{G}(H, K)=\left\{\varphi \in \operatorname{Hom}(H, K) \mid \varphi=c_{g}\right.$ for some $g \in G$ such that $\left.H^{g} \leq K\right\}$ and set $\operatorname{Aut}_{G}(H)=\operatorname{Hom}_{G}(H, H) \cong N_{G}(H) / C_{G}(H)$.

More generally, if $H, K \leq G$, we define $\operatorname{Aut}_{K}(H)=\left\{c_{k} \mid k \in K \cap N_{G}(H)\right\}$ to be the group of automorphisms of $H$ induced by conjugation by elements of $K$ which normalize $H$. Visibly $\operatorname{Aut}_{K}(H) \leq \operatorname{Aut}_{G}(H) \leq \operatorname{Aut}(H)$. Similarly, if $K \leq L$ and $A \leq \operatorname{Aut}(L)$, we write $\operatorname{Aut}_{A}(K)$ to represent the group of automorphisms of $K$ generated by the restriction of automorphisms in $N_{A}(K)=\{\gamma \in A \mid K \gamma=K\}$. In this case $\operatorname{Out}_{A}(K)=\operatorname{Aut}_{A}(K) \operatorname{Inn}(K) / \operatorname{Inn}(K)$. For groups $P$ and $Q, \operatorname{Inj}(P, Q)$ is the set of injective group homomorphisms from $P$ to $Q$.

Definition 5.2. A fusion system on a $p$-group $S$ is a category $\mathcal{F}$, with objects the set of all subgroups of $S$, and morphisms $\operatorname{Mor}_{\mathcal{F}}(P, Q)$ between objects $P$ and $Q$ which satisfy the following two properties:
(i) $\operatorname{Hom}_{S}(P, Q) \subseteq \operatorname{Mor}_{\mathcal{F}}(P, Q) \subseteq \operatorname{Inj}(P, Q)$; and
(ii) each $\varphi \in \operatorname{Mor}_{\mathcal{F}}(P, Q)$ is the composite of an $\mathcal{F}$-isomorphism followed by an inclusion.

If $\mathcal{F}$ is a fusion system and $P, Q \leq S$, then we write $\operatorname{Hom}_{\mathcal{F}}(P, Q)=\operatorname{Mor}_{\mathcal{F}}(P, Q)$ and $\operatorname{Aut}_{\mathcal{F}}(P)=\operatorname{Mor}_{\mathcal{F}}(P, P)$.

Definition 5.3. Suppose that $\mathcal{F}$ is a fusion system on a finite $p$-group $S$ and $P \leq S$. Then
(i) the $\mathcal{F}$-conjugacy class of $P$, is $P^{\mathcal{F}}=\left\{P \alpha \mid \alpha \in \operatorname{Hom}_{\mathcal{F}}(P, S)\right\}$;
(ii) $P$ is strongly $\mathcal{F}$-closed if and only if $Q^{\mathcal{F}} \subseteq P$ for all $Q \leq P$;
(iii) for $R \in P^{\mathcal{F}}, \alpha \in \operatorname{Hom}_{\mathcal{F}}(R, P), \alpha^{*}$ is the isomorphism between $\operatorname{Aut}_{\mathcal{F}}(R)$ and $\operatorname{Aut}_{\mathcal{F}}(P)$ defined by $\gamma \mapsto \alpha^{-1} \gamma \alpha$;
(iv) $P$ is fully $\mathcal{F}$-normalized if and only if $\left|N_{S}(P)\right| \geq\left|N_{S}(R)\right|$ for all $R \in P^{\mathcal{F}}$;
(v) $P$ is fully $\mathcal{F}$-centralized if and only if $\left|C_{S}(P)\right| \geq\left|C_{S}(R)\right|$ for all $R \in P^{\mathcal{F}}$;
(vi) $P$ is $S$-centric if and only if $C_{S}(P)=Z(P)$, and $P$ is $\mathcal{F}$-centric if and only if $R$ is $S$-centric for all $R \in P^{\mathcal{F}}$;
(vii) if $R \in P^{\mathcal{F}}$ and $\alpha \in \operatorname{Hom}_{\mathcal{F}}(R, P)$,

$$
N_{\alpha}=\left\{g \in N_{S}(R) \mid \alpha^{-1} c_{g} \alpha \in \operatorname{Aut}_{S}(P)\right\}
$$

is the $\alpha$-extension control subgroup of $S$;
(viii) $P$ is $\mathcal{F}$-receptive if and only if for all $R \in P^{\mathcal{F}}$ and $\alpha \in \operatorname{Hom}_{\mathcal{F}}(R, P)$, there exists $\widetilde{\alpha} \in \operatorname{Hom}_{\mathcal{F}}\left(N_{\alpha}, S\right)$ such that $\left.\widetilde{\alpha}\right|_{R}=\alpha ;$
(ix) $P$ is fully $\mathcal{F}$-automized if and only if $\operatorname{Aut}_{S}(P) \in \operatorname{Syl}_{p}\left(\operatorname{Aut}_{\mathcal{F}}(P)\right)$;
(x) $P$ is $\mathcal{F}$-saturated provided there exists $R \in P^{\mathcal{F}}$ such that $R$ is simultaneously
(a) fully $\mathcal{F}$-automized; and
(b) $\mathcal{F}$-receptive.
(xi) $\mathcal{F}$ is saturated if every subgroup of $S$ is $\mathcal{F}$-saturated.

If $\mathcal{F}$ is a fusion system and $X$ is a set of morphisms in $\mathcal{F},\langle X\rangle$ is the intersection of all the fusion systems on $S$ which contain $X$. We say that $\langle X\rangle$ is the fusion system generated by $X$. Obviously $\langle X\rangle$ is contained in $\mathcal{F}$.

In our arguments, an important role is played by normalizer fusion systems. Suppose that $\mathcal{F}$ is a saturated fusion system on $S, T \leq S$ and $K \leq \operatorname{Aut}(T)$. Then $N_{\mathcal{F}}^{K}(T)$ is the fusion system on $N_{S}(T)$ with, for $P, Q \leq N_{S}(T), \operatorname{Hom}_{N_{\mathcal{F}}(T)}(P, Q)$ consisting of morphisms $\alpha \in \operatorname{Hom}_{\mathcal{F}}(P, Q)$ such that there is $\widetilde{\alpha} \in \operatorname{Hom}_{\mathcal{F}}(P T, Q T)$ with $T \widetilde{\alpha}=T,\left.\widetilde{\alpha}\right|_{T} \in K$ and $\alpha=\left.\widetilde{\alpha}\right|_{P}$. Importantly, if $T$ is fully $\mathcal{F}$-normalized, then $N_{\mathcal{F}}^{K}(T)$ is saturated [5, Theorem I.5.5]. The two extreme cases $K=\operatorname{Aut}(T)$, and $K=1$ are of main interest. In the former case we have the $\mathcal{F}$-normalizer of $T$ and we write $N_{\mathcal{F}}(T)=N_{\mathcal{F}}^{\operatorname{Aut}(T)}(T)$ whereas in the latter we have the $\mathcal{F}$-centralizer of $T$ and we define $C_{\mathcal{F}}(T)=N_{\mathcal{F}}^{1}(T)$.

A subgroup $T \leq S$ is normal in $\mathcal{F}$ if and only if $\mathcal{F}=N_{\mathcal{F}}(T)$. The subgroup $O_{p}(\mathcal{F})$ is the product of all subgroups $T \leq S$ such that $T$ is normal in $\mathcal{F}$. It follows that $\mathcal{F}=N_{\mathcal{F}}\left(O_{p}(\mathcal{F})\right)$. A saturated fusion system $\mathcal{F}$ is constrained provided $O_{p}(\mathcal{F})$ is $\mathcal{F}$-centric.

Theorem 5.4 (The Model Theorem). Let $\mathcal{F}$ be a constrained fusion system on $S$. Then there exists a finite group $G$ with $O_{p^{\prime}}(G)=1, S \in \operatorname{Syl}_{p}(G), \mathcal{F}=\mathcal{F}_{S}(G)$ and $C_{G}\left(O_{p}(G)\right) \leq O_{p}(G)$.

Proof. See [5, Theorem III.5.10].
We now gather some elementary consequences of the definitions above. We have the following well-known fact and, as we shall use it several times, we provide the proof.

Lemma 5.5. Suppose that $\mathcal{F}$ is a fusion system on $S$ and $P \leq S$.
(i) If $\alpha \in N_{\operatorname{Aut}_{\mathcal{F}}(P)}\left(\operatorname{Aut}_{S}(P)\right)$, then $N_{\alpha}=N_{S}(P)$.
(ii) If $P$ is $\mathcal{F}$-receptive, then

$$
N_{\operatorname{Aut}_{\mathcal{F}}(P)}\left(\operatorname{Aut}_{S}(P)\right)=\left\{\left.\alpha\right|_{P} \mid \alpha \in N_{\operatorname{Aut}_{\mathcal{F}}\left(N_{S}(P)\right)}(P)\right\}
$$

Proof. Part (i) is just the definition of $N_{\alpha}$.
Suppose that $\alpha \in N_{\operatorname{Aut}_{\mathcal{F}}(P)}\left(\operatorname{Aut}_{S}(P)\right)$. Then $N_{\alpha}=N_{S}(P)$ by (i). Since $P$ is $\mathcal{F}$-receptive, $\alpha=\left.\widetilde{\alpha}\right|_{P}$ where $\widetilde{\alpha} \in \operatorname{Hom}_{\mathcal{F}}\left(N_{S}(P), S\right)$. As $P \widetilde{\alpha}=P \alpha=P$, we know $N_{S}(P) \widetilde{\alpha}=N_{S}(P \widetilde{\alpha})=N_{S}(P)$. Hence $\widetilde{\alpha} \in N_{\operatorname{Aut}_{\mathcal{F}}\left(N_{S}(P)\right)}(P)$ and so

$$
N_{\operatorname{Aut}_{\mathcal{F}}(P)}\left(\operatorname{Aut}_{S}(P)\right) \subseteq\left\{\left.\alpha\right|_{P} \mid \alpha \in N_{\operatorname{Aut}_{\mathcal{F}}\left(N_{S}(P)\right)}(P)\right\}
$$

Conversely, assume $g \in N_{S}(P)$. Then $c_{g} \in \operatorname{Aut}_{S}(P)$ and, for $\beta \in N_{\operatorname{Aut}_{\mathcal{F}}\left(N_{S}(P)\right)}(P)$, we have $\left.\left(\left.\beta\right|_{P}\right)^{-1} c_{g} \beta\right|_{P}=c_{g \beta} \in \operatorname{Aut}_{S}(P)$. Hence $\left.\beta\right|_{P} \in N_{\operatorname{Aut}_{\mathcal{F}}(P)}\left(\operatorname{Aut}_{S}(P)\right)$. This proves (ii).

Lemma 5.6. Suppose that $\mathcal{F}$ is a fusion system on $S, T \leq S$ and $\mathcal{K}=N_{\mathcal{K}}(T)$ is a subfusion system of $\mathcal{F}$ on $N_{S}(T)$. Assume that $R \leq T$ is Aut $\mathcal{K}_{\mathcal{K}}(T)$-invariant. Then $\mathcal{K} \subseteq N_{\mathcal{F}}(R)$. In particular, if $R \leq T$ is Aut $_{\mathcal{F}}(T)$-invariant, then $N_{\mathcal{F}}(T) \subseteq$ $N_{\mathcal{F}}(R)$.

Proof. Suppose that $X, Y \leq N_{S}(T)$ and $\theta \in \operatorname{Hom}_{\mathcal{K}}(X, Y)$. Then, as $R$ is $\operatorname{Aut}_{\mathcal{K}}(T)$-invariant, it is also $\operatorname{Aut}_{S}(T)$-invariant and thus $R$ is normalized by $N_{S}(T)$. In particular, $X, Y \leq N_{S}(T) \leq N_{S}(R)$ and so $X$ and $Y$ are objects in $N_{\mathcal{F}}(R)$. Since $\theta \in \operatorname{Hom}_{\mathcal{K}}(X, Y)$ and $\mathcal{K}=N_{\mathcal{K}}(T)$, the morphism $\theta$ extends to $\hat{\theta} \in \operatorname{Hom}_{\mathcal{K}}(X T, Y T)$ so that $T \hat{\theta}=T$. Thus $\left.\hat{\theta}\right|_{T} \in \operatorname{Aut}_{\mathcal{K}}(T)$ and $R \hat{\theta}=R$ because $R$
is $\operatorname{Aut}_{\mathcal{K}}(T)$-invariant. As $\hat{\theta}$ extends $\theta,\left.\hat{\theta}\right|_{X R} \in \operatorname{Hom}_{\mathcal{F}}(X R, Y R)$ also extends $\theta$ and this means that $\theta \in \operatorname{Hom}_{N_{\mathcal{F}}(R)}(X, Y)$. Hence $\operatorname{Hom}_{\mathcal{K}}(X, Y) \subseteq \operatorname{Hom}_{N_{\mathcal{F}}(R)}(X, Y)$ and this proves the main statement of the lemma.

Taking $\mathcal{K}=N_{\mathcal{F}}(T)$ and noting $\operatorname{Aut}_{N_{\mathcal{F}}(T)}(T)=\operatorname{Aut}_{\mathcal{F}}(T)$, yields the remaining statement.

If $Q$ is normal in $\mathcal{F}$, then the factor system $\mathcal{F} / Q$ has objects $\{T / Q \mid Q \leq S\}$ and, for $Q \leq T, R \leq S$, morphisms $\operatorname{Hom}_{\mathcal{F} / Q}(T / Q, R / Q)=\left\{\bar{\phi} \mid \phi \in \operatorname{Hom}_{\mathcal{F}}(T, R)\right\}$ where $(t Q) \bar{\phi}=t \phi Q$ for $\phi \in \operatorname{Hom}_{\mathcal{F}}(T, R)$.

Lemma 5.7. If $\mathcal{F}$ is saturated and $Q$ is normal in $\mathcal{F}$, then $\mathcal{F} / Q$ is saturated
Proof. This is [5, Lemma II.5.5].
Definition 5.8. Suppose that $\mathcal{F}$ is a fusion system. A subgroup $P$ of $S$ is $\mathcal{F}$-essential if $P \neq S, P$ is $\mathcal{F}$-centric, fully $\mathcal{F}$-normalized and $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly $p$-embedded subgroup. We write $\mathcal{E}_{\mathcal{F}}$ to denote the set of $\mathcal{F}$-essential subgroups of $\mathcal{F}$.

Note that if $E \in \mathcal{E}_{\mathcal{F}}$ then $O_{p}\left(\operatorname{Out}_{\mathcal{F}}(E)\right)=1$, that is, $O_{p}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)=\operatorname{Inn}(E)$. This fact will be used several times.

The main tool for classifying saturated fusion systems is provided by the following lemma.

THEOREM 5.9 (Alperin-Goldschmidt). If $\mathcal{F}$ is a saturated fusion system on the p-group $S$, then

$$
\mathcal{F}=\left\langle\operatorname{Aut}_{\mathcal{F}}(S), \operatorname{Aut}_{\mathcal{F}}(E) \mid E \in \mathcal{E}_{\mathcal{F}}\right\rangle
$$

Proof. See [5, Theorem I.3.5].
Lemma 5.10. Suppose that $\mathcal{F}$ is a saturated fusion system on $S$ and $E$ is an $\mathcal{F}$-essential subgroup. Then $O_{p}(\mathcal{F}) \leq E$ and $O_{p}(\mathcal{F})$ is Aut $\mathcal{F}(E)$-invariant.

Proof. Since $N_{O_{p}(\mathcal{F})}(E)$ is $\operatorname{Aut}_{\mathcal{F}}(E)$-invariant and $O_{p}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)=\operatorname{Inn}(E)$, we have $N_{O_{p}(\mathcal{F})}(E) \leq E$ by Lemma 2.6. Hence $O_{p}(\mathcal{F}) \leq E$ and so is normal in $E$ and $\operatorname{Aut}_{\mathcal{F}}(E)$-invariant.

We will also meet the fusion subsystems $O^{p}(\mathcal{F})$ and $O^{p^{\prime}}(\mathcal{F})$. We define the focal and hyperfocal subgroups as follows

$$
\left.\operatorname{foc}(\mathcal{F})=\langle[g, \alpha]| g \in Q \leq S \text { and } \alpha \in \operatorname{Aut}_{\mathcal{F}}(Q)\right\rangle
$$

$$
\left.\operatorname{hyp}(\mathcal{F})=\langle[g, \alpha]| g \in Q \leq S \text { and } \alpha \in O^{p}\left(\operatorname{Aut}_{\mathcal{F}}(Q)\right)\right\rangle
$$

where $[g, \alpha]=g^{-1}(g) \alpha$. The subfusion system $O^{p}(\mathcal{F})$ is a saturated fusion system on $\operatorname{hyp}(\mathcal{F})$ defined as follows:

$$
O^{p}(\mathcal{F})=\left\langle\operatorname{Inn}(\operatorname{hyp}(\mathcal{F})), O^{p}\left(\operatorname{Aut}_{\mathcal{F}}(Q)\right) \mid Q \leq \operatorname{hyp}(\mathcal{F})\right\rangle
$$

Lemma 5.11. Let $\mathcal{F}$ be a saturated fusion system on $S$. The following hold:
(i) $\operatorname{foc}(\mathcal{F})=\langle[g, \alpha]| g \in Q \leq S, Q$ is $\mathcal{F}$-essential or $\left.Q=S, \alpha \in \operatorname{Aut}_{\mathcal{F}}(Q)\right\rangle$.
(ii) $O^{p}(\mathcal{F})=\mathcal{F}$ if and only if $\operatorname{foc}(\mathcal{F})=S$ if and only if $\operatorname{hyp}(\mathcal{F})=S$.

Proof. This follows from Theorem 5.9 and [5, Corollary I.7.5].

The subfusion system $O^{p^{\prime}}(\mathcal{F})$ is more complicated to define, so we just settle for saying that it is the unique saturated subfusion system on $S$ minimal subject to containing $O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(Q)\right)$ for all $Q \leq S$ (see [5, Definition I.7.3]).

Definition 5.12. A saturated fusion system $\mathcal{F}$ is called reduced if and only if $O_{p}(\mathcal{F})=1$ and $\mathcal{F}=O^{p}(\mathcal{F})=O^{p^{\prime}}(\mathcal{F})$.

Normal subfusion systems are defined in [5, Definition I.6.1] and a simple saturated fusion system is a fusion system which has no proper normal subfusion system. Simple fusion systems are reduced as $O^{p}(\mathcal{F}), O^{p^{\prime}}(\mathcal{F})$ and $\mathcal{F}_{O_{p}(\mathcal{F})}\left(O_{p}(\mathcal{F})\right)$ are normal subsystems of $\mathcal{F}$.

Lemma 5.13. Suppose that $\mathcal{F}$ is a saturated fusion system on a p-group $S$. Assume that each $P \in \mathcal{E}_{\mathcal{F}}$ is minimal among all $\mathcal{F}$-centric subgroups. For each $P \in \mathcal{E}_{\mathcal{F}}$ define

$$
\left.\operatorname{Aut}_{\mathcal{F}}^{(P)}(S)=\left\langle\alpha \in \operatorname{Aut}_{\mathcal{F}}(S)\right| P \alpha=P,\left.\alpha\right|_{P} \in O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(P)\right)\right\rangle
$$

Then $O^{p^{\prime}}(\mathcal{F})=\mathcal{F}$ if and only if $\operatorname{Aut}_{\mathcal{F}}(S)=\left\langle\operatorname{Inn}(S), \operatorname{Aut}_{\mathcal{F}}^{(P)}(S) \mid P \in \mathcal{E}_{\mathcal{F}}\right\rangle$.
Proof. This is [42, Lemma 1.4].
In the next lemma, the containment $\operatorname{Aut}_{\mathcal{F}}(E) \subseteq \mathcal{G}$ means that $E$ is an object in $\mathcal{G}$ and that $\operatorname{Aut}_{\mathcal{F}}(E)=\operatorname{Aut}_{\mathcal{G}}(E)$. We will use this notation from here on.

Lemma 5.14. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are saturated fusion systems with $\mathcal{G} \subseteq \mathcal{F}$. Assume that $E$ is $\mathcal{F}$-essential and $\operatorname{Aut}_{\mathcal{F}}(E) \subseteq \mathcal{G}$. Then $E$ is $\mathcal{G}$-essential.

Proof. We have $\operatorname{Aut}_{\mathcal{F}}(E)=\operatorname{Aut}_{\mathcal{G}}(E)$ and $E^{\mathcal{G}} \subseteq E^{\mathcal{F}}$. Therefore, as $E$ is fully $\mathcal{F}$-normalized and $\mathcal{F}$-centric it is also fully $\mathcal{G}$-normalized and $\mathcal{G}$-centric. Since $\operatorname{Out}_{\mathcal{F}}(E)=\operatorname{Out}_{\mathcal{G}}(E)$, we know $\operatorname{Out}_{\mathcal{G}}(E)$ has a strongly p-embedded subgroup. Thus $E$ is $\mathcal{G}$-essential.

Recall that, following [5, Proposition I.3.3], if $P<S$ is fully $\mathcal{F}$-normalized, $H_{\mathcal{F}}(P)$ is defined to be the subgroup of $\operatorname{Aut}_{\mathcal{F}}(P)$ which is generated by those morphisms of $P$ which extend to $\mathcal{F}$-isomorphisms between strictly larger subgroups of $S$. Then the statement in [5, Proposition I.3.3], includes the fact that, if $P$ is $\mathcal{F}$-essential, then $H_{\mathcal{F}}(P) / \operatorname{Inn}(P)$ is strongly $p$-embedded in $\operatorname{Aut}_{\mathcal{F}}(P)$.

The next result is widely used to show that certain subgroups are not $\mathcal{F}$ essential.

Lemma 5.15. Suppose that $\mathcal{F}$ is a saturated fusion system on $S$ and $Q \leq S$. Assume that $Q_{s}<Q_{s-1}<\cdots<Q_{0}=Q$ are $\operatorname{Aut}_{\mathcal{F}}(Q)$-invariant with $Q_{s} \leq \Phi(Q)$. If $A \leq \operatorname{Aut}_{S}(Q)$ and $\left[Q_{i}, A\right] \leq Q_{i+1}$ for $0 \leq i \leq s-1$, then $A \leq O_{p}\left(\operatorname{Aut}_{\mathcal{F}}(Q)\right)$. In particular, if $A \not \leq \operatorname{Inn}(Q)$, then $Q$ is not $\mathcal{F}$-essential.

Proof. Set $B=\left\langle A^{\operatorname{Aut}_{\mathcal{F}}(Q)}\right\rangle$. Assume that $\beta \in B$ has order coprime to $p$. Then $[Q, \beta ; s] \leq Q_{s}$ and so $\left[\mathbf{2 4}\right.$, Theorem 5.3.6] implies that $[Q, \beta] \leq Q_{s} \leq \Phi(Q)$. Thus [24, Theorem 5.1.4] implies $\beta=1$ and so $B$ is a $p$-group. Therefore $A \leq$ $B \leq O_{p}\left(\operatorname{Aut}_{\mathcal{F}}(Q)\right)$ and this proves the first claim. Since $O_{p}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)=\operatorname{Inn}(E)$ for $E \in \mathcal{E}_{\mathcal{F}}$, if $A \not \leq \operatorname{Inn}(Q)$, then $Q$ is not $\mathcal{F}$-essential.

If $Q$ has a series of subgroups as in Lemma 5.15 and $P \leq S$ centralizes all quotients $Q_{i} / Q_{i+1}$, then we say that $P$ stabilizes the series $Q_{s}<Q_{s-1}<\cdots<$ $Q_{0}=Q$ and we conclude that, if $P \not \leq Q$, then $Q$ is not $\mathcal{F}$-essential.

The next lemma relies on Proposition 2.12 which provides a list of all nonabelian simple groups with a strongly $p$-embedded subgroup containing an elementary abelian subgroup of order $p^{2}$.

Lemma 5.16. Suppose that $\mathcal{F}$ is a saturated fusion system on $S$ and $E$ is $\mathcal{F}$ essential. Then $|E / \Phi(E)| \geq\left|N_{S}(E) / E\right|^{2}$.

Proof. This is [50, Proposition 4.6 (4)].
We now employ work of Henke [30].
Proposition 5.17. Suppose that $\mathcal{F}$ is a saturated fusion system on the $p$ group $S$ and $E$ is an $\mathcal{F}$-essential subgroup. Assume that $U$ and $W$ are $\operatorname{Aut}_{\mathcal{F}}(E)$ invariant subgroups of $E$ with $V=U / W$ centralized by $\operatorname{Inn}(E)$ and non-trivial as $a \operatorname{GF}(p) O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)$-module. If $\left|\left[V, \operatorname{Out}_{S}(E)\right]\right| \leq p$ or $\left|V: C_{V}\left(\operatorname{Out}_{S}(E)\right)\right| \leq p$, then

$$
O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right) / C_{O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)}(V) \cong \mathrm{SL}_{2}(p)
$$

and $V / C_{V}\left(O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)\right)$ is the natural $\operatorname{GF}(p) \operatorname{SL}_{2}(p)$-module.
Proof. The hypothesis that $V$ is a non-trivial $\operatorname{GF}(p) O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)$-module implies that $\mathrm{Aut}_{S}(E)$ does not centralize $V$. Hence

$$
\left|\left[V, \operatorname{Out}_{S}(E)\right]\right|=p \leq\left|\operatorname{Out}_{S}(E) / C_{\operatorname{Out}_{S}(E)}(V)\right|
$$

or

$$
\left|V: C_{V}\left(\operatorname{Out}_{S}(E)\right)\right|=p \leq\left|\operatorname{Out}_{S}(E) / C_{\operatorname{Out}_{S}(E)}(V)\right|
$$

This, by definition, means that either the dual of $V$ or $V$ is a failure of factorisation module for the group $O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right) / C_{O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)}(V)$. Using [30, Theorem 5.6], we obtain $O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right) / C_{O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)}(V) \cong \operatorname{SL}_{2}\left(p^{a}\right)$ and $V / C_{V}\left(O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)\right)$ is the natural $\operatorname{GF}(p) \mathrm{SL}_{2}\left(p^{a}\right)$-module. Since we know one of $\left|\left[V, \operatorname{Out}_{S}(E)\right]\right|=p$ or $\left|V: C_{V}\left(\operatorname{Out}_{S}(E)\right)\right|=p$, we deduce that $p=p^{a}$ and this completes the proof.

We continue this section by recalling important properties of $\mathcal{F}$-pearls (see Definition 1.1).

LEMMA 5.18. Suppose that $p$ is an odd prime, $\mathcal{F}$ is a saturated fusion system on a p-group $S$ and $P \in \mathcal{P}(\mathcal{F})$. Then
(i) $S$ has maximal class;
(ii) $\operatorname{Out}_{\mathcal{F}}(P)$ is isomorphic to a subgroup of $\mathrm{GL}_{2}(p)$ and $O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}(P)\right) \cong$ $\mathrm{SL}_{2}(p)$;
(iii) $P / \Phi(P)$ is a natural $\operatorname{GF}(p) \mathrm{SL}_{2}(p)$-module for $O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}(P)\right)$;
(iv) $\left[N_{S}(P): P\right]=p$; and
(v) every subgroup of $S$ that is $\mathcal{F}$-conjugate to $P$ is an $\mathcal{F}$-pearl.

Proof. This is a combination of [28, Lemma 1.5, Corollary 1.11 and Lemma 1.13].

Lemma 5.19. Suppose that $p$ is an odd prime and $\mathcal{F}$ is a saturated fusion system on a p-group $S$ of maximal class with $|S| \geq p^{4}$. If $E \leq S$ is an $\mathcal{F}$-essential subgroup of $S$, then the following are equivalent:
(i) $E$ is an $\mathcal{F}$-pearl;
(ii) $E$ is not contained in $\gamma_{1}(S)$ or in $C_{S}\left(Z_{2}(S)\right)$; and
(iii) there exists an element $x \in S \backslash C_{S}\left(Z_{2}(S)\right)$ of order $p$ such that either $E=\langle x\rangle Z(S)$ or $E=\langle x\rangle Z_{2}(S)$.

Proof. This is a restatement of [28, Lemma 2.4].
The next lemma helps when we apply Lemma 5.13.
Lemma 5.20. Suppose that $p$ is an odd prime and $\mathcal{F}$ is a saturated fusion system on a p-group $S$ with $|S| \geq p^{4}$. If $P \in \mathcal{P}(\mathcal{F})$, then no proper subgroup of $P$ is $\mathcal{F}$-centric.

Proof. There is nothing to do if $P \in \mathcal{P}_{a}(\mathcal{F})$ is abelian. Assume $P \in \mathcal{P}_{e}(\mathcal{F})$ is extraspecial. Since no subgroup of order $p$ is $\mathcal{F}$-centric we can assume $Q \leq P$ has order $p^{2}$. Then parts (ii) and (iii) of Lemma 5.18 imply that $Q$ is $\operatorname{Aut}_{\mathcal{F}}(P)$ conjugate to $Z_{2}(S) \leq P$. Since $|S| \geq p^{4}$, and $\left|S: C_{S}\left(Z_{2}(S)\right)\right|=p$, we conclude $Z_{2}(S)$ is not $\mathcal{F}$-centric. Hence $Q$ is not $\mathcal{F}$-centric.

For a subgroup $A$ of a group $B$, we define $N^{0}(A)=A$ and then, for $i>0$, $N^{i}(A)=N_{B}\left(N^{i-1}(A)\right)$. The ordered collection of these subgroups is called the normalizer tower of $A$ in $B$ and its length is the minimal $\ell$ such that $N^{\ell+1}(A)=$ $N^{\ell}(A)$.

Theorem 5.21. Suppose that $p$ is an odd prime, $\mathcal{F}$ is a saturated fusion system on a p-group $S$ and $P \in \mathcal{P}(\mathcal{F})$ is an $\mathcal{F}$-pearl with $|S: P|=p^{m}$. Then
(i) the members of the normalizer tower of $P$

$$
N^{0}(P)<N^{1}(P)<N^{2}(P)<\cdots<N^{m-1}(P)<N^{m}(P)=S
$$

are the only subgroups of $S$ which contain $P$; also $\left|N^{i}(P): N^{i-1}(P)\right|=p$ for every $1 \leq i \leq m$;
(ii) $P$ is not properly contained in any $\mathcal{F}$-essential subgroup of $S$; and
(iii) every morphism in $N_{\operatorname{Aut}_{\mathcal{F}}(P)}\left(\operatorname{Aut}_{S}(P)\right)$ is the restriction of an automorphism of $S$ that normalizes each $N^{i}(P)$.
Proof. This is an application of [28, Theorem 3.6].
Lemma 5.22. Suppose that $p$ is an odd prime and $\mathcal{F}$ is a saturated fusion system on a p-group $S$. Let $P \in \mathcal{P}(\mathcal{F})$ and let $m$ be such that $|S: P|=p^{m}$. For every $1 \leq i \leq m-1$ let $\mathcal{F}_{i}$ be the smallest fusion subsystem of $\mathcal{F}$ defined on $N^{i}(P)$ such that $\operatorname{Aut}_{\mathcal{F}_{i}}(P)=\operatorname{Aut}_{\mathcal{F}}(P)$ and $\operatorname{Aut}_{\mathcal{F}_{i}}\left(N^{i}(P)\right)=\operatorname{Inn}\left(N^{i}(P)\right) N_{\operatorname{Aut}_{\mathcal{F}}\left(N^{i}(P)\right)}(P)$. Then $\mathcal{F}_{i}$ is a saturated fusion subsystem of $\mathcal{F}$ and $P \in \mathcal{P}\left(\mathcal{F}_{i}\right)$.

Proof. This is [28, Lemma 3.7].
Lemma 5.23. Assume that $\mathcal{F}$ is a saturated fusion system on a p-group $S, \mathcal{C}$ is a set of $\mathcal{F}$-class representatives of $\mathcal{F}$-essential subgroups, and $P \in \mathcal{C}$. If $P$ is an $\mathcal{F}$-pearl, then $\mathcal{G}=\left\langle\operatorname{Aut}_{\mathcal{F}}(S), \operatorname{Aut}_{\mathcal{F}}(E) \mid E \in \mathcal{C} \backslash\{P\}\right\rangle$ is saturated.

Proof. This is [50, Lemma 6.5].
The next lemma is part of [28, Theorem 3.15].
Lemma 5.24. Suppose that $\mathcal{F}$ is a saturated fusion system on a p-group $S$ with $|S| \geq p^{4}$ and $P \in \mathcal{P}(\mathcal{F})$. Assume that the group $\gamma_{1}(S)$ is not abelian or extraspecial. Then
(i) $\operatorname{Out}_{\mathcal{F}}(S)$ is a Hall $p^{\prime}$-subgroup of $\operatorname{Out}(S)$, is cyclic of order $p-1$ and acts faithfully on $S / \gamma_{1}(S)$.
(ii) $\operatorname{Aut}_{\mathcal{F}}(S)=N_{\operatorname{Aut}_{\mathcal{F}}(S)}(P) \operatorname{Inn}(S)$.
(iii) $\operatorname{Out}_{\mathcal{F}}(P) \cong \mathrm{SL}_{2}(p)$.

Proof. Because $P \in \mathcal{P}(\mathcal{F}), S$ has maximal class. Since $\gamma_{1}(S)$ is not abelian or extraspecial, Corollary 3.12 yields the Hall $p^{\prime}$-subgroups of Out $(S)$ are cyclic of order at most $p-1$ and act faithfully on $S / \gamma_{1}(S)$. By Theorem 5.21 , every automorphism in $N_{\text {Aut }_{\mathcal{F}}(P)}\left(\operatorname{Aut}_{S}(P)\right)$ is the restriction of an automorphism of Aut $\mathcal{F}(S)$. Hence $\left|N_{\operatorname{Aut}_{\mathcal{F}}(P)}\left(\operatorname{Aut}_{S}(P)\right) / \operatorname{Aut}_{S}(P)\right| \leq p-1$. Since $P$ is an $\mathcal{F}$-pearl, Lemma 5.18 (ii) implies that $\operatorname{Out}_{\mathcal{F}}(P) \cong \mathrm{SL}_{2}(p)$ and we deduce that $\operatorname{Out}_{\mathcal{F}}(S)$ is cyclic of order $p-1$.

Theorem 5.25. Suppose that $p$ is an odd prime and $\mathcal{F}$ is a saturated fusion system on a p-group $S$ of order $p^{n}$ with $n \geq 4$. Assume that $P \in \mathcal{P}_{a}(\mathcal{F})$.
(i) If $T$ is strongly $\mathcal{F}$-closed in $S$, then either $T=S$ or $T=\gamma_{2}(S) P$ has index $p$ in $S$ and $\mathcal{P}_{a}(\mathcal{F})=P^{S}$.
(ii) If $\mathcal{E}_{\mathcal{F}}=\mathcal{P}_{a}(\mathcal{F})$ and $\gamma_{1}(S)$ is not abelian or extraspecial, then either
(a) $\mathcal{F}$ is simple; or
(b) $O^{p}(\mathcal{F})$ is simple, foc $(\mathcal{F})=\gamma_{2}(S) P, S$ is not exceptional and $n=$ $j(p-1)+1$ for some $j \geq 2$.
(iii) If $p \geq 5, \gamma_{1}(S)$ is not abelian or extraspecial and $\mathcal{F}$ is simple, then $\mathcal{F}$ is exotic.

Proof. Let $T \leq S$ be strongly $\mathcal{F}$-closed in $S$. Since $T$ is normal in $S$ and $|Z(S)|=p$, we have $Z(S) \leq T$. Recall that $P \in \mathcal{P}_{a}(\mathcal{F})$. Hence $Z(S) \leq P$, $P \not \leq C_{S}\left(Z_{2}(S)\right)$ and $P \not \leq \gamma_{1}(S)$ by Lemma 5.19. As $P$ is an abelian $\mathcal{F}$-pearl, there exists $\beta \in \operatorname{Aut}_{\mathcal{F}}(P)$ such that $P=Z(S) Z(S) \beta$. Since $T$ is strongly $\mathcal{F}$-closed, this implies $P \leq T$. Using $T$ is normal in $S$ yields, $T \in\left\{S, \gamma_{2}(S) P\right\}$. Assume that $P_{2} \in \mathcal{P}(\mathcal{F}) \backslash P^{S}$. Since $S$ has maximal class, $S$ acts transitively by conjugation on the subgroups of order $p^{2}$ containing $Z(S)$ which are not contained in $\gamma_{1}(S)$ but which are contained in $\gamma_{2}(S) P$. Hence $P_{2} \not \leq \gamma_{2}(S) P$ whereas we know $P_{2} \leq T$. Hence $T=S$ in this case. This proves (i).

Suppose that $\mathcal{E}_{\mathcal{F}}=\mathcal{P}_{a}(\mathcal{F})$ and that $\gamma_{1}(S)$ is not abelian or extraspecial. This implies $|S| \geq p^{5}$. Assume that $\mathcal{G}$ is a weakly normal subsystem of $\mathcal{F}$ defined on a non-trivial subgroup $T$ of $S$. Then $T$ is strongly $\mathcal{F}$-closed in $S$ and so $T \in$ $\left\{\gamma_{2}(S) P, S\right\}$ by (i). Since $\gamma_{1}(S)$ is not abelian or extraspecial, Lemma 5.24 implies that $\operatorname{Aut}_{\mathcal{F}}(P) \cong \mathrm{SL}_{2}(p)$ and $\operatorname{Out}_{\mathcal{F}}(S)$ is cyclic of order $p-1$. By [16, Lemma 5.33], $\operatorname{Aut}_{\mathcal{G}}(P)$ is a normal subgroup of $\operatorname{Aut}_{\mathcal{F}}(P) \cong \mathrm{SL}_{2}(p)$. Since $\left|\operatorname{Aut}_{S}(P)\right|=p$, we deduce that $\operatorname{Aut}_{\mathcal{G}}(P)=\operatorname{Aut}_{\mathcal{F}}(P)$ and this is true for all pearls $P \in \mathcal{P}(\mathcal{F})$. As $\mathcal{E}_{\mathcal{F}}=\mathcal{P}_{a}(\mathcal{F})$, for $R \leq S$ with $|R|>p^{2}$, we have $\operatorname{Aut}_{\mathcal{F}}(R)$ is generated by restrictions of automorphisms of $S$ by Theorem 5.9. Hence no such subgroup can be $\mathcal{G}$-essential. In particular, $\mathcal{E}_{\mathcal{F}}=\mathcal{E}_{\mathcal{G}}$. By Theorem 5.21 (iii), $\operatorname{Aut}_{\mathcal{G}}(T)$ has order divisible by $p-1$. On the other hand, $T$ is not an $\mathcal{F}$-essential subgroup by Theorem 5.21(ii) and thus every element of $\operatorname{Aut}_{\mathcal{G}}(T)$ is the restriction of some element of $\operatorname{Aut}_{\mathcal{F}}(T)$. We deduce that $\operatorname{Out}_{\mathcal{G}}(T)$ is cyclic of order $p-1$.

If $T=S$, we have $\operatorname{Aut}_{\mathcal{F}}(S)=\operatorname{Aut}_{\mathcal{G}}(S)$ and Theorem 5.9 implies that $\mathcal{F}=\mathcal{G}$. Therefore $\mathcal{F}$ is simple in this case and this is listed as (ii)(a).

Suppose that $T=\gamma_{2}(S) P$. Then, by (i), $\mathcal{P}_{a}(\mathcal{F})=P^{S}$. In particular, $\mathcal{P}_{a}(\mathcal{G})=$ $\mathcal{P}_{a}(\mathcal{F}) \neq P^{T}$. Since $\operatorname{Aut}_{\mathcal{G}}(T)$ is normal in $\operatorname{Aut}_{\mathcal{F}}(T)$ and $\operatorname{Out}_{\mathcal{G}}(T)$ has order $p-1$, we conclude that $\operatorname{Aut}_{\mathcal{G}}(T) \geq O^{p}\left(\operatorname{Aut}_{\mathcal{F}}(T)\right)$. Using (i), we now know foc $(\mathcal{F})=$ $\operatorname{hyp}(\mathcal{F})=\gamma_{2}(S) P$ and so $\mathcal{G}=O^{p}(\mathcal{F})$. Since $\mathcal{P}_{a}(\mathcal{G}) \neq P^{T}, T$ is the only strongly $\mathcal{G}$-closed subgroup of $\mathcal{G}$ by (i). Since $\operatorname{Out}_{\mathcal{G}}(T)$ has order $p-1$, we now have $\mathcal{G}$ is simple as before.

Continue to assume that $T=\gamma_{2}(S) P$. It remains to prove that $S$ is not exceptional and that $n \equiv 1(\bmod p-1)$. Let $\phi \in \operatorname{Aut}_{\mathcal{G}}(T)$ have order $p-1$ be such that $P \phi=P$ and $\left.\phi\right|_{P}$ acts as the diagonal matrix $\left(\begin{array}{cc}\lambda & 0 \\ 0 & \lambda^{-1}\end{array}\right)$. Then, choosing $x \in P \backslash \gamma_{1}(S)$ and using the notation introduced in Lemma 3.10, we have $x \phi \equiv x^{a}$ $\left(\bmod \gamma_{2}(S)\right)$, and $s_{n-1} \phi=s_{n-1} a^{-1}$ where $a \in \operatorname{GF}(p)$ has order $p-1$. Since $\phi \in O^{p}\left(\operatorname{Aut}_{\mathcal{F}}(T)\right)=\operatorname{Aut}_{\mathcal{G}}(T), \phi$ centralizes $S / T$ and so $s_{1} \phi \equiv s_{1}\left(\bmod \gamma_{2}(S)\right)$ ( $b=1$ in Lemma 3.10). If $S$ is exceptional, then $\phi$ leaves $C_{S}\left(Z_{2}(S)\right), T$ and $\gamma_{1}(S)$ invariant, contrary to $a \neq 1$. Hence $S$ is not exceptional. Now Lemma 3.10 applies to give

$$
s_{n-1}^{a^{-1}}=s_{n-1} \phi=s_{n-1}^{a^{n-2}} .
$$

Thus $a^{n-1}=1$ so that $n \equiv 1(\bmod p-1)$, that is, $n=j(p-1)+1$ for some $j \geq 1$. To complete the proof of (ii), note that if $j=1$ then $n=p$ and $S$ has a maximal subgroup that is abelian by $[\mathbf{2 8}$, Theorem A], a contradiction. Hence $j \geq 2$.

Finally, assume that $\mathcal{F}$ is simple and $p \geq 5$. If $\mathcal{F}$ is not exotic, then $\mathcal{F}$ is realised by a finite simple group by [16, Theorem 5.71]. But then Lemma 3.8 yields $\gamma_{1}(S)$ is extraspecial, a contradiction. This demonstrates (iii).

Semeraro's theorem of can be used to add pearls to saturated fusion systems without destroying saturation.

THEOREM 5.26. Let $\mathcal{F}_{0}$ be a saturated fusion system on a finite p-group $S$. Let $V \leq S$ be a fully $\mathcal{F}_{0}$-normalized subgroup, set $H=\operatorname{Out}_{\mathcal{F}_{0}}(V)$ and let $\widetilde{\Delta} \leq \operatorname{Out}(V)$ be such that $H$ is a strongly p-embedded subgroup of $\widetilde{\Delta}$. For $\Delta$ the full preimage of $\tilde{\Delta}$ in $\operatorname{Aut}(V)$, write

$$
\mathcal{F}=\left\langle\operatorname{Mor}\left(\mathcal{F}_{0}\right), \Delta\right\rangle
$$

Assume further that
(i) $V$ is $\mathcal{F}_{0}$-centric and minimal under inclusion amongst all $\mathcal{F}$-centric subgroups; and
(ii) no proper subgroup of $V$ is $\mathcal{F}_{0}$-essential.

Then $\mathcal{F}$ is saturated.
Proof. This is a statement of $[\mathbf{5 5}$, Theorem C] for the special case $m=1$.
In Section 13 we exploit results about fusion systems on $p$-groups with an abelian subgroup of index $p[\mathbf{1 9}, \mathbf{4 2}, \mathbf{4 5}]$. The tool for doing this is the following proposition.

Proposition 5.27. Suppose that $p$ is an odd prime, $\mathcal{F}$ is a saturated fusion system on $S, P \in \mathcal{P}(\mathcal{F})$ and $\operatorname{Aut}_{\mathcal{F}}(S)=\operatorname{Inn}(S) N_{\mathrm{Aut}_{\mathcal{F}}(S)}(P)$. Assume that $Z(S)<$ $V \leq Z\left(\gamma_{1}(S)\right)$ is $\operatorname{Aut}_{\mathcal{F}}(S)$-invariant, put $S_{1}=V P$,

$$
H=\operatorname{Inn}\left(S_{1}\right)\left\langle\left.\phi\right|_{S_{1}} \mid \phi \in N_{\operatorname{Aut}_{\mathcal{F}}(S)}(P)\right\rangle \leq \operatorname{Aut}_{\mathcal{F}}\left(S_{1}\right)
$$

and

$$
B=\left\{\left.\phi\right|_{V} \mid \phi \in \operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right\} \leq \operatorname{Aut}_{\mathcal{F}}(V)
$$

Then the fusion system

$$
\mathcal{G}=\left\langle\operatorname{Aut}_{\mathcal{F}}(P), B, H\right\rangle
$$

on $S_{1}$ is saturated and $\operatorname{Aut}_{\mathcal{G}}\left(S_{1}\right)=H$.
Proof. We have $P \leq S_{1}$ and so, if $\left|S_{1}: P\right|=p^{k}, S_{1}=N^{k}(P)$ by Theorem 5.21 (i). Let $\mathcal{F}_{k}=\left\langle\operatorname{Aut}_{\mathcal{F}}(P), H\right\rangle$. Then $\mathcal{F}_{k}$ is the smallest fusion subsystem of $\mathcal{F}$ defined on $S_{1}$ such that $\operatorname{Aut}_{\mathcal{F}_{k}}(P)=\operatorname{Aut}_{\mathcal{F}}(P)$ and $\operatorname{Aut}_{\mathcal{F}_{k}}\left(S_{1}\right)=\operatorname{Inn}\left(S_{1}\right) N_{\text {Aut }_{\mathcal{F}}\left(S_{1}\right)}(P)$. By Lemma $5.22, \mathcal{F}_{k}$ is saturated.

Since $V$ is normal in $S_{1}, V$ is fully $\mathcal{F}_{k}$-normalized and, as $V>Z(S), V$ is abelian and $\left[S_{1}: V\right]=p, V$ is an $\mathcal{F}_{k}$-centric subgroup of $S_{1}$. We have Aut $\mathcal{F}_{k}(V)=$ $\operatorname{Aut}_{H}(V)$ is a subgroup of $B \leq \operatorname{Aut}_{\mathcal{F}}(V)$. If $\operatorname{Aut}_{H}(V)=B$, then $\mathcal{G}=\mathcal{F}_{k}$ and $\mathcal{G}$ is saturated. So we may assume that $\operatorname{Aut}_{H}(V)<B$. In particular, as $\operatorname{Aut}_{\mathcal{F}}(S)=\operatorname{Inn}(S) N_{\operatorname{Aut}_{\mathcal{F}}(S)}(P), \operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)>\operatorname{Aut}_{\operatorname{Aut}_{\mathcal{F}}(S)}\left(\gamma_{1}(S)\right)$ and so $\gamma_{1}(S)$ is $\mathcal{F}$-essential and $\operatorname{Out}_{\operatorname{Aut}_{\mathcal{F}}(S)}\left(\gamma_{1}(S)\right)$ is strongly $p$-embedded in $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$. Therefore $\mathrm{Out}_{H}(V)$ is strongly $p$-embedded in $\mathrm{Out}_{B}(V)$. Furthermore, as $V$ is abelian, $V$ is minimal by inclusion amongst all $\mathcal{F}_{k}$-centric subgroups and no proper subgroup of $V$ is $\mathcal{F}_{k}$-essential. Now application of Theorem 5.26 delivers $\mathcal{G}$ is saturated. That $\operatorname{Aut}_{\mathcal{G}}\left(S_{1}\right)=H$ is apparent from its construction.

## 6. Fusion systems on groups of maximal class: generalities

This section begins the study of saturated fusion systems on maximal class p-groups.

Lemma 6.1. Suppose that $S$ is a non-abelian 2 -group of maximal class and $\mathcal{F}$ is a saturated fusion system on $S$. Then $S$ is dihedral, semidihedral or generalized quaternion of order at least 8 and $\mathcal{F}$ is known and realizable. In particular, if $E$ is an $\mathcal{F}$-essential subgroup of $S$, then $E$ is an $\mathcal{F}$-pearl and $\gamma_{1}(S)$ is cyclic.

Proof. If $|S|=8$, then $S$ is either dihedral or quaternion and it is an easy exercise to write down all the saturated fusion systems on $S$. That the maximal class 2 -groups of order at least $2^{4}$ are dihedral, semidihedral or generalized quaternion is well-known and can be found in [ $\mathbf{3 7}$, Corollary 3.3.4 (iii)] for example. In this case, the totality of the saturated fusion systems on such groups can be found in [5, Example I.3.8].

Lemma 6.2. Suppose that $\mathcal{F}$ is a saturated fusion system on a non-abelian maximal class 3-group. Then $\mathcal{F}$ is known, $\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{P}(\mathcal{F}) \cup\left\{\gamma_{1}(S)\right\}$ and, if $\gamma_{1}(S) \in \mathcal{E}_{\mathcal{F}}$, then $\gamma_{1}(S)$ is abelian.

Proof. This is Theorem B.5.
Because of Lemmas 6.1 and 6.2, for the remainder of this work we focus on the case $p \geq 5$.

Hypothesis 6.3. The prime $p$ is at least $5, S$ is a maximal class $p$-group of order at least $p^{4}$ and $\mathcal{F}$ is a saturated fusion system on $S$.

Throughout this section and Sections 7, 8, 9, 10 and 11, we assume that Hypothesis 6.3 holds sway and adopt its notation. We start with some lemmas which loosely locate the potential $\mathcal{F}$-essential subgroups within $S$. First we recall

Lemma 6.4. If $E$ is an $\mathcal{F}$-essential subgroup which is not an $\mathcal{F}$-pearl, then either $E \leq \gamma_{1}(S)$ or $E \leq C_{S}\left(Z_{2}(S)\right)$. Furthermore, there are no $\mathcal{F}$-pearls in $\gamma_{1}(S)$ or in $C_{S}\left(Z_{2}(S)\right)$.

Proof. This is just a restatement of Lemma 5.19.
The next result is included for completeness.
Corollary 6.5. If $\gamma_{1}(S)$ is abelian, then the candidates for $\mathcal{F}$-essential subgroups are $\gamma_{1}(S)$ and $\mathcal{F}$-pearls. In particular, if $O_{p}(\mathcal{F})=1$, then $\mathcal{F}$ has an $\mathcal{F}$-pearl.

Proof. (See also [42, Lemma 2.3 (a)]). Since $\gamma_{1}(S)$ is abelian, $S$ is not exceptional. Thus, if $E$ is $\mathcal{F}$-essential and not an $\mathcal{F}$-pearl, then $E \leq \gamma_{1}(S)$ by Lemma 6.4. As $E$ is $\mathcal{F}$-centric, this yields $E=\gamma_{1}(S)$ as claimed. In particular, if $\mathcal{F}$ has no $\mathcal{F}$-pearls, then $\gamma_{1}(S)$ is the only $\mathcal{F}$-essential subgroup in $\mathcal{F}$. Since $\gamma_{1}(S)$ is a characteristic subgroup of $S$, the Alperin-Goldschmidt Theorem yields $\gamma_{1}(S)=O_{p}(\mathcal{F})$. Hence, if $O_{p}(\mathcal{F})=1$, then $\mathcal{F}$ has an $\mathcal{F}$-pearl.

Lemma 6.6. If there exists a morphism $\varphi \in \operatorname{Aut}_{\mathcal{F}}\left(Z_{2}(S)\right)$ such that $Z(S) \varphi \neq$ $Z(S)$, then $C_{S}\left(Z_{2}(S)\right) \in \mathcal{E}_{\mathcal{F}}$.

Proof. Since $Z_{2}(S)$ is fully $\mathcal{F}$-normalized and so $\mathcal{F}$-receptive, there exists an automorphism $\bar{\varphi} \in \operatorname{Aut}_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right)$ such that $\left.\bar{\varphi}\right|_{Z_{2}(S)}=\varphi$. In particular $\bar{\varphi}$ does not normalize $Z(S)$ and so it cannot be the restriction of an automorphism of $S$. Since $C_{S}\left(Z_{2}(S)\right)$ is characteristic in $S$, by the Alperin-Goldschmidt fusion theorem we deduce that $C_{S}\left(Z_{2}(S)\right)$ is $\mathcal{F}$-essential.

Proposition 6.7. Suppose $E$ is $\mathcal{F}$-essential and $E$ is normal in $S$. Then $E$ is a maximal subgroup of $S$. Moreover, either $E \in\left\{\gamma_{1}(S), C_{S}\left(Z_{2}(S)\right)\right\}$ or $|S|=p^{4}$ and $E$ is a non-abelian $\mathcal{F}$-pearl.

Proof. Suppose for a contradiction that $E$ is not a maximal subgroup of $S$. Since $E \unlhd S$ and $S$ has maximal nilpotency class, $E$ is a member of the lower central series of $S$. Hence $E=\gamma_{k}(S)$ for some $k \geq 2$. In particular, $\operatorname{Out}_{S}(E) \cong S / E$ is a $p$-group of maximal class. Since $p$ is odd, $S / E$ contains an elementary abelian subgroup of order $p^{2}$ by $[\mathbf{2 4}, 5.4 .10$ (ii)]. Lemma 2.10 yields that $F^{*}\left(\operatorname{Out}_{\mathcal{F}}(E) / O_{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}(E)\right)\right)$ is a non-abelian simple group. Set $W=$ $F^{*}\left(\operatorname{Out}_{\mathcal{F}}(E) / O_{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}(E)\right)\right)$ and $X=W \operatorname{Out}_{S}(E) O_{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}(E)\right) / O_{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}(E)\right)$. Using Proposition 2.12 we know the candidates for $X$. Remember also that $p \geq 5$. Suppose first that $\operatorname{Out}_{S}(E)$ is abelian. Then $\operatorname{Out}_{S}(E) \cong S / S^{\prime}$ is elementary abelian of order $p^{2}$. The possibilities for $X$ are $\mathrm{PSL}_{2}\left(p^{2}\right)$, $\operatorname{Alt}(2 p)$ for arbitrary odd $p$, and ${ }^{2} \mathrm{~F}_{4}(2)^{\prime}$ or $\mathrm{Fi}_{22}$ with $p=5$. By a Frattini Argument and using [26, Theorem 7.8.1], $N_{\operatorname{Aut}_{\mathcal{F}}(E)}\left(\operatorname{Aut}_{S}(E)\right)$ acts irreducibly on $\operatorname{Out}_{S}(E)$. As every morphism in $N_{\text {Aut }_{\mathcal{F}}(E)}\left(\operatorname{Aut}_{S}(E)\right)$ is the restriction of an $\mathcal{F}$-automorphism of $S$ (because $E$ is $\mathcal{F}$ receptive), there exists an automorphism $\tau \in \operatorname{Aut}_{\mathcal{F}}(S)$ such that $\gamma_{1}(S) \tau \neq \gamma_{1}(S)$, which is absurd as this subgroup is characteristic in $S$. If Out ${ }_{S}(E)$ is non-abelian, then $\operatorname{Out}_{S}(E)$ is extraspecial and the possibilities for $X$ are $\mathrm{PSU}_{3}(p)$ for all $p \geq 5$, McL or ${ }^{2} \mathrm{~B}_{2}(32): 5$ with $p=5$, and $\mathrm{J}_{4}$ with $p=11$. This time [26, Theorem 7.6.2] shows that either $N_{\operatorname{Aut}_{\mathcal{F}}(E)}\left(\operatorname{Aut}_{S}(E)\right)$ acts irreducibly on $\operatorname{Out}_{S}(E) / Z\left(\operatorname{Out}_{S}(E)\right)$ or $X \cong{ }^{2} \mathrm{~B}_{2}(32): 5$. In the former case, we obtain a contradiction exactly as in the abelian case. Suppose that $X \cong{ }^{2} \mathrm{~B}_{2}(32): 5$. Then $\operatorname{Out}_{S}(E) \cong 5_{-}^{1+2} \cong S / \gamma_{3}(S)$ has exponent 25. Since $|S| \geq 5^{4}$, applying Lemma 3.2 (vi) to $S / \gamma_{4}(S)$ yields $S / \gamma_{3}(S)$ has exponent 5 , which is a contradiction. Therefore, if $E$ is normal in $S$, then $E$ is a maximal subgroup of $S$.

If $E \notin\left\{\gamma_{1}(S), C_{S}\left(Z_{2}(S)\right)\right\}$, then Lemma 5.19 implies $E$ is an $\mathcal{F}$-pearl and so $|S|=p^{4}$. This proves the lemma.

Lemma 6.8. Suppose that $|S| \geq p^{5}$ and $E$ is an $\mathcal{F}$-essential subgroup which is not an $\mathcal{F}$-pearl. Then $|E| \geq p^{4}$.

Proof. Suppose that $|E| \leq p^{3}$. Then, as $E$ is $\mathcal{F}$-essential and $E$ is not an $\mathcal{F}$-pearl, $|E|=p^{3}$ and $E$ is abelian. Furthermore, Lemma 6.4 indicates that either $E \leq \gamma_{1}(S)$ or $E \leq C_{S}\left(Z_{2}(S)\right)$. Since abelian groups of order $p^{3}$ and exponent at least $p^{2}$ cannot be $\mathcal{F}$-essential, we have $E$ is elementary abelian. Of course $Z(S) \leq E$. By [29, Corollary 1.23], every automorphism in $N_{\text {Aut }_{\mathcal{F}}(E)}\left(\operatorname{Aut}_{S}(E)\right)$ is the restriction of an $\mathcal{F}$-automorphism of $S$ and $\left|\operatorname{Aut}_{S}(E)\right|=p$. Since $E$ is not normal in $S$ by Proposition 6.7, $E \neq Z_{3}(S)$. If $E \leq \gamma_{1}(S)$, then $Z_{3}(S) \leq N_{S}(E)$ and, if $E \leq C_{S}\left(Z_{2}(S)\right)$, then $Z_{2}(S) \leq E$ and $Z_{3}(S) \leq N_{S}(E)$. In particular, as $E \neq Z_{3}(S)$ and $\left|N_{S}(E)\right|=p^{4}, N_{S}(E)=Z_{3}(S) E$ in both cases. Since $|S| \geq$ $p^{5}, Z_{3}(S)$ is abelian and so $Z\left(N_{S}(E)\right)=E \cap Z_{3}(S)$ is a maximal subgroup of $E$.

Proposition 5.17 (applied with $U=E$ and $W=1$ ) implies that $O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right) \cong$ $\mathrm{SL}_{2}(p)$ and, if we take $\tau \in Z\left(O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)\right)$ to be an involution, then $[E, \tau]$ is a natural $\operatorname{GF}(p) O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)$-module. Let $\hat{\tau} \in \operatorname{Aut}_{\mathcal{F}}(S)$ be such that $\left.\hat{\tau}\right|_{E}=\tau$ and $\hat{\tau}$ has $p^{\prime}$-order. Then

$$
\hat{\tau} \text { centralizes } Z_{3}(S) E / E \cong Z_{3}(S) /\left(Z_{3}(S) \cap E\right)
$$

Suppose that $Z_{2}(S)=E \cap Z_{3}(S)$. Then $Z_{2}(S)=C_{E}\left(Z_{3}(S)\right)$. If $\left[E, Z_{3}(S)\right]=$ $Z(S)$, then $\hat{\tau}$ inverts $Z(S)$ and centralizes $Z_{2}(S) / Z(S)$. Hence $\hat{\tau}$ centralizes the group $Z_{3}(S) / Z(S)$ and this contradicts Lemma 3.14. Hence $\left[E, Z_{3}(S)\right] \leq Z_{2}(S)$ but is not contained in $Z(S)$. Therefore $\hat{\tau}$ centralizes $Z(S)$ and inverts $Z_{2}(S) / Z(S)$. Since $\left[Z_{3}(S), \gamma_{1}(S)\right] \leq Z(S)$, we have $E \not \leq \gamma_{1}(S)$ and, in particular, $S$ is exceptional and $n$ is even by Lemma 3.3 (v). We apply Lemma 3.10 and use the notation from there with $\varphi=\hat{\tau}$. Thus, as $\hat{\tau}$ inverts $E \gamma_{1}(S) / \gamma_{1}(S) \cong E /\left(E \cap \gamma_{1}(S)\right)=E / Z_{2}(S)$, $a=-1$, we have $\hat{\tau}$ acts as

$$
(-1)^{n-3} b=-b=-1
$$

on $Z_{2}(S) / Z(S)$ and as

$$
(-1)^{n-3} b^{2}=-b^{2}=1
$$

on $Z(S)$. This is impossible and thus $Z_{2}(S) \not \leq E$. Now $N_{S}(E)=Z_{2}(S) E$ and $\left[E, Z_{2}(S)\right]=Z(S)$. It follows that $\hat{\tau}$ centralizes $Z_{2}(S) / Z(S)$ and $\left(Z_{3}(S) \cap E\right) / Z(S)$. Hence $\hat{\tau}$ centralizes $Z_{3}(S) / Z(S)$, which is impossible by Lemma 3.14. We have proved, that, if $|E|=p^{3}$ and $E$ is not an $\mathcal{F}$-pearl, then $E$ is not $\mathcal{F}$-essential.

Lemma 6.9. If $O_{p^{\prime}}\left(Z\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right)\right)$ is non-trivial, then either
(i) $\gamma_{1}(S)$ is abelian; or
(ii) $S$ is exceptional and $\gamma_{1}(S)$ is extraspecial.

Proof. Assume that $\tau \in \operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ projects to a non-trivial element of the group $O_{p^{\prime}}\left(Z\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right)\right)$. Then $\tau$ is the restriction of $\hat{\tau} \in \operatorname{Aut}_{\mathcal{F}}(S)$ of $p^{\prime}$-order and $[S, \hat{\tau}] \leq \gamma_{1}(S)$. Lemma 3.11 now gives the result.

## 7. Essential subgroups in exceptional maximal class groups

In this section, we start the investigation of $\mathcal{F}$-essential subgroups when $S$ is exceptional of order $p^{n}$. We assume that

Hypothesis 7.1. Hypothesis 6.3 holds with $S$ exceptional.

Because $S$ is exceptional, we know from Lemma 3.3 (v) that $n$ is even and

$$
p^{6} \leq|S|=p^{n} \leq p^{p+1}
$$

Our aim is to prove the following proposition.
Proposition 7.2. Suppose Hypothesis 7.1 holds. Then

$$
\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{P}_{a}(\mathcal{F}) \cup\left\{C_{S}\left(Z_{2}(S)\right)\right\} \cup\left\{E \mid Z(S)<E \leq \gamma_{1}(S)\right\}
$$

Furthermore,
(i) if $\mathcal{P}_{a}(\mathcal{F}) \neq \emptyset$, then $|S|=p^{p-1}$ and $\gamma_{1}(S)$ is extraspecial;
(ii) if $C_{S}\left(Z_{2}(S)\right) \in \mathcal{E}_{\mathcal{F}}$, then $|S|=p^{6}$, $O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right)\right) \cong \mathrm{SL}_{2}(p)$ and either
(a) $\gamma_{1}(S)$ is extraspecial; or
(b) $p=5, S=\operatorname{SmallGroup}\left(5^{6}, 661\right), \mathcal{E}_{\mathcal{F}}=\left\{C_{S}\left(Z_{2}(S)\right)\right\}$, $\operatorname{Out}_{\mathcal{F}}(S)$ is cyclic of order 4 , Out $_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right) \cong \mathrm{SL}_{2}(5)$ and $\mathcal{F}$ is unique.
and
(iii) if $\gamma_{1}(S)$ is extraspecial, then

$$
\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{P}_{a}(\mathcal{F}) \cup\left\{C_{S}\left(Z_{2}(S)\right), \gamma_{1}(S)\right\}
$$

We prove Proposition 7.2 via a series of lemmas.
Lemma 7.3. Assume Hypothesis 7.1 holds. Then $\mathcal{P}(\mathcal{F})=\mathcal{P}_{a}(\mathcal{F})$ (that is, every $\mathcal{F}$-pearl is abelian).

Proof. Aiming for a contradiction, suppose $E$ is an extraspecial $\mathcal{F}$-pearl. Then we have $Z(S)=Z(E)$ and the involution $\tau \in$ Aut $_{\mathcal{F}}(E)$ which maps into $Z\left(O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}(E)\right)\right)$ inverts $E / Z(S)$ and centralizes $Z(S)$. In addition, by Theo$\operatorname{rem} 5.21$ (iii), there is an automorphism $\hat{\tau} \in \operatorname{Aut}_{\mathcal{F}}(S)$ of $p^{\prime}$-order such that $\left.\hat{\tau}\right|_{E}=\tau$. Let $M$ be the maximal subgroup of $S$ containing $E$. Then $M \notin\left\{\gamma_{1}(S), C_{S}\left(Z_{2}(S)\right)\right\}$ by Lemma 5.19. Thus $M C_{S}\left(Z_{2}(S)\right)=\gamma_{1}(S) C_{S}\left(Z_{2}(S)\right)=S$ and so, as $\gamma_{1}(S)$ and $C_{S}\left(Z_{2}(S)\right)$ are characteristic in $S$,

$$
\gamma_{1}(S) / \gamma_{2}(S) \cong S / C_{S}\left(Z_{2}(S)\right) \cong M / \gamma_{2}(S)=E \gamma_{2}(S) / \gamma_{2}(S) \cong E / Z_{2}(S)
$$

as $\hat{\tau}$-groups. Hence $\gamma_{1}(S) / \gamma_{2}(S)$ and $M / \gamma_{2}(S)$ are inverted by $\hat{\tau}$. Therefore $\hat{\tau}$ inverts $S / \gamma_{2}(S)$ and so is an involution. Since $\hat{\tau}$ centralizes $Z(S)$, this contradicts Lemma 3.13. Thus every $\mathcal{F}$-pearl is abelian.

Lemma 7.4. Assume Hypothesis 7.1 holds with $\gamma_{1}(S) / Z(S)$ abelian. Then $\gamma_{1}(S)$ is extraspecial and no proper subgroup of $\gamma_{1}(S)$ is $\mathcal{F}$-essential.

Proof. Since $S$ is exceptional, $Z(S)=Z\left(\gamma_{1}(S)\right)$. By assumption we have $Z(S)=Z\left(\gamma_{1}(S)\right)=\left[\gamma_{1}(S), \gamma_{1}(S)\right]$. Hence $\gamma_{1}(S)$ is extraspecial.

Aiming for a contradiction, suppose there exists a subgroup $E<\gamma_{1}(S)$ that is $\mathcal{F}$-essential. Note that

$$
[E, E] \leq\left[E, \gamma_{1}(S)\right] \leq\left[\gamma_{1}(S), \gamma_{1}(S)\right]=Z(S) \leq E
$$

In particular, $E$ is normal in $\gamma_{1}(S)$. If $E$ is not elementary abelian, then $E$ has normal series $E>\Phi(E)=Z(S)$ which is stabilized by $\gamma_{1}(S)$. This contradicts Lemma 5.15. Hence $E$ is elementary abelian. Since $\gamma_{1}(S)$ is extraspecial of order $p^{n-1}$, this implies $|E| \leq p^{n / 2}$. In particular the quotient $\gamma_{1}(S) / E$ is elementary abelian of order $\left[\gamma_{1}(S): E\right] \geq p^{(n-2) / 2}$. On the other hand, Lemma 5.16 now yields that $|E| \geq p^{n-2}$. Thus $n=4$ and Lemma 3.3(v) contradicts the assumption that $S$ is exceptional. Therefore no proper subgroup of $\gamma_{1}(S)$ is $\mathcal{F}$-essential.

Lemma 7.5. Assume Hypothesis 7.1 holds with $C_{S}\left(Z_{2}(S)\right) \in \mathcal{E}_{\mathcal{F}}$. Then $|S|=$ $p^{6}, O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right)\right) \cong \mathrm{SL}_{2}(p)$ and

$$
\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{P}_{a}(\mathcal{F}) \cup\left\{C_{S}\left(Z_{2}(S)\right), \gamma_{1}(S)\right\}
$$

In addition, either $\gamma_{1}(S)$ is extraspecial or $p=5, S=\operatorname{SmallGroup}\left(5^{6}, 661\right), \mathcal{E}_{\mathcal{F}}=$ $\left\{C_{S}\left(Z_{2}(S)\right)\right\}$, $\operatorname{Out}_{\mathcal{F}}(S)$ is cyclic of order 4 , $\operatorname{Aut}_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right) \cong \mathrm{SL}_{2}(5)$ and $\mathcal{F}$ is unique.

Proof. Set $R=C_{S}\left(Z_{2}(S)\right)$. Then, as $R \neq \gamma_{1}(S)$ and $S / Z(S)$ is not exceptional by Lemma 3.3 (vi), we glean $\left[\gamma_{2}(S), R\right]=\gamma_{3}(S)$. It follows that $R^{\prime}=\Phi(R)=$ $\gamma_{3}(S)$ and $R / \Phi(R)$ has order $p^{2}$. Therefore Out $\mathcal{F}(R)$ embeds into $\mathrm{GL}_{2}(p)$ and

$$
O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}(R)\right)=\left\langle\operatorname{Out}_{S}(R)^{\operatorname{Out}_{\mathcal{F}}(R)}\right\rangle \cong \mathrm{SL}_{2}(p)
$$

acts irreducibly on $R / \gamma_{3}(S)$. In particular, $\operatorname{Inn}(R) \leq O^{p}\left(\operatorname{Aut}_{\mathcal{F}}(R)\right)$. Suppose that $|S| \geq p^{7}$. Then

$$
\gamma_{4}(S)=\left[\gamma_{3}(S), S\right]=\left[\gamma_{3}(S), \gamma_{1}(S) R\right] \geq\left[\gamma_{3}(S), \gamma_{1}(S)\right]\left[\gamma_{3}(S), R\right]
$$

Since $\left[\gamma_{3}(S), \gamma_{1}(S)\right] \leq \gamma_{5}(S)$ and $\left[\gamma_{3}(S), R\right]$ is normal in $S$, we deduce $\left[\gamma_{3}(S), R\right]=$ $\gamma_{4}(S)$ and similarly $\left[\gamma_{4}(S), R\right]=\gamma_{5}(S)$. Since $[R, R]=\gamma_{3}(S)$, we have $\gamma_{4}(S)=$ $[R, R, R]$ and $\gamma_{5}(S)=[R, R, R, R]$. Now we must have $O^{p}\left(\operatorname{Aut}_{\mathcal{F}}(R)\right)$ centralizes $\gamma_{3}(S) / \gamma_{5}(S)$. Since $\operatorname{Inn}(R) \leq O^{p}\left(\operatorname{Aut}_{\mathcal{F}}(R)\right)$, this means that $\left[\gamma_{3}(S), R\right] \leq \gamma_{5}(S)$, a contradiction. Hence $|S| \leq p^{6}$. Since $S$ is exceptional, Lemma 3.3 (v) implies that $|S|=p^{6}$.

We next show that there are no $\mathcal{F}$-essential subgroups properly contained in $R$. Aiming for a contradiction, suppose that $E<R$ is an $\mathcal{F}$-essential subgroup $E$. Then, as $E$ is not an $\mathcal{F}$-pearl, Lemma 6.8 implies $|E|=p^{4}$ and so $E$ is a maximal subgroup of $R$. Recall that $\operatorname{Aut}_{\mathcal{F}}(R)$ acts irreducibly on $R / \gamma_{3}(S)$. In particular, it acts transitively on the maximal subgroups of $R$ containing $\gamma_{3}(S)=\Phi(R)$. Thus $E$ is $\mathcal{F}$-conjugate to $\gamma_{2}(S)$. Since $E$ is fully $\mathcal{F}$-normalized, we deduce that $E=\gamma_{2}(S)$ is normal in $S$, contradicting Proposition 6.7. This proves the claim.

It remains to show that either $\gamma_{1}(S)$ is extraspecial or $p=5$ and, in the latter case, determine the structure of $\mathcal{F}$. Let $\varphi \in N_{O^{p^{\prime}\left(\operatorname{Out}_{\mathcal{F}}(R)\right)}}\left(\operatorname{Out}_{S}(R)\right)$ be the automorphism of order $p-1$ corresponding to the matrix $\left(\begin{array}{cc}\lambda^{-1} & 0 \\ 0 & \lambda\end{array}\right)$ for a fixed $\lambda \in \operatorname{GF}(p)$ of order $p-1$. Then by saturation there is an automorphism $\hat{\varphi} \in \operatorname{Aut}_{\mathcal{F}}(S)$ such that $\left.\hat{\varphi}\right|_{R}=\varphi$. In particular $\hat{\varphi}$ acts on $\gamma_{i}(S)$ for every $i \geq 1$. Thus for every $x, y \in S$ such that $R=\langle x\rangle \gamma_{2}(S)$ and $\gamma_{2}(S)=\langle y\rangle \gamma_{3}(S)$ we have

$$
\left(x \gamma_{2}(S)\right) \varphi=x^{\lambda^{-1}} \gamma_{2}(S) \quad \text { and } \quad\left(y \gamma_{3}(S)\right) \varphi=y^{\lambda} \gamma_{3}(S)
$$

Since $\left[R, \gamma_{1}(S)\right]=\gamma_{2}(S)$ we also deduce that $\hat{\varphi}$ raises the elements of $\gamma_{1}(S) / \gamma_{2}(S)$ to the power $\lambda^{2}$. By Lemma 3.10 we deduce that $\hat{\varphi}$ raises the elements of $Z_{2}(S) / Z(S)$ to the power $\lambda^{-1}$. Let $s_{1}, s_{2} \in S$ be such that $\gamma_{1}(S)=\left\langle s_{1}\right\rangle \gamma_{2}(S)$ and $\gamma_{2}(S)=$ $\left\langle s_{2}\right\rangle \gamma_{3}(S)$. Then $s_{1} \hat{\varphi}=s_{1}^{\lambda^{2}} u$ and $s_{2} \hat{\varphi}=s_{2}^{\lambda} v$ for some $u \in \gamma_{3}(S)$ and $v \in Z_{2}(S)$. Thus

$$
\left[s_{1}, s_{2}\right] \hat{\varphi}=\left[s_{1}^{\lambda^{2}} u, s_{2}^{\lambda} v\right]=\left[s_{1}, s_{2}\right]^{\lambda^{3}} \quad \bmod Z(S)
$$

Suppose $\left[s_{1}, s_{2}\right] \notin Z(S)$. Then, since $\left[s_{1}, s_{2}\right] \in\left[\gamma_{1}(S), \gamma_{2}(S)\right] \leq \gamma_{4}(S)=Z_{2}(S)$, we get

$$
\left[s_{1}, s_{2}\right] \hat{\varphi}=\left[s_{1}, s_{2}\right]^{\lambda^{-1}} \quad \bmod Z(S)
$$

from Lemma 3.10. Therefore $\lambda^{3} \equiv \lambda^{-1}(\bmod p)$ and so $\lambda^{4} \equiv 1(\bmod p)$. Hence either $\left[s_{1}, s_{2}\right] \in Z(S)$ or $p=5$. In the former case, as $\left[\gamma_{1}(S), \gamma_{3}(S)\right] \leq Z(S)$, we deduce that

$$
\left[\gamma_{1}(S), \gamma_{1}(S)\right]=\left[\gamma_{1}(S), \gamma_{2}(S)\right]=\left[\left\langle s_{1}\right\rangle \gamma_{2}(S),\left\langle s_{2}\right\rangle \gamma_{3}(S)\right] \leq Z(S)
$$

Thus $\gamma_{1}(S) / Z(S)$ is abelian and Lemma 7.4 implies $\gamma_{1}(S)$ is extraspecial. This proves that either $\gamma_{1}(S)$ is extraspecial or $p=5$.

Suppose that $p=5$ and assume that $\gamma_{1}(S)$ is not extraspecial. Then $|S|=5^{6}$. All the groups of order $5^{6}$ are known. We use Magma and the package from [50] to check that there are 39 maximal class 5 -groups, 16 of which are exceptional. Four of these groups have $\operatorname{Aut}(R)$ non-soluble and two of them have $\gamma_{1}(S)$ not extraspecial. This leaves two groups to consider. For one of the cases $\operatorname{Aut}(S)$ is a 5 -group and so this cannot support a fusion system with $\operatorname{Aut}_{\mathcal{F}}(R)$ non-soluble. In SmallGroup $\left(5^{6}, 661\right)$, we have $\operatorname{Out}_{\mathcal{F}}(S)$ has cyclic Sylow 2 -subgroups of order 4. In particular, $\operatorname{Aut}_{\mathcal{F}}(S)$ is uniquely determined up to isomorphism and by restriction we have a subgroup $Y$ of $\operatorname{Aut}_{\mathcal{F}}(R)$ of order $2^{2} .5^{4}$. Calculating in $\operatorname{Aut}_{\mathcal{F}}(R)$ we find two conjugacy classes of subgroups $X$ containing $\operatorname{Inn}(R)$ and with $X / \operatorname{Inn}(R) \cong \mathrm{SL}_{2}(5)$. Exactly one of these classes contains an $\operatorname{Aut}(R)$-conjugate of $Y$. We check that the corresponding fusion system is saturated using [50] (though it obviously is). This is the fusion system described in (ii). It remains to prove that there are no other candidates for $\mathcal{F}$-essential subgroups on $S=\operatorname{SmallGroup}\left(5^{6}, 661\right)$ when $R$ is $\mathcal{F}$-essential. Computer code to do this using [50] is described in Subsection C.3. However, we can also present an argument which does not require a computer. Suppose that $E \leq S$ is an $\mathcal{F}$-pearl. Then there exist $\phi \in N_{\operatorname{Aut}_{\mathcal{F}}(E)}\left(\operatorname{Out}_{S}(E)\right)$ of order 4 which is the restriction $\hat{\phi} \in \operatorname{Aut}_{\mathcal{F}}(S)$. Recall that the Sylow 2-subgroup of $\operatorname{Aut}(S)$ has order 4. It follows that $\left.\hat{\phi}\right|_{R} \in \operatorname{Aut}_{\mathcal{F}}(R)$. But then $\hat{\phi}$ normalizes $R$, $\gamma_{1}(S)$ and $E \gamma_{2}(S)$ and these are distinct maximal subgroups of $S$. In particular, $\phi$ acts on $S / \gamma_{2}(S)$ as a scalar. However, we know that $\left.\hat{\phi}\right|_{R} \in N_{\operatorname{Aut}_{\mathcal{F}}(R)}\left(\operatorname{Aut}_{S}(R)\right)$ and this element does not act as a scalar on $S / \gamma_{2}(S)$, a contradiction. Hence $\mathcal{F}$ has no $\mathcal{F}$-pearls. Now suppose that $E$ is $\mathcal{F}$-essential and $E \leq \gamma_{1}(S)$ with $E \not \leq$ R. Notice that $Z\left(\gamma_{1}(S)\right)=Z(S)$ and, as $\gamma_{1}(S)$ is not extraspecial, $\gamma_{1}(S)^{\prime}=$ $Z_{2}(S)$ and $\gamma_{3}(S) / Z(S)=Z\left(\gamma_{1}(S) / Z_{2}(S)\right)$. Moreover, $\gamma_{2}(S)=C_{\gamma_{1}(S)}\left(Z_{2}(S)\right)$. Thus $S$ stabilizes the characteristic series $\gamma_{1}(S)>\gamma_{2}(S)>\gamma_{3}(S)>\gamma_{4}(S)$ of $\gamma_{1}(S)$. Lemma 5.15 implies that $\gamma_{1}(S)$ is not $\mathcal{F}$-essential. Hence $E<\gamma_{1}(S)$. By Lemma 6.8 we deduce that $|E|=5^{4}$. Thus $E$ is a maximal subgroup of $\gamma_{1}(S)$. In particular $Z_{2}(S)=\gamma(S)^{\prime}=\left[\gamma_{1}(S), E\right] \leq E$. Since $E \not \leq R$ we deduce that $Z_{2}(S) \not \leq Z(E)$ and so $E$ is not abelian. Thus $1 \neq[E, E]<Z_{2}(S)$, that implies $|[E, E]|=p$. In particular $Z(S)[E, E] \leq Z(E)$. If $Z(S) \neq[E, E]$ then $Z(S)[E, E]=Z_{2}(S)$ and so $Z_{2}(S) \leq Z(E)$, a contradiction. Thus $Z(S)=[E, E]$. The group $\gamma_{3}(S)$
stabilizes the sequence $1<Z<E$ and by Lemma 5.15 we deduce that $\gamma_{3}(S) \leq$ $E$. Also $E$ is not extraspecial (it has order $5^{4}$ ), hence $|Z(E)|=5^{2}$. The group $\operatorname{Out}_{S}(E)$ acts non-trivially on $E / Z(E)$ which is elementary abelian of order 25. Hence by Proposition 5.17 we deduce that $O^{5^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}(E)\right) \cong \operatorname{SL}_{2}(5)$. Let $\tau \in$ $Z\left(O^{5^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}(E)\right)\right)$ be an involution. We have already proved that $\gamma_{1}(S)$ is not $\mathcal{F}$ essential, hence $E$ is maximal $\mathcal{F}$-essential in $S$ and so there is $\hat{\tau} \in \operatorname{Aut}_{\mathcal{F}}(S)$ such that $\left.\hat{\tau}\right|_{E}=\tau$. Note that $\hat{\tau}$ centralizes $\gamma_{2}(S) / \gamma_{3}(S) \cong \gamma_{1}(S) / E$ and $Z(E) / Z(S)$. Since $E$ is non abelian and $\gamma_{3}(S)$ is abelian, we deduce that $Z(E) \leq \gamma_{3}(S)$ and so $\gamma_{3}(S)=$ $Z_{2}(S) Z(E)$. Hence $\gamma_{3}(S) / Z_{2}(S)$ is congruent to $Z(E) / Z(S)$ as a $\hat{\tau}$-group and it is therefore centralized by $\hat{\tau}$. We showed that $\hat{\tau}$ centralizes $\gamma_{2}(S) / Z_{2}(S)$. Now, $\hat{\tau}$ acts non-trivially on $S / \gamma_{1}(S)$ and $\hat{\tau}$ is an automorphism of the group $S / Z(S)$, that is not exceptional. Therefore we get a contradiction from Lemma 3.14.

This proves that $\mathcal{E}_{\mathcal{F}}=\{R\}$ and completes the description of $\mathcal{F}$ in the case when $S=\operatorname{SmallGroup}\left(5^{6}, 661\right)$ and $R$ is $\mathcal{F}$-essential.

Lemma 7.6. Assume Hypothesis 7.1 holds. Suppose that $E \in \mathcal{E}_{\mathcal{F}}$ with $E \not 又$ $\gamma_{1}(S)$. Then either $E$ is an $\mathcal{F}$-pearl or $|S|=p^{6}$ and $E=C_{S}\left(Z_{2}(S)\right)$.

Proof. Suppose that the lemma is false. Set $R=C_{S}\left(Z_{2}(S)\right)$, and let $E$ be an $\mathcal{F}$-essential subgroup of $S$ chosen of maximal order with $E$ not contained in $\gamma_{1}(S)$. Then $E$ is not an $\mathcal{F}$-pearl. Lemma 6.4 and Lemma 6.8 together imply that $E \leq R$ and $|E| \geq p^{4}$. By Lemma 7.5 , we have $R \notin \mathcal{E}_{\mathcal{F}}$ and so $E<R$. Because $E<R$ and $E$ is $\mathcal{F}$-essential, we have $Z_{2}(S)<E$. By Lemma 3.3 (vi), $S / Z(S)$ is not exceptional. Since $E / Z(S)$ is not contained in $\gamma_{1}(S / Z(S))=\gamma_{1}(S) / Z(S)$, Lemma 3.6 implies that $E / Z(S)$ has maximal class. Since $|E| \geq p^{4}, E / Z(S)$ is not abelian. In particular, $Z(E / Z(S))=Z_{2}(S) / Z(S)$ and this implies $Z(E)=Z_{2}(S)$ is $\operatorname{Aut}_{\mathcal{F}}(E)$-invariant. Since $R \notin \mathcal{E}_{\mathcal{F}}$, Lemma 6.6 implies that Aut $\mathcal{F}_{\mathcal{F}}(E)$ leaves $Z(S)$ invariant. Now $C_{\operatorname{Aut}_{\mathcal{F}}(E)}(Z(S))$ has $p^{\prime}$-index in $\operatorname{Aut}_{\mathcal{F}}(E)$ and, since $E / Z(S)$ has maximal class, Theorem 3.9 implies that $|E / Z(S)|=p^{3}$. Therefore $O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)$ acts on $E / Z(E)$ as $\mathrm{SL}_{2}(p)$. Let $\tau \in \operatorname{Aut}_{\mathcal{F}}(E)$ project to $Z\left(O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}(E)\right)\right)$ be an involution. As $O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)$ centralizes $Z_{2}(S), \tau$ centralizes $Z_{2}(S)$. Then, the maximal choice of $E$ implies that there exists $\hat{\tau} \in \operatorname{Aut}_{\mathcal{F}}(S)$ so that $\left.\hat{\tau}\right|_{E}=\tau$. In particular, $\hat{\tau}$ has even order and inverts $E / Z(E)$. Since $Z(E)=Z_{2}(S)$ and $E \gamma_{1}(S)=S, \hat{\tau}$ inverts $S / \gamma_{1}(S)$. Assume that $\left(y \gamma_{2}(S)\right) \hat{\tau}=y^{b} \gamma_{2}(S)$ for some $b \in$ $\operatorname{GF}(p)^{\times}$. Then, in Lemma 3.10, we have $n$ is even and $a=-1$ and, as $\tau$ centralizes $Z_{2}(S)$, we obtain the unfathomable equations

$$
\begin{aligned}
a^{n-3} b & =-b=1 \\
a^{n-3} b^{2} & =-b^{2}=1
\end{aligned}
$$

This contradiction completes the proof of the lemma.
Proof of Proposition 7.2. Assume Hypothesis 7.1 holds. Let $E$ be an $\mathcal{F}$ essential subgroup. By Lemma 6.4 either $E$ is an $\mathcal{F}$-pearl, $E \leq \gamma_{1}(S)$ or $E \leq$ $C_{S}\left(Z_{2}(S)\right)$. If $E$ is an $\mathcal{F}$-pearl, then $E$ is abelian by Lemma 7.3. If $E$ is not an $\mathcal{F}$-pearl and it is not contained in $\gamma_{1}(S)$ then $E=C_{S}\left(Z_{2}(S)\right)$ by Lemma 7.6. This proves that

$$
\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{P}_{a}(\mathcal{F}) \cup\left\{C_{S}\left(Z_{2}(S)\right)\right\} \cup\left\{E \mid Z(S)<E \leq \gamma_{1}(S)\right\}
$$

which is the displayed statement of the proposition.

If $\mathcal{P}_{a}(\mathcal{F}) \neq \emptyset$, then $\left[\mathbf{2 8}\right.$, Theorem 3.14] implies $|S|=p^{p-1}$ and $\gamma_{1}(S)$ is extraspecial. Hence (i) holds.

If $C_{S}\left(Z_{2}(S)\right)$ is $\mathcal{F}$-essential, then Lemma 7.5 gives (ii).
If $\gamma_{1}(S)$ is extraspecial, then $\gamma_{1}(S) / Z(S)$ is abelian and Lemma 7.4 gives (iii).

## 8. The structure of $\gamma_{1}(S)$ when $\gamma_{1}(S)$ is $\mathcal{F}$-essential and $S$ is not exceptional

In this section we continue to assume Hypothesis 6.3. In addition, we assume that $S$ is not exceptional and $\gamma_{1}(S)$ is $\mathcal{F}$-essential. So we work with

Hypothesis 8.1. Hypothesis 6.3 holds with $S$ not exceptional and $\gamma_{1}(S) \in \mathcal{E}_{\mathcal{F}}$.
Our objective in this section is to explore the structure of $\Omega_{1}\left(\gamma_{1}(S)\right)$ when Hypothesis 8.1 holds.

Lemma 8.2. Assume that Hypothesis 8.1 holds. If $O_{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right.$ ) is not centralized by $\operatorname{Out}_{S}\left(\gamma_{1}(S)\right)$, then $\Omega_{1}\left(\gamma_{1}(S)\right)$ is elementary abelian of order either $p^{p-1}$ or $p^{p}$.

Proof. Assume that $O_{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right)$ is not centralized by $\operatorname{Out}_{S}\left(\gamma_{1}(S)\right)$. Set $R_{0}=O_{p, p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right)$ and

$$
R=\left[R_{0}, \operatorname{Aut}_{S}\left(\gamma_{1}(S)\right)\right] \operatorname{Inn}\left(\gamma_{1}(S)\right)
$$

Then $R_{0} / \operatorname{Inn}\left(\gamma_{1}(S)\right)=O_{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right), R>\operatorname{Inn}\left(\gamma_{1}(S)\right)$ and, by [24, Theorem 5.3.10], there exists an $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$-chief factor $V$ in $\Omega_{1}\left(\gamma_{1}(S)\right)$ which is not centralized by $R$. The definition of $R$ and coprime action implies that $R / C_{R}(V)$ is not centralized by $\operatorname{Aut}_{S}\left(\gamma_{1}(S)\right) C_{R}(V) / C_{R}(V)$. Applying Proposition 2.5 delivers $|V| \geq p^{p-1}$. Since $V$ is elementary abelian, either $n=p+1$ and $\gamma_{1}(S)=\Omega_{1}\left(\gamma_{1}(S)\right)$ or $V=\Omega_{1}\left(\gamma_{1}(S)\right)$ by Lemma 3.2 (iv). In the latter case, $\Omega_{1}\left(\gamma_{1}(S)\right)$ is elementary abelian and we are done. Assume $|S|=p^{p+1}$ and $\gamma_{1}(S)=\Omega_{1}\left(\gamma_{1}(S)\right)$ has order $p^{p}$. If $V=\gamma_{2}(S)$, then, as $V$ is irreducible, $V \leq Z\left(\gamma_{1}(S)\right)$ and $\Omega_{1}\left(\gamma_{1}(S)\right)$ is abelian. If $V=\gamma_{1}(S) / Z(S)$ and $\gamma_{1}(S)$ is extraspecial, then $C_{S}\left(Z_{2}(S)\right) \neq \gamma_{1}(S)$ and $S$ is exceptional, a contradiction. We conclude that $\Omega_{1}\left(\gamma_{1}(S)\right)$ is elementary abelian of order $p^{p-1}$ or $p^{p}$.

Remark 8.3. When $p=5$ and $|S|=5^{6}$, the baby Monster sporadic simple group provides an example which demonstrates that when $S$ is exceptional Out $_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ may have a non-central normal subgroup of $5^{\prime}$-order which is not centralized by Out $_{S}\left(\gamma_{1}(S)\right)$. See [47, Table 5.1] for example.

Recall that groups of $\mathrm{L}_{2}(p)$-type are defined in Definition 4.1.
Lemma 8.4. Assume that Hypothesis 8.1 holds. If $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ is not of $\mathrm{L}_{2}(p)$ type, then either
(i) $\Omega_{1}\left(\gamma_{1}(S)\right) \leq Z\left(\gamma_{1}(S)\right)$; or
(ii) $\left|\Omega_{1}\left(\gamma_{1}(S)\right): Z\left(\gamma_{1}(S)\right)\right|=p$.

In particular, $\Omega_{1}\left(\gamma_{1}(S)\right)$ is elementary abelian.
Proof. Assume that Out $_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ is not of $L_{2}(p)$-type. We have $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right| \leq$ $p^{p}$ by Lemma 3.2 (iv). Set $Z=Z\left(\gamma_{1}(S)\right)$. If $Z \nless \Omega_{1}\left(\gamma_{1}(S)\right)$, then, as $S$ has maximal class, $\Omega_{1}\left(\gamma_{1}(S)\right) \leq Z$ and (i) holds. So suppose that $Z<\Omega_{1}\left(\gamma_{1}(S)\right)$. Then $Z$ is elementary abelian because $\Omega_{1}\left(\gamma_{1}(S)\right)$ has exponent $p$.

Since $S$ is not exceptional, $Z \geq Z_{2}(S)$ and so $Z$ is not centralized by $S$. In particular, $Z$ admits a non-trivial action of $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ and $C_{\mathrm{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)}(Z)$ is a $p^{\prime}$-group. As $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ is not of $\mathrm{L}_{2}(p)$-type, $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right) / C_{\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)}(Z)$ is not of $\mathrm{L}_{2}(p)$-type. Therefore Theorem 4.2 implies $|Z| \geq p^{\frac{2}{3}(p-1)}$.

Define $X=\Omega_{1}\left(\gamma_{1}(S)\right) / Z$. Then, assuming that (ii) does not hold, $|X| \geq$ $p^{2}$. Since $\gamma_{1}(S)$ is the 2 -step centralizer, $V=X /\left[X, \gamma_{1}(S)\right] Z$ has order at least $p^{2}$. In particular, $V$ is a $\operatorname{GF}(p) \operatorname{Out}_{\mathcal{F}}(S)$-module and it is not centralized by $S$. Furthermore, $C_{\operatorname{Out}_{\mathcal{F}}(S)}(V)$ is a $p^{\prime}$-group and $\operatorname{Out}_{\mathcal{F}}(S) / C_{\mathrm{Out}_{\mathcal{F}}(S)}(V)$ is not of $\mathrm{L}_{2}(p)$ type. Applying Theorem 4.2 yields $|V| \geq p^{\frac{2}{3}(p-1)}$. Therefore, as $p \geq 5$, we obtain the contradiction

$$
p^{p} \geq\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right| \geq|Z||V| \geq p^{2 \frac{2}{3}(p-1)}=p^{\frac{4}{3}(p-1)}>p^{p}
$$

Hence $|X| \leq p$ and (ii) holds. This completes the proof.
Lemma 8.5. Assume that Hypothesis 8.1 holds. If $\Omega_{1}\left(\gamma_{1}(S)\right)$ is non-abelian, then $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right|=p^{p-1}$,

$$
Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right)=\Phi\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right)=\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S)\right]=Z\left(\gamma_{1}(S)\right)
$$

In particular, $\Omega_{1}\left(\gamma_{1}(S)\right)$ has nilpotency class 2. Furthermore,

$$
L=\left\langle\operatorname{Out}_{S}\left(\gamma_{1}(S)\right)^{\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)}\right\rangle \cong \operatorname{PSL}_{2}(p)
$$

and, if $H \leq \operatorname{Aut}_{\mathcal{F}}(S)$ has $p^{\prime}$-order and projects to a complement to $\operatorname{Out}_{S}\left(\gamma_{1}(S)\right)$ in $N_{L}$ (Out $\left._{S}(E)\right)$, then $\left|C_{\Omega_{1}\left(\gamma_{1}(S)\right)}(H)\right|=p^{2}$.

Proof. Set $V=\Omega_{1}\left(\gamma_{1}(S)\right) /\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S)\right]$ and

$$
V_{1}=\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S)\right] /\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S), \gamma_{1}(S)\right]
$$

Then, as $\gamma_{1}(S)$ is the 2-step centralizer, $|V| \geq p^{2}$ and so, $V$ and $V_{1}$ are not centralized by $S$ unless $V_{1}=\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S)\right]=Z(S)$ has order $p$.

By Lemmas 8.2, and 8.4, $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ is of $\mathrm{L}_{2}(p)$-type and $O_{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right)$ is centralized by $\operatorname{Out}_{S}\left(\gamma_{1}(S)\right)$. Since $O_{p}\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right)=1$, $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ is not $p$-soluble and so, setting $L=\left\langle\operatorname{Out}_{S}\left(\gamma_{1}(S)\right)^{\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)}\right\rangle$, we have $L$ has a quotient isomorphic to $\mathrm{PSL}_{2}(p)$ and $L$ is quasisimple. Hence $L \cong \mathrm{SL}_{2}(p)$ or $\mathrm{PSL}_{2}(p)$. Using Lemma 6.9, we obtain $Z(L)=1$ and so $L \cong \operatorname{PSL}_{2}(p)$. Let $\bar{H}$ be a complement to $\operatorname{Out}_{S}\left(\gamma_{1}(S)\right)$ in $N_{L}\left(\operatorname{Out}_{S}\left(\gamma_{1}(S)\right)\right)$ and $H$ be a preimage of $\bar{H}$ of $p^{\prime}$-order. Then $H$ is cyclic of order $(p-1) / 2$.

In $\Omega_{1}\left(\gamma_{1}(S)\right)$, assume that $W>U>T \geq 1$ are $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$-invariant subgroups with $\bar{W}=W / U$ and $\bar{U}=U / T$ both $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$-chief factors. Then $\bar{W}$ and $\bar{U}$ can be regarded as $\operatorname{GF}(p) L$-modules. If $|\bar{W}|=|\bar{U}|=p$, then, as $\gamma_{1}(S)$ is the 2step centralizer, $\operatorname{Aut}_{S}\left(\gamma_{1}(S)\right) \leq O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right) \operatorname{Inn}\left(\gamma_{1}(S)\right)$ centralizes $U / T$, contrary to $S$ having maximal class. Hence at least one of the chief factors has order greater then $p$. Write $|\bar{W}|=p^{w}$ and $|\bar{U}|=p^{u}$ with $p^{3} \leq p^{u+w} \leq\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right| \leq p^{p}$. Using Lemma 4.5 and $L \cong \operatorname{PSL}_{2}(p)$, gives $u$ and $v$ are odd and, as $u+w \leq p, H$ centralizes

$$
[\bar{W}, S ;(w-1) / 2] /[\bar{W}, S ;(w+1) / 2] \text { and }[\bar{U}, S ;(u-1) / 2] /[\bar{U}, S ;(u+1) / 2] .
$$

Because $S$ has maximal class, there exists $\ell \geq 1$ such

$$
[W, S ;(w-1) / 2]=\gamma_{\ell}(S) / U
$$

and

$$
[U, S ;(u-1) / 2]=\gamma_{\ell+(w+1) / 2+(u-1) / 2}(S) / T
$$

By Lemma 3.14,

$$
\ell-(\ell+(w+1) / 2+(u-1) / 2) \equiv 0 \quad(\bmod (p-1) / 2)
$$

Hence

$$
w+u=k(p-1)
$$

for some integer $k$. Therefore $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right| \geq p^{p-1}$. Assume $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right|=p^{p-1}$. Then $W=\Omega_{1}\left(\gamma_{1}(S)\right)$, and

$$
U=Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right)=\Phi\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right)=\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S)\right]=Z\left(\gamma_{1}(S)\right)
$$

and the statements in the lemma hold. Assume that $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right|>p^{p-1}$. Then Lemma 3.2 (iv) implies that $|S|=p^{p+1}$. This gives $\Omega_{1}\left(\gamma_{1}(S)\right)=\gamma_{1}(S)$ and there exists $\gamma_{1}(S) \geq W>U>T>A \geq 1$ with $W / U, U / T$ and $T / A$ each $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ chief factors. We assume that they have order $p^{u}, p^{w}$ and $p^{t}$ respectively. Then $u+$ $w=w+t=p-1$ and $u+w+t \leq p$. This means that $w=t=1$ and $u=p-2$. Notice that $Z\left(\gamma_{1}(S)\right)$ is normalized by $S$ and is $L$-invariant. So $Z\left(\gamma_{1}(S)\right) \in\{W, U, T\}$. Since $\gamma_{1}(S)$ is non-abelian by assumption, we must have $Z\left(\gamma_{1}(S)\right)=T=Z(S)$. Hence $S$ is exceptional, a contradiction.

We remark that we have constructed a saturated fusion system which satisfies the conclusion of Lemma 8.5 using Magma (see Subsection C.4). The example is realized by a group $G$ of shape $7^{3+3}: \mathrm{PGL}_{2}(7)$. Taking $S \in \operatorname{Syl}_{7}(G), S$ has exponent $7, \gamma_{1}(S)$ is special with centre of order $7^{3}$. Setting $\mathcal{F}=\mathcal{F}_{S}(G)$, we get $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right) \cong 7^{3}: \mathrm{PGL}_{2}(7)$. In this case, $\gamma_{1}(S)$ is the unique $\mathcal{F}$-essential subgroup and $\mathcal{F}$ cannot be further decorated with pearls to create a larger saturated fusion system $\mathcal{G}$ with $\mathcal{G}$-pearls.

## 9. The structure of $\gamma_{1}(S)$ when $\gamma_{1}(S)$ is $\mathcal{F}$-essential and $S$ is exceptional

We now consider Hypothesis 7.1 once again, applying results from Section 8 to fusion systems on the non-exceptional group $S / Z(S)$. The objective of this section is to prove

Proposition 9.1. Assume that Hypothesis 7.1 holds with $\gamma_{1}(S) \in \mathcal{E}_{\mathcal{F}}$. Then $\gamma_{1}(S)$ is extraspecial.

Proof. Assume that $S$ is exceptional and $\gamma_{1}(S)$ is $\mathcal{F}$-essential but is not extraspecial.

Set $Q=\gamma_{1}(S), V=Z_{2}(Q), Z=Z(Q)=Z(S)$ and $\mathcal{K}=N_{\mathcal{F}}(Z)$. Then $Z$ is fully $\mathcal{F}$-normalized and so $\mathcal{K}$ is saturated, $Q$ is a $\mathcal{K}$-essential subgroup and $Q / Z$ is a $\mathcal{K} / Z$-essential subgroup. By Lemma 3.3 (v) and (vi), $S / Z(S)$ is not exceptional and $|S| \leq p^{p+1}$ and Lemma 3.2 (vi) yields $S / Z(S)$ has exponent $p$. Because $Q$ is not extraspecial and $Q / Z=\Omega_{1}(Q / Z)$, we know $\Omega_{1}(Q / Z)$ is not elementary abelian. Hence Lemma 8.5 yields $(Q / Z)^{\prime}=Z(Q / Z)=V / Z$ and $Q$ has order $p^{p}$. Furthermore, $O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}(Q)\right) \cong \operatorname{PSL}_{2}(p)$.

Let $A$ be the preimage of $O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}(Q)\right)$ in $\operatorname{Aut}_{\mathcal{F}}(Q)$ and $a$ be a natural number such that $|V / Z|=p^{a}$. Then, as $V / Z$ is an irreducible $\operatorname{GF}(p) A$-module by Lemma 8.5, $a$ is odd and so $V$ is not extraspecial and, as $A$ acts irreducibly on $V / Z$, we deduce that $V$ is elementary abelian. Because $A$ acts irreducibly on $Q / V$ and $V$ is abelian, we also have $C_{Q}(V)=V$. Therefore, using $Z$ has order $p$ and $Q / Z(Q)$ and $V / Z$ are irreducible $\mathrm{GF}(p) A$-modules, we have $Q / V \cong \operatorname{Hom}_{\mathrm{GF}(p)}(V, Z)$ (the dual of $V$ ) which also has order $p^{a}$. Since $|Q|=p^{p}$, we infer that $a=(p-1) / 2$. As $a$ is odd, we also have $p \equiv 3(\bmod 4)$. Since $Q / Z$ is special, the commutator $\operatorname{map} \kappa: Q / V \times Q / V \rightarrow V / Z$ induces a non-trivial $\mathrm{GF}(p) A$-module homomorphism $\kappa^{*}: \Lambda^{2}(Q / V) \rightarrow V / Z$. In particular, $V / Z$ is isomorphic to a quotient of $\Lambda^{2}(Q / V)$ as $\operatorname{GF}(p) A$-modules. Writing $d=a-1=(p-1) / 2-1$, Proposition 4.6 yields $d \equiv 2$ $(\bmod 4)$. Hence $p \equiv 7(\bmod 8)$. Finally, we note that by Lemma 3.2(vi), $\gamma_{2}(S)$ has exponent $p$ and thus as $Q$ is regular by Lemma 3.2(ii), Lemma 2.1(ii) and the irreducible action of $A$ on $Q / V$ implies $Q$ has exponent $p$. We summarise what has been established.
(9.1.1) The following hold:
(i) $p \equiv 7(\bmod 8)$ and $d \equiv 2(\bmod 4)$.
(ii) $Q$ has exponent $p$ and nilpotency class $3, V=Q^{\prime}=C_{Q}(V)$ has order $p^{(p+1) / 2}$ and $Z=Z(Q)$ has order $p$.
(iii) $A / \operatorname{Inn}(Q) \cong \operatorname{PSL}_{2}(p)$ acts irreducibly on $Q / V \cong V / Z \cong \mathbf{V}_{\frac{p-3}{2}}$ and $A$ centralizes $Z$.

We intend to make an explicit calculation and so we begin by establishing some notation. Let $\epsilon$ be a generator of $\operatorname{GF}(p)^{\times}=(\mathbb{Z} / p \mathbb{Z})^{\times}$. Thus $\epsilon$ is an integer with $1 \leq \epsilon \leq p-1$. We regard $Q / V$ and $V / Z$ as unfaithful representations of $\mathrm{SL}_{2}(p)$ and we identify $\operatorname{Out}_{S}(Q)$ with $\left\langle\left(\begin{array}{l}1 \\ 0 \\ 1\end{array}\right)\right\rangle$ and let $\tau \in N_{A}\left(\operatorname{Aut}_{S}(Q)\right)$ have order $(p-1) / 2$ be $\left(\begin{array}{cc}\epsilon & 0 \\ 0 & \epsilon^{-1}\end{array}\right)$. The element $\iota$ has order 2 and corresponds to $\left\langle\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)\right\rangle$ in $A$. Hence $\iota$ inverts $\tau$ and does not normalize $\operatorname{Out}_{S}(Q)$.

We have $Q / V$ is semisimple as a $\operatorname{GF}(p)\langle\tau\rangle$-module and $\tau$ normalizes $\gamma_{j}(S)$ for all $j \geq 1$. The semisimple action of $\tau$ implies that we may select $t_{i} \in \gamma_{i}(S) \backslash \gamma_{i+1}(S)$, $1 \leq i \leq(p-1) / 2$ such that $t_{i} V$ is an eigenvector for $\tau$ on $Q / V$.

By (9.1.1)(iii) the $\mathrm{GF}(p) \mathrm{PSL}_{2}(p)$-modules $Q / V$ and $V / Z$ can identified with the $\mathrm{GF}(p) \mathrm{SL}_{2}(p)$-module $\mathbf{V}_{\frac{p-3}{2}}=\mathbf{V}_{d}$. When we make this identification, we may suppose that $t_{1}$ corresponds to $x^{d}, t_{(p-1) / 2}$ is $y^{d}$ and generally $t_{j}$ corresponds to $x^{d-j+1} y^{j-1}$ for $1 \leq j \leq(p-1) / 2=d+1$. We calculate that

$$
\left(t_{j} V\right) \tau=t_{j}^{\epsilon^{d-2 j+2}} V
$$

Define $m=(d+2) / 2$. Then, as $p \equiv 7(\bmod 8)$ by $(9.1 .1)(\mathrm{i}), m$ is even and we also have $\left(t_{m} V\right) \tau=t_{m} V$. Furthermore, we can calculate
(9.1.2) $\left(t_{j} V\right) \iota=\left(t_{d+2-j} V\right)^{(-1)^{(j-1)}}$.

For $1 \leq \rho_{i} \in V, 1 \leq i \leq d+1$ be such that $\rho_{i} Z$ are eigenvectors for $\tau$ with $\rho_{i} \in \gamma_{(p-1) / 2+i}(S) \backslash \gamma_{(p-1) / 2+i+1}(S)$. Lemma 3.11 implies that for each $j$ we have

$$
\left(\rho_{j} Z\right) \tau=\rho_{j}^{\epsilon^{d-2 j+2}} Z
$$

which is consistent with making the standard identification with $\mathbf{V}_{d}$. Just as above we have
(9.1.3) $\left(\rho_{j} Z\right) \iota=\left(\rho_{d+2-j} Z\right)^{(-1)^{(j-1)}}$.

Observe that, as $m$ is even, (9.1.2) and (9.1.3) demonstrate that $\iota$ inverts $t_{m} V$ and $\rho_{m} Z$.

We use the commutator relations as follows

$$
\left[\rho_{j}, t_{k}\right] \tau=\left[\rho_{j}^{\epsilon^{d-2 j+2}} z, t_{k}^{\epsilon^{d-2 k+2}} v\right]=\left[\rho_{j}^{\epsilon^{d-2 j+2}}, t_{k}^{\epsilon^{d-2 k+2}}\right]=\left[\rho_{j}, t_{k}\right]^{\epsilon^{d d-2(j+k)+4}}
$$

where $v \in V$ and $z \in Z$. Since $\tau$ centralizes $Z$ by (9.1.1)(iii), using the above commutator calculation and exploiting (9.1.1)(ii) to obtain equality demonstrates that the following statement holds.
(9.1.4) We have $C_{Q}\left(\rho_{j}\right)=V\left\langle t_{k} \mid k \neq d+2-j\right\rangle$.

Set $r_{j}=\left[t_{j}, t_{m}\right]$ for $1 \leq j \leq d+1$, then $r_{m}=1$ and the action of $\tau$ shows that $r_{j} \in\left\langle\rho_{j}\right\rangle Z$.

Since $Q$ has nilpotency class 3 and $p \geq 5$, Lemma 2.3 and Theorem 2.4 imply that $V \leq E_{2}(Q)<Q$ and the irreducible action of $A$ on $Q / V$ then yields $V=$ $E_{2}(Q)$. We obtain a contradiction by demonstrating that $t_{m} \in E_{2}(Q)$. Thus we calculate $\left[t_{m}, y, y\right]$ where $y=\prod_{i=1}^{d+1} t_{i}^{a_{i}}$ with $1 \leq a_{i} \leq p$ is a coset representative of $V$ in $Q$. Notice that in this next calculation, we have $\left[t_{m}, w_{1} w_{2}\right] \in\left[t_{m}, w_{1}\right]\left[t_{m}, w_{2}\right] Z$ and so $\left[t_{m}, w_{1} w_{2}, w\right]=\left[\left[t_{m}, w_{1}\right]\left[t_{m}, w_{2}\right], w\right]$ for $w, w_{1}, w_{2} \in Q$.

$$
\begin{aligned}
{\left[t_{m}, y, y\right] } & =\left[t_{m}, \prod_{i=1}^{d+1} t_{i}^{a_{i}}, y\right]=\left[\prod_{i=1}^{d+1}\left[t_{m}, t_{i}^{a_{i}}\right], y\right]=\left[\prod_{i=1}^{d+1} r_{i}^{a_{i}}, y\right]=\prod_{i=1}^{d+1}\left[r_{i}, y\right]^{a_{i}} \\
& =\prod_{i=1}^{d+1}\left[r_{i}, \prod_{k=1}^{d+1} t_{k}^{a_{k}}\right]^{a_{i}}=\prod_{i=1}^{d+1}\left[r_{i}, t_{d+2-i}\right]^{a_{i}+a_{d+2-i}}
\end{aligned}
$$

where the last equality follows from (9.1.4). Notice that, as $m$ is even by (9.1.1)(i), we apply $\iota$ as follows

$$
r_{i} \iota=\left[t_{i} \iota, t_{m} \iota\right]=\left[t_{d+2-i}^{(-1)^{(i-1)}}, t_{m}^{-1}\right]=r_{d+2-i}^{(-1)^{i}}
$$

Hence, as $d$ is even by (9.1.1)(i), (9.1.2) and (9.1.3) yield

$$
\left[r_{i}, t_{d+2-i}\right] \iota=\left[r_{d+2-i}^{(-1)^{(i)}}, t_{i}^{(-1)^{(d+2-i-1)}}\right]=\left[r_{d+2-i}, t_{i}\right]^{(-1)^{i+(d+2-i-1)}}=\left[r_{d+2-i}, t_{i}\right]^{-1}
$$

and so, as $r_{m}=1$, we can pair elements in the product $\prod_{i=1}^{d+1}\left[r_{i}, t_{d+2-i}\right]^{a_{i}+a_{d+2-i}}$ to obtain $\left[t_{m}, y, y\right]=1$ for all $y \in Q$. Hence $t_{m} \in E_{2}(Q)$ and this is our contradiction which establishes that if $Q$ is $\mathcal{F}$-essential, then $Q$ is extraspecial.

## 10. Locating $\mathcal{F}$-essential subgroups in groups of maximal class I

This section and the following section are devoted to the proof of Theorem D. To set the scene we repeat its statement.

Theorem D. Suppose that $p$ is a prime, $S$ is a p-group of maximal class and order at least $p^{4}$ and $\mathcal{F}$ is a saturated fusion system on $S$. If $E$ is an $\mathcal{F}$ essential subgroup, then either $E$ is an $\mathcal{F}$-pearl, or $E=\gamma_{1}(S)$ or $E=C_{S}\left(Z_{2}(S)\right)$. Furthermore, if $S$ is exceptional, then $\mathcal{P}(\mathcal{F})=\mathcal{P}_{a}(\mathcal{F})$.

Lemmas 6.1 and 6.2 show that Theorem D holds if $p \leq 3$. Hence we may and do assume that $p \geq 5$ and so Hypothesis 6.3 holds and we take our notation from there.

Lemma 10.1. Assume Hypothesis 6.3 holds. If $\gamma_{1}(S)$ is abelian and $E$ is $\mathcal{F}$ essential, then either $E$ is an $\mathcal{F}$-pearl or $E=\gamma_{1}(S)$.

Proof. This is a consequence of Corollary 6.5.
We now show that Theorem D holds when $|S|$ is small.
Lemma 10.2. Assume Hypothesis 6.3 holds. Suppose that $p^{4} \leq|S| \leq p^{5}$ and $\mathcal{F}$ is a saturated fusion system on $S$. If $E$ is $\mathcal{F}$-essential, then $E$ is either an $\mathcal{F}$-pearl or $E \in\left\{\gamma_{1}(S), C_{S}\left(Z_{2}(S)\right)\right\}$.

Proof. Suppose $E$ is not an $\mathcal{F}$-pearl. If $|S|=p^{4}$, then $E$ is normal in $S$ and the result follows from Proposition 6.7. Hence $|S|=p^{5}$ and $|E|=p^{4}$ by Lemma 6.8. In particular, $E$ is a normal subgroup of $S$. Again Proposition 6.7 yields the result.

Because of Corollary 3.12, Proposition 7.2 and Lemmas 10.1 and 10.2 we work under the following hypothesis until the proof of Theorem D is complete.

Hypothesis 10.3. Hypothesis 6.3 holds with $|S| \geq p^{6}$ and the saturated fusion system $\mathcal{F}$ is a minimal counterexample to Theorem D , first with respect to $|S|$ and second with respect to the number of morphisms in $\mathcal{F}$. Furthermore,
(i) $\gamma_{1}(S)$ is not abelian or extraspecial.
(ii) $\operatorname{Out}_{\mathcal{F}}(S)$ is cyclic of order dividing $p-1$ and acts faithfully on $S / \gamma_{1}(S)$.

We now assume that Hypothesis 10.3 is satisfied and say that an $\mathcal{F}$-essential subgroup which is not contained in $\mathcal{P}(\mathcal{F}) \cup\left\{\gamma_{1}(S), C_{S}\left(Z_{2}(S)\right)\right\}$ is a witness (to the fact that Theorem D is false).

Lemma 10.4. If $E$ is a witness, then $E$ is not normal in $S$ and $E<\gamma_{1}(S)$.
Proof. This follows from Lemma 6.4 and Propositions 6.7 and 7.2.
Lemma 10.5. The saturated fusion system $\mathcal{F}$ has no $\mathcal{F}$-pearls. Furthermore, every witness $E$ is properly contained in $\gamma_{1}(S),|E| \geq p^{4}$ and $\Omega_{1}(E)=E \cap \Omega_{1}\left(\gamma_{1}(S)\right)$.

Proof. The fact that $\mathcal{F}$ has no $\mathcal{F}$-pearls follows from Lemma 5.23 and the minimality of $\mathcal{F}$. The first part of the second statement is in Lemma 10.4. Since $\mathcal{F}$ has no $\mathcal{F}$-pearls, Lemma 6.8 implies $|E| \geq p^{4}$. Finally, we note that $\gamma_{1}(S)$ is regular by Lemma 3.2 (ii) and so Lemma 2.1 (iii) gives the final statement.

Our first major consequence of Hypothesis 10.3 is as follows.

Proposition 10.6. We have $O_{p}(\mathcal{F})=1$.
Proof. Let $Q=O_{p}(\mathcal{F})$ and assume that $Q \neq 1$. Let $E$ be a witness. As $Q$ is normal in $S$ and contained in $E$ by Lemma 5.10, Lemma 10.4 implies $Q<E$ with $|S: E| \geq p^{2}$ and so there exists $j \geq 3$ such that $Q=\gamma_{j}(S)$.

We first demonstrate that $Q$ is $\mathcal{F}$-centric. Suppose this is false. Then $C_{S}(Q) \not \leq$ $Q$ and so $C_{S}(Q)=\gamma_{k}(S)$ for some $k<j$ where we assume that $S=\gamma_{0}(S)$. In particular, $Q$ is abelian and $\operatorname{Aut}_{\mathcal{F}}(Q) \cong \operatorname{Out}_{\mathcal{F}}(Q)$. If $C_{S}(Q) \leq E$, then $C_{S}(Q)=$ $C_{E}(Q)$ is $\operatorname{Aut}_{\mathcal{F}}(E)$-invariant. Hence $N_{\mathcal{F}}(E) \subseteq N_{\mathcal{F}}\left(C_{S}(Q)\right) \subset \mathcal{F}$ as $C_{S}(Q)>Q$. Since $E$ is $\mathcal{F}$-essential and $\operatorname{Aut}_{\mathcal{F}}(E) \subset N_{\mathcal{F}}\left(C_{S}(Q)\right)$, Lemma 5.14 implies $E$ is $N_{\mathcal{F}}\left(C_{S}(Q)\right.$ )-essential and this contradicts the minimality of $\mathcal{F}$. Therefore $C_{S}(Q)=$ $\gamma_{k}(S) \not \leq E$ and we choose $k<\ell \leq j$ so that $\gamma_{\ell}(S) \leq E$ and $\gamma_{\ell-1}(S) \not \leq E$. Then $\gamma_{\ell-1}(S) \leq N_{S}(E)$ and $\gamma_{\ell-1}(S)$ centralizes $Q$.

Set $L=\left\langle\operatorname{Aut}_{\gamma_{\ell-1}(S)}(E)^{\operatorname{Aut}_{\mathcal{F}}(E)}\right\rangle$. Then $L$ centralizes $Q$ and $C_{L}(E / Q)$ is a $p$ group by coprime action $\left[\mathbf{2 4}\right.$, Theorem 5.3.6]. As $L$ is normal in $\operatorname{Aut}_{\mathcal{F}}(E)$, we have $O_{p}(L) \leq O_{p}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)=\operatorname{Inn}(E)$. Hence

$$
\operatorname{Inn}(E) \geq O_{p}(L) \geq C_{L}(E / Q)
$$

For $K \leq \operatorname{Aut}_{\mathcal{F}}(E)$, set

$$
\bar{K}=K \operatorname{Inn}(E) / \operatorname{Inn}(E) \leq \operatorname{Out}_{\mathcal{F}}(E)=\overline{\operatorname{Aut}_{\mathcal{F}}(E)}
$$

By Lemma $5.7, \mathcal{F} / Q$ is saturated on $S / Q$ and we know $S / Q$ has maximal class. Since $Q \neq 1$, Lemma 3.3(vi) implies that, if $|S / Q| \geq p^{4}, S / Q$ is not exceptional. We claim $E / Q$ is $\mathcal{F} / Q$-essential. Certainly $E / Q$ is fully $\mathcal{F} / Q$-normalized. Let $J=C_{\operatorname{Aut}_{\mathcal{F}}(E)}(E / Q)$. We have $[J, L] \leq J \cap L \leq C_{L}(E / Q) \leq \operatorname{Inn}(E)$. Hence $\bar{J}$ and $\bar{L}$ commute. Therefore $\bar{J}$ normalizes $\overline{\operatorname{Aut}_{\gamma \ell-1}(S)}(E)$.

Let $T=J \cap \operatorname{Aut}_{S}(E) \in \operatorname{Syl}_{p}(J)$ and assume that $\bar{T} \neq 1$. Then we obtain $\bar{J}=N_{\bar{J}}\left(\overline{\operatorname{Aut}_{\gamma_{\ell-1}(S)}(E)}\right)$ and, by the Frattini Argument,

$$
\operatorname{Out}_{\mathcal{F}}(E)=\bar{J} N_{\operatorname{Out}_{\mathcal{F}}(E)}(\bar{T})
$$

Thus

$$
\operatorname{Out}_{\mathcal{F}}(E)=N_{\operatorname{Out}_{\mathcal{F}}(E)}(\bar{T}) N_{\bar{J}}\left(\overline{\operatorname{Aut}_{\gamma_{\ell-1}(S)}(E)}\right),
$$

which contradicts $\operatorname{Out}_{\mathcal{F}}(E)$ having a strongly $p$-embedded subgroup. This proves that $T \leq \operatorname{Inn}(E)$ and that $\bar{J}$ is a $p^{\prime}$-group. Assume that $E / Q$ is not $\mathcal{F} / Q$-centric. Then $C_{S / Q}(E / Q) \not \leq E / Q$. Hence there exists $x \in N_{S}(E) \backslash E$ such that $c_{x} \in J$ and $\overline{c_{x}} \neq 1$, contrary to $\bar{J}$ having $p^{\prime}$-order. Hence $E / Q$ is $\mathcal{F} / Q$-centric. Finally, we note that $\operatorname{Aut}_{\mathcal{F} / Q}(E / Q) \cong \operatorname{Aut}_{\mathcal{F}}(E) / J$ and so $\operatorname{Out}_{\mathcal{F} / Q}(E / Q) \cong \operatorname{Out}_{\mathcal{F}}(E) / \bar{J}$. As $J$ has $p^{\prime}$-order and is centralized by $\bar{L}$, Lemma 2.11 implies $\operatorname{Out}_{\mathcal{F} / Q}(E / Q)$ has a strongly $p$-embedded subgroup. Thus $E / Q$ is $\mathcal{F} / Q$-essential as claimed. Therefore, if $Q$ is not $\mathcal{F}$-centric, then $E / Q$ is $\mathcal{F} / Q$-essential. In particular, $|E / Q| \geq p^{2}$ and $|S: E| \geq p^{2}$ as $E$ is not normal in $S$. Hence $|S / Q| \geq p^{4}$ and $E / Q<\gamma_{1}(S) / Q$. This contradicts the minimal choice of $\mathcal{F}$. Hence $Q$ is $\mathcal{F}$-centric.

By Theorem 5.4, there exists a group $G$ which is a model for $\mathcal{F}$. For this model, we have $Q=O_{p}(G)$ and $C_{G}(Q)=Z(Q)$. Furthermore, $C_{G}(Q / \Phi(Q))=Q$ and $E / \Phi(Q)$ is $\mathcal{F}_{S / \Phi(Q)}(G / \Phi(Q))$-essential. If $\Phi(Q) \neq 1$, we apply induction to obtain a contradiction as $\mathcal{F}_{S / \Phi(Q)}(G / \Phi(Q))$ has essential subgroups which are properly contained $\gamma_{1}(S / \Phi(Q))$. Hence $Q$ is elementary abelian. Since $Q$ is $\mathcal{F}$-centric, it is the largest normal abelian subgroup of $S$, say $Q=\gamma_{w}(S)$. Then, as $|S: E| \geq p^{2}$ and $E>Q, w \geq 3$ and $|Q| \leq\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right| \leq p^{p}$ by Lemma 3.2 (iv).

Since $Q=\gamma_{w}(S)$ is elementary abelian, Theorem 3.15 implies that $\gamma_{w-1}(S)$ has nilpotency class 2. Hence $\gamma_{w-1}(S)$ acts quadratically on $Q$. Consider $G / Q$. We have $O_{p}(G / Q)=1$. Since $\gamma_{w-1}(S)$ acts quadratically on $Q$, and $p>3, \gamma_{w-1}(S) / Q$ centralizes $O_{p, p^{\prime}}(G) / Q$ by [13, Lemma 1.2]. In particular, $G / Q$ has a component of order divisible by $p$. Suppose that $K$ is the preimage of such a component. Then $S \cap K>Q$. If $K$ is not normalised by $S$, then $\left|K^{S}\right| \geq p$ and $\left\langle\left(K^{S}\right\rangle \cap\right.$ $S) / Q$ contains an elementary abelian $p$-group of rank $p$. Since $|S| \geq \mid Q \|\left\langle K^{S}\right\rangle \cap$ $S\left|\left|S: N_{S}(K)\right| \geq p^{2} . p^{p} . p=p^{p+3}\right.$, this contradicts Lemma 3.2(v). Hence $K$ is normalized by $S$, it follows that $K \cap S$ is normal in $S$. If $L / Q \neq K / Q$ is a component of $G / Q$ with $S \cap L>Q$, then $L \cap S$ is normal in $S$ and we see that $Z(S / Q) \geq Z((L \cap S)(K \cap S) / Q)$ has order at least $p^{2}$. This is impossible as $S / Q$ has maximal class and $|S / Q| \geq p^{3}$. Hence $K / Q$ is the unique such component and we have $F^{*}(G / Q) \leq O_{p^{\prime}}(G / Q) K / Q$ with $\gamma_{w-1}(S)$ acting quadratically on $Q$. Since $\gamma_{w-1}(S) / Q$ has order $p$ and centralizes $O_{p^{\prime}}(G / Q), \gamma_{w-1}(S) / Q$ acts faithfully on $K / Q$. Because $S / Q$ has maximal class and order at least $p^{3}, \gamma_{w-1}(S) / Q$ is contained in every non-trivial normal subgroup of $S / Q$. Hence $S / Q$ acts faithfully on $K / Q$. As $p \geq 5,[\mathbf{1 3}$, Theorem A] implies that $K / Q$ is a group of Lie type defined in characteristic $p$. We refer to [26, Theorem 2.5.12] for properties of automorphism groups of groups of Lie type. Since $p \geq 5, K / Q$ has no graph automorphisms of order $p$ and so, if $S \not \leq K, S K / K$ is a cyclic group of field automorphisms of $K / Q$. If $S K / K>1$, it follows from [26, Theorem 3.3.1] that $\left|\Omega_{1}(Z((S \cap K) / Q))\right| \geq p^{p}$ and again we contradict Lemma 3.2(iv). Hence $S \leq K$. Since $S / Q$ is 2-generated and non-abelian, we now see that $K$ is a rank at most 2 Lie type group defined over $\mathrm{GF}(p)$. Hence $\left[\mathbf{2 6}\right.$, Theorem 3.3.1(b)] yields $K / Q \bmod$ its centre is one of $\operatorname{PSU}_{3}(p)$, $\mathrm{PSL}_{3}(p), \mathrm{PSp}_{4}(p), \mathrm{G}_{2}(p)$, and in each case $S / Q$ has maximal class. However in these groups we have $\left|N_{K / Q}(S / Q)\right| \geq(p-1)^{2} / 3>p-1$, contrary to Hypothesis 10.3 (ii). This contradiction illustrates that $Q=1$ and completes the proof of the proposition.

LEMMA 10.7. Assume that $E$ is a witness and suppose there is $n>j \geq 1$ such that $\gamma_{j}(S) \leq E$. Then $\gamma_{j}(S)$ is not $\operatorname{Aut}_{\mathcal{F}}(E)$-invariant.

Proof. Note that $\gamma_{j}(S)$ is fully $\mathcal{F}$-normalized since it is normal in $S$. Consider the saturated fusion system $N_{\mathcal{F}}\left(\gamma_{j}(S)\right)$ on $S$. Aiming for a contradiction, suppose $\operatorname{Aut}_{\mathcal{F}}(E) \subseteq N_{\mathcal{F}}\left(\gamma_{j}(S)\right)$, Lemma 5.14 implies that $E$ is $N_{\mathcal{F}}\left(\gamma_{j}(S)\right)$-essential. Thus $N_{\mathcal{F}}\left(\gamma_{j}(S)\right)$ is a counterexample to Theorem $\mathbf{D}$ and by the minimality of $\mathcal{F}$ we deduce that $\mathcal{F}=N_{\mathcal{F}}\left(\gamma_{j}(S)\right)$. But then $O_{p}(\mathcal{F}) \neq 1$, contrary to Proposition 10.6.

In the next three lemmas we exploit Proposition 10.6 to provide both lower and upper bounds for the order of $S$.

Proposition 10.8. If $S$ is exceptional then $|S| \geq p^{8}$. In particular, if $S$ is exceptional, then $C_{S}\left(Z_{2}(S)\right)$ is not $\mathcal{F}$-essential.

Proof. Because $S$ is exceptional, Hypothesis 7.1 holds. Aiming for a contradiction, suppose that $S$ has order $p^{6}$ and let $E$ be a witness. By Hypothesis 10.3 and Lemma 10.4 we know $E<\gamma_{1}(S)$ and $\gamma_{1}(S)$ is not extraspecial. Note that $Z(S)$ is not $\operatorname{Aut}_{\mathcal{F}}(E)$-invariant by Lemma 10.7. In particular, $Z(S)<Z(E)$ and so $|Z(E)| \geq p^{2}$. By Lemmas 6.8 and $10.4,|E|=p^{4}$ and $E \neq \gamma_{2}(S)$ as $E$ is not normal in $S$. So $\gamma_{1}(S)=E E^{s}$ for some $s \in S$. Also, as $\gamma_{2}(S)=C_{S}\left(Z_{2}(S)\right) \cap \gamma_{1}(S)$, $E \neq \gamma_{2}(S)$ implies $Z_{2}(S) \cap Z(E)=Z(S)$. If $E$ is abelian then $Z\left(\gamma_{1}(S)\right)=E \cap E^{s}$
has order $p^{3}$, contradicting the fact that $Z\left(\gamma_{1}(S)\right)=Z(S)$ because $S$ is exceptional. Hence $E$ is non-abelian and $|Z(E)|=p^{2}$. This implies

$$
[E, E] \leq\left[\gamma_{1}(S), \gamma_{1}(S)\right] \cap Z(E) \leq Z_{2}(S) \cap Z(E)=Z(S)
$$

Since $E$ is not abelian, we get that $[E, E]=Z(S)$ and so $Z(S)$ is $\operatorname{Aut}_{\mathcal{F}}(E)$-invariant, a contradiction. This proves $|S| \geq p^{8}$ and the last statement follows from Proposition 7.2.

Lemma 10.9. We have $|S| \geq p^{7}$.
Proof. By Hypothesis 10.3 we have $|S| \geq p^{6}$ and $\gamma_{1}(S)$ is not abelian. Aiming for a contradiction, suppose $|S|=p^{6}$. By Lemma 10.8 the group $S$ is not exceptional. Let $E<\gamma_{1}(S)$ be a witness. By Lemmas 6.8 and 10.4, $|E|=p^{4}$, $\gamma_{1}(S)=N_{S}(E)$ and $Z_{2}(S) \leq Z\left(\gamma_{1}(S)\right)<E$. In addition $E \neq \gamma_{2}(S)$ as $E$ is not normal in $S$. The group $Z_{2}(S)$ is not $\operatorname{Aut}_{\mathcal{F}}(E)$-invariant by Lemma 10.7, hence $Z_{2}(S)<Z(E)$. So $|E: Z(E)| \leq p$ and we conclude that $E$ is abelian. Since $S^{p} \leq Z(S)$ by Lemma $3.2(\mathrm{vi})$ and $Z(S)$ is not $\operatorname{Aut}_{\mathcal{F}}(E)$-invariant by Lemma 10.7, we also get that $E$ has exponent $p$, that is, $E$ is elementary abelian. Let $s \in$ $S \backslash N_{S}(E)$. Then $\gamma_{1}(S)=E E^{s}$ and so $Z\left(\gamma_{1}(S)\right)=E \cap E^{s}$ has order $p^{3}$ and $\gamma_{1}(S)=\Omega_{1}\left(\gamma_{1}(S)\right)$ is non-abelian. Also, $\gamma_{3}(S)=Z\left(\gamma_{1}(S)\right)=C_{E}\left(\gamma_{1}(S)\right)$ and $\left[\gamma_{1}(S): E\right]=p=\left[E: C_{E}\left(\gamma_{1}(S)\right)\right]$. Proposition 5.17 applied with $V=E$ implies $O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right) \cong \operatorname{SL}_{2}(p)$ and $E / C_{E}\left(O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)\right)$ is a natural module for $O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)$. In particular the group $K=C_{E}\left(O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}(E)\right)\right)$ has order $p^{2}$ and it is contained in $\gamma_{3}(S)=Z\left(\gamma_{1}(S)\right)$.

If $\gamma_{1}(S)$ is $\mathcal{F}$-essential, then Hypothesis 8.1 holds and Lemma 8.5 implies that $p^{5}=p^{p-1}$, a contradiction. Hence $\gamma_{1}(S)$ is not $\mathcal{F}$-essential and $E$ is not properly contained in any $\mathcal{F}$-essential subgroup of $S$. Let $\tau \in Z\left(O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(E)\right)\right)$ be an involution. Then there is $\hat{\tau} \in \operatorname{Aut}_{\mathcal{F}}(S)$ such that $\left.\hat{\tau}\right|_{E}=\tau$ and $\hat{\tau}$ has $p^{\prime}$-order. So $C_{\gamma_{3}(S)}(\hat{\tau})=C_{E}(\hat{\tau})=K$ has order $p^{2}$ and $\left[\gamma_{1}(S), \hat{\tau}\right] \leq E$. Hence $\left|C_{\gamma_{1}(S)}(\hat{\tau})\right|=p^{3}$. The only way this is compatible with Lemma 3.10 is if $\hat{\tau}$ centralizes $\gamma_{1}(S) / \gamma_{2}(S)$, $\gamma_{3}(S) / \gamma_{4}(S)$ and $\gamma_{5}(S)=Z(S)$ and inverts $\gamma_{2}(S) / \gamma_{3}(S)$ and $\gamma_{4}(S) / Z(S)$. As $\gamma_{1}(S) / E \cong \gamma_{2}(S) / \gamma_{3}(S)$ which is inverted by $\hat{\tau}$, we have a contradiction. Hence $|S| \geq p^{7}$.

We now turn to determining an upper bound for $|S|$.
Lemma 10.10. Suppose that $E$ is a witness. Then $Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right) \leq E$ and $\Omega_{1}\left(\gamma_{1}(S)\right) \not \leq E$. In particular, $\Omega_{1}\left(\gamma_{1}(S)\right)$ is not abelian, $|S|<p^{2 p-4}$ and $p \geq 7$.

Proof. Using Lemma 2.1(iii), we have

$$
\left[N_{Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right)}(E), E\right] \leq \Omega_{1}\left(\gamma_{1}(S)\right) \cap E \leq \Omega_{1}(E) \leq \Omega_{1}\left(\gamma_{1}(S)\right)
$$

and so the group $N_{Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right)}(E)$ stabilizes the characteristic series $1<\Omega_{1}(E)<$ $E$. Since $E$ is $\mathcal{F}$-essential, Lemmas 2.6 and 5.15 imply that $Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right) \leq E$.

If $\Omega_{1}\left(\gamma_{1}(S)\right) \leq E$, then $\Omega_{1}\left(\gamma_{1}(S)\right)=\Omega_{1}(E)$ is $\operatorname{Aut}_{\mathcal{F}}(E)$-invariant, contrary to Lemma 10.7. Hence $\Omega_{1}\left(\gamma_{1}(S)\right) \not \leq E$. This proves the first two statements. In particular, as $Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right) \leq E, \Omega_{1}\left(\gamma_{1}(S)\right)$ is not abelian.

Assume that $|S| \geq p^{2 p-4}$. Then $|S|>p^{p+1}$ since $2 p-4>p+1$ unless $p=5$ and in the latter case we know $|S|>5^{6}$ by Lemma 10.9. Thus $S$ has positive degree of commutativity by Lemma 3.3(iv) and (v), and $\Omega_{1}\left(\gamma_{1}(S)\right)=\gamma_{n-p+1}(S)$
by Lemma 3.2 (iii). Since $\Omega_{1}\left(\gamma_{1}(S)\right)$ is not abelian and $S$ has positive degree of commutativity we get

$$
\begin{aligned}
Z(S) & =\gamma_{n-1}(S) \leq\left[\gamma_{n-p+1}(S), \gamma_{n-p+1}(S)\right] \\
& =\left[\gamma_{n-p+1}(S), \gamma_{n-p+2}(S)\right] \leq \gamma_{2 n-2 p+3+1}(S)
\end{aligned}
$$

This implies that $n \leq 2 p-4-1$, contradicting our assumptions. Therefore $|S|<$ $p^{2 p-4}$. Now, if $p=5$ we obtain $|S|<5^{6}$, contradicting Lemma 10.9. Hence $p \geq 7$.

We conclude this section with a characterization of the proper subsystems of $\mathcal{F}$.

Lemma 10.11. Suppose that $\mathcal{K}$ is a proper saturated fusion subsystem of $\mathcal{F}$ on S. Then $\mathcal{K} \subseteq N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$.

Proof. Suppose $\mathcal{K} \nsubseteq N_{\mathcal{F}}(S)$. Then there is a $\mathcal{K}$-essential subgroup $P$. Since $\mathcal{K}$ is properly contained in $\mathcal{F}$, it is not a counterexample to Theorem D. Hence either $P$ is a $\mathcal{K}$-pearl or $P \in\left\{C_{S}\left(Z_{2}(S)\right), \gamma_{1}(S)\right\}$. If $P$ is a $\mathcal{K}$-pearl then $P$ is contained in some $\mathcal{F}$-essential subgroup $E^{*}$. By Lemma 10.5 we have $P \leq E^{*} \leq \gamma_{1}(S)$ or $P \leq E^{*} \leq C_{S}\left(Z_{2}(S)\right)$ both of which are impossible by Lemma 6.4 applied to $P$. Thus $P$ is not a $\mathcal{K}$-pearl. If $P=C_{S}\left(Z_{2}(S)\right) \neq \gamma_{1}(S)$, then $S$ is exceptional and $p$ is $\mathcal{F}$-essential. This contradicts Lemma 10.8. Hence the only option is $P=\gamma_{1}(S)$. This proves the statement.

## 11. Locating $\mathcal{F}$-essential subgroups in groups of maximal class II

In this section, we continue to prepare for the proof of Theorem D. In particular, we continue to work under Hypothesis 10.3. We start by creating a compendium facts that we have established about saturated fusion systems which satisfy Hypothesis 10.3.

Lemma 11.1. Suppose that $E \in \mathcal{E}_{\mathcal{F}}$ is a witness.
(i) $E<\gamma_{1}(S), E$ is not normal in $S$ and $|E| \geq p^{4}$;
(ii) if $S$ is exceptional, then $C_{S}\left(Z_{2}(S)\right)$ is not $\mathcal{F}$-essential and $\gamma_{1}(S)$ is not extraspecial;
(iii) $p \geq 7$ and $p^{7} \leq|S|<p^{2 p-4}$;
(iv) $\Omega_{1}\left(\gamma_{1}(S)\right)$ is not abelian;
(v) $O_{p}(\mathcal{F})=1$;
(vi) $\operatorname{Out}_{\mathcal{F}}(S)$ acts faithfully on $S / \gamma_{1}(S)$ and is cyclic of order dividing $p-1$;
(vii) if $\gamma_{1}(S)$ is $\mathcal{F}$-essential, then $S$ is not exceptional and $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right) \cong$ $\mathrm{PSL}_{2}(p)$ or $\mathrm{PGL}_{2}(p)$.

Proof. Part (i) follows from Lemmas 10.4 and 10.5. Part (ii) is Proposition 10.8 and, as in this case Hypothesis 7.1 holds, Lemma 7.4. Part (iii) is a combination of Lemmas 10.9 and 10.10. Part (iv) follows from Lemma 10.10.

Part (v) is precisely Proposition 10.6. Using part (iv), (vi) follows from Corollary 3.12. Finally, for part (vii), if $S$ is exceptional, then Proposition 9.1 implies $\gamma_{1}(S)$ is extraspecial, which is against (iv). Hence $S$ is not exceptional and Hypothesis 8.1 holds. Thus $O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right) \cong \operatorname{PSL}_{2}(p)$ by Lemma 8.5. Using (vi) gives $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right) \cong \operatorname{PSL}_{2}(p)$ or $\operatorname{PGL}_{2}(p)$.

Suppose that $E$ is a witness. Then $E<\gamma_{1}(S)$ by Lemma 11.1 (i). If $N_{\mathcal{F}}(E) \subseteq$ $N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$, then $E$ is $N_{\mathcal{F}}\left(\gamma_{1}(S)\right.$ )-essential by Lemma 5.14 and we obtain the contradiction $E<\gamma_{1}(S) \leq O_{p}\left(N_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right) \leq E$ from Lemma 5.10. Hence $N_{\mathcal{F}}(E) \nsubseteq$ $N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$. Hence the set of subgroups $T$ of $\gamma_{1}(S)$ with $N_{\mathcal{F}}(T) \nsubseteq N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ is non-empty. We now set up the notation which shall be used in the remainder of this section.

Notation 11.2. Set $\mathcal{G}=N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$. From among all non-trivial subgroups $T \leq \gamma_{1}(S)$, select one which satisfies the following conditions in the specified order.
(i) $N_{\mathcal{F}}(T) \nsubseteq \mathcal{G}$;
(ii) $\left|N_{S}(T)\right|$ is maximal; and
(iii) $|T|$ is maximal.

Observe that $N_{\mathcal{F}}(S) \subseteq \mathcal{G}$ with equality if and only if $\gamma_{1}(S)$ is not $\mathcal{F}$-essential.
The discussion before Notation 11.2 shows that a witness satisfies the first condition and so we can conclude that a subgroup $T$ as specified in Notation 11.2 exists.

Lemma 11.3. The subgroup $T$ is fully $\mathcal{F}$-normalized and $N_{\mathcal{F}}(T)$ is saturated.
Proof. By [5, Lemma I. 2.6 (c)], there exists $\alpha \in \operatorname{Hom}_{\mathcal{F}}\left(N_{S}(T), S\right)$ such that $T \alpha$ is fully $\mathcal{F}$-normalized. By the Alperin-Goldschmidt Theorem, $\alpha$ is a product of maps from $\operatorname{Aut}_{\mathcal{F}}(A), A \in \mathcal{E}_{\mathcal{F}}$ with $\left|N_{S}(T)\right| \leq|A|$ and from $\operatorname{Aut}_{\mathcal{F}}(S)$. Then, as $A \leq \gamma_{1}(S)$ by Lemmas 10.4 and 11.1 (ii), the selection method of $T$ shows that each $\operatorname{Aut}_{\mathcal{F}}(A)$ is contained in $\mathcal{G}$ as is $\operatorname{Aut}_{\mathcal{F}}(S)$. Let $X, Y \leq N_{S}(T)$. Then
$\operatorname{Hom}_{N_{\mathcal{F}}(T \alpha)}(X \alpha, Y \alpha) \supseteq \operatorname{Hom}_{N_{\mathcal{F}}(T)}(X, Y) \alpha^{*}$. Since $\alpha \in \mathcal{G}, \operatorname{Hom}_{N_{\mathcal{F}}(T \alpha)}(X \alpha, Y \alpha) \subset$ $\mathcal{G}$ if and only if $\operatorname{Hom}_{N_{\mathcal{F}}(T)}(X, Y) \alpha^{*} \subset \mathcal{G}$. We conclude that $N_{\mathcal{F}}(T \alpha) \not \subset \mathcal{G}$. Since $\left|N_{S}(T \alpha)\right| \geq\left|N_{S}(T)\right|$, the maximal choice of $N_{S}(T)$ implies that $T$ is fully $\mathcal{F}$ normalized.

Lemma 11.4. Assume that $1 \neq K \leq T$. Then following statements hold.
(i) If $K$ is $\operatorname{Aut}_{\mathcal{F}}(T)$-invariant, then $N_{S}(K)=N_{S}(T)$.
(ii) If $T$ is not normal in $S$ and $K$ is characteristic in $N_{S}(T)$, then $K$ is not Aut $\mathcal{F}^{(T) \text {-invariant. }}$

Proof. Suppose that $K$ is $\operatorname{Aut}_{\mathcal{F}}(T)$-invariant. Then $K$ is invariant under $\operatorname{Aut}_{S}(T)$ and so is normal in $N_{S}(T)$. Lemma 5.6 states that $N_{\mathcal{F}}(T) \subseteq N_{\mathcal{F}}(K)$. Therefore, if $N_{S}(K)>N_{S}(T)$, the maximal choice of $\left|N_{S}(T)\right|$ implies $N_{\mathcal{F}}(T) \subseteq$ $N_{\mathcal{F}}(K) \subseteq \mathcal{G}$ which contradicts the choice of $T$. This proves (i).

Part (ii) follows from (i).
Lemma 11.5. If $K$ is a normal subgroup of $S$ which is contained in $T$, then $K$ is not $\operatorname{Aut}_{\mathcal{F}}(T)$-invariant. In particular, $T$ is not normal in $S$.

Proof. Suppose that $K \leq T$ is $\operatorname{Aut}_{\mathcal{F}}(T)$-invariant and normal in $S$. Then Lemma 11.4 (i) implies that $T$ is normal in $S$. Hence $N_{\mathcal{F}}(T)$ is a fusion system on $S$. Since $T \leq O_{p}\left(N_{\mathcal{F}}(T)\right)$ and $O_{p}(\mathcal{F})=1, N_{\mathcal{F}}(T) \neq \mathcal{F}$. Application of Lemma 10.11 yields $N_{\mathcal{F}}(T) \subseteq \mathcal{G}$, a contradiction.
11.1. The case $T$ is $S$-centric. In this subsection we assume

Hypothesis 11.6. Hypothesis 10.3 holds and adopting Notation 11.2 we have $T$ is $S$-centric; that is $C_{S}(T) \leq T$.

As $T$ is fully $\mathcal{F}$-normalized by Lemma 11.3 , Hypothesis 11.6 implies $T$ is $\mathcal{F}$ centric. Since $Z(S) \leq T$ and $T$ is not normal in $S$ by Lemma 11.5, Lemma 3.7 implies $N_{S}(T) \leq \gamma_{1}(S)$. By Theorem 5.4 there exists a model $G$ for $N_{\mathcal{F}}(T)$. Choose $G_{1}$ such that $G \geq G_{1}>N_{S}(T)$ and $G_{1}$ has minimal order such that $\mathcal{F}_{N_{S}(T)}\left(G_{1}\right) \nsubseteq$ $\mathcal{G}$.

Lemma 11.7. Assume that Hypothesis 11.6 holds. We have $T=O_{p}\left(G_{1}\right)=$ $O_{p}(G)$ and $C_{G_{1}}(T) \leq T$.

Proof. Notice that $N_{S}\left(O_{p}\left(G_{1}\right)\right) \geq N_{S}(T)$ and $O_{p}\left(G_{1}\right) \geq T$. As $\mathcal{F}_{N_{S}(T)}\left(G_{1}\right)=$ $N_{\mathcal{F}_{N_{S}(T)}\left(G_{1}\right)}\left(O_{p}\left(G_{1}\right)\right) \subseteq N_{\mathcal{F}}\left(O_{p}\left(G_{1}\right)\right)$, we get $N_{\mathcal{F}}\left(O_{p}\left(G_{1}\right)\right) \nsubseteq \mathcal{G}$. The maximal choice of $T$ now yields $T=O_{p}\left(G_{1}\right)$. We now have $T \leq O_{p}(G) \leq O_{p}\left(G_{1}\right)=T$ and so $T=O_{p}(G)=O_{p}\left(G_{1}\right)$. As $G$ is a model for $N_{\mathcal{F}}(T), C_{G_{1}}(T) \leq C_{G}(T) \leq T$.

Lemma 11.8. Assume that Hypothesis 11.6 holds. Then the following hold.
(i) $\Omega_{1}\left(\gamma_{1}(S)\right) \not \leq T$; and
(ii) $Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right) \leq \Omega_{1}(T)$.

Proof. If $\Omega_{1}\left(\gamma_{1}(S)\right) \leq T$, then $\Omega_{1}\left(\gamma_{1}(S)\right)=\Omega_{1}(T)$ is normalized by $S$ and is Aut $_{\mathcal{F}}(T)$-invariant contrary to Lemma 11.5. Hence $\Omega_{1}\left(\gamma_{1}(S)\right) \not \leq T$. This proves (i).

Let $R=N_{Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right)}(T)$. Then, as $\gamma_{1}(S)$ is regular by Lemma 3.2 (ii), using Lemma 2.1(iii) yields

$$
[T, R] \leq T \cap \Omega_{1}\left(\gamma_{1}(S)\right) \leq \Omega_{1}(T)
$$

and $\left[\Omega_{1}(T), R\right] \leq\left[\Omega_{1}\left(\gamma_{1}(S)\right), R\right]=1$. Hence $R$ centralizes $T / \Omega_{1}(T)$ and $\Omega_{1}(T)$. Now Lemmas 5.15 and 11.8 imply $\operatorname{Aut}_{R}(T) \leq O_{p}\left(\operatorname{Aut}_{\mathcal{F}}(T)\right)=\operatorname{Inn}(T)$. Hence $R \leq T$. Applying Lemma 2.6 yields $Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right) \leq T$, as claimed.

Before reading the next lemma the following example is worthy of some consideration. Let $X=2^{6}: 3 \operatorname{Sym}(6)$. Then $X$ is isomorphic to a 2-local subgroup in $\mathrm{M}_{24}$. Let $R \in \operatorname{Syl}_{2}(X)$. Then $\mathcal{F}_{R}(X)$ has three $\mathcal{F}_{R}(X)$-essential subgroups $E_{0}=R \cap X^{\prime}, E_{1}$ and $E_{2}$ where $E_{1} / O_{2}(X) \cong E_{2} / O_{2}(X)$ are elementary abelian of order 8. We have $\operatorname{Out}_{\mathcal{F}_{R}(X)}\left(E_{i}\right) \cong \operatorname{Sym}(3)$ for $i=0,1,2$. Here is the point: $X=\left\langle N_{X}\left(E_{1}\right), N_{X}\left(E_{2}\right)\right\rangle$, however $\mathcal{F}_{R}(X) \neq\left\langle\mathcal{F}_{R}\left(N_{X}\left(E_{1}\right)\right), \mathcal{F}_{R}\left(N_{X}\left(E_{2}\right)\right)\right\rangle$ by [50, Lemma 3.13].

Lemma 11.9. Assume that Hypothesis 11.6 holds. Then $N_{S}(T)$ is contained in a unique maximal subgroup of $G_{1}$.

Proof. Suppose that $M_{1}$ and $M_{2}$ are maximal subgroups of $G_{1}$ with $N_{S}(T) \leq$ $M_{1} \cap M_{2}$ and $M_{1} \neq M_{2}$. By the minimal choice of $G_{1}$,

$$
\left\langle\mathcal{F}_{N_{S}(T)}\left(M_{1}\right), \mathcal{F}_{N_{S}(T)}\left(M_{2}\right)\right\rangle \subseteq \mathcal{G}
$$

In particular,

$$
\begin{aligned}
\operatorname{Aut}_{\mathcal{G}}(T) & \supseteq\left\langle\operatorname{Aut}_{\mathcal{F}_{N_{S}(T)}\left(M_{1}\right)}(T), \operatorname{Aut}_{\mathcal{F}_{N_{S}(T)}\left(M_{2}\right)}(T)\right\rangle \\
& =\left\langle\operatorname{Aut}_{M_{1}}(T), \operatorname{Aut}_{M_{2}}(T)\right\rangle=\operatorname{Aut}_{G_{1}}(T)
\end{aligned}
$$

as $G_{1}=\left\langle M_{1}, M_{2}\right\rangle$. Since $T$ is $\mathcal{F}$-centric, $Z\left(\gamma_{1}(S)\right) \leq T$ and $\operatorname{Aut}_{\mathcal{G}}(T)$ leaves $Z\left(\gamma_{1}(S)\right)$ invariant. Since $N_{\mathcal{F}}\left(Z\left(\gamma_{1}(S)\right)\right)$ is a fusion system on $S$ and $O_{p}(\mathcal{F})=1$, $N_{\mathcal{F}}\left(Z\left(\gamma_{1}(S)\right)\right) \subseteq \mathcal{G}$ by Lemma 10.11. Therefore

$$
\operatorname{Aut}_{\mathcal{F}_{N_{S}(T)}\left(G_{1}\right)}(T)=\operatorname{Aut}_{G_{1}}(T) \subseteq \operatorname{Aut}_{\mathcal{G}}(T) \subseteq \mathcal{G}=N_{\mathcal{F}}\left(Z\left(\gamma_{1}(S)\right)\right)
$$

Since $\mathcal{F}_{N_{S}(T)}\left(G_{1}\right)=N_{\mathcal{F}_{N_{S}(T)}\left(G_{1}\right)}(T)$, application of Lemma 5.6 gives $\mathcal{F}_{N_{S}(T)}\left(G_{1}\right) \subseteq$ $\mathcal{G}$, a contradiction. Hence $N_{S}(T)$ is contained in a unique maximal subgroup of $G_{1}$.

Let $U$ be the unique maximal subgroup of $G_{1}$ which contains $N_{S}(T)$.
Lemma 11.10. Assume that Hypothesis 11.6 holds. If $K$ is a non-trivial characteristic subgroup of $N_{S}(T)$, then $N_{G_{1}}(K) \leq U$.

Proof. We have $N_{G_{1}}(K) \geq N_{S}(T)$ and so, as $U$ is the unique maximal subgroup of $G_{1}$ which contains $N_{S}(T)$, if $U$ does not contain $N_{G_{1}}(K)$, then $K$ is normalized by $G_{1}$. Suppose that $K$ is normal in $G_{1}$. Then, by Lemma 11.7, $K \leq O_{p}\left(G_{1}\right)=$ $T \leq \gamma_{1}(S)$ and $\mathcal{F}_{N_{S}(T)}\left(G_{1}\right) \subseteq N_{\mathcal{F}}(K)$ which means that $N_{\mathcal{F}}(K) \nsubseteq \mathcal{G}$. It follows from the maximal choice of $\left|N_{S}(T)\right|$ that $\left|N_{S}\left(N_{S}(T)\right)\right| \leq\left|N_{S}(K)\right| \leq\left|N_{S}(T)\right|$, and so we must have that $T$ is normal in $S$. This contradicts Lemma 11.5 and demonstrates $N_{G_{1}}(K) \leq U$.

Recall from [11] that for a finite group $X$ and $R \in \operatorname{Syl}_{p}(X)$, $C(X, R)=\left\langle N_{X}(K)\right| 1 \neq K$ a characteristic subgroup of $\left.R\right\rangle$.
In [11], they also define

$$
\left.C^{*}(X, R)=\left\langle N_{X}(K)\right| 1 \neq K \text { a characteristic subgroup of } B(R) \text { or } \Omega_{1}(Z(R))\right\rangle,
$$

where $B(R)$ is the Baumann subgroup (see [11, Definition 1.1 and just before]). Since $B(R)$ and $\Omega_{1}(Z(R))$ are characteristic in $R$, we have $C^{*}(X, R) \leq C(X, R)$.

Lemma 11.11. Assume that Hypothesis 11.6 holds. Then there is a natural number $b \geq 1$ such that $O^{p}\left(G_{1}\right) / O_{p}\left(O^{p}\left(G_{1}\right)\right) \cong \mathrm{SL}_{2}\left(p^{b}\right)$ and $O_{p}\left(O^{p}\left(G_{1}\right)\right)$ is a natural $\operatorname{GF}(p) O^{p}\left(G_{1}\right) / O_{p}\left(O^{p}\left(G_{1}\right)\right)$-module. Furthermore, $G_{1} / T \cong \operatorname{SL}_{2}\left(p^{b}\right)$.

Proof. By Lemma 11.10, we have $C^{*}\left(G_{1}, N_{S}(T)\right) \leq C\left(G_{1}, N_{S}(T)\right) \leq U$. Lemma 11.7 states $C_{G_{1}}(T) \leq T$ and $T=O_{p}\left(G_{1}\right)$. Since $U$ is the unique maximal subgroup of $G_{1}$ containing $N_{S}(T)$, we may apply [11, Corollary 1.9] to obtain $B\left(N_{S}(T)\right.$-blocks $B_{1}, \ldots, B_{r}$ such that the product $B_{1} \ldots B_{r}$ is normal in $G_{1},\left[B_{i}, B_{j}\right]=1$ for $1 \leq i<j \leq r$ and $G_{1}=\left(B_{1} \ldots B_{r}\right) C\left(G_{1}, N_{S}(T)\right)$. Furthermore, as $U$ is the unique maximal subgroup of $G_{1}$ containing $N_{S}(T)$, we have $G_{1}=\left(B_{1} \ldots B_{r}\right) N_{S}(T)$ and $N_{S}(T)$ permutes $\left\{B_{1}, \ldots, B_{r}\right\}$ transitively by conjugation. In particular, $r$ is a power of $p$. The definition of a $B\left(N_{S}(T)\right)$ block and the fact that $p \geq 5$ yields $B_{1}=O^{p}\left(B_{1}\right), B_{1} / O_{p}\left(B_{1}\right) \cong \mathrm{SL}_{2}\left(p^{b}\right)$ for some $b \geq 1, O_{p}\left(B_{1}\right)=\Omega_{1}\left(Z\left(O_{p}\left(B_{1}\right)\right)\right)$ and $O_{p}\left(B_{1}\right)$ is a natural $\operatorname{GF}(p) B_{1} / O_{p}\left(B_{1}\right)$ module. Assume $r \geq p$. Then $|S| \geq\left|N_{S}(T)\right| \geq p^{3 p}$ which contradicts Lemma 11.1 (iii). Hence $r<p$, and, as $r$ is a power of $p$, we must have $r=1$. It follows that $B_{1}$ is normal in $G_{1}$ and thus $B_{1}=O^{p}\left(G_{1}\right)$. Finally, we note that if $G_{1} / T=G_{1} / O_{p}\left(G_{1}\right) \not \approx \mathrm{SL}_{2}\left(p^{b}\right)$, then some $p$ element of $G_{1}$ must induce a nontrivial field automorphism on $B_{1} / O_{p}\left(B_{1}\right)$. But then $p \geq b$ and $\left|N_{S}(T) \cap B_{1}\right|=p^{3 p}$, which once again contradicts Lemma 11.1 (iii). Hence $G_{1} / O_{p}\left(G_{1}\right) \cong \mathrm{SL}_{2}\left(p^{b}\right)$ and this proves the lemma.

We define

$$
B_{1}=O^{p}\left(G_{1}\right) \text { and } W=O_{p}\left(B_{1}\right)
$$

Lemma 11.12. Assume that Hypothesis 11.6 holds. Then $G_{1} / T \cong \mathrm{SL}_{2}(p)$ and $W=\left[T, B_{1}\right]$ has order $p^{2}$.

Proof. By Lemma 11.11, there is a natural number $b$ such that $B_{1} / O_{p}\left(B_{1}\right) \cong$ $G_{1} / T \cong \mathrm{SL}_{2}\left(p^{b}\right)$ and $G_{1}=B_{1} T$. Select $g$ of order $p^{b}-1$ in $N_{G_{1}}\left(N_{S}(T)\right)$ and set $\theta=c_{g}$. Then, by saturation, $\theta$ is the restriction of a morphism $\theta^{*} \in \operatorname{Aut}_{\mathcal{F}}\left(N_{S}(T)\right)$. The maximal choice of $T$ implies that $\theta^{*}$ is the restriction of a morphism $\hat{\theta} \in$ $\operatorname{Aut}_{\mathcal{G}}\left(\gamma_{1}(S)\right)$. In particular, $\hat{\theta}$ has order a multiple of $p^{b}-1$. We shall show that $b=1$. Since $\operatorname{Out}_{\mathcal{F}}(S)$ has order dividing $p-1$ by Lemma 11.1 (vi), we may suppose that $\mathcal{G} \neq N_{\mathcal{F}}(S)$. Hence $\gamma_{1}(S)$ is $\mathcal{F}$-essential and $\mathcal{G}=N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$. In this case $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right) \cong \mathrm{PSL}_{2}(p)$ or $\mathrm{PGL}_{2}(p)$ by Lemma 11.1 (vii), and, as the maximal order of a $p^{\prime}$-element of $\mathrm{PGL}_{2}(p)$ is $p+1$, we conclude that $b=1$. Therefore $G_{1} / T \cong \mathrm{SL}_{2}(p)$ and $W=\left[T, B_{1}\right]$ has order $p^{2}$.

Lemma 11.13. Assume that Hypothesis 11.6 holds. Suppose that $\mathcal{G}=N_{\mathcal{F}}(S)$ and let $t \in G_{1}$ be an involution. Then $c_{t}$ is the restriction of $\tau \in \operatorname{Aut}_{\mathcal{F}}(S), W=$ $\left[N_{S}(T), \tau\right], Z(S) \leq W$ and $Z(S)$ is inverted by $\tau$.

Proof. Since $t$ is an involution, $t T \in Z\left(G_{1} / T\right)$ and so $t \in N_{G_{1}}\left(N_{S}(T)\right)$. Let $\tilde{\tau}$ be the element of $\operatorname{Aut}_{\mathcal{F}}\left(N_{S}(T)\right)$ which restricts to $c_{t}$. The maximality of $T$ implies $N_{\mathcal{F}}\left(N_{S}(T)\right) \subseteq \mathcal{G}$ and so $\tilde{\tau}$ is the restriction of a morphism $\tau$ in $\operatorname{Aut}_{\mathcal{F}}(S)$. Since $t$ centralizes $N_{S}(T) / T,[T, t] \leq\left[T, O^{p^{\prime}}\left(G_{1}\right)\right]=W$ and $t$ inverts $W$, we have $W=\left[N_{S}(T), t\right]=\left[N_{S}(T), \tau\right]$. If $\bar{Z}(S) \not \leq W$ then, as $Z(S) \leq T, V=Z(S) W$ is normalized by $B_{1} N_{S}(T)=G_{1}$ and is contained in $Z(T)$. As $W$ is the unique non-central $G_{1}$-chief factor in $V$, we get $V=C_{V}(\tau) \times[V, \tau]$ and $C_{V}(\tau)=C_{V}\left(G_{1}\right)$. Hence, as $Z(S)$ is $\langle\tau\rangle$-invariant $Z(S) \leq C_{V}(\tau)=C_{V}\left(G_{1}\right)$. But then $\mathcal{F}_{N_{S}(T)}\left(G_{1}\right) \subseteq$
$N_{\mathcal{F}}(Z(S)) \subseteq \mathcal{G}$ by Lemma 10.11. Since this is not the case, we conclude that $Z(S) \leq W$ and $Z(S)$ is inverted by $\tau$.

Lemma 11.14. Assume that Hypothesis 11.6 holds. Then $\gamma_{1}(S)$ is $\mathcal{F}$-essential and $S$ is not exceptional. In particular, Hypothesis 8.1 holds.

Proof. Assume that $\gamma_{1}(S)$ is not $\mathcal{F}$-essential. Then $\mathcal{G}=N_{\mathcal{F}}(S)$. Let $t \in G_{1}$ and $\tau$ be as in Lemma 11.13. Then

$$
Z(S) \leq\left[N_{S}(T), \tau\right]=W
$$

By Lemma 3.3 (iii) and (vi) the group $\gamma_{1}(S) / Z(S)$ is the unique 2-step centralizer in $S / Z(S)$. Hence

$$
\left[T, Z_{3}(S)\right] \leq\left[\gamma_{1}(S), Z_{3}(S)\right] \leq Z(S)
$$

Since $Z(S) \leq T$, we deduce that $Z_{3}(S) \leq N_{S}(T)$ and consequently we also have $\left[Z_{3}(S), \tau\right] \leq Z_{3}(S) \cap W$.

By Lemma $3.10, \tau$ does not centralize $Z_{3}(S) / Z(S)$. Hence, as $\tau$ inverts $Z(S)$, we have $\left[Z_{3}(S), \tau\right] \leq[T, \tau]=W$. As $|S| \geq p^{7}$ by Lemma 11.1 (iii), $Z_{3}(S)$ is abelian and so $Z_{3}(S) \leq C_{N_{S}(T)}(W)=T$. Furthermore, $Z_{5}(S) \leq \gamma_{1}(S)$ and it follows that $Z_{5}(S) \leq N_{S}(T)$. Therefore

$$
\left[Z_{5}(S), \tau\right] \leq\left[N_{S}(T), \tau\right] \leq W \leq Z_{3}(S)
$$

and this contradicts Lemma 3.10. Hence $\gamma_{1}(S)$ is $\mathcal{F}$-essential. Lemma 11.1(vii) yields $S$ is not exceptional.

Lemma 11.15. Assume that Hypothesis 11.6 holds. Then the following hold:
(i) $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right) \cong \operatorname{PSL}_{2}(p)$ or $\mathrm{PGL}_{2}(p)$;
(ii) $\gamma_{1}(S)=N_{S}(T)$; and
(iii) $\gamma_{1}(S)=\Omega_{1}\left(\gamma_{1}(S)\right)$ has class 2 and order $p^{p-1}$.

Proof. By Lemma 11.14, we have Hypothesis 8.1 is satisfied. Part (i) is just Lemma 11.1(vii). The group $\Omega_{1}\left(\gamma_{1}(S)\right)$ is not abelian by Lemma 11.1(iv). From Lemma 8.5 we obtain $\Omega_{1}\left(\gamma_{1}(S)\right)$ has order $p^{p-1}$ and nilpotency class 2 and

$$
\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S)\right] \leq Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right)
$$

Hence, using Lemma 11.8,

$$
\left[\Omega_{1}(T), \gamma_{1}(S)\right] \leq\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S)\right] \leq Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right) \leq \Omega_{1}(T)
$$

In particular, $\gamma_{1}(S)$ normalizes $\Omega_{1}(T)$. Therefore, as $N_{\mathcal{F}}\left(\Omega_{1}(T)\right) \supseteq N_{\mathcal{F}}(T) \nsubseteq \mathcal{G}$, the maximal choice of $N_{S}(T)$ implies that $N_{S}(T) \geq \gamma_{1}(S)$. As $T$ is not normal in $S$ by Lemma 11.5, we now have $N_{S}(T)=\gamma_{1}(S)$ which is part (ii).

By Lemma $11.12, G_{1} / T \cong \mathrm{SL}_{2}(p)$ and so $T$ is a maximal subgroup of $N_{S}(T)$. As $\Omega_{1}\left(\gamma_{1}(S)\right) \not \pm T$ by Lemma 11.8, we have $\gamma_{1}(S)=\Omega_{1}\left(\gamma_{1}(S)\right) T$. Hence Lemmas 2.2 and 3.2 (ii) imply $\mho^{1}(T)=\mho^{1}\left(\gamma_{1}(S)\right)$. If $\mho^{1}\left(\gamma_{1}(S)\right) \neq 1$, then $N_{\mathcal{F}}(T) \leq$ $N_{\mathcal{F}}\left(\mho^{1}\left(\gamma_{1}(S)\right)\right) \subseteq \mathcal{G}$ and this contradicts the choice of $T$. Therefore $\mho^{1}\left(\gamma_{1}(S)\right)=1$ and we conclude $\gamma_{1}(S)=\Omega_{1}\left(\gamma_{1}(S)\right)$. Thus part (iii) holds.

Proposition 11.16. If Hypothesis 10.3 holds, then $T$ is not $S$-centric.
Proof. Suppose that $T$ is $S$-centric. Then Hypothesis 11.6 holds. From Lemma 11.14 we know Hypothesis 8.1 holds. Lemma 11.15 implies that $N_{S}(T)=$ $\gamma_{1}(S)$ has exponent $p$, order $p^{p-1}$ and nilpotency class 2 . We also know that
$\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right) \cong \operatorname{PSL}_{2}(p)$ or $\operatorname{PGL}_{2}(p)$. Let $L=\left\langle\operatorname{Aut}_{S}\left(\gamma_{1}(S)\right)^{\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)}\right\rangle$ be the preimage of $O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right)$.

As in Lemma 11.14, take $\theta \in N_{G_{1}}\left(N_{S}(T)\right)$ of order $p-1$ and let $\rho$ the element of $\operatorname{Aut}_{\mathcal{F}}\left(N_{S}(T)\right)=\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ which restricts to $\theta$. Set $H=\langle\rho\rangle$. Then, as $H$ centralizes $T / W$, we have $\left|\left[N_{S}(T), H\right]\right|=\left|\left[\gamma_{1}(S), H\right]\right|=p^{3}$. Since $H \cap L$ has index at most 2 in $H$, we have $\left|(H \cap L) \operatorname{Inn}\left(\gamma_{1}(S)\right) / \operatorname{Inn}\left(\gamma_{1}(S)\right)\right|=(p-1) / 2$ and subgroups of this order normalize a Sylow $p$-subgroup of $L$. We calculate, using $\left|\gamma_{1}(S)\right|=p^{p-1}$, that

$$
\left|C_{\gamma_{1}(S)}\left(\rho^{2}\right)\right|=\left|\gamma_{1}(S) /\left[\gamma_{1}(S), \rho^{2}\right]\right|=p^{p-4}
$$

However, Lemma 8.5 states that $\left|C_{\gamma_{1}(S)}\left(\rho^{2}\right)\right|=p^{2}$ and so we conclude that $p<7$. This contradicts Lemma 11.1 (iii) and proves that $T$ is not $\mathcal{F}$-centric.
11.2. The case $T$ is not $S$-centric. In this subsection, we continue the notation so far established and, in addition, assume that

Hypothesis 11.17. Hypothesis 10.3 holds and adopting Notation 11.2 we have $C_{S}(T) \not \leq T$.

Lemma 11.18. Assume that Hypothesis 11.17 holds. Then $\mathcal{E}_{N_{\mathcal{F}}(T)} \neq \emptyset$.
Proof. Suppose that $\mathcal{E}_{N_{\mathcal{F}}(T)}=\emptyset$. Then, because $N_{\mathcal{F}}(T)$ is saturated by Lemma 11.3, $N_{\mathcal{F}}(T) \subseteq N_{\mathcal{F}}\left(N_{S}(T)\right)$. Hence $N_{\mathcal{F}}\left(N_{S}(T)\right) \nsubseteq \mathcal{G}$ and, as $T \neq S$, this contradicts the maximal choice of $T$. We conclude $\mathcal{E}_{N_{\mathcal{F}}(T)} \neq \emptyset$.

Since $N_{\mathcal{F}}(T)$ is saturated and $\mathcal{E}_{N_{\mathcal{F}}(T)} \neq \emptyset$, the Alperin-Goldschmidt fusion theorem implies there is an $N_{\mathcal{F}}(T)$-essential subgroup $P$ such that $\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P) \nsubseteq$ $\mathcal{G}$.

Notation 11.19. The subgroup $P \leq N_{S}(T)$ is an $N_{\mathcal{F}}(T)$-essential subgroup of maximal order such that $\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P) \nsubseteq \mathcal{G}$.

Lemma 11.20. If Hypothesis 11.17 holds, then
(i) $T<P$;
(ii) $N_{S}(P) \leq \gamma_{1}(S)$;
(iii) $\left|N_{S}(P)\right|<\left|N_{S}(T)\right|$; and
(iv) $P<N_{N_{S}(T)}(P)<N_{S}(T)$.

Proof. By Lemma $5.10, T \leq O_{p}\left(N_{\mathcal{F}}(T)\right) \leq P$. Since $T$ is not $S$-centric, $T$ is not $N_{S}(T)$-centric and so $T<P$. This is (i).

Suppose that $P \not \leq \gamma_{1}(S)$. Then Lemma 11.1 (i) implies that $P$ is not contained in any $\mathcal{F}$-essential subgroups. Therefore the elements of $\operatorname{Aut}_{\mathcal{F}}(P)$ are all restrictions of elements in $\operatorname{Aut}_{\mathcal{F}}(S)$ and this means that $O_{p}\left(\operatorname{Out}_{\mathcal{F}}(P)\right)=\operatorname{Out}_{S}(P)$. In particular, $\operatorname{Aut}_{\mathcal{F}}(P)$ has a unique Sylow $p$-subgroup. As $\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)$ is a subgroup of $\operatorname{Aut}_{\mathcal{F}}(P)$, we have $\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)$ has a unique Sylow $p$-subgroup and this contradicts $P$ being $N_{\mathcal{F}}(T)$-essential. Hence $P \leq \gamma_{1}(S)$. Since $Z(S) \leq N_{S}(T)$, we have $Z(S) \leq C_{N_{S}(T)}(P) \leq P$ and so, either, $P$ is normal in $S$ or $N_{S}(P) \leq \gamma_{1}(S)$ by Lemma 3.7. As $T$ is not normal in $S$ by Lemma 11.5, the maximal choice of $T$ implies $N_{S}(P) \neq S$. Hence $N_{S}(P) \leq \gamma_{1}(S)$. This proves (ii). If $\left|N_{S}(P)\right| \geq\left|N_{S}(T)\right|$ then as $P>T$, we have a contradiction to the maximal choice of $T$. Thus $\left|N_{S}(P)\right|<\left|N_{S}(T)\right|$ so (iii) holds. Part (iii) yields $P<N_{N_{S}(T)}(P)<N_{S}(T)$ which is (iv).

Recall from Section 5, that the subgroup $H_{N_{\mathcal{F}}(T)}(P)$ of $\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)$ is generated by those $N_{\mathcal{F}}(T)$-automorphisms of $P$ which extend to $N_{\mathcal{F}}(T)$-isomorphisms between strictly larger subgroups of $N_{S}(T)$.

Lemma 11.21. Assume that Hypothesis 11.17 holds. Let $A, B \leq N_{N_{S}(T)}(P)$ with $P \leq A \cap B$. Then $\operatorname{Hom}_{N_{\mathcal{F}}(T)}(A, B) \subseteq \mathcal{G}$ and

$$
H_{N_{\mathcal{F}}(T)}(P)=N_{\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)}\left(\operatorname{Aut}_{N_{S}(T)}(P)\right) \subseteq \mathcal{G}
$$

Proof. From the maximal choice of $P$, we know that any $N_{\mathcal{F}}(T)$-essential subgroup $P_{1}$ containing $A$ has $\operatorname{Aut}_{N_{\mathcal{F}}(T)}\left(P_{1}\right) \subset \mathcal{G}$. It follows that $\operatorname{Hom}_{N_{\mathcal{F}}(T)}(A, B) \subseteq \mathcal{G}$. In particular, the elements of $\operatorname{Hom}_{N_{\mathcal{F}}(T)}(A, B)$ are restrictions of morphisms in $N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$. If $\alpha \in \operatorname{Hom}_{N_{\mathcal{F}}(T)}(A, B)$ and $P=P \alpha$, then taking any $\hat{\alpha} \in \operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ which restricts to $\alpha$, we have $P \hat{\alpha}=P, T=T \hat{\alpha}$ and, because of Lemma 11.20,

$$
N_{N_{S}(T)}(P) \hat{\alpha}=N_{N_{\gamma_{1}(S)}(T)}(P) \hat{\alpha}=N_{N_{\gamma_{1}(S)}(T)}(P)=N_{N_{S}(T)}(P)
$$

Hence $H_{N_{\mathcal{F}}(T)}(P)=N_{\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)}\left(\operatorname{Aut}_{N_{S}(T)}(P)\right)$, as stated.
Lemma 11.22. Assume that Hypothesis 11.17 holds. Then $C_{S}(T) \not \leq P$. In particular, $\operatorname{Out}_{C_{S}(T)}(P) \neq 1$.

Proof. If $C_{S}(T) \leq P$, then $C_{S}(T) T=C_{P}(T) T$ is normalized by $N_{S}(T)$ and is invariant under the action of $\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P) \nsubseteq \mathcal{G}$. Thus the maximal choice of $T$ yields $C_{S}(T) T=T$. Hence $C_{S}(T) \leq T$, a contradiction. Therefore $C_{S}(T) \not \leq P$. Since $P C_{S}(T)$ is a subgroup of $S$, we have $N_{C_{S}(T)}(P) \not \leq P$ by Lemma 2.6 and so $\operatorname{Out}_{C_{S}(T)}(P) \neq 1$.

Lemma 11.23. Assume that Hypothesis 11.17 holds. Then $N_{\mathcal{F}}(Z(S)) \subseteq \mathcal{G}$ and $Z(S) \not \leq T$. In particular, $\Omega_{1}(P) \not \leq T$.

Proof. By Lemma 10.11, $N_{\mathcal{F}}(Z(S)) \subseteq \mathcal{G}$.
As $T$ is $\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)$-invariant, the group $K=\left\langle\operatorname{Aut}_{C_{S}(T)}(P)^{\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)}\right\rangle$ centralizes $T$. Aiming for a contradiction, suppose $Z(S) \leq T$. Then $K$ centralizes $Z(S)$ and so

$$
K \operatorname{Inn}(P) \subseteq N_{\mathcal{F}}(Z(S)) \subseteq \mathcal{G}
$$

Since $C_{S}(T) \not \leq P$ by Lemma $11.22, K \not \leq \operatorname{Inn}(P)$ and $\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)=K H_{N_{\mathcal{F}}(T)}(P)$ by the Frattini Argument. Hence we get $\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P) \subseteq \mathcal{G}$ from Lemma 11.21, a contradiction. Thus $Z(S) \not \leq T$. Since $Z(S) \leq C_{N_{S}(T)}(P) \leq P$, we have $Z(S) \leq$ $\Omega_{1}(P)$ and so $\Omega_{1}(P) \not \leq T$.

Lemma 11.24. Assume that Hypothesis 11.17 holds. Then $\mathcal{G}=N_{\mathcal{F}}(S)$.
Proof. Suppose that $\mathcal{G} \neq N_{\mathcal{F}}(S)$. Then $\gamma_{1}(S)$ is $\mathcal{F}$-essential and Lemma 11.1 (vii) says that $S$ is not exceptional. Also Lemmas 3.7 and 11.4 imply $N_{S}(T) \leq$ $\gamma_{1}(S)$. As $\Omega_{1}\left(\gamma_{1}(S)\right)$ is not abelian by Lemma 10.10 , Lemma 8.5 implies that $\Omega_{1}\left(\gamma_{1}(S)\right)$ has nilpotency class 2 and order $p^{p-1}$ with $Z\left(\gamma_{1}(S)\right)=Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right)=$ $\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S)\right]$. In particular, $Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right) \leq N_{S}(T)$ and $Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right) \leq$ $C_{N_{S}(T)}(P) \leq P$. Thus

$$
\left[\Omega_{1}(P), \gamma_{1}(S)\right] \leq\left[\Omega_{1}\left(\gamma_{1}(S)\right), \gamma_{1}(S)\right]=Z\left(\Omega_{1}\left(\gamma_{1}(S)\right)\right) \leq \Omega_{1}(P)
$$

and so $\Omega_{1}(P)$ is normal in $\gamma_{1}(S)$. In particular, as $N_{S}(T) \leq \gamma_{1}(S), N_{S}(T) \leq$ $N_{S}\left(T \Omega_{1}(P)\right)$. By Lemma 5.6 we have $N_{N_{\mathcal{F}}(T)}(P) \subseteq N_{N_{\mathcal{F}}(T)}\left(T \Omega_{1}(P)\right)$ and so, as
$N_{N_{\mathcal{F}}(T)}(P) \nsubseteq \mathcal{G}, N_{\mathcal{F}}\left(T \Omega_{1}(P)\right) \nsubseteq \mathcal{G}$. The maximal choice of $T$ now implies that $T=T \Omega_{1}(P)$. Therefore $\Omega_{1}(P) \leq T$, and this contradicts Lemma 11.23.

Notation 11.25. Define

$$
L=O^{p^{\prime}}\left(\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)\right)=\left\langle\operatorname{Aut}_{N_{S}(T)}(P)^{\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)}\right\rangle
$$

Then the Frattini Argument yields

$$
\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)=L N_{\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)}\left(\operatorname{Aut}_{N_{S}(T)}(P)\right)
$$

Lemma 11.26. Assume that Hypothesis 11.17 holds. Then $Z(S)$ is not $L$ invariant.

Proof. Suppose that $Z(S)$ is normalized by $L$. Then, as $\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)=$ $L N_{\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)}\left(\operatorname{Aut}_{N_{S}(T)}(P)\right)$ and $N_{\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)}\left(\operatorname{Aut}_{N_{S}(T)}(P)\right) \subseteq \mathcal{G}=N_{\mathcal{F}}(S)$ by Lemmas 11.21 and 11.24, we deduce from Lemma 11.23 that $\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P) \subseteq$ $N_{\mathcal{F}}(Z(S)) \subseteq \mathcal{G}$, which is a contradiction.

The next three lemmas limit the structure of $L / \operatorname{Inn}(P)$.
Lemma 11.27. Assume that Hypothesis 11.17 holds. Then $O_{p^{\prime}}\left(\operatorname{Out}_{N_{\mathcal{F}}(T)}(P)\right)$ is centralized by $\operatorname{Out}_{N_{S}(T)}(P)$. In particular, $L / \operatorname{Inn}(P)$ centralizes $O_{p^{\prime}}\left(\operatorname{Out}_{N_{\mathcal{F}}(T)}(P)\right)$.

Proof. Suppose false. Then there exists an $x \in \operatorname{Aut}_{N_{S}(T)}(P)$ such that $x \operatorname{Inn}(P)$ does not centralize $O_{p^{\prime}}\left(\operatorname{Out}_{N_{\mathcal{F}}(T)}(P)\right)$. Let $K \leq \operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)$ be the preimage of $\left[O_{p^{\prime}}\left(\operatorname{Out}_{N_{\mathcal{F}}(T)}(P)\right), x \operatorname{Inn}(P)\right]$. Then $K\langle x\rangle$ is $p$-soluble and, since $p$ is odd, there exists a non-central $K\langle x\rangle$-chief factor $V$ in $\Omega_{1}(P)$ by [24, Theorem 5.3.10]. By Proposition 2.5, $|V| \geq p^{p-1}$. Since $\Omega_{1}(P)<\Omega_{1}\left(\gamma_{1}(S)\right)$ and $1<\Omega_{1}(T) \leq \Omega_{1}(P)$ is $L$-invariant, Lemma 3.2 (iv) implies $\Omega_{1}(P)=\Omega_{1}(T) \leq T$ and this contradicts Lemma 11.23.

Lemma 11.28. Assume that Hypothesis 11.17 holds. Then $\operatorname{Out}_{N_{S}(T)}(P)$ is cyclic.

Proof. If $\operatorname{Out}_{N_{S}(T)}(P)$ is not cyclic, then, as $p$ is odd, $\operatorname{Out}_{N_{S}(T)}(P)$ has an elementary abelian subgroup of order $p^{2}$. Using Lemma 2.10, we obtain $K=$ $\operatorname{Out}_{N_{\mathcal{F}}(T)}(P) / O_{p^{\prime}}\left(\operatorname{Out}_{N_{\mathcal{F}}(T)}(P)\right)$ is an almost simple group. Because $p \geq 7$ by Lemma 11.1 (iii), using Proposition 2.12 we obtain that $K \cong \operatorname{PSL}_{2}\left(p^{a}\right)$ with $a \geq 2$, $\operatorname{PSU}_{3}\left(p^{a}\right)$ with $a \geq 1$, $\operatorname{Alt}(2 p)$ or $p=11$ and $L \cong \mathrm{~J}_{4}$. Since $H_{N_{\mathcal{F}}(T)}(P) \subset \mathcal{G}=$ $N_{\mathcal{F}}(S)$ by Lemmas 11.21 and 11.24 , we get $N_{L}\left(\operatorname{Aut}_{N_{S}(T)}(P)\right)$ is cyclic of order dividing $p-1$ from Lemma 11.1 (vi). Using [26, Theorem 7.6.2], we find this is not compatible with any of the candidates for $O^{p}(L / \operatorname{Inn}(P)) / Z(L / \operatorname{Inn}(P))$. Thus $\operatorname{Out}_{N_{S}(T)}(P)$ is cyclic.

Lemma 11.29. Assume that Hypothesis 11.17 holds. Then $L / \operatorname{Inn}(P) \cong \operatorname{PSL}_{2}(p)$ or $\mathrm{SL}_{2}(p)$.

Proof. Since $L / \operatorname{Inn}(P)$ centralizes $O^{p^{\prime}}\left(\operatorname{Out}_{N_{\mathcal{F}}(T)}(P)\right)$ by Lemma 11.27 and $O_{p}(L)=\operatorname{Inn}(P), L / \operatorname{Inn}(P)$ centralizes the Fitting subgroup of $\operatorname{Out}_{N_{\mathcal{F}}(T)}(P)$ and so $E\left(\operatorname{Out}_{N_{\mathcal{F}}(T)}(P)\right) \neq 1$ and, by Lemma 11.28 , there is a unique component $K / \operatorname{Inn}(P)$ of $\operatorname{Out}_{N_{\mathcal{F}}(T)}(P)$ which has order divisible by $p$. Since $\operatorname{Aut}_{N_{S}(T)}(P) \leq L, K$ is a normal subgroup of $L$ and $\operatorname{Out}_{N_{\mathcal{F}}(T)}(P)=H_{\mathcal{F}}(P) K$. Since $H_{\mathcal{F}}(P)$ leaves $Z(S)$ invariant, and $L \leq H_{\mathcal{F}}(P) K, K$ does not leave $Z(S)$ invariant by Lemma 11.26.

Assume that $K / \operatorname{Inn}(P)$ does not have a quotient isomorphic to $\mathrm{PSL}_{2}(p)$. Then $K$ is not of $\mathrm{L}_{2}(p)$-type and $K / \operatorname{Inn}(P)$ is quasisimple. Hence, as $Z(S)$ is not $K$ invariant, setting $U=\left\langle Z(S)^{K}\right\rangle \leq \Omega_{1}(Z(P)), U$ has a non-central $K$-chief factor. Since $U$ is $\operatorname{Aut}_{N_{S}(T)}(P) K$-invariant, Theorem 4.2 yields $|U| \geq p^{2(p-1) / 3}$, $\left|\operatorname{Out}_{N_{S}(T)}(P)\right|=p$ and $U$ is indecomposable as a $\operatorname{GF}(p) \operatorname{Out}_{N_{S}(T)}(P)$-module. In particular, $\left[U, \operatorname{Out}_{N_{S}(T)}(P) ;\lceil 2(p-1) / 3\rceil-1\right] \neq 1$. This with Lemma 3.5 gives $\lceil 2(p-1) / 3\rceil-1<(p+1) / 2$ and yields $p=7$. But then, as $p=7$, Lemma 3.5 additionally tells us that $S$ is exceptional. As $|S| \geq 7^{7}$ by Lemma 11.1 (iii), Lemma 3.3 (v) implies $|S|=7^{8}$. Since $T \leq P$ and $U \not \leq T$, we also have $|P| \geq 7^{5}$. By Lemma 11.20, $P<N_{N_{S}(T)}(P)<N_{S}(T)$. Hence $N_{S}(T)$ is a maximal subgroup of $S$. Since $S$ is exceptional, either $Z\left(N_{S}(T)\right)=Z(S)$ or $N_{S}(T)=C_{S}\left(Z_{2}(S)\right)$. In the former case, as $T$ is normal in $N_{S}(T)$, we have $Z(S) \leq T$ which contradicts Lemma 11.23. Hence $N_{S}(T)=C_{S}\left(Z_{2}(S)\right)$ and consequently $Z_{2}(S) \leq P$. Since $P \leq \gamma_{1}(S)$ by Lemma 11.20, $P \leq \gamma_{1}(S) \cap C_{S}\left(Z_{2}(S)\right)=\gamma_{2}(S)$. Now $\left[\gamma_{2}(S), \gamma_{2}(S)\right] \leq \gamma_{5}(S)=Z_{2}(S) \leq P$ and so $\gamma_{2}(S)=N_{N_{S}(T)}(P)$ and $\gamma_{2}(S)$ acts quadratically on $P$ contrary to $\left[U, \operatorname{Out}_{N_{S}(T)}(P) ;\lceil 2(p-1) / 3\rceil-1\right] \neq 1$. We conclude that $K / \operatorname{Inn}(P)$ is of $\mathrm{L}_{2}(p)$-type and it follows that $K / \operatorname{Inn}(P) \cong \mathrm{PSL}_{2}(p)$ or $\mathrm{SL}_{2}(p)$. Since $O_{p}(L)=\operatorname{Inn}(P)$ and $\operatorname{Out}_{N_{S}(T)}(P)$ is cyclic, we further deduce that $L=K$.

By Lemma 11.29, $\operatorname{Aut}_{N_{S}(T)}(P)$ has order $p$. Hence Lemma 11.22 implies

$$
\operatorname{Aut}_{C_{S}(T)}(P) \operatorname{Inn}(P)=\operatorname{Aut}_{N_{S}(T)}(P)
$$

We now establish some notation which will play an important role in the remaining lemmas of this section. Let $\langle\theta\rangle$ be a complement in $L$ to $\operatorname{Aut}_{N_{S}(T)}(P)=$ $\operatorname{Aut}_{C_{S}(T)}(P)$ chosen so that $\langle\theta\rangle \leq\left\langle\operatorname{Aut}_{C_{S}(T)}(P)^{\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)}\right\rangle \leq C_{\mathcal{F}}(T)$. We know $\theta$ has order $(p-1) / 2$ when $L / \operatorname{Inn}(P) \cong \mathrm{PSL}_{2}(p)$ and order $p-1$ when $L / \operatorname{Inn}(P) \cong$ $\mathrm{SL}_{2}(p)$. Since $\theta \in\left\langle\operatorname{Aut}_{C_{S}(T)}(P)^{\operatorname{Aut}_{N_{\mathcal{F}}(T)}(P)}\right\rangle$, we also know $\theta$ centralizes $T$.

Because $\theta$ normalizes $\operatorname{Aut}_{N_{S}(T)}(P), \theta$ is the restriction of a morphism

$$
\tilde{\theta} \in \operatorname{Aut}_{N_{\mathcal{F}}(T)}\left(N_{N_{S}(T)}(P)\right) \subseteq H_{N_{\mathcal{F}}(T)}(P) \subseteq \mathcal{G}=N_{\mathcal{F}}(S)
$$

by Lemmas 11.21 and 11.24. Hence $\theta$ is in fact the restriction of an element $\hat{\theta}$ of $\operatorname{Aut}_{\mathcal{F}}(S)$ and we may assume that $\hat{\theta}$ has $p^{\prime}$-order.

If $\theta$ has order $p-1$, then Lemma 11.1 (vi) shows that $\hat{\theta}$ has order $p-1$ and so $\hat{\tau}=\hat{\theta}^{(p-1) / 2}$ is an involution which restricts to $\tau=\theta^{(p-1) / 2}$. Recall that the action for $\hat{\theta}$ on $\gamma_{1}(S)$ is the subject of Lemmas 3.10 and 3.14.

Lemma 11.30. Assume that Hypothesis 11.17 holds. Then $Z(S) \not \leq C_{\Omega_{1}(P)}(\theta)$.
Proof. Suppose that $Z(S)$ is centralized by $\theta$ and define $V=\left\langle Z(S)^{L}\right\rangle \Omega_{1}(T)$. Since $Z(S) \not \leq C_{V}\left(O^{p}(L)\right)$ by Lemma 11.26 , we can select $U$ maximal such that

$$
1 \neq \Omega_{1}(T) \leq C_{V}\left(O^{p}(L)\right) \leq U<V
$$

is $L$-invariant. Then $V / U$ is irreducible as a $\mathrm{GF}(p) L$-module and, by the definition of $V, Z(S) \not \leq U$. Set $\bar{V}=V / C_{V}\left(O^{p}(L)\right)$. Then $|\bar{V}|<p^{p}$ by Lemma 3.2 (iv). Combining Lemma 11.29 and Corollary 4.5 yields $V=Z(S) U$ and $\bar{V}=\overline{Z(S)} \bar{U}>$ $\bar{U}$. Lemma 2.7 implies $\bar{V}=C_{\bar{V}}(L) \bar{U}$. By coprime action, $C_{\bar{V}}(L) \leq C_{\bar{V}}\left(O^{p}(L)\right)=1$. Hence $\bar{V}=\bar{U}$, a contradiction. Therefore $Z(S)$ is not centralized by $\theta$.

Lemma 11.31. Assume that Hypothesis 11.17 holds. If $\Omega_{1}(P) / \Omega_{1}(T)$ is abelian, then $Z_{2}(S) \leq C_{N_{S}(T)}\left(\Omega_{1}\left(N_{S}(T)\right)\right)$. In particular, $Z_{2}(S) \leq N_{S}(T)$.

Proof. If $S$ is not exceptional, then, as $T$ is not normal in $S$ by Lemma 11.5, Lemma 3.7 implies $N_{S}(T) \leq \gamma_{1}(S)$. The claim then follows as

$$
Z_{2}(S) \leq Z\left(\gamma_{1}(S)\right) \leq Z\left(N_{\gamma_{1}(S)}(T)\right)=Z\left(N_{S}(T)\right)
$$

Suppose that $S$ is exceptional. Then $Z\left(\gamma_{1}(S)\right)=Z(S)$. If $\Omega_{1}\left(N_{S}(T)\right) \leq$ $C_{S}\left(Z_{2}(S)\right)$, then

$$
Z_{2}(S) \leq C_{S}\left(\Omega_{1}(T)\right) \leq N_{S}\left(\Omega_{1}(T)\right)=N_{S}(T)
$$

by Lemma 11.4. Thus $Z_{2}(S) \leq C_{N_{S}(T)}\left(\Omega_{1}\left(N_{S}(T)\right)\right)$ in this case.
So assume that $\Omega_{1}\left(N_{S}(T)\right) \not \leq C_{S}\left(Z_{2}(S)\right)$. Then $N_{S}(T) \not \leq C_{S}\left(Z_{2}(S)\right)$ and, as $T$ is not normal in $S$ by Lemma 11.5, Lemma 3.7 implies $N_{S}(T) \leq \gamma_{1}(S)$. Since $\Omega_{1}\left(N_{S}(T)\right) \not \leq C_{S}\left(Z_{2}(S)\right), \Omega_{1}\left(\gamma_{1}(S)\right) \not \leq \gamma_{2}(S)$ and, as $S$ has maximal class, we deduce $\gamma_{1}(S)=\Omega_{1}\left(\gamma_{1}(S)\right)$ which has exponent $p$ by Lemmas 3.2 (ii) and 2.1 (ii). In particular, $P=\Omega_{1}(P)$ and $T=\Omega_{1}(T)$. By Lemmas 11.20 and 11.29, $N_{N_{S}(T)}(P)<N_{S}(T)$ and $N_{N_{S}(T)}(P) / P$ has order $p$. Therefore we can pick $x \in N_{N_{S}(T)}\left(N_{N_{S}(T)}(P)\right) \backslash N_{N_{S}(T)}(P)$ such that $N_{N_{S}(T)}(P)=P P^{x}$ and $P \cap$ $P^{x}$ has index $p$ in $P$. Using Lemma 11.29 we can find $\ell \in L$ such that $L=$ $\left\langle\operatorname{Inn}(P), \operatorname{Aut}_{P^{x}}(P), \operatorname{Aut}_{P^{x}}(P)^{\ell}\right\rangle$. Then, as $\Omega_{1}(P) / \Omega_{1}(T)=P / T$ is abelian, $O^{p}(L)$ centralizes $\left(P \cap P^{x} \cap P^{x \ell}\right) / T$ and centralizes $T$ and so coprime action implies $\left|P / C_{P}\left(O^{p}(L)\right)\right| \leq p^{2}$. It follows that $L / \operatorname{Inn}(P) \cong \mathrm{SL}_{2}(p)$, and $\theta$ has order $p-1$. By Lemmas $3.3(\mathrm{vi})$ and $3.14,\left|C_{\gamma_{1}(S) / Z(S)}(\hat{\theta})\right| \leq p$. As $Z(S)$ is not $L$-invariant by Lemma 11.26, it follows that $\left|C_{P}\left(O^{p}(L)\right)\right| \leq p$ and so $T=C_{P}\left(O^{p}(L)\right)$ has order $p$ and $|P|=p^{3}$. In particular, we have $\left|N_{N_{S}(T)}(P)\right|=p^{4}$ and $P / T$ is an $\left(N_{\mathcal{F}}(T) / T\right)$ pearl.

Since $p^{8} \leq|S| \leq p^{p+1}$ by Lemmas $3.3(\mathrm{v})$ and 11.1 (iii) and $\gamma_{1}(S)$ has exponent $p$, we have $Z_{4}(S)$ is elementary abelian. We establish some notation for the action of $\hat{\theta}$ which we know has order $p-1$. So let $x \in C_{S}\left(Z_{2}(S)\right) \backslash \gamma_{1}(S)$ and $s_{1} \in \gamma_{1}(S) \backslash \gamma_{2}(S)$. Then we may suppose that

$$
\begin{array}{rll}
x \hat{\theta} & \equiv x^{a} & \bmod \gamma_{2}(S) \\
s_{1} \hat{\theta} & \equiv s_{1}^{b} & \bmod \gamma_{2}(S)
\end{array}
$$

where $a, b \in \operatorname{GF}(p)^{\times}$. We also know that $\hat{\theta}$ centralizes $T$. By Lemma 3.10,

$$
s_{n-1} \hat{\theta}=s_{n-1}^{c}
$$

where $c=a^{n-3} b^{2}$ and, as $P / T$ is a natural $\operatorname{GF}(p) \mathrm{SL}_{2}(p)$-module for $y \in P / Z(S) T$ we have

$$
y \hat{\theta} \equiv y^{c^{-1}} \quad \bmod Z(S) T
$$

Set $V=Z_{4}(S) / Z(S)$. Then, by Lemma 3.10, we may select elements $v_{n-4}, v_{n-3}$ and $v_{n-2}$ of $V$ such that $v_{j} \hat{\theta}=v_{j}^{a^{j-1} b}$. It particular, the action of $\hat{\theta}$ on each of these elements is different and they correspond to eigenvectors of $\left.\hat{\theta}\right|_{V}$ on $V$ with different eigenvalues. Pick $t \in T^{\#}$, then $\left.\hat{\theta}\right|_{V}$ and $\left.c_{t}\right|_{V}$ commute, and so $T$ normalizes the eigenspaces of $\theta$ on $V$. Since the $\theta$-eigenspaces on $V$ have order $p$, we conclude that $V=C_{V}(T)$. This shows that $\left[Z_{4}(S), T\right] \leq Z(S)$ and so $\left|C_{Z_{4}(S)}(T)\right| \geq p^{3}$.

Suppose that $\left[P, Z_{4}(S)\right] \leq Z(S)$. Then $C_{Z_{4}(S)}(T) \leq N_{N_{S}(T)}(P)$ as $Z(S) \leq P$. Since $\left|N_{N_{S}(T)}(P)\right|=p^{4}$ and $N_{N_{S}(T)}(P)$ is non-abelian, $T C_{Z_{4}(S)}(T)$ has order $p^{3}$.

Hence $T \leq Z_{4}(S)$. But then $Z_{4}(S)=N_{N_{S}(T)}(P)$, a contradiction as $Z_{4}(S)$ is abelian. We have proved that

$$
\left[P, Z_{4}(S)\right] \not \leq Z(S)
$$

In particular, as $S / Z(S)$ has positive degree of commutativity, $P \not \leq \gamma_{2}(S)$.
Suppose that $T \not \leq \gamma_{2}(S)$. Then $\left\langle T^{S}\right\rangle=\gamma_{1}(S)$ and so, as $\left[Z_{4}(S), T\right] \leq Z(S)$, we have

$$
\begin{aligned}
{\left[Z_{4}(S), P\right] } & \leq\left[Z_{4}(S), \gamma_{1}(S)\right]=\left[Z_{4}(S),\left\langle T^{S}\right\rangle\right] \\
& =\left\langle\left[Z_{4}(S), T\right]^{S}\right\rangle \leq Z(S)
\end{aligned}
$$

which is a contradiction. Hence $T \leq \gamma_{2}(S)$. Since $Z(S) \leq \gamma_{2}(S)$ and $P \not \leq \gamma_{2}(S)$, for $y \in P \backslash Z(S) T$, we have $y \notin \gamma_{2}(S)$. Hence

$$
y \hat{\theta} \equiv y^{b} \quad \bmod \gamma_{2}(S)
$$

and

$$
y \hat{\theta} \equiv y^{c^{-1}} \quad \bmod \gamma_{2}(S)
$$

Therefore $c=b^{-1}$. Suppose that $T \not \leq \gamma_{3}(S)$, then as $\hat{\theta}$ centralizes $T$, we have $a b=1$ by Lemma 3.10. Now applying Lemma 3.10 again yields

$$
s_{3} \hat{\theta} \equiv s_{3}^{b^{-1}} \quad \bmod \gamma_{3}(S)=s_{3}^{c} \quad \bmod \gamma_{3}(S)
$$

and also

$$
s_{n-1} \hat{\theta}=s_{n-1}^{b^{-1}}=s_{n-1}^{c} .
$$

Lemma 3.14 yields $p-1$ divides $n-1-3$. Hence $n \geq p+3$ and this contradicts $n \leq p+1$. Therefore $T \leq \gamma_{3}(S)$.

Since $\left[Z_{3}(S), \gamma_{3}(S)\right]=1$, we now have $Z_{3}(S) \leq N_{N_{S}(T)}(P)$. As $P \not \leq \gamma_{2}(S)$, $P$ does not centralize $Z_{3}(S)$. Hence $P Z_{3}(S)=N_{N_{S}(T)}(P)$ and $Z_{3}(S) \cap P=$ $Z\left(N_{N_{S}(T)}(P)\right)=Z(S) T$. In particular, $T \leq Z_{3}(S)$ and $Z_{5}(S) \leq \gamma_{3}(S) \leq N_{S}(T)$. Now $\left|Z_{5}(S) / Z_{3}(S)\right|=p^{2}$ and is centralized by $N_{S}(T) \leq \gamma_{1}(S)$. Therefore $N_{S}(T) / T$ does not have maximal class. However, $P / T$ is an $\left(N_{\mathcal{F}}(T) / T\right)$-pearl and consequently Lemma 5.18 (i) implies that $N_{S}(T) / T$ does have maximal class. We have derived a contradiction and this proves the lemma.

Lemma 11.32. Assume that Hypothesis 11.17 holds. Then we have $L / \operatorname{Inn}(P) \not \approx$ $\operatorname{PSL}_{2}(p)$.

Proof. Suppose that $L / \operatorname{Inn}(P) \cong \mathrm{PSL}_{2}(p)$. We start by considering the case $\left|C_{\Omega_{1}(P)}(\theta)\right| \leq p^{2}$. In this case, as $\theta$ centralizes $\Omega_{1}(T) \leq \Omega_{1}(P),\left|C_{\Omega_{1}(P) / \Omega_{1}(T)}(\theta)\right| \leq$ $p$. Since $Z(S) \leq \Omega_{1}(P)$ and $O^{p}(L)$ does not centralize $Z(S)$ by Lemma 11.26, $\Omega_{1}(P) / \Omega_{1}(T)$ contains at least one non-central $L$-chief factor contained in $\left\langle Z(S)^{L}\right\rangle$. As $\left|\Omega_{1}(P) / \Omega_{1}(T)\right|<p^{p}$, Lemma 4.5 implies each $L$-chief factor in $\Omega_{1}(P) / \Omega_{1}(T)$ contributes $p$ to $\left|C_{\Omega_{1}(P) / \Omega_{1}(T)}(\theta)\right|$ and therefore $\Omega_{1}(P) / \Omega_{1}(T)$ is a non-central $L$ chief factor. In particular, Lemma 4.5 implies $\Omega_{1}(P) / \Omega_{1}(T)$ has order $p^{a}$ with $a$ odd in the range $3 \leq a \leq p-2$, $\operatorname{Out}_{N_{N_{S}(T)}(P)}(P)$ acts indecomposably on $\Omega_{1}(P) / \Omega_{1}(T)$ and, as $\left|C_{\Omega_{1}(P)}(\theta)\right| \leq p^{2},\left|\Omega_{1}(T)\right|=p$.

We will frequently use

$$
\left[\Omega_{1}(P) / \Omega_{1}(T), N_{N_{S}(T)}(P), N_{N_{S}(T)}(P)\right] \neq 1
$$

which is a consequence of $a \geq 3$ and $\operatorname{Out}_{N_{N_{S}(T)}(P)}(P)$ acts indecomposably on $\Omega_{1}(P) / \Omega_{1}(T)$. We also remark that

$$
\Omega_{1}(T)\left\langle Z(S)^{L}\right\rangle \leq Z\left(\Omega_{1}(P)\right) \leq \Omega_{1}(P)
$$

and, as $\left\langle Z(S)^{L}\right\rangle$ has a non-central $L$-chief factor, we obtain $\Omega_{1}(P)=Z\left(\Omega_{1}(P)\right)$ is abelian.

By Lemma 11.31, $Z_{2}(S) \leq C_{N_{S}(T)}\left(\Omega_{1}\left(N_{S}(T)\right)\right) \leq N_{S}(T)$. Since $\left[P, Z_{2}(S)\right] \leq$ $Z(S) \leq P, Z_{2}(S) \leq N_{N_{S}(T)}(P)$. If $Z_{2}(S) \not \leq P$, then, as $\left|N_{N_{S}(T)}(P) / P\right|=p$, $N_{N_{S}(T)}(P)=P Z_{2}(S)$ and we obtain

$$
\left[P, N_{N_{S}(T)}(P), N_{N_{S}(T)}(P)\right]=\left[P, Z_{2}(S), Z_{2}(S)\right] \leq\left[Z(S), Z_{2}(S)\right]=1
$$

which is a contradiction. Thus $Z_{2}(S) \leq P$ and so $Z_{2}(S) \leq \Omega_{1}(P)$.
Since $Z_{2}(S) \leq C_{N_{S}(T)}\left(\Omega_{1}\left(N_{S}(T)\right)\right), Z_{2}(S)$ is centralized by $N_{\Omega_{1}\left(N_{S}(T)\right)}(P)$. Assume that $N_{\Omega_{1}\left(N_{S}(T)\right)}(P) \leq P$. Then, by Lemma 2.6, $\Omega_{1}\left(N_{S}(T)\right) \leq P$ and so $\Omega_{1}(P)=\Omega_{1}\left(N_{S}(T)\right)$. But then $\Omega_{1}(P) T=\Omega_{1}\left(N_{S}(T)\right) T, N_{\mathcal{F}}\left(\Omega_{1}(P) T\right) \supseteq N_{\mathcal{F}}(P) \nsubseteq$ $\mathcal{G}$ and $N_{S}\left(\Omega_{1}(P) T\right) \geq N_{S}(T)$. The maximal choice of $T$ implies $T \geq \Omega_{1}(P) \geq Z(S)$, and this contradicts Lemma 11.23. Hence $N_{\Omega_{1}\left(N_{S}(T)\right)}(P) \not \leq P$ and $N_{N_{S}(T)}(P)=$ $P N_{\Omega_{1}\left(N_{S}(T)\right)}(P)$. As $N_{\Omega_{1}\left(N_{S}(T)\right)}(P)$ centralizes $Z_{2}(S)$ and $\operatorname{Out}_{N_{S}(T)}(P)$ acts indecomposably on $\Omega_{1}(P) / \Omega_{1}(T)$, we deduce that

$$
Z_{2}(S) \Omega_{1}(T) / \Omega_{1}(T) \leq C_{\Omega_{1}(P) / \Omega_{1}(T)}\left(N_{N_{S}(T)}(P)\right)=Z(S) \Omega_{1}(T) / \Omega_{1}(T)
$$

Hence, as $\Omega_{1}(T)$ has order $p$, we have $\Omega_{1}(T) \leq Z_{2}(S)$. In particular, $\gamma_{2}(S)$ normalizes $T$ and $Z_{2}(S)=Z(S) \Omega_{1}(T) \leq P$.

Since $|S| \geq p^{7}$ by Lemma 11.1(iii), $Z_{4}(S) \leq \gamma_{2}(S) \leq N_{S}(T)$. Hence $\left[P, Z_{4}(S)\right] \leq$ $Z_{2}(S) \leq P$ implies $Z_{4}(S) \leq N_{N_{S}(T)}(P)$. In addition

$$
\left[P, Z_{4}(S), Z_{4}(S)\right] \leq\left[\gamma_{1}(S), Z_{4}(S), Z_{4}(S)\right] \leq\left[Z_{2}(S), Z_{4}(S)\right]=1
$$

Since $N_{N_{S}(T)}(P)$ does not act quadratically on $\Omega_{1}(P) / \Omega_{1}(T)$, we have $Z_{4}(S) \leq P$. Since $Z_{4}(S)$ has exponent $p$ by Lemma 3.2 (iii) and (vi), $Z_{4}(S) \leq \Omega_{1}(P)$. Now

$$
\begin{aligned}
{\left[Z_{4}(S) / \Omega_{1}(T), N_{N_{S}(T)}(P)\right] } & \leq\left[Z_{4}(S) / \Omega_{1}(T), \gamma_{1}(S)\right] \leq Z_{2}(S) / \Omega_{1}(T) \\
& =Z(S) \Omega_{1}(T) / \Omega_{1}(T)
\end{aligned}
$$

and, as $\left|Z_{4}(S) / Z(S) \Omega_{1}(T)\right|=p^{2}$, we have a contradiction to the indecomposable action of $\operatorname{Out}_{N_{S}(T)}(P)$ on $\Omega_{1}(P) / \Omega_{1}(T)$. This contradiction shows that $\left|C_{\Omega_{1}(P)}(\theta)\right|>$ $p^{2}$ 。

By Lemma 11.30 the morphism $\theta$ does not centralize $Z(S)$. Hence

$$
\left|C_{\Omega_{1}(P / Z(S))}(\theta)\right|>p^{2}
$$

Consider the group $S / Z(S)$ and note that $S$ is not exceptional. Since $\hat{\theta}$ has order divisible by $(p-1) / 2$, and $\left|C_{\Omega_{1}\left(\gamma_{1}(S / Z(S))\right)}(\hat{\theta})\right| \geq\left|C_{\Omega_{1}(P / Z(S))}(\theta)\right|>p^{2}$, Lemma 3.14 implies $\left|\Omega_{1}\left(\gamma_{1}(S / Z(S))\right)\right|=p^{p}$. By Lemma $3.2(\mathrm{iv})$ we get $|S / Z(S)|=p^{p+1}$, and so $|S|=p^{p+2}$ and $\mho^{1}\left(\gamma_{1}(S)\right) \leq Z(S)=\gamma_{p+1}(S)$. However this contradicts Lemma 3.2(iii) which states that $\mho^{1}\left(\gamma_{1}(S)\right)=\gamma_{p}(S)>Z(S)$.

Lemma 11.33. Assume that Hypothesis 11.17 holds. Then $L / \operatorname{Inn}(P) \not \neq \mathrm{SL}_{2}(p)$.
Proof. Suppose that $L / \operatorname{Inn}(P) \cong \mathrm{SL}_{2}(p)$. Then, $\hat{\theta}$ has order $p-1$ and acts faithfully on $S / \gamma_{1}(S)$. If $\left|C_{\Omega_{1}\left(\gamma_{1}(S)\right)}(\theta)\right| \geq p^{2}$, then Lemma 3.14 applied to $S / Z(S)$ implies that $Z(S)$ is centralized by $\theta$. This is impossible by Lemma 11.30. Hence $\left|C_{\Omega_{1}\left(\gamma_{1}(S)\right)}(\theta)\right| \leq p$. Applying Lemma 4.5 (i) and (ii), delivers all the non-central
$L$-chief factors in $\Omega_{1}(P) / \Omega_{1}(T)$ are faithful $\mathrm{GF}(p) L / \operatorname{Inn}(P)$-modules. In particular, $\tau$ inverts $\Omega_{1}(P) / \Omega_{1}(T)$ and so $\Omega_{1}(P) / \Omega_{1}(T)$ is abelian and $\Omega_{1}(T)$ has order $p$.

Suppose that $Z_{3}(S) \leq N_{S}(T)$. Then, as $\left[Z_{3}(S), P\right] \leq Z(S)$, we obtain $Z_{3}(S) \leq$ $N_{N_{S}(T)}(P)$.

Assume that $Z_{3}(S) \not \leq P$. Then $\left[P, Z_{3}(S)\right] \leq\left[Z_{3}(S), \gamma_{1}(S)\right] \leq Z(S)$. Hence $\Omega_{1}(P) / \Omega_{1}(T)$ has exactly one non-central $L$-chief factor and $\left|P / \Omega_{1}(T)\right|=p^{2}$ by Lemma 4.4. Hence $\left|Z_{3}(S) \cap P\right|=p^{2}$. If $\Omega_{1}(T) \nsubseteq Z_{3}(S)$, then $P=\Omega_{1}(T)(P \cap$ $Z_{3}(S)$ ) and this means that $Z_{3}(S)$ centralizes $P$. Since $P$ is $N_{\mathcal{F}}(T)$-centric, this is impossible. Thus $Z_{3}(S) P / P \cong Z_{3}(S) /\left(Z_{3}(S) \cap P\right)$ and $\Omega_{1}(T)$ are both centralized by $\tau$. As $Z(S) \leq P$ and $Z(S) \cap \Omega_{1}(T)=1$, we have $Z_{3}(S) / Z(S)$ is centralized by $\tau$. This is impossible by Lemma 3.14. Hence $Z_{3}(S) \leq P$ and so $Z_{3}(S) \leq \Omega_{1}(P)$ by Lemma 3.2(iii) and (vi). Now Lemma 3.14 implies $1 \neq C_{Z_{3}(S)}(\tau) \leq C_{\Omega_{1}(P)}(\tau)=$ $\Omega_{1}(T)$. As $\Omega_{1}(T)$ has order $p$, this means that $\Omega_{1}(T) \leq Z_{3}(S)$.

We conclude that, if $Z_{3}(S) \leq N_{S}(T)$, then $\Omega_{1}(T) \leq Z_{3}(S) \leq \Omega_{1}(P)$ and, in particular, $N_{S}(T) \geq \gamma_{3}(S)$.

Continue to assume that $Z_{3}(S) \leq N_{S}(T)$. Then $Z_{4}(S) \leq \gamma_{3}(S) \leq N_{S}(T)$ by Lemma 11.1 (iii). Since $Z_{3}(S) \leq P, Z_{4}(S) \leq N_{N_{S}(T)}(P)$. Suppose that $Z_{4}(S) \not \leq$ $P$. Then $\tau$ centralizes $Z_{4}(S) P / P$ and so $\tau$ centralizes $Z_{4}(S) / Z_{3}(S)$. Lemma 3.10 implies $\tau$ inverts $Z_{3}(S) / Z_{2}(S)$ and so $\Omega_{1}(T) \leq Z_{2}(S)$. Since $\left[P, Z_{4}(S)\right] \leq Z_{2}(S)$, $\Omega_{1}(P) / \Omega_{1}(T)$ has order $p^{2}$ by Lemma 4.4. Hence $\Omega_{1}(P)=Z_{3}(S)$ and we then have $\left[\Omega_{1}(P), Z_{4}(S)\right] \leq\left[Z_{3}(S), \gamma_{3}(S)\right]=1$, a contradiction. Therefore $Z_{4}(S) \leq$ $P$. Since $Z_{4}(S)$ has exponent $p$ by Lemma 3.2 (iii) and (vi), $Z_{4}(S) \leq \Omega_{1}(P)$ and $Z_{4}(S) / \Omega_{1}(T)$ is inverted by $\tau$. Lemma 3.10 yields $\Omega_{1}(T)^{\#} \subseteq Z_{3}(S) \backslash Z_{2}(S)$ and $Z_{2}(S)$ is inverted by $\tau$. In particular, $S$ is exceptional and so $|S| \geq p^{8}, \gamma_{2}(S)$ has exponent $p$ by Lemma 3.2 (vi) and 3.3(v). Therefore $Z_{5}(S) \leq \gamma_{3}(S) \leq N_{S}(T)$ and $\left[P, Z_{5}(S)\right] \leq Z_{3}(S) \leq P$. If $Z_{5}(S) \leq P$, then $Z_{5}(S) \leq \Omega_{1}(P)$ and $C_{Z_{5}(S)}(\tau)=$ $C_{\Omega_{1}(P)}(\tau)=\Omega_{1}(T)$ has order $p$. This contradicts Lemma 3.10. Hence $Z_{5}(S) \not \leq P$, $Z_{4}(S) \leq \Omega_{1}(P)$ and, in particular, $\left|\Omega_{1}(P)\right| \geq p^{4}$. As

$$
\left[P, Z_{5}(S), Z_{5}(S)\right] \leq\left[Z_{3}(S), Z_{5}(S)\right]=1
$$

and $Z_{3}(S) / Z(S)$ has order $p^{2}$, we have that $\Omega_{1}(P) / \Omega_{1}(T)$ has two $L$-chief factors and they are both natural $\operatorname{GF}(p) L / P$-modules (of order $p^{2}$ ) by Lemma 4.4. Hence $|P|=p^{5}$ and $\left|P: C_{P}\left(Z_{5}(S)\right)\right| \geq p^{2}$. If $Z_{5}(S)$ is abelian, then $Z_{4}(S) \leq C_{P}\left(Z_{5}(S)\right)$ and $\left|P: C_{P}\left(Z_{5}(S)\right)\right| \leq p$, a contradiction. Hence $Z_{5}(S)$ is non-abelian and consequently $|S|=p^{8}$. Since $\left[Z_{4}(S), \Omega_{1}(P)\right] \leq \Omega_{1}(P)^{\prime} \cap Z_{2}(S) \leq \Omega_{1}(T) \cap Z_{2}(S)=1$, $Z_{4}(S) \leq Z\left(\Omega_{1}(P)\right)$ and we deduce that $\Omega_{1}(P)$ is abelian. On the other hand, $C_{\gamma_{1}(S)}\left(Z_{4}(S)\right)=Z_{k}(S)$ for some $k \geq 4$. As $Z_{5}(S)$ is non-abelian, we have that $C_{\gamma_{1}(S)}\left(Z_{4}(S)\right)=Z_{4}(S)$ which means that $Z_{4}(S)<P \leq Z_{4}(S)$, a contradiction. We have demonstrated that $Z_{3}(S)$ does not normalize $T$.

Assume that $S$ is not exceptional. Then, as $Z_{2}(S) \leq Z\left(\gamma_{1}(S)\right), Z_{2}(S) \leq N_{S}(T)$ and so $Z_{2}(S) \leq P$. Since $\tau$ inverts $Z_{2}(S) \Omega_{1}(T) / \Omega_{1}(T)$, Lemma 3.14 implies that $Z_{2}(S) \geq \Omega_{1}(T)$. Hence $\Omega_{1}(T)$ is centralized by $\gamma_{1}(S)$ and Lemma 11.4 (i) implies $\gamma_{1}(S)$ normalizes $T$. Since $Z_{3}(S) \leq \gamma_{1}(S)$, this is impossible. Thus $S$ is exceptional.

By Lemma 11.31, $Z_{2}(S) \leq C_{N_{S}(T)}\left(\Omega_{1}\left(N_{S}(T)\right)\right)$ and so, in particular, $Z_{2}(S) \leq$ $C_{N_{S}(T)}\left(\Omega_{1}(P)\right)$. Because $L / P$ acts faithfully on $\Omega_{1}(P) / \Omega_{1}(T)$, we deduce that $Z_{2}(S) \leq \Omega_{1}(P)$. If $\Omega_{1}(T) \leq Z_{2}(S)$, then $N_{S}(T) \geq \gamma_{2}(S)$ and then $Z_{3}(S)$ normalizes $T$, a contradiction. Hence $Z_{2}(S) \cong Z_{2}(S) \Omega_{1}(T) / \Omega_{1}(T)$ is inverted by $\tau$.

It follows that $\hat{\tau}$ centralizes $S / C_{S}\left(Z_{2}(S)\right)$ and so

$$
\left[\gamma_{1}(S), \hat{\tau}\right] \leq \gamma_{1}(S) \cap C_{S}\left(Z_{2}(S)\right)=\gamma_{2}(S)
$$

As $T$ is not normalized by $Z_{3}(S), \Omega_{1}(T)$ is not centralized by $Z_{3}(S)$ by Lemma 11.4 (i). Hence $\Omega_{1}(T) \not \leq \gamma_{3}(S)$, but $\Omega_{1}(T)$ does centralize $Z_{2}(S)$ and so $\Omega_{1}(T) \leq$ $\gamma_{2}(S)$. Thus $\gamma_{2}(S) / \gamma_{3}(S)=\Omega_{1}(T) \gamma_{3}(S) / \gamma_{3}(S)$ is centralized by $\hat{\tau}$. Therefore $\hat{\tau}$ centralizes $\gamma_{1}(S) / \gamma_{3}(S)$ and this finally contradicts Lemma 3.10. We have shown that $L / \operatorname{Inn}(P) \not \models \mathrm{SL}_{2}(p)$.

Proposition 11.34. If Hypothesis 10.3 holds. Then $T$ is $S$-centric.
Proof. If $T$ is not $S$-centric, then Hypothesis 11.17 holds. Recall the definitions of $P$ and $L$ from Notation 11.19 and Notation 11.25. Then Lemma 11.29 yields $L / \operatorname{Inn}(P) \cong \operatorname{PSL}_{2}(p)$ or $\mathrm{SL}_{2}(p)$, whereas Lemmas 11.32 asserts that $L / \operatorname{Inn}(P) \not \approx$ $\mathrm{PSL}_{2}(p)$ and Lemma 11.33 states that $L / \operatorname{Inn}(P) \nsubseteq \mathrm{SL}_{2}(p)$. This is impossible. Thus $T$ is $S$-centric.

Proof of Theorem D. As we remarked at the beginning of Section 10, Lemmas 6.1 and 6.2 show that Theorem D holds if $p \leq 3$ as in this case $S$ is not exceptional by Lemma 3.3(v). Hence we may assume that Hypothesis 6.3 holds. Assume Theorem D is false. Then Hypothesis 10.3 and Notation 11.2 hold. Now we obtain a contradiction as Proposition 11.16 yields $T$ is not $S$-centric while Proposition 11.34 asserts that $T$ is $S$-centric. We conclude that the main statement in Theorem D is true. If $S$ is exceptional, then we obtain $\mathcal{P}(\mathcal{F})=\mathcal{P}_{a}(\mathcal{F})$ from Proposition 7.2. This concludes the proof.

## 12. The saturated fusion systems on exceptional maximal class groups: the proof of Theorem B

The objective of this section is to prove Theorem B, which for convenience we repeat below. Recall that in the introduction we defined $S(p)$ as the unique split extension of an extraspecial group of exponent $p$ and order $p^{p-2}$ by a cyclic group of order $p$ which has maximal class [41, Proposition 8.1].

Theorem B. Suppose that $p \geq 5, S$ is an exceptional maximal class p-group of order at least $p^{4}$ and $\mathcal{F}$ is a saturated fusion system on $S$. Assume that $\mathcal{F} \neq N_{\mathcal{F}}(S)$. Then one of the following holds.
(i) $\gamma_{1}(S)$ is extraspecial, and, if $\mathcal{F} \neq N_{\mathcal{F}}\left(\gamma_{1}(S)\right.$ ), then one of the following holds:
(a) $S$ is isomorphic to a Sylow p-subgroup of $\mathrm{G}_{2}(p)$ and either
$(\alpha) \mathcal{F}=N_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right), O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right)\right) \cong \mathrm{SL}_{2}(p)$;
( $\beta$ ) $p=5,1 \neq O_{p}(\mathcal{F}) \leq \gamma_{2}(S), \mathcal{F} \cong \mathcal{F}_{S}\left(5^{3} \cdot \mathrm{SL}_{3}(5)\right)$;
$(\gamma) p \geq 5$ and $\mathcal{F}=\mathcal{F}_{S}\left(\mathrm{G}_{2}(p)\right)$;
( $\delta) ~ p=5$ and $\mathcal{F}=\mathcal{F}_{S}(G)$ where $G=\mathrm{Ly}, \mathrm{HN}, \operatorname{Aut}(\mathrm{HN})$ or B ; or
(ع) $p=7$ and either $\mathcal{F}$ is exotic (27 examples) or $\mathcal{F}=\mathcal{F}_{S}(\mathrm{M})$.
(b) $p \geq 11, S \cong S(p), \mathcal{P}(\mathcal{F})=\mathcal{P}_{a}(\mathcal{F}) \neq \emptyset$ and, if $\gamma_{1}(S)$ is $\mathcal{F}$-essential, then $\operatorname{Out}_{\mathcal{F}}(S) \cong \operatorname{GF}(p)^{\times} \times \operatorname{GF}(p)^{\times}, O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{2}(S)\right)\right) \cong \operatorname{SL}_{2}(p)$ and $\gamma_{1}(S) / Z\left(\gamma_{1}(S)\right)$ is the unique $(p-3)$-dimensional irreducible $\mathrm{GF}(p) \mathrm{SL}_{2}(p)$-module.
(ii) $p=5, S=\operatorname{SmallGroup}\left(5^{6}, 661\right), O_{5}(\mathcal{F})=C_{S}\left(Z_{2}(S)\right)$ is the unique $\mathcal{F}$ essential subgroup, Out $\mathcal{F}(S)$ is cyclic of order 4, $\operatorname{Out}_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right) \cong$ $\mathrm{SL}_{2}(5)$ and $\mathcal{F}$ is unique.
In particular, if $\mathcal{F} \neq N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$, then $\mathcal{F}=O^{p}(\mathcal{F})$ and, in addition, $O_{p}(\mathcal{F})=1$ in all cases other than parts (i) (a) ( $\alpha$ ), (i) (a) ( $\beta$ ) and part (ii).

Proof. Let $\mathcal{F}$ be a saturated fusion system on $S$ and assume that $\mathcal{F} \neq N_{\mathcal{F}}(S)$. Then $\mathcal{E}_{\mathcal{F}}$ is non-empty and

$$
\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{P}_{a}(\mathcal{F}) \cup\left\{C_{S}\left(Z_{2}(S)\right), \gamma_{1}(S)\right\}
$$

by Theorem D.
Suppose $\mathcal{E}_{\mathcal{F}}=\left\{\gamma_{1}(S)\right\}$. Then $\mathcal{F}=N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ and $\gamma_{1}(S)$ is extraspecial by Proposition 9.1. From now on, suppose $\mathcal{E}_{\mathcal{F}} \neq\left\{\gamma_{1}(S)\right\}$. Then by Proposition 7.2 either $\gamma_{1}(S)$ is extraspecial or conclusion (ii) holds. So suppose $\gamma_{1}(S)$ is extraspecial and let $R=O_{p}(\mathcal{F})$.

Assume $R \neq 1$. Then $R$ is normal in $S$ and, for $E \in \mathcal{E}_{\mathcal{F}}, R \leq E$ and $R$ is Aut $_{\mathcal{F}}(E)$-invariant by Lemma 5.10. In particular, $\mathcal{F}$ has no $\mathcal{F}$-pearls as all the $\mathcal{F}$ pearls are abelian. Since $\mathcal{E}_{\mathcal{F}} \neq\left\{\gamma_{1}(S)\right\}$ by assumption, we deduce that $C_{S}\left(Z_{2}(S)\right) \in$ $\mathcal{E}_{\mathcal{F}}$. Thus Proposition 7.2 implies that $|S|=p^{6}$ and $O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right)\right) \cong$ $\mathrm{SL}_{2}(p)$ acts transitively on the maximal subgroups of $C_{S}\left(Z_{2}(S)\right)$. As $\gamma_{2}(S)$ has exponent $p$ by Lemma 3.2 (vi) and $\Phi\left(C_{S}\left(Z_{2}(S)\right)\right)=\gamma_{4}(S), C_{S}\left(Z_{2}(S)\right)$ has exponent $p$. In particular, there exists an element $x$ of order $p$ in $C_{S}\left(Z_{2}(S)\right)$ such that $S=\gamma_{1}(S)\langle x\rangle$. Hence [41, Proposition 8.1] implies that $S$ is isomorphic to a Sylow $p$-subgroup of $\mathrm{G}_{2}(p)$. If $\left|\mathcal{E}_{\mathcal{F}}\right|=1$ then $\mathcal{E}_{\mathcal{F}}=C_{S}\left(Z_{2}(S)\right)$ and we are in case (i) (a) ( $\alpha$ ). If $\left|\mathcal{E}_{\mathcal{F}}\right|=2$, then $\mathcal{E}_{\mathcal{F}}=\left\{\gamma_{1}(S), C_{S}\left(Z_{2}(S)\right)\right\}$. We have $R \leq C_{S}\left(Z_{2}(S)\right) \cap$ $\gamma_{1}(S)=\gamma_{2}(S)$ and, as $R$ is $\operatorname{Aut}_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right.$ )-invariant and $\operatorname{Aut}_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right)$ acts irreducibly on $C_{S}\left(Z_{2}(S)\right) / \Phi\left(C_{S}\left(Z_{2}(S)\right)\right)$, we obtain $R \leq \Phi\left(C_{S}\left(Z_{2}(S)\right)\right)=$
$\gamma_{3}(S)$. Since Aut $\mathcal{F}^{( }\left(C_{S}\left(Z_{2}(S)\right)\right)$ does not normalize $Z(S)$, we have $R=Z_{2}(S)$ or $R=Z_{3}(S)=\gamma_{3}(S)$. Assume that $R=Z_{2}(S)$. Then $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ normalizes $C_{\gamma_{1}(S)}\left(Z_{2}(S)\right)=\gamma_{2}(S)$ and $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ acts on $\gamma_{2}(S) / Z_{2}(S)$ which has order $p^{2}$. Since $\gamma_{2}(S) / \gamma_{3}(S), Z_{2}(S) / Z(S)$ and $Z(S)$ are centralized by $O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}(S)\right.$ ), we have $O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right) \cong \operatorname{SL}_{2}(p)$. Now using an element $\tau \in \operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ which maps to the involution in $O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right)$ we have $\left[\gamma_{1}(S), \tau\right]$ is extraspecial of order $p^{3}$ and is normalized by $S$. Since $\left[\gamma_{1}(S), \tau\right] \neq \gamma_{3}(S)$, this is impossible. Hence $R=\gamma_{3}(S)$ and $R$ is $\mathcal{F}$-centric. It follows that there is a model $H$ for $N_{\mathcal{F}}(R)$. From the structure of $\operatorname{Aut}(R) \cong \mathrm{GL}_{3}(p)$, we deduce that $O^{p^{\prime}}(H) / R \cong \mathrm{SL}_{3}(p)$ as $\left|\mathcal{E}_{\mathcal{F}}\right| \geq 2$. Let $P=C_{O_{p^{\prime}(H)}}(Z(S))$, then $P / \gamma_{1}(S) \cong \mathrm{SL}_{2}(p)$ and $O^{p^{\prime}}(H)$ acts on the natural $\mathrm{GF}(p) \mathrm{SL}_{2}(p)$-modules $\gamma_{1}(S) / R$ and $R / Z(S)$. Since $S$ has maximal class, $\gamma_{1}(S) / Z(S)$ is a non-split extension for $P / \gamma_{1}(S)$. Now we may apply $[\mathbf{1 9}$, Proposition 4.2] and obtain $p=5$. This, together with the model theorem gives (i)(a)( $\beta$ ).

Hence we may now suppose that $R=1$. [41, Main Theorem] implies that either $S$ is isomorphic to a Sylow $p$-subgroup of $\mathrm{G}_{2}(p)$ or $S \cong S(p)$. In the first case, [47, Theorem 1.1] shows that $\mathcal{F}$ is known and it is one of the fusion systems described in $(\mathrm{i})(\mathrm{a})(\gamma)$, $(\delta)$ and $(\epsilon)$. If $S \cong S(p)$, then $\mathcal{F}$ is known by [41, Theorem 2 ] and corresponds to case (i)(b).

## 13. The saturated fusion systems on non-exceptional maximal class groups

In this section we will lay the ground work for the proof of Theorem C which will be presented in the next section. So suppose that $p$ is an odd prime and $S$ is a maximal class $p$-group of order at least $p^{4}$. Following [19], we define

$$
\Delta=\mathrm{GF}(p)^{\times} \times \mathrm{GF}(p)^{\times}
$$

which has order $(p-1)^{2}$ and, for $i \in \mathbb{Z}$, we define the diagonal subgroups

$$
\Delta_{i}=\left\{\left(r, r^{i}\right) \mid r \in \operatorname{GF}(p)^{\times}\right\}
$$

Most important here are $\Delta_{0}$ and $\Delta_{-1}$. Indeed, these diagonals originate from the existence of $\mathcal{F}$-pearls, as we shall see in Lemma 13.4. Suppose that $\alpha \in \operatorname{Aut}(S)$, $x \in S \backslash \gamma_{1}(S)$ and $z \in Z(S)^{\#}$. Then there exist $r, s \in \operatorname{GF}(p)^{\times}$which are independent of $x$ and $z$ such that

$$
x \alpha \gamma_{1}(S)=x^{r} \gamma_{1}(S)
$$

and

$$
z \alpha=z^{s} .
$$

With this notation established, define the homomorphism

$$
\begin{aligned}
\mu: \operatorname{Aut}(S) & \rightarrow \Delta \\
\alpha & \mapsto(r, s) .
\end{aligned}
$$

We also use the following definitions from [19].
Definition 13.1. Suppose that $p$ is a prime, $G$ is a group, and $U \in \operatorname{Syl}_{p}(G)$. Then $G$ is in class $\mathcal{G}_{p}^{\wedge}$ if and only if $O_{p}(G)=1,|U|=p$, and $\left|\operatorname{Aut}_{G}(U)\right|=p-1$.

Definition 13.2. Suppose that $G \in \mathcal{G}_{p}^{\wedge}$ and let $V$ be a faithful $\operatorname{GF}(p) G$ module. Then $G$ is minimally active on $V$ if and only if the matrix representing $u \in U^{\#}$ in its action on $V$ has one non-trivial Jordan block.

We will need the following consequence of the results of Craven, Oliver and Semeraro [19] as listed in Appendix A.

Proposition 13.3. Suppose that $p$ is an odd prime, $\mathcal{G}$ is a reduced fusion system on a p-group $S$ of maximal class of order at least $p^{4}$ with $\gamma_{1}(S)$ elementary abelian and $\mathcal{G}$-essential. Assume that $\operatorname{Out}_{\mathcal{G}}(S) \cong \operatorname{Aut}_{\mathcal{G}}(S) \mu=\Delta_{0}$ or $\Delta_{-1}$ and $\left|\gamma_{1}(S)\right| \leq p^{p-1}$. Then $p \geq 5$, $\operatorname{Aut}_{\mathcal{G}}\left(\gamma_{1}(S)\right) \cong \operatorname{Sym}(p)$ or $\operatorname{PGL}_{2}(p), \gamma_{1}(S)$ is irreducible as a $\operatorname{GF}(p) \operatorname{Aut}_{\mathcal{G}}\left(\gamma_{1}(S)\right)$-module and $\left|\gamma_{1}(S)\right|=p^{p-2}$. Furthermore, $\operatorname{Aut}_{\mathcal{G}}(S) \mu=\Delta_{-1}$ and $\mathcal{P}(\mathcal{G})=\mathcal{P}_{a}(\mathcal{G})$ is non-empty.

Proof. Set $V=\gamma_{1}(S), G=\operatorname{Aut}_{\mathcal{G}}(V)$ and $G_{0}=F^{*}(G)$. By Theorem D we have

$$
\mathcal{E}_{\mathcal{G}}=\mathcal{P}(\mathcal{G}) \cup\{V\}
$$

and since $\mathcal{G}$ is reduced, $\mathcal{P}(\mathcal{G}) \neq \emptyset$. Because $\operatorname{Out}_{\mathcal{G}}(S) \cong \operatorname{Aut}_{\mathcal{G}}(S) \mu=\Delta_{0}$ or $\Delta_{-1}$, we know $Z(G)=1$ from the definition of $\Delta_{0}$ and $\Delta_{-1}$. Now, as $Z(G)=1$ and $\operatorname{Aut}_{\mathcal{G}}(S) \mu=\Delta_{0}$ or $\Delta_{-1}$, applying Theorem A. 1 we have that $\operatorname{Aut}_{\mathcal{G}}\left(\gamma_{1}(S)\right) \cong$ $\operatorname{Sym}(p)$ or $\mathrm{PGL}_{2}(p)$ and that $\mathcal{G}$ is described in lines 3, 4, 29, 30, 33 or 34 of Table 2. If $\operatorname{Aut}_{\mathcal{G}}(S) \mu=\Delta_{0}$, then inspection of the consequence of column six of Table 2 as read from Table 1 yields a contradiction to the structure of $\operatorname{Aut}_{\mathcal{G}}(S) \mu$. If $\operatorname{Aut}_{\mathcal{G}}(S) \mu=\Delta_{-1}$, then, again using Tables 1 and 2 , we obtain cases 3 and 4 and that III from Table 1 holds. Thus $|V|=p^{p-2}$ and, additionally, $\mathcal{P}(\mathcal{G})=\mathcal{P}_{a}(\mathcal{G})$.

From here on we assume that $\mathcal{F}$ is a saturated fusion system on $S$ and that $S$ is not exceptional. By Theorem D , the $\mathcal{F}$-essential subgroups are contained in $\mathcal{P}(\mathcal{F}) \cup\left\{\gamma_{1}(S)\right\}$. If $\mathcal{P}(\mathcal{F})$ is empty, then $\mathcal{F}=N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ and, if $\mathcal{P}(\mathcal{F})$ is non-empty, then $O_{p}(\mathcal{F}) \leq Z(S)$. Hence $\mathcal{F} \neq N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ if and only if $\mathcal{P}(\mathcal{F})$ is non-empty.

Lemma 13.4. Assume that $\gamma_{1}(S)$ is not abelian and $P \in \mathcal{P}(\mathcal{F})$.
(i) $\operatorname{Out}_{\mathcal{F}}(S)$ is a Hall $p^{\prime}$-subgroup of $\operatorname{Out}(S)$, is cyclic of order $p-1$ and acts faithfully on $S / \gamma_{1}(S)$.
(ii) $\operatorname{Aut}_{\mathcal{F}}(S)=N_{\operatorname{Aut}_{\mathcal{F}}(S)}(P) \operatorname{Inn}(S)$.
(iii) $\operatorname{Out}_{\mathcal{F}}(P) \cong \mathrm{SL}_{2}(p)$.
(iv) If $P$ is abelian, then $\operatorname{Aut}_{\mathcal{F}}(S) \mu=\Delta_{-1}$.
(v) If $P$ is extraspecial, then $\operatorname{Aut}_{\mathcal{F}}(S) \mu=\Delta_{0}$.
(vi) Either $\mathcal{P}(\mathcal{F})=\mathcal{P}_{a}(\mathcal{F})$ or $\mathcal{P}(\mathcal{F})=\mathcal{P}_{e}(\mathcal{F})$.

In particular, if $\gamma_{1}(S)$ is $\mathcal{F}$-essential, then $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right.$ ) is a member of $\mathcal{G}_{p}^{\wedge}$ and $Z\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right)=1$.

Proof. Parts (i), (ii) and (iii) are Lemma 5.24.
Suppose that $P$ is abelian. Then Lemma 5.19 (iii) implies that $P \cap \gamma_{1}(S)=Z(S)$ and $S=P \gamma_{1}(S)$. Let $\tau \in \operatorname{Aut}_{\mathcal{F}}(S)$ be such that $\tau$ restricts to a generator of $N_{\text {Aut }_{\mathcal{F}}(P)}\left(\operatorname{Aut}_{S}(P)\right)$. As Aut $\mathcal{F}(P) \cong \mathrm{SL}_{2}(p)$ by (iii), the element $\tau$ raises elements of $S / \gamma_{1}(S) \cong P / Z(S)$ to the power $r$ and elements of $Z(S)$ to the power $r^{-1}$ for some $r \in \operatorname{GF}(p)^{\times}$. Hence $\operatorname{Aut}_{\mathcal{F}}(S) \mu=\Delta_{-1}$ in this case. If $P$ is extraspecial, then, as $\operatorname{Aut}_{\mathcal{F}}(P) \cong \mathrm{SL}_{2}(p), Z(S)$ is centralized by $\operatorname{Aut}_{\mathcal{F}}(P)$. Thus $\operatorname{Aut}_{\mathcal{F}}(S) \mu=\Delta_{0}$. As $\left|\operatorname{Out}_{\mathcal{F}}(S)\right|=p-1$ by (i), we have shown parts (iv) and (v) hold. Finally (vi) follows from (iv) and (v).

For the rest of this section, we assume that

## Hypothesis 13.5.

(i) $p \geq 5, S$ has maximal class, is not exceptional and has order at least $p^{4}$;
(ii) $\mathcal{F}$ is a saturated fusion system on $S$;
(iii) $\gamma_{1}(S)$ is $\mathcal{F}$-essential and is not abelian; and
(iv) $P \in \mathcal{P}(\mathcal{F})$.

To ease notation we set

$$
Q=\gamma_{1}(S), G=\operatorname{Out}_{\mathcal{F}}(Q) \text { and } G_{0}=F^{*}(G)
$$

We also put

$$
V=\Omega_{1}(Z(Q)) \text { and } S_{1}=V P
$$

As $S$ is not exceptional, $V \geq Z_{2}(S)$ and so $|V| \geq p^{2}$. Set

$$
H=\operatorname{Inn}\left(S_{1}\right)\left\langle\left.\phi\right|_{S_{1}} \mid \phi \in N_{\operatorname{Aut}_{\mathcal{F}}(S)}(P)\right\rangle \leq \operatorname{Aut}_{\mathcal{F}}\left(S_{1}\right)
$$

and

$$
B=\left\{\left.\phi\right|_{V} \mid \phi \in \operatorname{Aut}_{\mathcal{F}}(Q)\right\} \leq \operatorname{Aut}_{\mathcal{F}}(V)
$$

Proposition 5.27 implies that

$$
\mathcal{F}_{0}=\left\langle\operatorname{Aut}_{\mathcal{F}}(P), B, H\right\rangle
$$

is a saturated fusion system on $S_{1}$ and, by construction, $V$ has index $p$ in $S_{1}$ which has maximal class. Furthermore $P$ is an $\mathcal{F}_{0}$-pearl. Set

$$
V_{0}=V \cap \operatorname{hyp}\left(\mathcal{F}_{0}\right)
$$

Lemma 13.6. Assume that Hypothesis 13.5 (i),(ii) and (iv) hold with $Q$ nonabelian but not necessarily $\mathcal{F}$-essential. Suppose that $\tau \in \operatorname{Aut}_{\mathcal{F}}(S)$ has order $p-1$ and let $s_{i} \in \gamma_{i}(S) \backslash \gamma_{i+1}(S)$ for $1 \leq i \leq n-1$. Then either
(i) $P$ is abelian, $\tau$ acts fixed-point-freely on $V_{0}$ and centralizes $V / V_{0}$ and for all $j$,

$$
s_{n-j} \gamma_{n-j+1}(S) \tau=s_{n-j}^{r^{-j}} \gamma_{n-j+1}(S)
$$

where $\tau \mu=\left(r, r^{-1}\right) \in \Delta_{-1}$; or
(ii) $P$ is extraspecial, $\tau$ acts fixed-point-freely on $V / Z(S)$ and centralizes $Z(S)$ and, for all $j$,

$$
s_{n-(j+1)} \gamma_{n-j}(S) \tau=s_{n-(j+1)}^{r^{-j}} \gamma_{n-j}(S)
$$

where $\tau \mu=(r, 1) \in \Delta_{0}$.
Proof. Recall that $|S|=p^{n}$. Assume that $P$ is abelian. Then Lemma 13.4 (iv) gives $\operatorname{Aut}_{\mathcal{F}}(S) \mu=\Delta_{-1}$. Hence we take $\tau=\left(r, r^{-1}\right)$. Then, for $x \in P \backslash Q$ we have $x Q \tau=x^{r} Q$, and $s_{n-1} \tau=s_{n-1}^{r^{-1}}$. We may write $s_{1} \tau \gamma_{2}(S)=s_{1}^{b} \gamma_{2}(S)$. Then, Lemma 3.10 shows that $r^{-1}=r^{n-2} b$,

$$
s_{n-j} \gamma_{n-j+1}(S) \tau=s_{n-j}^{r^{n-j-1} b} \gamma_{n-j+1}(S) \tau=s_{n-j}^{r^{-j}} \gamma_{n-j+1}(S)
$$

Since $r^{-1}$ has order $p-1$, we now have $\tau$ acts fixed-point-freely on $V_{0}$ which has order $p^{p-2}$ and $\tau$ centralizes $V / V_{0}=\gamma_{n-(p-1)}(S) / \gamma_{n-(p-2)}(S)$. Since $O^{p^{\prime}}(G)$ centralizes $V / V_{0}, G$ centralizes $V / V_{0}$. Thus (i) holds.

The proof of (ii) follows similarly.
Lemma 13.7. Assume that Hypothesis 13.5 holds. Then
(i) $G$ is faithful and minimally active on $V$.
(ii) $\operatorname{Aut}_{\mathcal{F}_{0}}(V) \cong G$.
(iii) Either $O_{p}\left(\mathcal{F}_{0}\right)=1$ or $P$ is extraspecial and $O_{p}\left(\mathcal{F}_{0}\right)=Z(S)$.
(iv) If $P$ is abelian, then $\operatorname{Out}_{O^{p}\left(\mathcal{F}_{0}\right)}\left(\operatorname{hyp}\left(\mathcal{F}_{0}\right)\right) \cong \operatorname{Aut}_{O^{p}\left(\mathcal{F}_{0}\right)}\left(\operatorname{hyp}\left(\mathcal{F}_{0}\right)\right) \mu=\Delta_{-1}$ and, if $P$ is extraspecial $\operatorname{Out}_{O^{p}\left(\mathcal{F}_{0}\right)}\left(\operatorname{hyp}\left(\mathcal{F}_{0}\right)\right) \cong \operatorname{Aut}_{O^{p}\left(\mathcal{F}_{0}\right)}\left(\operatorname{hyp}\left(\mathcal{F}_{0}\right)\right) \mu=$ $\Delta_{0}$.
(v) If $P$ is extraspecial and $O_{p}\left(\mathcal{F}_{0}\right)=Z(S)$, then

$$
\operatorname{Aut}_{O^{p}\left(\mathcal{F}_{0} / Z(S)\right)}\left(\operatorname{hyp}\left(\mathcal{F}_{0}\right) / Z(S)\right) \mu=\Delta_{-1}
$$

(vi) If $P$ is abelian, then $O_{p}(\mathcal{F})=O_{p}\left(\mathcal{F}_{0}\right)=1$ and $O^{p}\left(\mathcal{F}_{0}\right)$ is reduced.
(vii) If $P$ is extraspecial, then either $O_{p}(\mathcal{F})=O_{p}\left(\mathcal{F}_{0}\right)=1$ and $O^{p}\left(\mathcal{F}_{0}\right)$ is reduced or $O_{p}(\mathcal{F})=O_{p}\left(\mathcal{F}_{0}\right)=Z(S)$ and $O^{p}\left(\mathcal{F}_{0} / Z(S)\right)$ is reduced.
Proof. Since $V$ is centralized by $Q, V$ is a $\operatorname{GF}(p) G$-module. As $C_{V}(S)=Z(S)$ has order $p, S / Q$ acts with a unique Jordan block on $V$ and as $|V| \geq p^{2}$ this block is not trivial. Therefore $G / C_{G}(V)$ is minimally active on $V$ and $C_{G}(V)$ is a $p^{\prime}$-group. Assume that $C_{G}(V) \neq 1$. Let $K \leq \operatorname{Aut}_{\mathcal{F}}(Q)$ be of $p^{\prime}$-order satisfy $K \operatorname{Inn}(Q) / \operatorname{Inn}(Q)=C_{G}(V)$. By Lemma $13.4(\mathrm{i}), C_{K \operatorname{Inn}(Q) / \operatorname{Inn}(Q)}(S / Q)=1$. In particular, $\left[K, N_{\operatorname{Aut}_{S}(Q)}(K)\right]=K$ by coprime action. Using [24, Theorem 5.3.10], we obtain that $K$ acts faithfully on $\Omega_{1}(Q)$. Since $K$ centralizes $V, K N_{\text {Aut }_{S}(Q)}(K)$ acts faithfully on $\Omega_{1}(Q) / V$ which has order at most $p^{p-2}$ by Lemma 3.2 (vi). However, Proposition 2.5 implies that $\left|\Omega_{1}(Q) / V\right| \geq p^{p-1}$, which is a contradiction. Hence $C_{G}(V)=1$ and $G$ acts faithfully on $V$. This proves (i).

To see (ii), just note that the restriction map $\operatorname{Aut}_{\mathcal{F}}(Q) \rightarrow \operatorname{Aut}_{\mathcal{F}}(V)$ has kernel $Q$ as $G$ acts faithfully on $V$. Thus (ii) follows immediately from (i).

Since $O_{p}\left(\mathcal{F}_{0}\right) \leq P$, is normal in $S$ and Aut $_{\mathcal{F}_{0}}(P)$-invariant by Lemma 5.10, if $O_{p}\left(\mathcal{F}_{0}\right) \neq 1$, then $O_{p}\left(\mathcal{F}_{0}\right)=Z(P)=Z(S)$ and $P$ is extraspecial. This proves (iii).

Because $\operatorname{Aut}_{\mathcal{F}_{0}}(P V)=H$, and $\operatorname{Aut}_{\mathcal{F}_{0}}(P)=O^{p}\left(\operatorname{Aut}_{\mathcal{F}_{0}}(P)\right) \leq O^{p}\left(\mathcal{F}_{0}\right)$ by Lemma 5.11, we have $\operatorname{Aut}_{O^{p}\left(\mathcal{F}_{0}\right)}(P V) \mu=N_{\operatorname{Aut}_{\mathcal{F}}\left(\operatorname{hyp}\left(\mathcal{F}_{0}\right)\right)}(P) \mu$ and so (iv) follows from Lemma 13.4 (iv) and (v).

Assume that $O_{p}\left(\mathcal{F}_{0}\right)=Z(S)$. Then $P / Z(S)$ is an abelian $\mathcal{F}_{0} / Z(S)$-pearl. The argument which proves Lemma 13.4 (iv), also establishes part (v).

As $\operatorname{Aut}_{\mathcal{F}_{0}}^{(P)}\left(\operatorname{hyp}\left(\mathcal{F}_{0}\right)\right)=\operatorname{Aut}_{\mathcal{F}_{0}}\left(\operatorname{hyp}\left(\mathcal{F}_{0}\right)\right)$, Lemma 5.20 and the fact that $V$ is abelian provide us the hypothesis of Lemma 5.13 and this yields $O^{p^{\prime}}\left(O^{p}\left(\mathcal{F}_{0}\right)\right)=$ $O^{p}\left(\mathcal{F}_{0}\right)$. Hence $O^{p}\left(\mathcal{F}_{0}\right)$ is reduced whenever $O_{p}\left(\mathcal{F}_{0}\right)=1$ and otherwise we have $O^{p}\left(\mathcal{F}_{0}\right) / Z(S)$ is reduced.

Lemma 13.8. Assume that Hypothesis 13.5 holds. If $O_{p}(\mathcal{F})=1$, then $P$ is abelian, $V_{0}$ is irreducible as a $\mathrm{GF}(p) G$-module, $\left|V_{0}\right|=p^{p-2}$ and $G \cong \operatorname{Sym}(p)$ or $\mathrm{PGL}_{2}(p)$.

Proof. Because of Lemma 13.7 (iv), (vi) and (vii), we have $O^{p}\left(\mathcal{F}_{0}\right)$ is reduced and, as $\gamma_{1}\left(S_{1}\right)=V$, we may apply Proposition 13.3 to $O^{p}\left(\mathcal{F}_{0}\right)$ to obtain the result.

Lemma 13.9. Assume that Hypothesis 13.5 holds. If $O_{p}(\mathcal{F}) \neq 1$, then $P$ is extraspecial, $G \cong \operatorname{Sym}(p)$ or $\mathrm{PGL}_{2}(p), O_{p}(\mathcal{F})=Z(S)=Z(P)$ and $V=V_{0}$ is abelian with $G$-composition factors $O_{p}(\mathcal{F})=Z(S)$ which is centralized by $G$ and $V / Z(S)$ which is an irreducible $\mathrm{GF}(p) G$-module of order $p^{p-2}$.

Proof. By Lemma 13.7 (iii) and (vii), $P$ is extraspecial and $O^{p}\left(\mathcal{F}_{0} / Z(S)\right)$ is reduced. In addition, Lemma 13.7 (v) gives $\operatorname{Aut}_{O^{p}\left(\mathcal{F}_{0} / Z(S)\right)}\left(\operatorname{hyp}\left(\mathcal{F}_{0}\right) / Z(S)\right) \mu=$ $\Delta_{-1}$. Now Proposition 13.3 yields $G \cong \operatorname{Sym}(p)$, or $\mathrm{PGL}_{2}(p)$ and $V_{0} / Z(S)$ has order $p^{p-2}$ is irreducible as a $\operatorname{GF}(p) G$-module. Thus $\left|V_{0}\right|=p^{p-1}$ and $O^{p^{\prime}}(G)$ centralizes $Z(S)$. In particular, $V=V_{0}$. Since $\operatorname{Aut}_{\mathcal{F}}(P)$ also centralizes $Z(S)$, we have $G$ centralizes $Z(S)=O_{p}(\mathcal{F})$ and this completes the proof of the lemma.

Lemma 13.10. Assume that Hypothesis 13.5 holds. Then $V=\Omega_{1}(Q)$ has order $p^{p-1}$ and $|S|>p^{p+1}$. In particular, $\Omega_{1}(Q)<Q$.

Proof. We know $\left|V_{0}\right| \geq p^{p-2}$ by Lemmas 13.8 and 13.9. Suppose that $|S| \leq$ $p^{p+1}$. If $|S|=p^{p}$, then $V_{0}$ has index $p$ in $Q$ and as $V_{0} \leq Z(Q)$, we deduce that $Q$ is abelian, a contradiction. Suppose that $|S|=p^{p+1}$. If $P$ is extraspecial, then Lemma 13.9 implies that $V_{0}=V$ has order $p^{p-1}$. Thus, in this case, $V$ has index $p$ in $Q$. Since $V \leq Z(Q)$, we have $Q$ is abelian, a contradiction. So suppose that $P$ is abelian. Then $Q / V_{0}$ has order $p^{2}$ and $Q / V_{0}$ is abelian. Since $G$ does not embed in $\mathrm{GL}_{2}(p)$, we must have $\left[Q / V_{0}, O^{p^{\prime}}(G)\right]=1$. But then $S / V_{0}$ is abelian, a contradiction. Thus $|S|>p^{p+1}$ and so $\Omega_{1}(Q)$ has order $p^{p-1}$ by Lemma 3.2 (iv). In particular, $V_{0}$ has index at most $p$ in $\Omega_{1}(Q)$ with equality if $P$ is extraspecial. So assume $P$ is abelian. Then, as $Q$ is the 2-step centralizer in $S$ and $V_{0}$ is an irreducible $\mathrm{GF}(p) G$-module, we deduce that $V_{0}>\left[\Omega_{1}(Q), Q\right]=1$ and so $\Omega_{1}(Q) \leq Z(Q)$. Hence $V=\Omega_{1}(Q)$ in both cases. This proves the lemma.

There are two possibilities for the structure of $V$ as a $\mathrm{GF}(p) G$-module dependent upon the type of $P$. If $P$ is abelian, then $V$ is a non-split extension of a submodule $V_{0}$ of order $p^{p-2}$ (that is $Z_{p-2}(S)$ ) by a quotient of order $p$, whereas, if $P$ is extraspecial, then there is a submodule of order $p$ (that is $Z(S)$ ) and a
quotient of order $p^{p-2}$ which is isomorphic to $V_{0}$. In each case there is a unique proper submodule and a unique non-trivial quotient.

In the next lemma, we set $\Omega_{0}(Q)=1$. Also, its useful to remember that if $\left|Q / \Omega_{j-1}(Q)\right|>p^{p}$ for some $j \geq 0$, then $\left|\Omega_{j}(Q) / \Omega_{j-1}(Q)\right|=p^{p-1}$ by Lemma 3.2 (iii) applied to $S / \Omega_{j-1}(S)$.

Lemma 13.11. Assume that Hypothesis 13.5 holds. Suppose that $j \geq 1$ is maximal such that $\Omega_{j}(Q) \leq Q$ and $\left|\Omega_{j}(Q) / \Omega_{j-1}(Q)\right|=p^{p-1}$. Then, for $1 \leq k \leq j$, as $\operatorname{GF}(p) \operatorname{Aut}_{\mathcal{F}}(Q)$-modules, $\Omega_{k}(Q) / \Omega_{k-1}(Q) \cong V$ and, either $Q=\Omega_{j}(Q)$ or, as $\mathrm{GF}(p) \mathrm{Aut}_{\mathcal{F}}(Q)$-modules

$$
Q / \Omega_{j}(Q) \cong \begin{cases}Z(S) & P \text { extraspecial } \\ V_{0} & P \text { abelian } .\end{cases}
$$

In particular, $\Omega_{m}(Q) / \Omega_{m-1}(Q)$ is centralized by $Q$ for all $m$ and the (ascending) Aut $_{\mathcal{F}}(Q)$-chief factors in the unique $\operatorname{Aut}_{\mathcal{F}}(Q)$-chief series in $Q$ alternate between having order $p$ and order $p^{p-2}$ until it reaches $Q$.

Proof. From Lemma 13.10 we know that $Q>\Omega_{1}(Q)=V$ and $V \leq Z(Q)$. In particular, $\Omega_{2}(Q)>V$. Suppose that $\ell \geq 0$ is minimal such that $\Omega_{\ell+1}(Q) / \Omega_{\ell}(Q)$ is not isomorphic to $V$. By Lemma $13.10, \ell \geq 1$.

Let $\Omega_{\ell}(Q)<W \leq \Omega_{\ell+1}(Q)$ be defined as

$$
W / \Omega_{\ell}(Q)=C_{\Omega_{\ell+1}(Q) / \Omega_{\ell}(Q)}(Q)
$$

Notice that $W^{\prime} \Omega_{\ell-1}(Q) / \Omega_{\ell-1}(Q) \leq \Omega_{\ell}(Q) / \Omega_{\ell-1}(Q)$ and so $W^{\prime} \Omega_{\ell-1}(Q) / \Omega_{\ell-1}(Q)$ has exponent $p$. Also, as $\Omega_{\ell}(Q) / \Omega_{\ell-1}(Q) \cong V$ as a $\operatorname{GF}(p)$ Aut $_{\mathcal{F}}(Q)$-module by the definition of $\ell, \Omega_{\ell}(Q) / \Omega_{\ell-1}(Q)$ is centralized by $Q$ and so is $W^{\prime} \Omega_{\ell-1}(Q) / \Omega_{\ell-1}(Q)$.

Consider the map $\theta: W \rightarrow \Omega_{\ell}(Q) / \Omega_{\ell-1}(Q)$ defined by $w \mapsto w^{p} \Omega_{\ell-1}(Q)$. Then, for $x, y \in W$, using $\left[\mathbf{2 4}\right.$, Lemma 2.2 (ii)] together with $[x, y]^{p} \in \Omega_{\ell-1}(Q)$, we have

$$
(x y) \theta=(x y)^{p} \Omega_{\ell-1}(Q)=x^{p} y^{p}[x, y]^{\frac{1}{2} p(p-1)} \Omega_{\ell-1}(Q)=x^{p} y^{p} \Omega_{\ell-1}(Q)=x \theta y \theta
$$

as $p$ is odd. In addition, for $\alpha \in \operatorname{Aut}_{\mathcal{F}}(Q)$,

$$
(x y) \alpha \theta=(x \alpha)^{p}(y \alpha)^{p} \Omega_{\ell-1}(Q)=x^{p} \alpha y^{p} \alpha \Omega_{\ell-1}(Q)=\left(\left(x^{p} y^{p}\right) \Omega_{\ell-1}(Q)\right) \alpha=(x y) \theta \alpha
$$

Hence $\theta$ is $\operatorname{Aut}_{\mathcal{F}}(Q)$-invariant. As $Q$ centralizes $\Omega_{\ell}(Q) / \Omega_{\ell-1}(Q)$ and $W / \Omega_{\ell}(Q)$, the $\operatorname{map} \theta$ is a $\operatorname{GF}(p) G$-module homomorphism and $\operatorname{ker} \theta=\Omega_{\ell}(Q)$ as $Q$ is regular.

Suppose that $\ell+1 \leq j$. Then $\Omega_{\ell+1}(Q) / \Omega_{\ell}(Q)$ has order $p^{p-1}$ and, as $Q$ is the 2-step centralizer, $\left|\left[\Omega_{\ell+1}(Q), Q\right] \Omega_{\ell}(Q) / \Omega_{\ell}(Q)\right| \leq p^{p-3}$ and $\left|W / \Omega_{\ell}(Q)\right| \geq p^{2}$. Since $\operatorname{ker} \theta=\Omega_{\ell}(Q), W \theta$ has order at least $p^{2}$. If $P$ is extraspecial, we know that the unique proper submodule of $V$ has order $p$ and so, in this case, $\theta$ is onto and $W=\Omega_{\ell+1}(Q)$, which is a contradiction. If $P$ is abelian, then $W \theta$ has order at least $p^{p-2}$ and so $\left|W / \Omega_{\ell}(Q)\right| \geq p^{p-2}$. Since $W \neq \Omega_{\ell+1}(Q)$, we know that $W / \Omega_{\ell}(Q)$ is isomorphic to the unique proper submodule $V_{0}$ of $V$. In particular, $W / \Omega_{\ell}(Q)$ is an irreducible $\operatorname{GF}(p) G$-module. As $\left|\left[\Omega_{\ell+1}(Q), Q\right] \Omega_{\ell}(Q) / \Omega_{\ell}(Q)\right| \leq p^{p-3}$, this is a contradiction. We have proved that, for $1 \leq k \leq j$, as $\operatorname{GF}(p) \operatorname{Aut}_{\mathcal{F}}(Q)$-modules, $\Omega_{k}(Q) / \Omega_{k-1}(Q) \cong V$. In particular, the result is proved if $Q=\Omega_{j}(Q)$. Hence we assume that $Q>\Omega_{j}(Q)$ which means $\ell+1>j$.

As $\ell+1>j$ and, as $j \geq \ell$ by the maximal choice of $j$, we get $\ell=j$. By the maximal choice of $j, \Omega_{j+1}=Q$ and $\left|Q / \Omega_{j}(Q)\right|<p^{p-1}$. If $P$ is abelian, then $W / \Omega_{j}(Q) \cong W \theta \cong V_{0}$ which has order $p^{p-2}$ and so $W=Q$. If $P$ is extraspecial,
then, as $|W \theta|<p^{p-1}$, we conclude $W \theta$ has order $p$ and $W=Q$ as $Q$ is a 2-step centralizer. This concludes the proof.

Lemma 13.12. Assume that Hypothesis 13.5 holds. Suppose that $R_{2}<R_{1} \leq$ $Q$ with $R_{1}$ and $R_{2}$ both $\operatorname{Aut}_{\mathcal{F}}(Q)$-invariant. If $\left|R_{1} / R_{2}\right| \leq p^{p-1}$, then $R_{1} / R_{2}$ is elementary abelian and centralized by $Q$.

Proof. We may assume that $R_{1} / R_{2}$ is not a $\operatorname{Aut}_{\mathcal{F}}(Q)$-chief factor as this case is clear. Hence Lemma 13.11 implies that $\left|R_{1} / R_{2}\right|=p^{p-1}$ and that there are exactly two $\operatorname{Aut}_{\mathcal{F}}(Q)$-chief factors in $R_{1} / R_{2}$ one of order $p$ and one of order $p^{p-2}$. Since $Q$ is a 2 -step centralizer and $S$ has maximal class we know that $\left|R_{1}:\left[R_{1}, Q\right]\right| \geq p^{2}$ and $\left|C_{R_{1} / R_{2}}(Q)\right| \geq p^{2}$. If follows that $Q$ centralizes $R_{1} / R_{2}$ and, in particular, $R_{1} / R_{2}$ is abelian. Since $\mho^{1}\left(R_{1} / R_{2}\right)$ and $\Omega_{1}\left(R_{1} / R_{2}\right)$ are $\operatorname{Aut}_{\mathcal{F}}(Q)$-invariant and $\mho^{1}\left(R_{1} / R_{2}\right) \leq \Omega_{1}\left(R_{1} / R_{2}\right)$, it follows that $R_{1} / R_{2}$ is elementary abelian.

For $\operatorname{Alt}(p)$ there is a unique irreducible $\mathrm{GF}(p)$-module of dimension $p-2$ and this is the irreducible heart of the degree $p$ permutation module. For $\mathrm{PSL}_{2}(p)$, again there is a unique irreducible module of dimension $p-2$ and this time it is the module that we denoted by $\mathbf{V}_{p-3}$ in Section 4 which consists of homogeneous polynomials in $\operatorname{GF}(p)[x, y]$ of degree $p-3$. Lemma 13.6 now gives us a unique action of $G=G_{0}\langle\tau \operatorname{Inn}(Q)\rangle$ on $V$. In the case that $G \cong \operatorname{Sym}(p)$, this module is in fact the quotient of the $p$-dimensional permutation module by the sum of all vectors in the natural basis. In the case of $\mathrm{PGL}_{2}(p)$, it is less natural to define.

Lemma 13.13. Assume that Hypothesis 13.5 holds. Suppose that $Q=Q_{1}>$ $Q_{2}>\cdots>Q_{\ell}=1$ is an $\operatorname{Aut}_{\mathcal{F}}(Q)$-chief series in $Q$. Then $O^{p}(\mathcal{F}) \neq \mathcal{F}$ if and only if $\left|Q_{1} / Q_{2}\right|=p$. Furthermore, if $O^{p}(\mathcal{F}) \neq \mathcal{F}$, then $\operatorname{hyp}(\mathcal{F})=P \gamma_{2}(S)$ and $\gamma_{1}(\operatorname{hyp}(\mathcal{F}))=\gamma_{2}(S)$.

Proof. By Lemma 5.11(i)

$$
\left.\operatorname{foc}(\mathcal{F})=\langle[g, \alpha]| g \in R \in \mathcal{E}_{\mathcal{F}} \cup\{S\} \text { and } \alpha \in \operatorname{Aut}_{\mathcal{F}}(R)\right\rangle
$$

Lemma 13.4 implies that $P \leq \operatorname{foc}(\mathcal{F})$ and $\operatorname{so} \operatorname{foc}(\mathcal{F})=P\left[Q, \operatorname{Aut}_{\mathcal{F}}(Q)\right]$.
Lemma 13.11 says that, as $\operatorname{GF}(p)$ Aut $_{\mathcal{F}}(Q)$-modules, either $Q / Q_{2} \cong V_{0}$ or $Q / Q_{2} \cong Z(S)$ is centralized by $\operatorname{Aut}_{\mathcal{F}}(Q)$. If $Q / Q_{2} \cong V_{0}$, then $\left.Q=\left[Q, \operatorname{Aut}_{\mathcal{F}}(Q)\right)\right] Q_{2}$ and so, as $S$ has maximal class, $Q=\left[Q, \operatorname{Aut}_{\mathcal{F}}(Q)\right]$ and $S=\operatorname{foc}(\mathcal{F})$. Hence, using Lemma 5.11(ii) yields $\mathcal{F}=O^{p}(\mathcal{F})$. Thus, if $\left|Q_{1} / Q_{2}\right|>p$, then $\mathcal{F}=O^{p}(\mathcal{F})$.

Conversely, if $\left|Q / Q_{2}\right|$ has order $p$, then $Q_{2}=\gamma_{2}(S)$ and $Q_{2} / Q_{3} \cong V_{0}$ by Lemma 13.11. Hence $\left[Q, \operatorname{Aut}_{\mathcal{F}}(Q)\right]=\gamma_{2}(S)$ in this case. This means that $\operatorname{foc}(\mathcal{F})=$ $P \gamma_{2}(S)=\operatorname{hyp}(\mathcal{F})$ and $\gamma_{1}(\operatorname{hyp}(\mathcal{F}))=\gamma_{2}(S)$. In particular, Lemma 5.11(ii) implies $\mathcal{F} \neq O^{p}(\mathcal{F})$.

Lemma 13.14. Assume that Hypothesis 13.5 holds. Then $\mathcal{F} \neq O^{p}(\mathcal{F})$.
Proof. Assume that $\mathcal{F}=O^{p}(\mathcal{F})$ and let $Q=Q_{1}>Q_{2}>\cdots>Q_{\ell}=1$ be an $\operatorname{Aut}_{\mathcal{F}}(Q)$-chief series in $Q$. Then Lemmas 13.11 and 13.13 imply that $Q / Q_{2} \cong V_{0}$ as a $\operatorname{GF}(p) G$-module.

By hypothesis $Q$ is non-abelian. We intend to show that this cannot be the case. Choose $N \leq Q$ maximal in $Q$ such that $N$ is $\operatorname{Aut}_{\mathcal{F}}(Q)$-invariant and $Q / N$ is non-abelian. Since $\operatorname{Aut}_{\mathcal{F}}(Q)$ acts on $Q / N$, it is sufficient to show that $Q / N$ cannot admit $\operatorname{Aut}_{\mathcal{F}}(Q)$. Thus we work with the quotient $Q / N$ and to make the notation lighter we assume that $N=1$. Set $M=Q^{\prime}$. Then, the maximal choice of $N$ implies
that $M$ is the smallest non-trivial $\operatorname{Aut}_{\mathcal{F}}(Q)$-invariant subgroup of $Q$. In particular $M$ is elementary abelian, $M \leq Z(Q)$ and by Lemma 13.11 either $M$ has order $p$ and is centralized by $G$ or $M \cong V_{0}$ as a $\operatorname{GF}(p) G$-module. Since $Q$ is now a quotient of $\gamma_{1}(S)$, we need to argue that $|Q| \geq p^{p+1}$. Note that $[Q: Z(Q)] \geq\left[Q: Q_{2}\right]=p^{p-2}$ since $Q$ is non-abelian. Also, $|Z(Q)|>p$ because $Q$ is the 2 -step centralizer and so Lemma 13.11 implies $|Z(Q)| \geq p^{p-2}$. Hence, as $p \geq 5,|Q| \geq p^{2(p-2)} \geq p^{p+1}$.

Set $\bar{Q}=Q / \mho^{1}(Q)$. As $|Q| \geq p^{p+1}$, Lemmas 2.1 (iv) and 3.2 (iii) yield $|\bar{Q}|=\left|Q / \mho^{1}(Q)\right|=\left|\Omega_{1}(Q)\right|=p^{p-1}$. Lemma 13.12 implies that $\bar{Q}$ and $\Omega_{1}(Q)$ are elementary abelian.

Define

$$
\kappa: \bar{Q} \times \bar{Q} \rightarrow M
$$

by $\left(x \mho^{1}(Q), y \mho^{1}(Q)\right) \kappa=[x, y]$. Since $M$ is elementary abelian and central in $Q$, we have $\left[x^{p}, y\right]=[x, y]^{p}=1$ for all $x, y \in Q$. In particular, $\mho^{1}(Q) \leq Z(Q)$ and $\kappa$ is a well-defined, surjective, $G$-invariant, alternating, bilinear map. Therefore there is a unique $\operatorname{GF}(p) G$-module homomorphism from $\Lambda^{2}(\bar{Q})$ onto $M$. Notice that $M$ either has order $p$ or $p^{p-2}$. Suppose that $G \cong \operatorname{Sym}(p)$. In this case, $\bar{Q} / C_{\bar{Q}}\left(G^{\prime}\right)$ is the unique $\operatorname{GF}(p) \operatorname{Alt}(p)$-module of dimension $p-2$ and $\mathrm{H}^{1}\left(\operatorname{Alt}(p), \bar{Q} / C_{\bar{Q}}\left(G^{\prime}\right)\right)$ has dimension 1 by Lemma 4.11. Hence $\bar{Q} / C_{\bar{Q}}\left(G^{\prime}\right)$ is uniquely determined as the submodule of dimension $p-1$ of the natural permutation module for $\operatorname{Alt}(p)$ on $p$ points. However, this means that $\Lambda^{2}(\bar{Q})$ is the module described in Lemma 4.10 and this has no quotients isomorphic to $M$.

Suppose that $G=\mathrm{PGL}_{2}(p)$. Then $\bar{Q}$ is an indecomposable $\operatorname{GF}(p) \mathrm{PSL}_{2}(p)$ module with socle the trivial 1-dimensional module and quotient of dimension $p-2$. By Lemma 4.8, there are unique irreducible quotients of $\Lambda^{2}(\bar{Q})$ of dimensions 1 and of dimension $p-2$ only if the same is true for $\Lambda^{2}\left(\bar{Q} / C_{\bar{Q}}(G)\right)$. Since $\left|\bar{Q} / C_{\bar{Q}}(G)\right|=$ $p^{p-2}$ and $p-2$ is odd, $\bar{Q} / C_{\bar{Q}}(G)$ cannot support an alternating bilinear form and thus $|M|=p^{p-2}$.

By Lemma 13.6, $\tau$ acts fixed-point-freely on $M$ and on $\bar{Q} / C_{\bar{Q}}(G)$. Since $\mho^{1}(Q) \leq Z(Q)$ and $\mho^{1}(Q)$ has index $p$ in the preimage of $C_{\bar{Q}}(G)$, we get that the preimage of $C_{\bar{Q}}(G)$ is abelian. Let $Q^{*}$ represent the quotient of $\bar{Q}$ by $C_{\bar{Q}}(G)$ and we consider the map from $\kappa^{*}: \Lambda^{2}\left(Q^{*}\right) \rightarrow M$. Let $t_{1}, \ldots, t_{p-2}$ be eigenvectors for the action of $\tau$ on $Q^{*}$ where $t_{j} \in \gamma_{j}(S) \backslash \gamma_{j+1}(S)$. Since the action of $G$ on $M$ and on $Q^{*}$ is isomorphic to the action of $G$ on $V_{0}$, Lemma 13.6 implies $t_{i} \tau=t_{i}^{r^{i}}$ for $1 \leq i \leq p-2$. It follows that, as $r^{p-1}=1$,

$$
\left[t_{1}, t_{p-2}\right] \tau=\left[t_{1}^{r}, t_{p-2}^{r^{(p-2)}}\right]=\left[t_{1}, t_{p-2}\right]^{\left(r r^{p-2}\right)}=\left[t_{1}, t_{p-2}\right]
$$

Since $C_{M}(\tau)=1$, we deduce that $\left[t_{1}, t_{p-2}\right]=1$. Thus $\kappa^{*}\left(t_{1} \wedge t_{p-2}\right)=\left[t_{1}, t_{p-2}\right]=$ 1 and this contradicts Lemma 4.9. This contradiction proves that $Q$ is abelian. However, $Q$ is not abelian by hypothesis and so we deduce that $\mathcal{F} \neq O^{p}(\mathcal{F})$ as claimed.

Lemma 13.15. Assume that Hypothesis 13.5 holds. Then
(i) $\gamma_{2}(S)$ is abelian and is $\operatorname{Aut}_{\mathcal{F}}(Q)$-invariant;
(ii) $Z(Q)=\mho^{1}(Q),|Q: Z(Q)|=p^{p-1}$ and $Q$ has nilpotency class 2;
(iii) $P$ is abelian and $[Q, Q]=V_{0}$;
(iv) $|S|=p^{j(p-1)+1}$ for some $j \geq 2$ and $S$ has sectional rank $p-1$; and
(v) $\mathcal{P}(\mathcal{F})=P^{\mathcal{F}}$.

Proof. By Lemmas 13.13 and 13.14 we have $\mathcal{F} \neq O^{p}(\mathcal{F}), \operatorname{hyp}(\mathcal{F})=P \gamma_{2}(S)$ and $\gamma_{1}(\operatorname{hyp}(\mathcal{F}))=\gamma_{2}(S)$. As $O^{p}\left(O^{p}(\mathcal{F})\right)=O^{p}(\mathcal{F})$ and $O^{p}(\mathcal{F})$ satisfies Hypothesis 13.5 except $\gamma_{1}(\operatorname{hyp}(\mathcal{F}))$ being abelian, Lemma 13.14 implies that $\gamma_{2}(S)$ is abelian. This is (i).

Set $X=\left\langle r^{p} \mid r \in Q \backslash \gamma_{2}(S)\right\rangle$. Let $t \in Q \backslash \gamma_{2}(S)$. Then $t^{p}$ is centralized by $t$ and $\gamma_{2}(S)$. Hence $t^{p} \in Z(Q)$ and so $X \leq Z(Q)$ is a normal subgroup of $S$ and $t \in$ $\Omega_{1}(Q / X)$. Since the preimage of $\Omega_{1}(Q / X)$ is normal in $S$ and $S$ has maximal class we deduce that $Q / X$ has exponent $p$. Hence $\mho^{1}(Q) \leq X \leq \mho^{1}(Q)$ which means that $X=\mho^{1}(Q)$. It follows that $Z(Q) \geq \mho^{1}(Q)$. Since $\left|Q / \mho^{1}(Q)\right|=\left|\Omega_{1}(Q)\right|=p^{p-1}$ by Lemmas 2.1 (iii) and 13.10 we can use Lemma 13.11 to conclude $Z(Q)=\mho^{1}(Q)$ as surely $Z(Q) \neq \gamma_{2}(S)$. Also, $Q / \mho^{1}(Q)$ is elementary abelian by Lemma 13.12 . Hence $Q^{\prime} \leq Z(Q)$ and $Q$ has nilpotency class 2. This proves (ii).

Let $x, y \in Q$, then, by (ii), $x^{p} \in Z(Q)$ and so $1=\left[x^{p}, y\right]=[x, y]^{p}$. Hence $Q^{\prime} \leq \Omega_{1}(Q)=V$. Now the commutator map $\kappa: Q / X \times Q / X \rightarrow V$ again determines a $\operatorname{GF}(p) G$-module homomorphisms $\kappa^{*}: \Lambda^{2}(Q / X) \rightarrow V$. Now $Q / X$ has $\gamma_{2}(S) / X \cong$ $V_{0}$ as its unique submodule and we know that $\Lambda^{2}\left(\gamma_{2}(S) / X\right)$ is in the kernel of $\kappa^{*}$. We have $\Lambda^{2}(Q / X) / \Lambda^{2}\left(\gamma_{2}(S) / X\right) \cong \gamma_{2}(X) \cong V_{0}$ as $\mathrm{GF}(p) G$-modules. Hence the image of $\kappa^{*}$ in $V$ is isomorphic to $V_{0}$. If $P$ is extraspecial, then $Z(S)$ is the unique proper subgroup of $V$ which is $G$-invariant. Hence $P$ is abelian and $Q^{\prime}=V_{0}$. Thus (iii) holds.

As for part (iv), if $Q=Q_{1}>Q_{2}>\cdots>Q_{\ell-1}>Q_{\ell}=1$ is an $\operatorname{Aut}_{\mathcal{F}}(Q)$ chief series in $Q$, then, by part (i), $Q_{2}=\gamma_{2}(S)$ and, by part (iii), $Q_{\ell-1}=Q^{\prime}$ has order $p^{p-2}$. Hence Lemma 13.11 implies that $|Q|=p^{j(p-1)}$ for some $j \geq 1$. Thus Lemma 13.10 yields $|S|=p^{j(p-1)+1}$ for some $j \geq 2$. In particular $|S| \geq p^{2 p}$ and $[\mathbf{2 8}$, Theorem A] implies that $S$ has sectional rank $p-1$.

Note that $P \gamma_{2}(S)$ is $\operatorname{Aut}_{\mathcal{F}}(S)$-invariant and $\operatorname{Aut}_{\mathcal{F}}(S)$ centralizes $\gamma_{1}(S) / \gamma_{2}(S)$. Hence $P \gamma_{2}(S)$ and $\gamma_{1}(S)$ are the only $\operatorname{Aut}_{\mathcal{F}}(S)$-invariant maximal subgroups of $S$. Therefore every $\mathcal{F}$-pearl is contained in $P \gamma_{2}(S)$ and from this it follows that $\mathcal{P}(\mathcal{F})=\left\{P^{x} \mid x \in S\right\}=P^{\mathcal{F}}$. Hence (v) holds.

## 14. The proofs of Theorems $\mathbf{A}$ and $\mathbf{C}$

In this short section we prove Theorems A and C as well as Corollary 1.3. We begin with Theorem C.

Theorem C. Suppose that $p$ is an odd prime, $S$ is a maximal class p-group of order at least $p^{4}$ and $\mathcal{F}$ is a saturated fusion system on $S$. Assume that $S$ is not exceptional, $\gamma_{1}(S)$ is not abelian and $\mathcal{F} \neq N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$. Then one of the following holds:
(i) $\mathcal{E}_{\mathcal{F}}=\mathcal{P}_{a}(\mathcal{F}),|S: \operatorname{hyp}(\mathcal{F})| \leq p$ with $|S: \operatorname{hyp}(\mathcal{F})|=p$ if and only if $|S|=p^{j(p-1)+1}$ for some $j \geq 2$. Furthermore, either $O^{p}(\mathcal{F})$ is simple and exotic or $p=3$ and $O^{3}(\mathcal{F})$ is realized by $\mathrm{PSL}_{3}(q)$ for suitable prime powers $q$.
(ii) $p \geq 5, \mathcal{E}_{\mathcal{F}}=\mathcal{P}_{e}(\mathcal{F}), O_{p}(\mathcal{F})=Z(S),|S: \operatorname{hyp}(\mathcal{F})| \leq p$ with $\mid S:$ $\operatorname{hyp}(\mathcal{F}) \mid=p$ if and only if $|S|=p^{j(p-1)+2}$ for some $j \geq 2$. Furthermore, $O^{p}(\mathcal{F} / Z(S))$ is simple and exotic.
(iii) $p \geq 5, \mathcal{E}_{\mathcal{F}}=\mathcal{P}_{a}(\mathcal{F}) \cup\left\{\gamma_{1}(S)\right\}, O_{p}(\mathcal{F})=1, \mathcal{F} \neq O^{p}(\mathcal{F})$ and
(a) $\mathcal{P}_{a}(\mathcal{F})$ is a single $\mathcal{F}$-class, $|S|=p^{j(p-1)+1}$ for some $j \geq 2$ and $S$ has sectional rank $p-1$;
(b) $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right) \cong \operatorname{Sym}(p)$ or $\mathrm{PGL}_{2}(p)$;
(c) $Z\left(\gamma_{1}(S)\right)=\mho^{1}\left(\gamma_{1}(S)\right)$ has index $p^{p-1}$ in $\gamma_{1}(S), \gamma_{1}(S)^{\prime}<\Omega_{1}\left(\gamma_{1}(S)\right)$ has order $p^{p-2}$ and $\gamma_{2}(S)$ is abelian but not elementary abelian;
(d) every composition factor of $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ on $\gamma_{1}(S)$ has order $p$ or $p^{p-2}$ and the composition factors of order $p$ are centralized by the automorphism group $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$;
(e) for $P \in \mathcal{P}_{a}(\mathcal{F})$, $\operatorname{hyp}(\mathcal{F})=P \gamma_{2}(S)$, $O^{p}(\mathcal{F})$ is a saturated fusion system on $P \gamma_{2}(S)$, and $\operatorname{Aut}_{O^{p}(\mathcal{F})}\left(\gamma_{2}(S)\right) \cong \operatorname{Sym}(p)$ or $\mathrm{PGL}_{2}(p)$.
Furthermore, in all cases $\operatorname{Out}_{\mathcal{F}}(S)$ is a Hall $p^{\prime}$-subgroup of $\operatorname{Out}(S)$ and is cyclic of order $p-1$ and, if $|S|=p^{n}$, and $P \in \mathcal{P}(\mathcal{F})$, then either $\mathcal{P}(\mathcal{F})=P^{S}$ or $\mathcal{E}_{\mathcal{F}}=\mathcal{P}(\mathcal{F})$ and $n \equiv \epsilon(\bmod p-1)$ where $\epsilon=0$ if $P \in \mathcal{P}_{a}(\mathcal{F})$ and $\epsilon=1$ if $P \in \mathcal{P}_{e}(\mathcal{F})$.

Proof. Suppose that $\mathcal{F}$ is a saturated fusion system on $S$ with $|S| \geq p^{4}$. In addition, suppose $S$ has maximal nilpotency class, is not exceptional and has $Q=$ $\gamma_{1}(S)$ non-abelian. Assume that $\mathcal{F} \neq N_{\mathcal{F}}(Q)$. By Theorem D, $\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{P}(\mathcal{F}) \cup\{Q\}$ and so $\mathcal{P}(\mathcal{F})$ is non-empty.

Assume that $p=3$. Then the saturated fusion systems are presented in Appendix B. Since we require $\gamma_{1}(S)$ to be non-abelian, the discussion after Theorem B. 2 says we only need to inspect Table 4 for $S=\mathrm{B}(2 \ell ; 1,0,2), \mathrm{B}(2 \ell, 1,0,0), \ell \geq 3$ and $\mathrm{B}(2 k+1,1,0,0)$ with $k \geq 2$. This shows that $\mathcal{E}_{\mathcal{F}}=\mathcal{P}_{a}(\mathcal{F}), O_{p}(\mathcal{F})=1$, and $|S: \operatorname{hyp}(\mathcal{F})|=3$ if and only if $|S|=3^{2 k+1}$. Furthermore, either $S$ is one of $\mathrm{B}(2 \ell ; 1,0,2)$ or $\mathrm{B}(2 \ell, 1,0,0)$ with $\ell \geq 3$ and $\mathcal{F}$ is simple and exotic or $S=\mathrm{B}(2 k+1,1,0,0)$ and $O^{3}(\mathcal{F})$ is realised by $\mathrm{PSL}_{3}(q)$ for suitable $q$. Hence (i) holds when there are no saturated fusion systems $\mathcal{F}$ with extraspecial $\mathcal{F}$-pearls or with $\gamma_{1}(S) \in \mathcal{E}_{\mathcal{F}}$.

Assume from now on that $p \geq 5$. By Lemma 13.4 (vi), either $\mathcal{P}(\mathcal{F})=\mathcal{P}_{a}(\mathcal{F})$ or $\mathcal{P}(\mathcal{F})=\mathcal{P}_{e}(\mathcal{F})$. Furthermore, if $\mathcal{E}_{\mathcal{F}}=\mathcal{P}_{e}(\mathcal{F}) \cup\{Q\}$, then Hypothesis 13.5 holds and this contradicts Lemma 13.15 (iii). Thus $\mathcal{E}_{\mathcal{F}} \neq \mathcal{P}_{e}(\mathcal{F}) \cup\{Q\}$. By Lemma 13.4, $\operatorname{Out}_{\mathcal{F}}(S)$ is a Hall $p^{\prime}$-subgroup of $\operatorname{Out}(S)$ and is cyclic of order $p-1$. This, together with [ $\mathbf{2 8}$, Theorem 3.15] establishes the chaser to the theorem.

Suppose that $\mathcal{E}_{\mathcal{F}}=\mathcal{P}_{a}(\mathcal{F})$. Then Lemma 5.25 gives (i).

If $\mathcal{E}_{\mathcal{F}}=\mathcal{P}_{e}(\mathcal{F})$, then $O_{p}(\mathcal{F})=Z(S)$. Since $P \leq \operatorname{hyp}(\mathcal{F})$ and $\operatorname{hyp}(\mathcal{F})$ is normal in $S,|S: \operatorname{hyp}(\mathcal{F})| \leq p$. Furthermore, $|S: \operatorname{hyp}(\mathcal{F})|=p$ if and only if $s_{1} \gamma_{2}(S)$ is centralized by $\operatorname{Aut}_{\mathcal{F}}(S)$ which is if and only if $n-2 \equiv 0(\bmod p-1)$ by Lemma 13.6 (ii). Hence $O^{p}(\mathcal{F}) \subset \mathcal{F}$ if and only if $|S|=p^{j(p-1)+2}$ with $j \geq 1$. In addition, $P / Z(S)$ is an abelian $\mathcal{F} / Z(S)$-pearl. In the case that $Q / Z(S)$ is non-abelian, $\mathcal{F} / Z(S)$ satisfies (i) with $|S / Z(S)|=p^{n-1} \geq p^{4}$. So, in particular, $O^{p}(\mathcal{F} / Z(S))$ is simple and exotic and $j \geq 2$ when $O^{p}(\mathcal{F}) \neq \mathcal{F}$. On the other hand, if $Q / Z(S)$ is abelian, then applying [42, Theorem 2.8 (a)(i) and a(iv)] delivers $O^{p}(\mathcal{F} / Z(S))$ is simple and exotic. Suppose that $O^{p}(\mathcal{F}) \subset \mathcal{F}$ and $j=1$. Then $|S|=p^{p+1}$. We have $Q^{\prime}=Z(S)$. In addition, $O^{p}(\mathcal{F})$ has extraspecial pearls and $|\operatorname{hyp}(\mathcal{F})|=p^{p}$ and so [28, Theorem A] applied $O^{p}(\mathcal{F})$ yields $\gamma_{1}(\operatorname{hyp}(\mathcal{F}))=\gamma_{2}(S)$ is elementary abelian. Let $\tau \in \operatorname{Out}_{\mathcal{F}}(S)$ have order $p-1$. Then $\tau$ centralizes $s_{1} \gamma_{2}(S)$ and $Z(S)$. Choose $k$ maximal so that $\left[s_{1}, s_{k}\right] \neq 1$. Then $\left[s_{1}, s_{k}\right] \in Z(S)$ and

$$
\left[s_{1}, s_{k}\right]=\left[s_{1}, s_{k}\right] \tau=\left[s_{1} \tau, s_{k} \tau\right]
$$

Since $1<k<n-1=p$ and $\tau$ has order $p-1, s_{k} \tau=s_{k}^{b} g_{k+1}$ for some $b \in$ $\operatorname{GF}(p)^{\times} \backslash\{1\}$ and $g_{k+1} \in \gamma_{k+1}(S)$ by Lemma 3.14. We also have $s_{1} \tau=s_{1} g_{2}$ for some $g_{2} \in \gamma_{2}(S)$. Since $\gamma_{2}(S)$ is abelian, the maximal choice of $k$ gives

$$
\left[s_{1}, s_{k}\right]=\left[s_{1} \tau, s_{k} \tau\right]=\left[s_{1} g_{2}, s_{k}^{b} g_{k+1}\right]=\left[s_{1}, s_{k}\right]^{b}
$$

a contradiction.
Suppose that $\mathcal{E}_{\mathcal{F}}=\mathcal{P}_{a}(\mathcal{F}) \cup\{Q\}$. Then Hypothesis 13.5 holds. That $\mathcal{F} \neq$ $O^{p}(\mathcal{F})$ is just Lemma 13.14. Now part (iii)(a) is Lemma 13.15 (iv) and (v), part (iii)(b) is Lemma 13.8, part (iii)(c) is Lemma 13.15(i), (ii), (iii) and (iv), part (iii)(d) is Lemma 13.11, finally part (iii)(e) follows from Lemmas 13.13 and 13.15(i) as clearly $\operatorname{Aut}_{O^{p}(\mathcal{F})}\left(\gamma_{2}(S)\right) \cong \operatorname{Sym}(p)$ or $\mathrm{PGL}_{2}(p)$ by (iii)(b).

Theorem A. Suppose that $\mathcal{F}$ is a reduced saturated fusion system on a p-group $S$ of maximal class of order at least $p^{4}$. Then one of the following statements holds.
(i) $\gamma_{1}(S)$ is non-abelian, $S$ is not exceptional, $\mathcal{E}_{\mathcal{F}}=\mathcal{P}_{a}(\mathcal{F})$, and $\mathcal{F}$ is simple and exotic.
(ii) $\gamma_{1}(S)$ is non-abelian, $S$ is exceptional and either (a) $p \geq 5$ and $\mathcal{F}=\mathcal{F}_{S}\left(\mathrm{G}_{2}(p)\right)$;
(b) $p=5, S$ is isomorphic to a Sylow 5 -subgroup of $\mathrm{G}_{2}(5)$ and $\mathcal{F}=$ $\mathcal{F}_{S}(G)$ where $G$ is one of the sporadic simple groups $\mathrm{Ly}, \mathrm{HN}$ or B ;
(c) $p=7, S$ is isomorphic to a Sylow 7 -subgroup of $\mathrm{G}_{2}(7)$ and either $\mathcal{F}$ is exotic (20 examples) or $\mathcal{F}=\mathcal{F}_{S}(\mathrm{M})$ where M denotes the monster; or
(d) $p \geq 11, S$ is uniquely determined of order $p^{p-1}, \mathcal{P}(\mathcal{F})=\mathcal{P}_{a}(\mathcal{F}) \neq \emptyset$ and, if $\gamma_{1}(S)$ is $\mathcal{F}$-essential, then $\operatorname{Out}_{\mathcal{F}}(S) \cong \operatorname{GF}(p)^{\times} \times \operatorname{GF}(p)^{\times}$, $O^{p^{\prime}}\left(\operatorname{Out}_{\mathcal{F}}\left(\gamma_{2}(S)\right)\right) \cong \operatorname{SL}_{2}(p)$ and $\gamma_{1}(S) / Z\left(\gamma_{1}(S)\right)$ is the $(p-3)$ dimensional irreducible $\mathrm{GF}(p) \mathrm{SL}_{2}(p)$-module.
(iii) $\gamma_{1}(S)$ is abelian and $\mathcal{F}$ is described by Theorem A.1.

Proof. Suppose that $\mathcal{F}$ is reduced $\left(O_{p}(\mathcal{F})=1\right.$ and $\left.\mathcal{F}=O^{p}(\mathcal{F})=O^{p^{\prime}}(\mathcal{F})\right)$. In addition we may assume that $\gamma_{1}(S)$ is non-abelian. If $S$ is not exceptional, then, as $\mathcal{F}$ is reduced, Theorem C (ii) and (iii) cannot hold as in the first case $O_{p}(\mathcal{F}) \neq 1$ and in the second case $\mathcal{F} \neq O^{p}(\mathcal{F})$. Hence Theorem C (ii) holds and in particular $\mathcal{E}_{\mathcal{F}}$ consists of abelian $\mathcal{F}$-pearls. This is point (i) of the theorem.

Suppose that $S$ is exceptional. Then Theorem B applies. In particular, this immediately gives parts (ii)(a) and (ii)(d) of the theorem, completing the proof for $p \geq 11$. For $p=5$ and $\mathcal{F} \neq \mathcal{F}_{S}\left(\mathrm{G}_{2}(5)\right)$, Theorem B (i)(a)( $\alpha$ ), (i)(a)( $\beta$ ) and (ii) cannot hold, as $O_{5}(\mathcal{F}) \neq 1$ in these cases. Hence Theorem B (i)(a)( $\delta$ ) holds. We deduce that $\mathcal{F}$ is not realized by $\operatorname{Aut}(\mathrm{HN})$ and this gives point (ii)(b) of the theorem.

Finally, when $p=7$ and $S$ is isomorphic to a Sylow 7 -subgroup of $\mathrm{G}_{2}(7)$, then we use [47, Table 5.1] to determine how many of the examples are reduced. There are 20 exotic examples all appearing as subsystems of $\mathcal{F}_{S}(\mathrm{M})$. This is point (ii)(c) of the theorem and completes the proof.

Corollary 1.3. Let $p$ be a prime, $S$ be a p-group of maximal class and let $\mathcal{F}$ be a saturated fusion system on $S$ with $O_{p}(\mathcal{F})=1$. If $\mathcal{P}(\mathcal{F})$ is empty, then $S$ is isomorphic to a Sylow p-subgroup of $\mathrm{G}_{2}(p)$ and either
(i) $\mathcal{F}=\mathcal{F}_{S}\left(\mathrm{G}_{2}(p)\right)$;
(ii) $p=5$ and $\mathcal{F}=\mathcal{F}_{S}(G)$ where $G=\mathrm{Ly}, \mathrm{HN}$, $\operatorname{Aut}(\mathrm{HN})$ or B ;
(iii) $p=7, \mathcal{F}$ is exotic and the $\mathcal{F}$-essential subgroups are $C_{S}\left(Z_{2}(S)\right)$ and $\gamma_{1}(S)$, with $\operatorname{Out}_{\mathcal{F}}\left(C_{S}\left(Z_{2}(S)\right)\right) \cong \operatorname{GL}_{2}(7)$, $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right) \cong 3 \times 2 \cdot \operatorname{Sym}(7)$, and $\operatorname{Out}_{\mathcal{F}}(S) \cong \mathrm{GF}(7)^{\times} \times \mathrm{GF}(7)^{\times}$.
Proof of Corollary 1.3. Suppose $\mathcal{F}$ has no $\mathcal{F}$-pearls and $O_{p}(\mathcal{F})=1$. Theorem C implies that either $\gamma_{1}(S)$ is abelian or $S$ is exceptional. If $\gamma_{1}(S)$ is abelian, then $\mathcal{P}(\mathcal{F})$ is non-empty by [42, Lemma 2.3]. So assume that $S$ is exceptional. Then examining Theorem B we see that part (i)(a) must hold. In particular, $S$ is isomorphic to a Sylow $p$-subgroup of $\mathrm{G}_{2}(p)$. Using [47, Theorem 1.1 and Table 5.1] yields the result.

## 15. A series of examples with non-abelian 2-step centralizer

Let $p$ be an odd prime, then by Dirichlet's Theorem [22] there exists a prime $r$ such that $r \equiv 1\left(\bmod p^{k}\right)$. Let $T$ be a Sylow $p$-subgroup of $\operatorname{GF}\left(r^{p}\right)^{\times}$and $M$ be the monomial subgroup of $\mathrm{GL}_{p}\left(r^{p}\right)$ which has all matrix entries in $T$. Then $T$ is a cyclic group and $|M|=|T|^{p} p$ !. Notice that $Z(M) \cong T$ and consists of scalar matrices. We denote by $D$ the subgroup of diagonal matrices of $M$. Let $R \in \operatorname{Syl}_{p}(M)$. We claim $R / Z(M)$ has maximal class. Let $\pi$ be the permutation matrix corresponding to the permutation $(1,2, \ldots, p)$ :

$$
\pi=\left(\begin{array}{ccccc}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{array}\right) .
$$

We may assume that $\pi \in R$. Then a typical element of $R$ has the form

$$
\operatorname{diag}\left(d_{1}, \ldots, d_{p}\right) \pi^{j} \in D\langle\pi\rangle
$$

where $d_{i} \in T$ and $1 \leq j \leq p$. Now a calculation shows that the set of matrices in $R$ which centralize $\pi \bmod Z(M)$ is

$$
C=\left\{\operatorname{diag}\left(d, d e, d e^{2}, \ldots, d e^{p-1}\right) \pi^{j} \mid d, e \in T, e^{p}=1, j \in \mathbb{Z}\right\} \leq \mathrm{GL}_{p}(r)
$$

Because $(C \cap D) / Z(M)=\left\{\operatorname{diag}\left(1, e, \ldots, e^{p-1}\right) Z(M) \mid e \in T, e^{p}=1\right\}$ has order $p$, Lemma 3.4 yields $R / Z(M)$ has maximal class. Since $\pi \in \mathrm{SL}_{p}(r)$, and

$$
\operatorname{det}\left(\operatorname{diag}\left(1, e, \ldots, e^{p-1}\right)\right)=e^{p(p-1) / 2}=1
$$

we see the image of $C$ in $\mathrm{PGL}_{p}(r)$ is contained in $\mathrm{PSL}_{p}(r)$. Using [2, Section 4] we record the well-known fact:

Lemma 15.1. If p divides $r-1$, then the Sylow $p$-subgroups of $G=\mathrm{PGL}_{p}(r)$ and $G=\mathrm{PSL}_{p}(r)$ have maximal class. Furthermore, in both cases, we have $\mathrm{Aut}_{G}(\widetilde{C}) \cong$ $\mathrm{SL}_{2}(p)$ where $\widetilde{C}$ is the image of $C$ in $\mathrm{PGL}_{2}(r)$.

Since $p^{k}$ is the highest power of $p$ which divides $r-1, p^{k+1}$ exactly divides $r^{p}-1$ (see [39, Lemmas 4.1(iv) and 4.2(ii)]). Let $\sigma: \operatorname{GF}\left(r^{p}\right) \rightarrow \operatorname{GF}\left(r^{p}\right)$ be the field automorphism given by $x \mapsto x^{r}$ and extend $\sigma$ to the standard Frobenius automorphism of $\mathrm{GL}_{p}\left(r^{p}\right)$ which acts as $\sigma$ on each matrix entry. We denote this automorphism by $\sigma$ as well. Define $M^{*}$ to be the semidirect product of $M$ and $\langle\sigma\rangle$, identify $M$ and $\langle\sigma\rangle$ with their images in $M^{*}$ and let $R^{*}=R\langle\sigma\rangle$. Consider the subgroup $C^{*}$ of $R^{*}$ which centralizes $\pi \bmod Z(M)$. Since $\pi \in \mathrm{GL}_{p}(r)$, we see

$$
C^{*}=C\langle\sigma\rangle=\left\{\operatorname{diag}\left(d, d e, d e^{2}, \ldots, d e^{p-1}\right) \pi^{j} \sigma^{k} \mid d, e \in T, e^{p}=1, j, k \in \mathbb{Z}\right\}
$$

and so $\left|C^{*}\right| Z(M) \mid=p^{3}$. Let $R>R_{0} \geq Z(M)$ be such that $R_{0} / Z(M)$ is a Sylow $p$-subgroup of $\mathrm{PSL}_{p}\left(r^{p}\right)$ and put $D_{0}=D \cap R_{0}$. Then $R^{*}>R>R_{0}$ and $R^{*} / R_{0}=\left\langle R_{0} \sigma, R_{0} \operatorname{diag}(d, 1, \ldots, 1)\right\rangle$ which has order $p|T|$. Now let $R_{1}=$ $R_{0}\langle\operatorname{diag}(d, 1, \ldots, 1) \sigma\rangle$. Then $\pi \in R_{1}, C^{*} \cap R_{1}=C$ and so $R_{1} / Z(M)$ has maximal class by Lemma 3.4 and

$$
\gamma_{1}\left(R_{1} / Z(M)\right)=\left\langle D_{0} / Z(M), Z(M) \operatorname{diag}(d, 1, \ldots, 1) \sigma\right\rangle
$$

is not abelian as $\sigma$ does not centralize $D_{0} / Z(M)$.
Example 15.2. Assume that $p>3$ is a prime. Put $X_{1}=\mathrm{SL}_{p}\left(r^{p}\right), X=X_{1} R_{1}$ and $\bar{X}=X / Z(X)$. Set $\mathcal{F}=\mathcal{F}_{\bar{R}_{1}}(\bar{X})$. Then
(i) $\overline{R_{1}}$ has maximal class and $\gamma_{1}\left(\overline{R_{1}}\right)$ is non-abelian;
(ii) $\bar{C}$ and $\gamma_{1}\left(\overline{R_{1}}\right)$ are $\mathcal{F}$-essential with $\bar{C}$ an abelian $\mathcal{F}$-pearl;
(iii) $\operatorname{Aut}_{\mathcal{F}}(\bar{C}) \cong \operatorname{SL}_{2}(p)$, $^{\operatorname{Out}_{\mathcal{F}}}\left(\gamma_{1}\left(\overline{R_{1}}\right)\right) \cong \operatorname{Sym}(p)$; and (iv) $O_{p}(\mathcal{F})=1$.

In particular, there exist realizable saturated fusion systems which satisfy Theorem $C$ (iv) with $\gamma_{1}(S)$ non-abelian and $\operatorname{Out}_{\mathcal{F}}\left(\gamma_{1}(S)\right) \cong \operatorname{Sym}(p)$. By [50, Theorem 6.2], the subfusion systems generated by $N_{\operatorname{Aut}_{\mathcal{F}}\left(\overline{\left.R_{1}\right)}\right.}(\bar{C})$ and $\operatorname{Aut}_{\mathcal{F}}(\bar{C})$ is also saturated and this gives an example of Theorem $C$ (i).

## A. Saturated fusion systems on maximal class $p$-groups with $\gamma_{1}(S)$ abelian

In this appendix we curate a list of the reduced fusion systems on maximal class $p$-groups which have $\gamma_{1}(S)$ abelian. Thus we present a synopsis of the main results from $[\mathbf{1 9}, \mathbf{4 2}, \mathbf{4 5}]$. For this, we first establish some further notation from $[\mathbf{1 9}, 42,45]$ which is honed to our special situation. First there are the sets of $\mathcal{F}$-essential subgroups in the sets $\mathcal{H}$ and $\mathcal{B}$ which are introduced in [42]. We continue to use the notation introduced in Subsection 3.1 so that $S / \gamma_{2}(S)=\left\langle x \gamma_{2}(S), s_{1} \gamma_{2}(S)\right\rangle$. We also recall the definition of the set $\mathcal{P}(\mathcal{F})=\mathcal{P}_{a}(\mathcal{F}) \cup \mathcal{P}_{e}(\mathcal{F})$ of $\mathcal{F}$-pearls from Definition 1.1. If $\mathcal{P}(\mathcal{F})$ is non-empty, then we may additionally assume that $x$ has order $p$. For $0 \leq i \leq p-1$, define

$$
H_{i}=\left\langle x s_{1}^{i}\right\rangle Z(S) \text { and } B_{i}=\left\langle x s_{1}^{i}\right\rangle Z_{2}(S)
$$

Then $\mathcal{H}=\bigcup_{i=0}^{p-1} H_{i}^{S}$ and $\mathcal{B}=\bigcup_{i=0}^{p-1} B_{i}^{S}$. In our notation we have

$$
\begin{aligned}
& \mathcal{P}_{a}(\mathcal{F})=\{H \in \mathcal{H} \mid H \text { is } \mathcal{F} \text {-essential }\} \\
& \mathcal{P}_{e}(\mathcal{F})=\{B \in \mathcal{B} \mid B \text { is } \mathcal{F} \text {-essential }\}
\end{aligned}
$$

We also let $\mathcal{P}_{a}^{i}(\mathcal{F})$ consist of those $\mathcal{F}$-pearls which are $S$-conjugate to $H_{i}$ and $\mathcal{P}_{e}^{i}(\mathcal{F})$ contain the $\mathcal{F}$-pearls $S$-conjugate to $B_{i}$. Finally, for $I \subseteq \mathbb{Z} / p \mathbb{Z}$ and $b \in\{a, e\}$ define

$$
\mathcal{P}_{b}^{I}(\mathcal{F})=\bigcup_{i \in I} \mathcal{P}_{b}^{i}(\mathcal{F}) \text { and } \mathcal{P}_{b}^{*}(\mathcal{F})=\bigcup_{i=1}^{p-1} \mathcal{P}_{b}^{i}(\mathcal{F})
$$

Set $G=\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$. Then, we define

$$
\mu_{1}: N_{G}\left(\operatorname{Aut}_{S}\left(\gamma_{1}(S)\right)\right) \rightarrow \Delta
$$

by

$$
\alpha \mu_{1}=\beta \mu
$$

where $\beta$ in $\operatorname{Aut}_{\mathcal{F}}(S),\left.\beta\right|_{\gamma_{1}(S)}=\alpha$ and $\mu$ is as defined at the beginning of Section 13 . In [19, page 218] it is explained why $\mu_{1}\left(\right.$ denoted $\left.\mu_{A}\right)$ is well-defined. If $X$ is a finite cyclic group and $n$ divides $|X|$, then $\frac{1}{n} X$ denotes the unique subgroup of $X$ which has index $n$. Almost all the other notation that we require can be found in the introduction to Section 13 one exception being the cyclic subgroups of $\Delta$ defined as

$$
\Delta_{k / \ell}=\left\{\left(u^{\ell}, u^{k}\right) \mid u \in \mathbb{Z} / p \mathbb{Z}^{\times}\right\}
$$

whenever $k$ and $\ell$ are coprime.
Suppose that $\mathcal{F}$ is a reduced saturated fusion system on $S$ where $S$ has maximal class, $\gamma_{1}(S)$ is abelian and $|S|=p^{n}>p^{3}$. Since we are primarily focussed on the cases when $\gamma_{1}(S)$ is $\mathcal{F}$-essential, we can sift through the results in $[\mathbf{1 9}, \mathbf{4 5}]$.

By Lemma 3.2 (iv) we know that either $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right| \leq p^{p-1}$ or $|S|=p^{p+1}$ and $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right|=p^{p}$. Furthermore, Lemma 3.4 implies that Aut ${ }_{S}\left(\gamma_{1}(S)\right)$ acts on $\Omega_{1}\left(\gamma_{1}(S)\right)$ with a single Jordan block. By [19, Proposition 3.7], if $\Omega_{1}\left(\gamma_{1}(S)\right)$ has order at most $p^{p}$, then $\operatorname{Aut}_{S}\left(\gamma_{1}(S)\right)$ operates on $\Omega_{1}\left(\gamma_{1}(S)\right)$ with a single Jordan block and so from the results in $[\mathbf{1 9}, \mathbf{4 5}]$ we just need to collate the ones with $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right| \leq p^{p}$. If $\gamma_{1}(S)>\Omega_{1}\left(\gamma_{1}(S)\right)$, then $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right| \neq p^{p}$ and so [45, Theorem A] implies $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right|=p^{p-1}$. In particular, if $\Omega_{1}\left(\gamma_{1}(S)\right)$ is irreducible, then $\gamma_{1}(S)$ is homocyclic. In Table 1 we present conditions on the structure of $\operatorname{Out}_{\mathcal{F}}(S)$ which determine the various possibilities for constellations of $\mathcal{F}$-pearls in a reduced fusion system. This table is transcribed from [19, Theorem 2.8, Table 2.1].

|  | $\left(N_{G}\left(\operatorname{Aut}_{S}\left(\gamma_{1}(S)\right) \mu_{1}\right.\right.$ | $G=O^{p^{\prime}}(G) X$ | $\left\|\gamma_{1}(S)\right\|=p^{m}$ | Pearls |
| :---: | :---: | :---: | :---: | :---: |
| I | $\Delta$ | $X=N_{G}\left(\operatorname{Aut}_{S}\left(\gamma_{1}(S)\right)\right.$ ) | $m \equiv 0(\bmod p-1)$ | $\mathcal{P}_{a}^{0}(\mathcal{F}) \cup \mathcal{P}_{e}^{*}(\mathcal{F})$ |
| II | $\Delta$ | $X=N_{G}\left(\operatorname{Aut}_{S}\left(\gamma_{1}(S)\right)\right.$ ) | $m \equiv p-2(\bmod p-1)$ | $\mathcal{P}_{a}^{*}(\mathcal{F}) \cup \mathcal{P}_{e}^{0}(\mathcal{F})$ |
| III | $\begin{aligned} & \geq \Delta_{-1} \end{aligned}$ | $\begin{aligned} & X=\left(\Delta_{-1}\right) \mu_{1}^{-1} \\ & X=\left(\Delta_{-1}\right) \mu_{1}^{-1} \\ & \hline \end{aligned}$ | $m \equiv p-2(\bmod p-1)$ | $\begin{gathered} \mathcal{P}_{a}^{I}(\mathcal{F}), I \subseteq \mathbb{Z} / p \mathbb{Z} \\ \mathcal{P}_{a}^{0}(\mathcal{F}) \\ \hline \end{gathered}$ |
| IV | $\begin{aligned} & \geq \Delta_{0} \\ & \geq \Delta_{0} \\ & \hline \end{aligned}$ | $\begin{aligned} & X=\left(\Delta_{0}\right) \mu_{1}^{-1} \\ & X=\left(\Delta_{0}\right) \mu_{1}^{-1} \end{aligned}$ | $m \equiv 0(\bmod p-1)$ | $\begin{gathered} \mathcal{P}_{e}^{I}(\mathcal{F}), I \subseteq \mathbb{Z} / p \mathbb{Z} \\ \mathcal{P}_{e}^{0}(\mathcal{F}) \\ \hline \end{gathered}$ |
| $\begin{aligned} & \text { TABLE 1. Configurations of } \quad \text { pearls de } \\ & \left(N_{G}\left(\operatorname{Aut}_{S}\left(\gamma_{1}(S)\right)\right) \mu_{1} \text { where } G=\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right.\right. \\ & N_{G}\left(\operatorname{Aut}_{S}\left(\gamma_{1}(S)\right)\right) \text {. } \end{aligned}$ |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |

Theorem A.1. Suppose that $p$ is an odd prime, $S$ has maximal class of order at least $p^{4}$ and $\gamma_{1}(S)$ is abelian. If $\mathcal{F}$ is a reduced saturated fusion system on $S$, then $\mathcal{E}_{\mathcal{F}} \subseteq\left\{\gamma_{1}(S)\right\} \cup \mathcal{P}(\mathcal{F})$ and $\mathcal{P}(\mathcal{F})$ is non-empty. Furthermore, one of the following holds:
(i) $\mathcal{E}_{\mathcal{F}}=\mathcal{P}(\mathcal{F})$.
(ii) $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right|<p^{p-1}, \gamma_{1}(S)=\Omega_{1}\left(\gamma_{1}(S)\right)$, the candidates for $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ and the configurations of $\mathcal{F}$-pearls are listed in the first section of Table 2.
(iii) $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right|=p^{p}, \gamma_{1}(S)=\Omega_{1}\left(\gamma_{1}(S)\right),|S|=p^{p+1}$ and the possibilities for $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ and the configurations of $\mathcal{F}$-pearls are listed in the second section of Table 2.
(iv) $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right|=p^{p-1}, \Omega_{1}\left(\gamma_{1}(S)\right)$ is irreducible as a $\operatorname{GF}(p) \operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ module, $\gamma_{1}(S)$ is homocyclic of order $p^{a(p-1)}$ for some $a \geq 1$ and the possibilities for $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ and the configurations of $\mathcal{F}$-pearls are listed in the third section of Table 2.
(v) $\left|\Omega_{1}\left(\gamma_{1}(S)\right)\right|=p^{p-1}, \Omega_{1}\left(\gamma_{1}(S)\right)$ is indecomposable but not irreducible as $a \operatorname{GF}(p) \operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$-module, $\gamma_{1}(S)$ not necessarily homocyclic and the possibilities for $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ and the configurations of $\mathcal{F}$-pearls are listed in the fourth section of Table 2.

Proof. This mostly follows from our previous discussion and by combining the results of [19, Corollary 2.10 and Theorem 4.1] with [45, Theorem A]. However, lines 29,30 and 33 and 34 in Table 2 need further explanation to confirm column 6 , where the possibilities for $\mathcal{F}$-pearls are described. So suppose that $\mathcal{F}$ is reduced. For lines 29 and 33, we note that if IV from Table 1 holds, then $\mathcal{P}(\mathcal{F})=\mathcal{P}_{e}(\mathcal{F})$ and $Z(S)=O_{p}(\mathcal{F})$ which is impossible. The possibility that III holds when considering lines 30 and 34 leads to $\operatorname{Aut}_{\mathcal{F}}(S) \mu=\Delta_{-1}$. Using Lemma 3.10 we ob$\operatorname{tain}\left[\gamma_{1}(S), \operatorname{Aut}_{\mathcal{F}}(S)\right] \leq \gamma_{2}(S)$ and so, as $\left[\gamma_{1}(S), O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right]=\gamma_{2}(S)\right.$, we have $\left[\gamma_{1}(S)\right.$, $\left.\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right]=\gamma_{2}(S)$. Now applying [19, Lemma 2.7 (b)] shows that $\mathcal{F} \neq O^{p}(\mathcal{F})$, which contradicts $\mathcal{F}$ being reduced.

| Row | $p$ | $Y=O^{p^{\prime}}\left(\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right)\right)$ | $\Omega_{1}\left(\gamma_{1}(S)\right)$ | $\left(N_{Y}\left(\operatorname{Aut}_{\mathcal{F}}(S)\right)\right)^{\prime}$ | Pearls |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $p$ | $\mathrm{SL}_{2}(p)$ | $p^{e+1} \cong \mathbf{V}_{e}$ | $\left\{\left(u^{2}, u^{e}\right) \mid u \in \mathbb{Z} / p \mathbb{Z}^{\times}\right\}$ | III or IV |
|  |  |  | $e \leq p-4$, odd |  |  |
| 2 | $p$ | $\mathrm{PSL}_{2}(p)$ | $p^{e+1} \cong \mathbf{V}_{e}$ | $\left\{\left(u^{2}, u^{e}\right) \mid u \in \mathbb{Z} / p \mathbb{Z}^{\times}\right\}$ | III or IV |
|  |  |  | $e \leq p-5$, even |  |  |
| 3 | $p$ | $\mathrm{PSL}_{2}(p)$ | $p^{p-2} \cong \mathbf{V}_{p-3}$ | $\frac{1}{2} \Delta_{-1}$ | II, III or IV |
| 4 | $p$ | Alt(p) | $p^{p-2}$ | $\frac{1}{2} \Delta_{-1}$ | II, III or IV |
| 5 | 7 | $2 \cdot \mathrm{Alt}(7)$ | $7^{4}$ | $\Delta_{3 / 2}$ | III or IV |
| 6 | 11 | $\mathrm{J}_{1}$ | $11^{7}$ | $\Delta_{3}$ | III or IV |
| 7 | $p$ | $\mathrm{PSL}_{2}(p)$ | $p^{p}=\mathbf{V}_{p-1}$ | $\frac{1}{2} \Delta_{1}$ | III or IV |
| 8 | $p$ | $\mathrm{PSL}_{2}(p)$ | $p^{p}=\underset{\substack{\mathbf{V}_{p-3} \\ \mathbf{V}_{0} \\ \mathbf{V}_{0}}}{ }$ | $\frac{1}{2} \Delta_{0}$ | III |
| 9 | $p$ | Alt $(p)$ | $p^{p}=p_{1}^{1-2}$ | $\frac{1}{2} \Delta_{0}$ | III |
| 10 | $p$ | $\operatorname{Alt}(p+1)$ | $p^{p}$ | $\frac{1}{2} \Delta_{0}$ | $\begin{aligned} & \text { III or IV } \\ & \text { III or IV } \end{aligned}$ |
| 11 | $p$ | $\begin{gathered} Y \leq O^{p^{\prime}}((p-1) 2 \operatorname{Sym}(p)) \\ Y / O_{p^{\prime}}(Y) \cong \operatorname{Alt}(p) \end{gathered}$ | $p^{p}$ |  |  |
| 12 | $p$ | $\begin{aligned} Y \leq & O^{p^{\prime}}((p-1) \imath \operatorname{Sym}(p)) \\ & \left\|Y / O_{p^{\prime}}(Y)\right\|=p \end{aligned}$ | $p^{p}$ |  | III or IV |
|  |  |  |  |  |  |
| 13 | 7 | $\mathrm{PSU}_{3}(3)$ | $7^{7}$ | $\begin{gathered} \frac{1}{2} \Delta_{0} \\ \frac{1}{3} \Delta_{1} \\ \Delta_{3} \\ \hline \hline \end{gathered}$ | III or IV |
| 14 | 7 | $\mathrm{SL}_{2}$ (8) | $7^{7}$ |  | III or IV |
| 15 | 7 | $\mathrm{Sp}_{6}(2)$ | $7^{7}$ |  | III or IV |
| 16 | $p$ | $\mathrm{SL}_{2}(p)$ | $p^{p-1}=\mathbf{V}_{p-2}$ | $\left\{\left(u^{2}, u^{-1}\right) \mid u \in \mathbb{Z} / p \mathbb{Z}^{\times}\right\}$ | I, III or IV |
| 17 | 5 | $2 \cdot \operatorname{Alt}(6)$ | $5^{4}$ | $\Delta_{1 / 2}$ | I, III or IV |
| 18 | 5 | $4 \circ 2^{1+4} \cdot \operatorname{Alt}(6)$ | $5^{4}$ |  | I, III or IV |
| 19 | 5 | $2_{-}^{1+4}$. $\operatorname{Alt}$ (5) | $5^{4}$ |  | I, III or IV |
| 20 | 5 | $4 \circ 2_{-}^{1+4} . \operatorname{Alt}(5)$ | $5^{4}$ |  | I, III or IV |
| 21 | 5 | $2_{-}^{1+4} .5$ | $5^{4}$ |  | I, III or IV |
| 22 | 7 | $6 \cdot \mathrm{PSL}_{3}(4)$ | $7^{6}$ | $\left\{\left(u^{2}, w\right) \mid u, w \in \mathbb{Z} / p \mathbb{Z}\right\}$ | I, III or IV |
| 23 | 7 | $6_{1} \mathrm{PSU}_{4}(3)$ | $7^{6}$ | $\left\{\left(u^{2}, w\right) \mid u, w \in \mathbb{Z} / p \mathbb{Z}\right\}$ | I, III or IV |
| 24 | 7 | $\mathrm{PSU}_{3}(3)$ | $7^{6}$ | $\frac{1}{2} \Delta_{1}$ | I, III or IV |
| 25 | 11 | $\mathrm{PSU}_{5}(2)$ | $11^{10}$ | $\frac{1}{2} \Delta_{2}$ | I, III or IV |
| 26 | 11 | $2 \cdot \mathrm{M}_{12}$ | $11^{10}, 11^{10}$ | $\Delta_{1 / 2}, \Delta_{7 / 2}$ | I, III or IV |
| 27 | 11 | $2 \cdot \mathrm{M}_{22}$ | $11^{10}, 11^{10}$ | $\Delta_{1 / 2}, \Delta_{7 / 2}$ | I, III or IV |
| 28 | 13 | $\mathrm{PSU}_{3}(4)$ | $13^{12}$ | $\frac{1}{3} \Delta_{1}$ | I, III or IV |
| 29 | $p$ | Alt ( $p$ ) | $p^{p-1}={ }_{1}^{p-2}$ | $\frac{1}{2} \Delta_{0}$ | I or III |
| 30 | $p$ | Alt $(p)$ | $p^{p-1}=p^{1} 2$ | $\frac{1}{2} \Delta_{-1}$ | I or IV |
| 31 | $p$ | $\mathrm{SL}_{2}(p)$ | $p^{p-1}=\mathbf{V}_{\mathbf{V}_{e}}^{\mathbf{V}^{\prime}}$ | $\left\{\left(u^{2}, u^{e}\right) \mid u \in \mathbb{Z} / p \mathbb{Z}^{\times}\right\}$ | I, III or IV |
| 32 | $p$ | $\mathrm{PSL}_{2}(p)$ | $e+f=p-3, e$ odd $p^{p-1}=\mathbf{V}_{f}$ $\mathbf{V}_{e}$ | $\left\{\left(u^{2}, u^{e}\right) \mid u \in \mathbb{Z} / p \mathbb{Z}^{\times}\right\}$ | I, III or IV |
|  |  |  | $e+f=p-3$, ef $\neq 0, e$ even |  |  |
| 33 | $p$ | $\mathrm{PSL}_{2}(p)$ | $p^{p-1}=\underset{\mathbf{V}_{p-3}}{\mathbf{V}_{0}}={ }_{1}^{p-2}$ | $\frac{1}{2} \Delta_{0}$ | I or III |
| 34 | $p$ | $\mathrm{PSL}_{2}(p)$ | $p^{p-1}=\mathbf{V}_{p-3}^{\mathbf{V}_{0}}={ }_{p-2}^{1}$ | $\frac{1}{2} \Delta_{-1}$ | I or IV |

TABLE 2. The reduced saturated fusion systems $\mathcal{F}$ on maximal class $p$-groups of order at least $p^{4}, p$ odd, with an abelian $\mathcal{F}$ essential subgroup of index $p$. Where expressions are of the form $p-j$ for some natural number $j$, we require $p$ to be large enough to ensure that $p-j \geq 2$.

Finally, we trim [19, Table 2.2] and provide the list of realizable reduced fusion systems on maximal class $p$-groups $S$ with $\gamma_{1}(S)$ abelian and $|S| \geq p^{4}$. In this table $\nu_{p}(m)$ denotes that exponent of the highest power of $p$ which divides $m$. All the fusion systems not listed in Table 3 but listed in Table 2 are exotic.

| Line | $G$ | $p$ | Conditions | $\operatorname{Rank}\left(\gamma_{1}(S)\right)$ | $e$ | $\left\|\gamma_{1}(S)\right\|$ | $\operatorname{Aut}_{G}\left(\gamma_{1}(S)\right)$ | Pearls |
| :--- | :--- | :--- | :--- | :---: | :--- | :---: | :---: | :---: |
| 2, IV | $\mathrm{PSp}_{4}(p)$ | $p$ | - | 3 | 1 | $p^{3}$ | $\mathrm{GL}_{2}(p) /\{ \pm I\}$ | $\mathcal{P}_{e}^{0}(\mathcal{F})$ |
| 11, III | $\operatorname{Alt}^{2}\left(p^{2}\right)$ | $p$ | - | $p$ | 1 | $p^{p}$ | $\frac{1}{2}(p-1) 2 \operatorname{Sym}(p)$ | $\mathcal{P}_{a}^{0}(\mathcal{F})$ |
| 4, III | $\operatorname{PSL}_{p}(q)$ | $p$ | $\nu_{p}(q-1)=1, p>3$ | $p-2$ | 1 | $p^{p-2}$ | $\operatorname{Sym}(p)$ | $\mathcal{P}_{a}^{0}(\mathcal{F}) \cup \mathcal{P}_{a}^{*}(\mathcal{F})$ |
| 34, III | $\operatorname{PSL}_{p}(q)$ | $p$ | $\nu_{p}(q-1) \geq 2, p>3$ | $p-1$ | $\nu_{p}(q-1)$ | $p^{e(p-1)-1}$ | $\operatorname{Sym}(p)$ | $\mathcal{P}_{a}^{0}(\mathcal{F}) \cup \mathcal{P}_{a}^{*}(\mathcal{F})$ |
| 10, IV | $\mathrm{PSL}_{p+1}(q)$ | $p$ | $\nu_{p}(q-1)=1$ | $p$ | 1 | $p^{p}$ | $\operatorname{Sym}(p+1)$ | $\mathcal{P}_{e}^{0}(\mathcal{F})$ |
| 11, IV | $\mathrm{P}_{2 p}^{+}(q)$ | $p$ | $\nu_{p}(q-1)=1$ | $p$ | 1 | $p^{p}$ | $2^{p-1}: \operatorname{Sym}(p)$ | $\mathcal{P}_{e}^{0}(\mathcal{F})$ |
| 16, IV | ${ }^{2} \mathrm{~F}_{4}\left(2^{2 n+1}\right)$ | 3 | $2 n+1 \geq 3$ | 2 | $\nu_{3}(q+1)$ | $3^{2 e}$ | $\operatorname{GL}_{2}(3)$ | $\mathcal{P}_{e}^{0}(\mathcal{F}) \cup \mathcal{P}_{e}^{*}(\mathcal{F})$ |
| 15, IV | $\mathrm{E}_{7}(q)$ | 7 | $\nu_{7}(q-1)=1$ | 7 | 1 | $7^{7}$ | $\mathrm{~W}\left(\mathrm{E}_{7}\right)=2 \times \operatorname{Sp}_{6}(2)$ | $\mathcal{P}_{e}^{0}(\mathcal{F})$ |
| 18, I | $\mathrm{E}_{8}(q)$ | 5 | $\nu_{5}\left(q^{2}+1\right) \geq 1$ | 5 | $\nu_{5}\left(q^{2}+1\right)$ | $5^{4 e}$ | $\left(4 \circ 2^{1+4}\right) \cdot \operatorname{Sym}(6)$ | $\mathcal{P}_{a}^{0} \cup \mathcal{P}_{e}^{*}(\mathcal{F})$ |
| 3, II | $\mathrm{Co}_{1}$ | 5 |  | 3 | 1 | 3 | $4 \times \operatorname{Sym}(5)$ | $\mathcal{P}_{e}^{0} \cup \mathcal{P}_{a}^{*}(\mathcal{F})$ |

Table 3. Realizable, reduced fusion systems $\mathcal{F}=\mathcal{F}_{S}(G)$ on maximal class $p$-groups $S, p \geq 3,|S| \geq p^{4}$ and $\gamma_{1}(S)$ abelian of exponent $e$.

## B. The saturated fusion systems on maximal class 3-groups

In this appendix, we bring together the outcome of various investigations into saturated fusion systems on maximal class 3 -groups. These results are mainly extracted from $[\mathbf{2 0}, \mathbf{4 8}, 54]$.

There are two maximal class 3 -groups of order $3^{3}$, they are both extraspecial one of exponent 3 and one of exponent 9 .

Theorem B.1. Suppose that $S$ is extraspecial of order $3^{3}$ and $\mathcal{F}$ is a saturated fusion system on $S$. Then either $\mathcal{F}=N_{\mathcal{F}}(S)$ or $S$ has exponent $3, \mathcal{E}_{\mathcal{F}}=\mathcal{P}_{a}(\mathcal{F})$ and one of the following holds.
(i) $\mathcal{F}=\mathcal{F}_{S}\left(3^{2}: \mathrm{SL}_{2}(3)\right)$ or $\mathcal{F}_{S}\left(3^{2}: \mathrm{GL}_{2}(3)\right)$;
(ii) $\mathcal{F}=\mathcal{F}_{S}\left(\mathrm{PSL}_{3}(3)\right)$ or $\mathcal{F}=\mathcal{F}_{S}\left(\mathrm{PSL}_{3}(3): 2\right)$;
(iii) $\mathcal{F}=\mathcal{F}_{S}\left({ }^{2} \mathrm{~F}_{4}(2)^{\prime}\right)$; or
(iv) $\mathcal{F}=\mathcal{F}_{S}\left(\mathrm{~J}_{4}\right)$.

Proof. The extraspecial group of order $3^{3}$ and exponent 9 is meatacyclic. Hence, in this case, $\mathcal{F}=N_{\mathcal{F}}(S)$ by [56, Proposition 5.4] (see also [16, Theorem 7.5]). If $S$ is extraspecial of exponent 3, we read the result from [54, Theorem 1.1].

The classification of maximal class 3 -groups is due to Blackburn [8]. We take their presentations from [20, Theorem A.2]. For $n \geq 4$, and $\beta, \gamma, \delta \in\{0,1,2\}$, define

$$
\mathrm{B}(n ; \beta, \gamma, \delta)=\left\langle x, s_{1}, \ldots, s_{n-1} \mid \mathbf{R 1}, \mathbf{R 2}, \mathbf{R 3}, \mathbf{R 4}, \mathbf{R} 5, \mathbf{R 6}\right\rangle
$$

where, understanding that $s_{n}=s_{n+1}=1$, the relations are as follows:
R1: $s_{i}=\left[s_{i-1}, x\right]$ for $i \in\{2, \ldots, n-1\} ;$
R2: $\left[s_{1}, s_{i}\right]=1$ for $i \in\{3, \ldots, n-1\}$;
R3: $s_{i}^{3} s_{i+1}^{3} s_{i+2}=1$ for $i \in\{2, \ldots, n-1\}$;
R4: $\left[s_{1}, s_{2}\right]=s_{n-1}^{\beta}$;
R5: $s_{1}^{3} s_{2}^{3} s_{3}=s_{n-1}^{\gamma}$; and
R6: $x^{3}=s_{n-1}^{\delta}$.
ThEOREM B. 2 (Blackburn). Suppose that $S$ is a maximal class 3-group of order $3^{n}$ with $n \geq 4$. Then $S \cong \mathrm{~B}(n ; \beta, \gamma, \delta)$ for some $\beta, \gamma, \delta \in\{0,1,2\}$. Furthermore, $S$ is metabelian and, unless $S \cong \mathrm{~B}(4 ; 0,1,0)$, $S$ has rank 2 .

Proof. This comes from the text before and after [8, Theorems 4.2 and 4.3].

There are isomorphisms between some of the Blackburn group. Using the discussion after [8, Theorem 4.3], the full list of groups of order $3^{4}$ is given as $\mathrm{B}(4 ; 0, \gamma, \delta)$ where

$$
(0, \gamma, \delta) \in \Sigma_{4}=\{(0,1,0),(0,2,0),(0,0,0),(0,0,1)\}
$$

Since these groups have order $3^{4}$, they each have an abelian subgroup of index 3. The group $\mathrm{B}(4 ; 0,1,0)$ is the unique maximal class 3 -group of rank 3 . Thus $\mathrm{B}(4 ; 0,1,0)$ is isomorphic to a Sylow 3-subgroup of $\operatorname{Sym}(9)$. For $n \geq 5$, the groups with no abelian maximal subgroups are given by

$$
(\beta, \gamma, \delta) \in \Theta=\{(1,0,0),(1,0,1),(1,0,2)\}
$$

With this we can write down the a full irredundant list of maximal class 3-groups of order at least $3^{5}$ :
(i) for $n$ odd,

$$
(\beta, \gamma, \delta) \in \Theta \cup\{(0,1,0),(0,0,1),(0,0,0)\}
$$

(ii) for $n$ even,

$$
(\beta, \gamma, \delta) \in \Theta \cup \Sigma_{4}
$$

So, for $n \geq 5$, there are six maximal class 3 -groups when $n$ is odd, and seven when $n$ is even.

Lemma B.3. Assume that $S=\mathrm{B}(4 ; 0,1,0)$ is a Sylow 3 -subgroup of $\operatorname{Sym}(9)$ and $\mathcal{F}$ is a saturated fusion system on $S$ with $\mathcal{F} \neq N_{\mathcal{F}}(S)$. Set $A=\left\langle x, s_{3}\right\rangle$ and $E=\left\langle s_{2}, A\right\rangle$. Then $\gamma_{1}(S)=\left\langle s_{1}, s_{2}, s_{3}\right\rangle$ and $\mathcal{F}$ is described as follows:
(i) $\mathcal{E}_{\mathcal{F}}=\left\{\gamma_{1}(S)\right\}, \mathcal{F}=N_{\mathcal{F}}\left(\gamma_{1}(S)\right)$ and $\operatorname{Aut}_{\mathcal{F}}\left(\gamma_{1}(S)\right) \cong \operatorname{Frob}(39)$, $\operatorname{Frob}(39) \times$ 2 , $\operatorname{Alt}(4), 2 \times \operatorname{Alt}(4), \operatorname{Sym}(4)$ two different actions, or $2 \times \operatorname{Sym}(4)$;
(ii) $\mathcal{E}_{\mathcal{F}}=\mathcal{P}_{e}(\mathcal{F})=\{E\}, \mathcal{F}=N_{\mathcal{F}}(E)$ and either $\operatorname{Aut}_{\mathcal{F}}(E) \cong 3^{2}: \mathrm{SL}_{3}(3)$ or $3^{2}: \mathrm{GL}_{2}(3)$;
(iii) $\mathcal{E}_{\mathcal{F}}=\left\{E, \gamma_{1}(S)\right\}=\mathcal{P}_{e}(\mathcal{F}) \cup\left\{\gamma_{1}(S)\right\}$ and $\mathcal{F}=\mathcal{F}_{S}(G)$ with $G \cong \operatorname{PSp}_{4}(3)$ or $\mathrm{PSp}_{4}(3): 2$;
(iv) $\mathcal{E}_{\mathcal{F}}=A^{\mathcal{F}} \cup\left\{\gamma_{1}(S)\right\}=\mathcal{P}_{a}(\mathcal{F}) \cup\left\{\gamma_{1}(S)\right\}$ and $\mathcal{F}=\mathcal{F}_{S}(G)$ with $G \cong \operatorname{Alt}(9)$ or $\operatorname{Sym}(9)$; or
(v) $\mathcal{E}_{\mathcal{F}}=\mathcal{P}_{a}(\mathcal{F})=A^{\mathcal{F}}, \operatorname{Aut}_{\mathcal{F}}(A) \cong \mathrm{SL}_{3}(3)$ or $\mathrm{GL}_{2}(3)$, and $O^{3^{\prime}}(\mathcal{F})$ is simple and exotic.

In particular, $\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{P}(\mathcal{F}) \cup\left\{\gamma_{1}(S)\right\}$.
Proof. The examples have been enumerated by computer using the procedures from [50]. The code is in Subsection C.5. This confirms that there are 15 examples.

Theorem B.4. Suppose that $S=\mathrm{B}(n ; \beta, \gamma, \delta)$ is a maximal class 3 -group of rank 2 and order $3^{n}$ with $n \geq 4$. Assume $\mathcal{F}$ is a saturated fusion system on $S$ with $\mathcal{F} \neq N_{\mathcal{F}}(S)$. Then

$$
S \cong \begin{cases}\mathrm{~B}(2 k ; 0,0,0), \mathrm{B}(2 k ; 0,2,0) & k \geq 2 \\ \mathrm{~B}(2 \ell ; 0,1,0), \mathrm{B}(2 \ell ; 1,0,0), \mathrm{B}(2 \ell ; 1,0,2) & \ell \geq 3 \\ \mathrm{~B}(2 k+1 ; 0,0,0), \mathrm{B}(2 k+1 ; 1,0,0) & k \geq 2\end{cases}
$$

and $\mathcal{F}$ is as described in Table 4. Furthermore, for each of the fusion systems tabulated, $\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{P}(\mathcal{F}) \cup\left\{\gamma_{1}(S)\right\}$ and, if $\gamma_{1}(S) \in \mathcal{E}_{\mathcal{F}}$, then $S \cong \mathrm{~B}(2 k+1 ; 0,0,0)$ with $k \geq 2$ and $\gamma_{1}(S)$ is homocyclic of rank 2 .

Proof. This is a compilation of [20, Theorem 5.10] and [48, Theorem 1.1]. The final statement is obtained by inspection of Table 4.

| Group | $\left\|\mathrm{Out}_{\mathcal{F}}(S)\right\|$ | Out ${ }_{\mathcal{F}}\left(A_{0}\right)$ | $\mathrm{Out}_{\mathcal{F}}\left(A_{1}\right)$ | $\mathrm{Out}_{\mathcal{F}}\left(A_{-1}\right)$ | $\mathrm{Out}_{\mathcal{F}}\left(E_{0}\right)$ | $\mathrm{Out}_{\mathcal{F}}\left(E_{1}\right)$ | $\mathrm{Out}_{\mathcal{F}}\left(E_{-1}\right)$ | $\mathrm{Out}_{\mathcal{F}}\left(\gamma_{1}\right)$ | Example |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{B}(2 k ; 0,0,0)$ | 2 | $\mathrm{SL}_{2}(3)$ |  |  |  |  |  |  | $\mathcal{F}_{\text {DRV }}\left(3^{2 k}, 1\right)$ |
| - $\overline{\mathrm{B}}(\underline{2} \overline{\mathrm{k}} ; \mathbf{0}, \overline{0}, 0,0)$ | 2 |  | $\overline{S o L}_{2} \overline{2}(\overline{3})$ | $\mathrm{S}_{\mathrm{L}}^{2}-\overline{-}^{(\overline{3})}$ |  |  |  |  | $\bar{F}_{\text {DRV }}^{-\bar{s}}{\left.\overline{( }{ }^{2 k}-2\right)}^{2}$ |
| - $\overline{\mathrm{B}}(\underline{2} \bar{k} ; \overline{0}, \overline{0}, 0,0)$ | 2 | $\overline{S L L}_{2}^{-}(\overline{3})$ | $\overline{S o L}_{2}(\overline{3}$ (3) | $\mathrm{SSL}_{2}(\overline{3})$ |  |  |  |  | $\overline{\mathrm{PSL}}_{3}^{-}\left(\bar{q}_{1}\right)$ |
| $\left.{ }^{-} \overline{\mathrm{B}} \overline{2} \overline{\mathrm{k}} ; \overline{0}, \overline{0}, \overline{0}\right)$ | 2 |  |  |  | SL $\bar{L}_{2}(\overline{3})$ |  |  |  | $\begin{gathered} \mathrm{B}^{2} \overline{\mathrm{P}} \overline{\mathrm{G}} \overline{\mathrm{~L}}^{-}\left(\bar{q}_{2}\right), k \\ N_{\mathcal{F}}\left(E_{0}\right), k=2 \end{gathered}$ |
| - $\overline{\mathrm{B}}(2 \overline{2} \bar{k} ; \overline{0}, 0,0,0)$ | $2^{2}$ | $\mathrm{CLL}_{2}^{-} \overline{(3)}$ |  |  |  |  |  |  | - $\mathcal{F}_{\text {DRV }}\left(3^{2 k}, 1\right) \cdot 2-$ |
| - $\overline{\mathrm{B}}(\overline{2} \bar{k} ; \overline{0}, \overline{0}, \overline{0})$ | $\overline{2}^{2}$ |  | $\overline{S o L}_{2} \overline{-}(\overline{3})$ | $\bar{A}_{1}^{-} \bar{\sim}_{\mathcal{I}}^{-} \bar{A}_{-1}^{-1}$ |  |  |  |  | $\left.F_{\text {DRV }}-\overline{(32 \pi}, \overline{2}\right) \cdot 2^{-}$ |
| - $\overline{\mathrm{B}}(\underline{2} \bar{k} ; \overline{0}, 0,0)$ | $\overline{2}^{2}$ | $\mathrm{GLL}_{2} \overline{(3)}$ | $\bar{S}_{\text {L }}^{2} \overline{2}_{2}(\overline{3})$ | $\bar{A}_{1}^{-} \sim_{\mathcal{F}} \bar{A}_{-1}^{-1}$ |  |  |  |  | ${ }^{\text {P }} \overline{\mathrm{S}} \bar{L}_{3}\left(q_{1}\right)_{1}^{\prime} \cdot 2^{-}$ |
| - $\overline{\mathrm{B}} \overline{2} \overline{\mathrm{k}} ; \overline{0}, \overline{0}, \overline{0})$ | $\overline{2}^{2}$ |  |  |  | $\left.\overline{\mathrm{GL}}_{2}^{-} \overline{( } \overline{3}\right)$ |  |  |  | $\begin{gathered} 3 \mathrm{PGL}_{3}\left(q_{2}\right), 2, k>\overline{2} \\ N_{\mathcal{F}}\left(E_{0}\right), k=2 \end{gathered}$ |
| ${ }^{-} \overline{\mathrm{B}}(2 \bar{k} ; \overline{0}, \overline{0}, \underline{0})$ | $\overline{2}^{2}$ |  | $\overline{S o L}_{2} \overline{(1)}$ | $\overline{A_{1}^{\prime}} \bar{\sim}_{\mathcal{F}}^{-} \bar{A}_{-1}^{-}$ | $\left.\overline{\mathrm{GLL}}_{2} \overline{( } \overline{3}\right)$ |  |  |  |  |
| B (2k; $0,2,0$ ) | 2 | $\mathrm{SL}_{2}(3)$ |  | * |  | * | * |  | $\begin{aligned} & \mathcal{F}_{\mathrm{DRV}}\left(3^{2 k}, 4\right), k>2 \\ & \mathcal{F}_{\mathrm{DRV}}\left(3^{4}, 3\right), k=2 \end{aligned}$ |
| ${ }^{-} \overline{\mathrm{B}}(2 \bar{k} ; \overline{0}, \overline{2, ~}-0)^{\prime}$ | 2 |  |  | * | $\left.\mathrm{SL}_{2} \bar{\sim}^{-} \overline{3}\right)^{-}$ | * | * |  | $\begin{gathered} 3 \mathrm{P} \overline{\mathrm{G}}_{3}\left(q_{2}\right), k>2 \\ N_{\mathcal{F}}\left(E_{0}\right), k=2 \end{gathered}$ |
| - $\overline{\mathrm{B}}\left(2 \overline{2} \bar{\beta} ; \overline{0}, \overline{2,0} \overline{0}^{\prime}\right.$ | $2^{2}$ | $\mathrm{CLL}_{2}^{-}(\overline{3})$ |  | * |  |  |  |  | $\begin{aligned} & \mathcal{F}_{\mathrm{FRRV}}\left(3^{2 k}, \overline{4}\right) \cdot \mathbf{2}^{-2} \\ & \left.\mathcal{F}^{4}, 3\right) \cdot 2, k=2 \end{aligned}$ |
| $\left.{ }^{-} \overline{\mathrm{B}} \overline{2} \overline{\mathrm{k}} ; \overline{0}, \overline{2}, \overline{0}\right)^{-}$ | $\overline{2}^{2}$ |  | * | * | $\overline{\mathrm{GL}}_{2}{ }^{-}(\overline{3})^{-}$ |  |  |  | $\begin{gathered} 3 \mathrm{PGL} \mathrm{P}_{3}\left(q_{2}\right) \cdot 2, k>2 \\ N_{\mathcal{F}}\left(E_{0}\right), k=2 \end{gathered}$ |
| $\mathrm{B}(2 \ell ; 0,1,0)$ | - | $\mathrm{SL}_{2}(3)$ |  | * |  |  | * |  | $\mathcal{F}_{\text {DRV }}\left(3^{2 \ell}, 3\right)$ |
| - $\overline{\mathrm{B}}(\underline{2} \bar{\ell} ; \overline{0}, \overline{1}, \overline{0})$ | 2 |  |  | ${ }^{*}$ | $\mathrm{SL}_{2} \overline{2}^{-}(\overline{3})$ |  |  |  | $\left.\overline{3} \overline{\mathrm{P}} \overline{\mathrm{G}}_{\overline{\mathrm{L}}}^{3} \overline{\left(q_{2}\right.} \overline{q_{2}}\right)$ |
| - $\overline{\mathrm{B}}(\underline{2} \bar{\chi} \bar{\chi} \overline{0}, 1,1,0)$ | - | $\mathrm{CLL}_{2} \overline{(3)}$ |  |  |  |  |  |  |  |
| - $\overline{\mathrm{B}}(\underline{2} \bar{\ell} ; \overline{0}, \overline{1}, 0)^{\prime}$ | $\overline{2}^{2}$ |  | * | * | $\left.\overline{\mathrm{GL}}_{2} \overline{\mathrm{C}}^{-} \overline{3}\right)$ |  | * |  | ${ }^{-} \overline{\mathrm{PGL}} \overline{3}_{3}\left(\overline{q_{2}}\right) \cdot \overline{2}$ |
| B $(2 \ell ; 1,0,2)$ |  | * | $\mathrm{SL}_{2}(3)$ |  | * |  |  |  | $\mathcal{F}\left(3^{2 \ell}, 7\right)$ |
| - $\overline{\mathrm{B}}(\underline{2} \bar{\ell}, \overline{1}, \overline{1}, \overline{2}, \overline{2})$ | 2 |  |  | S'ī ${ }_{2}^{-}(\overline{3})$ |  |  |  |  | $\overline{\mathcal{F}}\left(\overline{3}^{\left.3 \chi^{2 \ell}, 8\right)}\right.$ |
| - $\overline{\mathrm{B}}\left(\underline{2} \bar{\ell} \overline{1}, \overline{0}, \overline{2} \overline{2}^{\prime}\right.$ | 2 | - ${ }^{-}$ | $\overline{S N L}_{2} \overline{(3)}$ | $\overline{S L L}_{2}(\overline{3})$ | * |  |  |  | $\overline{\mathcal{F}}\left(\overline{3}^{3 \ell}, 9\right)$ |
| $\mathrm{B}(2 \ell ; 1,0,0)$ |  | $\mathrm{SL}_{2}(3)$ | * | * |  | * | * |  | $\mathcal{F}\left(3^{2 \ell}, 6\right)$ |
| B $(2 k+1 ; 0,0,0)$ | 2 |  |  |  | $\mathrm{SL}_{2}(3)$ |  |  |  | $3 \cdot \mathcal{F}_{\text {Rev }}\left(3^{2 k}, 1\right)$ |
| $\overline{\mathrm{B}}(\underline{2} \bar{k}+\overline{1} ; \overline{0}, \overline{0}, \overline{0})$ | 2 |  |  |  |  | $\bar{S}_{\underline{L}}^{2}-\overline{2}(\overline{3})$ | $\bar{S}_{\underline{L}}^{\bar{L}} \overline{-}(\overline{3})$ |  | $\overline{3} \mathcal{F}_{\text {DRV }}\left(\overline{3^{2}}, \overline{2}\right)$ |
| - $\overline{\mathrm{B}}(2 \bar{k}+\overline{1} ; 0,0,0,0)$ | - |  |  |  | SL ${ }_{2}^{-}(\overline{3})$ | $\overline{S i L}_{2} \mathrm{~L}_{2}(3)$ | $\bar{S}_{\underline{L}}^{L_{2}}(\underline{\text { a }}$ |  | $\left.\bar{S}^{-} \bar{L}_{3} \overline{q_{1}}\right)^{-}$ |
| $\overline{\mathrm{B}}(2 \bar{k}+\overline{1} ; 0,0,0,0)$ | 2 |  |  |  |  |  |  | - $\left.\bar{S} \bar{L}_{2} \overline{( }\right)$ (3) | $\left.\overline{N_{\mathcal{F}}} \overline{\left(\gamma_{1}\right.} \overline{(S)}\right)$ |
| $\overline{\mathrm{B}}(2 \bar{k}+\overline{1} ; 0,0,0,0)$ |  | $\mathrm{SL}_{2}^{2}(\overline{3})$ |  |  |  |  |  |  | $\mathrm{PGL}_{3}\left(\overline{q_{1}}\right)$ |
| $\overline{\mathrm{B}}(2 \bar{k}+\overline{1} ; 0,0,0)$ |  |  |  |  |  |  |  | - $\overline{\mathrm{G}}_{\underline{\mathrm{L}}}^{2} \overline{2}(3)-$ | $\left.\hat{N}_{\mathcal{F}} \overline{\gamma_{1}} \overline{( } \bar{S}\right){ }^{\text {a }}$ |
| $\overline{\mathrm{B}}(2 \bar{k}+\overline{1} ; \overline{0}, \overline{0}, 0)$ | $\overline{2}^{2}$ |  |  |  |  | $\bar{S}_{\underline{L}}^{2}-\overline{2}(\overline{3})$ | $\bar{E}_{1}^{-} \sim_{\sim}^{-} \bar{E}_{-1}^{-1}$ |  | 3. $\bar{F}_{\text {DRV }}^{-}\left(\overline{3}^{2 k} ; 2\right) .2$ |
| - $\overline{\mathrm{B}}(2 \bar{k}+\overline{1} ; 0,0,0,0)$ | $2^{2}$ |  |  |  |  | $\bar{S}_{-} \bar{L}_{2} \overline{(3)}$ | $\bar{E}_{1}^{-} \sim_{\mathcal{F}} \bar{E}_{-1}^{-1}$ | $\overline{\mathrm{G}}_{\underline{2}}^{2}-\overline{3}\left(\overline{3}{ }^{-}\right.$ |  |
| - $\overline{\mathrm{B}}(2 \bar{k}+\overline{1} ; \overline{0}, \overline{0}, 0)$ | $\overline{2}^{2}$ |  |  |  | GLī ${ }^{-}(\overline{3})$ |  |  |  | 3. $\bar{F}_{\text {DRV }}^{-}\left(3^{2 k}-1\right) \cdot \overline{2}-$ |
| - $\overline{\mathrm{B}}(\underline{2} \bar{k}+\overline{1} ; 0,0,0)$ | $2^{2}$ |  |  |  | $\mathrm{GL}_{2}(\overline{3})$ |  |  | - $\overline{\mathrm{G}}_{\mathrm{L}} \overline{\mathrm{L}}(\overline{\text { a }}$ - | $-\mathcal{F}_{\mathrm{DR}}\left(3^{2 k+\mathrm{I}}, 2\right)-2-1$ |
| $\overline{\mathrm{B}}(\overline{2} \bar{k}+\overline{1} ; 0,0,0)$ | - |  |  |  | $\mathrm{CLL}_{2}(\overline{3})^{-}$ | $\overline{\mathrm{S}}_{2} \overline{-}(3)$ | $E_{1} \sim_{\mathcal{F}} E_{-1}$ |  |  |
| - $\overline{\mathrm{B}}(\underline{2} \overline{\mathrm{k}}+\overline{1} ; \overline{0}, \overline{0}, \overline{0})$ |  |  |  |  | $\mathrm{GL}_{2}(\overline{3})^{-}$ |  | $\bar{E}_{1}^{-} \sim_{\sim}^{-} \bar{E}^{-1}$ | ${ }^{-\overline{\mathrm{G}} \overline{\mathrm{L}}_{2}(\overline{3})}{ }^{-}$ | ${ }^{2} \overline{1}^{1} \bar{F}_{4}\left(\overline{q_{4}}\right)$ |
| - $\overline{\mathrm{B}}(\underline{2} \bar{k}+\overline{1} ; 0,0,0,0)$ |  | - $\left.\mathrm{GLL}_{2} \overline{( } \overline{3}\right)$ |  |  |  |  |  |  | $\mathrm{P}^{\mathrm{P}} \overline{\mathrm{G}} \overline{\mathrm{L}}_{3}\left(q_{1}\right) \cdot 2$ |
| - $\overline{\mathrm{B}}(2 \bar{k}+\overline{1} ; \overline{0} \overline{0} 0,0)$ | $\overline{2}^{2}$ | - $\mathrm{GLL}_{2}^{-}(3)$ |  |  |  |  |  | - ${\overline{\mathrm{G}} \overline{\mathrm{L}}_{2} \overline{2}(3)-}^{-}$ |  |
| - $\overline{\mathrm{B}}(\underline{2} \bar{k}+\overline{1} ; \overline{0}, \overline{0}, 0)$ | $\overline{2}^{2}$ | - $\mathrm{GL}_{2}{ }^{-}(3)$ |  |  |  | $\overline{S o}_{2} \bar{L}_{2}(3)$ | $\bar{E}_{1}^{-} \sim_{\sim}^{-} \bar{E}_{-1}^{-}$ |  |  |
| $\overline{\mathrm{B}}(\underline{2} \bar{k}+\overline{1} ; \overline{0}, \overline{0}, 0)$ | $\overline{2}^{2}$ | $\mathrm{GLL}_{2} \overline{3}(3)$ |  |  |  | $\bar{S}_{\underline{L}}^{2} \overline{(3)}$ | $\bar{E}_{1}^{-\sim_{\mathcal{F}}}{ }^{-} \bar{E}_{-1}^{-}$ | $\left.\overline{\mathrm{G}}_{\underline{2}} \overline{( } \overline{3}\right)$ | $\overline{\mathcal{F}_{\text {DRV }}} \overline{\left(3^{\overline{2} k+1}, \overline{4}\right)}$ |
| $\mathrm{B}(2 k+1 ; 1,0,0)$ | 2 | $\mathrm{SL}_{2}(3)$ | * | * |  | * | * |  | $\mathrm{PSL}_{3}\left(q_{1}^{3}\right) \cdot 3$ |

TABLE 4. The saturated fusion systems on maximal class 3-groups of rank 2 and order at least $3^{4}$.

In Table $4, q_{1}, q_{2}, q_{3}$ and $q_{4}$ are prime powers with $\nu_{3}\left(q_{1}-1\right)=\nu_{3}\left(q_{4}^{2}-1\right)=k$ and $\nu_{3}\left(q_{2}-1\right)=\nu_{3}\left(q_{3}^{2}-1\right)=k-1$ where $\nu_{3}(m)$ denotes that exponent of the highest power of 3 which divides $m$. Of course, $q_{4}$ is an odd power of 2 .

To understand the data presented in Table 4 , let $S=\mathrm{B}(n ; \beta, \gamma, \delta)$ with $n \geq 4$ and assume that $S$ has rank 2 . Let $\mathcal{F}$ be a saturated fusion system on $S$. It is easy to see that $Z(S)=\left\langle s_{n-1}\right\rangle, Z_{2}(S)=\left\langle s_{n-2}, s_{n-1}\right\rangle$ and $\gamma_{1}(S)=\left\langle s_{i} \mid 1 \leq i \leq n-1\right\rangle$. For $i \in\{0,1,-1\}$ define

$$
A_{i}=\left\langle x s_{1}^{i}, s_{n-1}\right\rangle
$$

and

$$
E_{i}=\left\langle A_{i}, s_{n-2}\right\rangle
$$

When the subgroups $A_{i}, E_{i}$ have exponent 3 , then up to $S$-conjugacy they are the candidates to be $\mathcal{F}$-pearls. If there is a $*$ in Table 4 , this indicates that for the given $S$, the corresponding subgroups $A_{i}$ and $E_{i}$ do not have exponent 3 (see [48, Table 1]). We do not make this indication for $\gamma_{1}(S)$ as it does not have exponent 3. The notation $A_{1} \sim_{\mathcal{F}} A_{-1}, E_{1} \sim_{\mathcal{F}} E_{-1}$ means that these subgroups are $\mathcal{F}$-conjugate.

We only show this in the case when $A_{1}$ or $E_{1}$ is an $\mathcal{F}$-pearl. Reading across a row of Table 4, first comes the Blackburn name of $S$, then $\left|\operatorname{Out}_{\mathcal{F}}(S)\right|$. As we move a long the row, an entry displays the structure of $\operatorname{Out}_{\mathcal{F}}\left(A_{i}\right)$, $_{\operatorname{Out}_{\mathcal{F}}}\left(E_{i}\right)$ or $^{\operatorname{Out}_{\mathcal{F}}}\left(\gamma_{1}(S)\right)$ in the case that the corresponding subgroup is $\mathcal{F}$-essential. A group or $N_{\mathcal{F}}\left(E_{0}\right)$ in the final column indicates that the fusion system is realizable, all the other tabulated fusion systems are exotic. Where the exotic fusion system is in [20, Theorem 5.10], this is indicated by a subscript DRV and we have adhered to their names.

The fusion systems $\mathcal{F}_{\mathrm{DRV}}\left(3^{2 k+1}, 5\right)$ are exotic and incorrectly labeled in $[\mathbf{2 0}$, Table 6] (hence the strange numbering of the system). The fusion systems on $S=\mathrm{B}(2 k+1 ; 1,0,0)$ included as the final row of Table 4 are claimed to be exotic in [48]. In fact they can be constructed by the methods in Example 15.2.

Theorem B.5. Suppose that $\mathcal{F}$ is a saturated fusion system on a non-abelian maximal class 3-group. Then $\mathcal{F}$ is known, $\mathcal{E}_{\mathcal{F}} \subseteq \mathcal{P}(\mathcal{F}) \cup\left\{\gamma_{1}(S)\right\}$ and, if $\gamma_{1}(S) \in \mathcal{E}_{\mathcal{F}}$, then $\gamma_{1}(S)$ is abelian.

Proof. Combine Theorems B. 1 and B. 4 and Lemma B. 3 .

## C. Computer code

In this appendix we provide the Magma code that we have used for the work. We have used the functionality developed in [50]. To use these subroutines, the intrinsics provide in [49] must be available. For this you will be required to attach the package file following the instructions provided here
https://magma.maths.usyd.edu.au/magma/handbook/text/24\#168.
C.1. The Magma code for Example 1.2. This code requires the fusion system package written by Parker and Semeraro $[\mathbf{4 9}, 50]$.

```
X:= ASL(2,5); S5:= Sylow(X,5);
BA:= Normalizer(X,S5);
Y:= PSp(4,5); S5:= Sylow(Y,5);
B:= Normalizer(Y,Centre(S5));
B1:= DerivedSubgroup(B);
BE:=Normalizer(B1,S5);
// BA is isomorphic to the normalizer of
//a Sylow 5-subgroup of 5^2:\SL_2(5) and BE
//is isomorphic to the normalizer of a
//Sylow 5-subgroup of 5^{1+2}:\SL_2(5)
AP:= [];
EP:=[];
SS:= SmallGroups(5^7,IsMaximalClass);
#SS;
SSA:= [] ;
for x in SS do
L:= LowerCentralSeries(x);
G1:=Centralizer(x,L[2],L[4]);
if not IsAbelian(G1) then Append(~ SSA,x);
end if;
end for;
#SSA;
for x in SSA do MakeAutos(x); end for;
SSAA:= [x : x in SSA |#Sylow(x‘autoperm,2) eq 4];
#SSAA;
for X in SSAA do
L:= LowerCentralSeries(X);
G1:=Centralizer(X,L[2],L[4]);
AP[Index(SSAA,X)] :=[];
```

```
EP[Index(SSAA,X)] := [];
    SAP:= Subgroups(X:OrderEqual:= 25);
    SAP:= {x'subgroup: x in SAP| not x'subgroup subset G1};
    SEP:= Subgroups(X:OrderEqual:= 125);
        SEP:= {x'subgroup: x in SEP|not IsAbelian(x'subgroup)
        and not x'subgroup subset G1 and Exponent(x'subgroup) eq 5};
H:= Sylow(X'autoperm,2);
H:= SubInvMap(X`autopermmap,X`autogrp,H);
AFS:= sub<X`autogrplInn(X),H>;
SAP:={x: x in SAP|IsOdd(#AutOrbit(X,x,AFS))};
SEP:={x: x in SEP|IsOdd(#AutOrbit(X,x,AFS))};
for x in SAP do
NSx:= Normalizer(X,x);
A, B, C:= AutOrbit(X,x,AFS);
MakeAutos(NSx);
W:= Sylow(SubMap(X`autopermmap,X`autoperm, B) ,2);
K:= SubInvMap(X`autopermmap,X`autogrp,W);
K:= sub<NSx`autogrp|K>;
B := Holomorph(NSx,K);
a:= IsIsomorphic(B,BA);
if a then Append(~AP[Index(SSAA,X)] ,x); end if;
end for;
for x in SEP do
NSx:= Normalizer(X,x);
A, B, C:= AutOrbit(X,x,AFS);
MakeAutos(NSx);
W:= Sylow(SubMap(X`autopermmap,X`autoperm,B) ,2);
K:= SubInvMap(X`autopermmap,X`autogrp,W);
K:= sub<NSx`autogrp|K>;
B := Holomorph(NSx,K);
a:= IsIsomorphic(B,BE);
if a then Append(~EP[Index(SSAA,X)] ,x); end if;
end for;
end for;
IA:={i:i in [1..#SSAA]|#AP[i] ne 0};
IE:={i:i in [1..#SSAA]|#EP[i] ne 0};
print "there are", #{x:x in AP|#x ne 0},
"maximal class groups of order 5^7
with abelian pearls.";
print "there are", #{x:x in EP|#x ne 0},
"maximal class groups of order 5^7
with extraspecial pearls.";
JJ:={1297, 1308,1321, 1360, 1363,1374,1384};
```

```
for i in IA do
for j in JJ do
if IsIsomorphic(SmallGroup(5^7,j),SSAA[i]) then
i,j;end if;
end for;
end for;
for i in IE do
for j in JJ do
if IsIsomorphic(SmallGroup(5^7,j),SSAA[i]) then
i,j; end if;
end for;
end for;
JJ:={1297,1308,1321,1360,1363,1374,1384};
A:=[];
for j in JJ do
A[j]:= AllFusionSystems(SmallGroup(5^7,j):
    OpTriv:=false, pPerfect:=false);
end for;
for j in JJ do
if IsDefined(A,j) then A[j]; end if;
end for;
for j in JJ do
S:= SmallGroup(5^7,j);
L:= LowerCentralSeries(S);
G1:=Centralizer(S,L[2],L[4]);
j, NilpotencyClass(G1);
end for;
```

C.2. The Magma code required to verify Lemma 4.10. This code verifies that when $p \in\{5,7\}$, the exterior square of the module $S^{p-1,1}$ for $\operatorname{GF}(p) \operatorname{Sym}(p)$ has two composition factors and no quotient factor which is 1-dimensional or isomorphic to $D^{p-1,1}$.

```
for p in {5,7} do
G:= Sym(p);
V:= PermutationModule(G,GF(p));
W:= sub<V|V.1-V.1*G.1>;
DPminus11:= W/Socle(W);
U:= ExteriorSquare(W);
C:=CompositionFactors(U);
C;
IsIrreducible(U/Socle(U));
IsIsomorphic(U/Socle(U),DPminus11);
IsIsomorphic(Socle(U),DPminus11);
end for;
```

C.3. The Magma code for Lemma 7.5. This code requires the fusion system package written by Parker and Semeraro [49, 50].

```
SS:= SmallGroups(5^6,IsMaximalClass);
#SS;
TT:=[];
for S in SS do
L:= LowerCentralSeries(S);
gamma1:= Centralizer(S,L[2],L[4]);
CSZ2:= Centralizer(S,L[4]);
if gamma1 ne CSZ2 then Append(~TT,S);
end if;
end for;
#TT;
TT2:= [];
for S in TT do
L:= LowerCentralSeries(S);
gamma1:= Centralizer(S,L[2],L[4]);
CSZ2:= Centralizer(S,L[4]);
MakeAutos(CSZ2);
if not IsExtraSpecial(gamma1) and
    not IsSoluble(CSZ2'autoperm) then
Append(~TT2,S);
end if;
end for;
#TT2;
TT3:=[];
for S in TT2 do MakeAutos(S);
if not IsNilpotent(S'autoperm) then Append(*TT3,S);
end if;
end for;
#TT3;
S:= TT3[1];
A:=AllFusionSystems(S:OpTriv:= false,pPerfect:= false);
A;
```

C.4. The group providing an example for Lemma 8.5. This code constructs a group $G$ of shape $7^{3+3}: \mathrm{PGL}_{2}(7)$ which has Sylow 7 -subgroup $S$ of maximal class and $\gamma_{1}(S)$ special with centre of order $7^{3}$. It further shows that $S$ cannot support a saturated fusion system $\mathcal{G}$ containing $\mathcal{G}$-pearls.

```
G:= FreeGroup(5);
R<r1,r2,r3,s,t>:= FreeGroup(5);
R:= quo<R|r1^7,r2^7,r3^7,((r1,r2),r1),((r1,r2),r2),
((r1,r2),r3),((r1,r3),r1),((r1,r3),r2),((r1,r3),r3),
((r2,r3),r1),((r2,r3),r2),((r2,r3),r3),
s^3, t^2, (s,t)^4,(t*s)^8,
```

```
r1^s = r1^5*r2^2*(r1,r2)^2*(r1,r3)*(r2,r3)^3,
r2^s =r1*r2*r3*(r1,r2)^5*(r1,r3)^4*(r2,r3)^2,
r3^s = r1^4*r2^2*r3*(r1,r3)^3*(r2,r3)^6,
r1^t = r2*r3*(r1,r2)^3*(r1,r3)^2*(r2,r3),
r2^t =r1^ 5*r2^3*r3^2*(r1,r2)^4*(r1,r3)^3*(r2,r3)^5,
r3^t =r1^3*r2^4*r3^5*(r1,r2)*(r1,r3)^5*(r2,r3)^4>;
G:= CosetImage(R,sub<R|s,t>);
ChiefFactors(G);
S:=Sylow(G,7);
IsMaximalClass(S);
//We now check if G can be decorated with pearls
p:= FactoredOrder(S) [1] [1];
L:= LowerCentralSeries(S);
gamma1:= Centralizer(S,L[2],L[4]);
Z2:= L[#L-2];
R:= Centralizer(S,Z2);
M:= MaximalSubgroups(S);
PotAbelianPearls:=[];
PotExtraspecialPearls:=[];
for x in M do y:= x'subgroup;
if y ne gamma1 and y ne R then
if exists(xx){z:z in yl
not z in gamma1 and Order(z) eq p and not z in R}
then
            Append(~PotExtraspecialPearls,sub<S|Z2,xx>);
            Append(~PotAbelianPearls,sub<S|L[#L-1],xx>);
end if;
end if;
end for;
J1:= Centralizer(PSp(4,p),Centre(Sylow(PSp(4,p),7)));
BE:= Normalizer(J1,Sylow(J1,p));
BA:= Normalizer(ASL(2,p),Sylow(ASL(2,p),p));
for x in PotExtraspecialPearls do
    if IsIsomorphic(Normalizer(G,x),BE) then
        "S can be decorated with extraspecial pearls";
        else "this potential extraspecial pearl,
                        is not a pearl."; end if;
end for;
for x in PotAbelianPearls do
    if IsIsomorphic(Normalizer(G,x),BA) then
        "S can be decorated with abelian pearls";
        else "this potential abelian pearl,
```

    is not a pearl."; end if;
    end for;
    C.5. The Magma code for Lemma B.3. This code requires the fusion system package written by Parker and Semeraro $[49,50]$.
S:= Sylow (Sym (9), 3) ;
$A:=A l l F u s i o n S y s t e m s(S: O p T r i v:=$ false,pPerfect:= false);
\#A;

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