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Reducing the projection onto the monotone extended second-order cone to the pool-adjacent-violators algorithm of isotonic regression

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ABSTRACT
This paper introduces the monotone extended second-order cone (MESOC), which is related to the monotone cone and the second-order cone. Some properties of the MESOC are presented and its dual cone is computed. Projecting onto the MESOC is reduced to the pool-adjacent-violators algorithm (PAVA) of isotonic regression. An application of MESOC to portfolio optimisation is provided. Some broad descriptions of possible MESOC-regression models are also outlined.

KEYWORDS
Extended second-order cone; isotonic regression; dual cone; metric projection.

1. Introduction

The purpose of this paper is to introduce a new second-order cone, which we call the monotone extended second-order cone. Some properties of the MESOC are studied and formulas for projecting onto it are presented. We will follow the ideas used in [1] for projecting onto a non-monotone extension of the second-order cone. It is worth to note that the projection in this paper is considerably more difficult to find, because it is partly based on projecting onto the monotone nonnegative cone, which is a nontrivial problem compared to the projection onto the nonnegative orthant, see [2,3]. The definition of the MESOC relates two well-known cones, namely, the monotone cone and a second-order cone, known as Lorentz cone. The monotone cone has connections with the isotonic regression problem, in fact it is the constraint set of this problem, see for example [4]. This cone arises in statistics and has also connections with finance [5]. In [6] some properties of the weighted version of the monotone cone have been also considered. The Lorentz cone is an important object in theoretical physics, and it is commonly used in optimization, a good survey paper with a wide range of applications of second-order cone programming is [7]. Various connections of second-order cone programming and second-order cone complementarity problem with physics, mechanics, economics, game theory, robotics, optimization and neural networks have been considered in [8–16].

The structure of the paper is as follows: In Section 2 we fix the notations and the terminology used throughout the paper. In Section 3 we introduce the MESOC and

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compute its dual cone, and in Section 4 we find the complementarity set of the MESOC. The formulas for projecting onto the pair of mutually dual monotone extended second-order cones are derived in Section 5. In Section 6 we have presented an application of the MESOC to portfolio optimisation via a conic optimization problem related to the mean-absolute deviation model [17]. Finally, we make some remarks in the last section, including some broad descriptions about how could the projection onto the MESOC occur directly in modelling some practical problems.

2. Preliminaries

Here, we recall some notations, definitions, and basic properties of convex cones and projections onto it. Let ℓ, m, p, q be positive integers such that $m = p + q$. We identify the vectors of $\mathbb{R}^\ell$ with $\ell \times 1$ matrices with real entries. The scalar product in $\mathbb{R}^\ell$ and the corresponding norm are defined, respectively, by $\mathbb{R}^\ell \times \mathbb{R}^\ell \ni (x, y) \mapsto \langle x, y\rangle := x^\top y \in \mathbb{R}$ and $\mathbb{R}^\ell \ni x \mapsto \|x\| := \sqrt{\langle x, x\rangle} \in \mathbb{R}$. The equality $\langle x, y\rangle = 0$ is denoted by $x \perp y$. We identify the elements of $\mathbb{R}^p \times \mathbb{R}^q$ with the elements of $\mathbb{R}^m$ through the correspondence $\mathbb{R}^p \times \mathbb{R}^q \ni (x, y) \mapsto (x^\top, y^\top)^\top$. Through this identification the scalar product in $\mathbb{R}^p \times \mathbb{R}^q$ is defined by $\langle (x, y), (u, v)\rangle := \langle x, u\rangle + \langle y, v\rangle$. A closed set $\mathcal{K} \subseteq \mathbb{R}^\ell$ with nonempty interior is called a *proper cone* if $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$, $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$ and $\lambda\mathcal{K} \subseteq \mathcal{K}$, for any λ positive real number. The *dual cone* of a proper cone $K \subseteq \mathbb{R}^\ell$ is a proper cone defined by $\mathcal{K}^* := \{x \in \mathbb{R}^\ell \ : \ \langle x, y\rangle \geq 0, \ \forall y \in \mathcal{K}\}$. For a proper cone $\mathcal{K} \in \mathbb{R}^\ell$, the *complementarity set* of $\mathcal{K}$ is defined by $C(\mathcal{K}) := \{(x, y) \in \mathcal{K} \times \mathcal{K}^* : \ x \perp y\}$. Let $C \in \mathbb{R}^\ell$ be a closed convex set. The projection mapping $P_C \colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ onto C is defined by $P_C(x) := \operatorname{argmin}\{\|x - y\| : y \in C\}$, which is piecewise linear whenever C is a polyhedral cone; see [18, Definition 4.1.3 and Proposition 4.1.4]. We recall here Moreau's decomposition theorem [19] (stated here for proper cones only):

Theorem 2.1. *Let $\mathcal{K} \subseteq \mathbb{R}^\ell$ be a proper cone, $\mathcal{K}^*$ its dual cone and $z \in \mathbb{R}^\ell$. Then, the following two statements are equivalent:*

(i) $z = x - y$ *and* $(x, y) \in C(\mathcal{K})$,
(ii) $x = P_{\mathcal{K}}(z)$ *and* $y = P_{\mathcal{K}^*}(-z)$.

In particular, Theorem 2.1 implies that

$$P_{\mathcal{K}}(z) \perp P_{\mathcal{K}^*}(-z), \qquad z = P_{\mathcal{K}}(z) - P_{\mathcal{K}^*}(-z).$$

For $z \in \mathbb{R}^\ell$ we denote $z = (z_1, \dots, z_\ell)^\top$. Denote by $\mathbb{R}^\ell_+ = \{x \in \mathbb{R}^\ell \ : \ x \geq 0\}$ the nonnegative orthant. The proper cone $\mathbb{R}^\ell_+$ is self-dual, i.e., $\mathbb{R}^\ell_+ = (\mathbb{R}^\ell_+)^*$. For a real number $\alpha \in \mathbb{R}$ denote $\alpha^+ := \max(\alpha, 0)$ and $\alpha^- := \max(-\alpha, 0)$. For a vector $z \in \mathbb{R}^\ell$ denote $z^+ := (z_1^+, \dots, z_\ell^+)^\top$, $z^- := (z_1^-, \dots, z_\ell^-)^\top$ and $|z| := (|z_1|, \dots, |z_\ell|)^\top$. Therefore, $z^+ = P_{\mathbb{R}^\ell_+}(z)$, $z^- = P_{\mathbb{R}^\ell_+}(-z)$, $z = z^+ - z^-$ and $|z| = z^+ + z^-$. In particular, we denote $P_{\mathcal{K}}(z)^+ = x^+$ and $P_{\mathcal{K}}(z)^- = x^-$, where $\mathcal{K} \subseteq \mathbb{R}^\ell$ is a proper cone and $x = P_{\mathcal{K}}(z)$. Thus, $P_{\mathcal{K}}(z) = P_{\mathcal{K}}(z)^+ - P_{\mathcal{K}}(z)^-$. Without leading to any confusion, depending on the context, we will denote by 0 the vector in $\mathbb{R}^\ell$ or a scalar zero and by $e^i \in \mathbb{R}^p$ the i-th canonical unit vector, i.e., the vector with all coordinates 0 except the i-th coordinate which is 1. The *monotone cone* $\mathbb{R}^p_\geq$ is defined as follows:

$$\mathbb{R}^p_\geq := \{x \in \mathbb{R}^p : \ x_1 \geq x_2 \geq \cdots \geq x_p\}. \tag{1}$$

Let $j \in \{1, ..., p-1\}$. To simplify the notations we define

$$e^{1:j} := e^1 + \cdots + e^j = (\underbrace{1, \ldots, 1}_{j \text{ times}}, \underbrace{0, \ldots, 0}_{p-j \text{ times}}) \in \mathbb{R}^p, \qquad e := e^1 + \cdots + e^p = (\underbrace{1, \ldots, 1}_{p \text{ times}}) \in \mathbb{R}^p.$$

The *dual* of the cone $\mathbb{R}^p_{\geq}$ is given by

$$(\mathbb{R}^p_{\geq})^* := \left\{ y \in \mathbb{R}^p : \ \left\langle y, e^{1:j} \right\rangle \geq 0, \ j = 1, \ldots, p-1, \ \langle y, e \rangle = 0 \right\}. \tag{2}$$

The *monotone nonnegative cone*, is defined by

$$\mathbb{R}^p_{\geq +} := \{ x \in \mathbb{R}^p : \ x_1 \geq x_2 \geq \cdots \geq x_p \geq 0 \}. \tag{3}$$

The *dual* of the cone $\mathbb{R}^p_{\geq +}$ is given by

$$(\mathbb{R}^p_{\geq +})^* := \left\{ y \in \mathbb{R}^p : \ \left\langle y, e^{1:j} \right\rangle \geq 0, \ j = 1, \ldots, p-1, \ \langle y, e \rangle \geq 0 \right\}. \tag{4}$$

3. The monotone extended second-order cone

In this section we introduce the monotone extended second-order cone, which generalize the well known Lorentz cone. We also compute the dual cone of the monotone extended second-order cone. The *monotone extended second-order cone* $\mathcal{L}_{p,q} \subseteq \mathbb{R}^m := \mathbb{R}^{p+q}$ is defined as follows:

$$\mathcal{L}_{p,q} := \left\{ (x, u) \in \mathbb{R}^p \times \mathbb{R}^q : \ x_1 \geq x_2 \geq \cdots \geq x_p \geq \|u\| \right\}. \tag{5}$$

Remark 1. If $p, q \geq 1$, then the cone $\mathcal{L}_{p,q}$ is a proper cone. Letting $p = 1$ in (5), the cone $\mathcal{L}_{p,q}$ becomes $\mathcal{L}_{1,p} = \{(t, u) \in \mathbb{R} \times \mathbb{R}^q : t \geq \|u\|\}$, which is the second-order cone in $\mathbb{R}^{1+q} \equiv \mathbb{R} \times \mathbb{R}^q$ known as Lorentz cone. The cone $\mathcal{L}_{p,q}$ is polyhedral, if and only if $q = 0$ or $q = 1$. If $q = 0$, then the cone $\mathcal{L}_{p,q}$ becomes the monotone nonnegative cone $\mathbb{R}^p_{\geq +}$ defined in (3).

Before proceeding with our presentation, let us state *Abel's partial summation formula* that will be useful to study the properties of the MESOC:

$$\langle x, y \rangle = \sum_{i=1}^{p-1} (x_i - x_{i+1}) \sum_{j=1}^{i} y_j + x_p \sum_{i=1}^{p} y_i, \qquad \forall x, y \in \mathbb{R}^p. \tag{6}$$

Interesting applications of this formula can be found in [20,21]. Next we present the dual cone of the MESOC.

Proposition 3.1. *The dual cone $\mathcal{L}^*_{p,q}$ of the monotone extended second-order cone $\mathcal{L}_{p,q}$ is*

$$\mathcal{L}^*_{p,q} := \left\{ (y, v) \in \mathbb{R}^p \times \mathbb{R}^q : \ \left\langle y, e^{1:j} \right\rangle \geq 0, \ j = 1, \ldots, p-1, \ \langle y, e \rangle \geq \|v\| \right\}. \tag{7}$$

Proof. To simplify the notations, denote by M the right hand side of (7). Our task is to prove that $M = \mathcal{L}^*_{p,q}$, this will be done by proving that $M \subseteq \mathcal{L}^*_{p,q}$ and $\mathcal{L}^*_{p,q} \subseteq M$.

We proceed to prove the first inclusion, for that take $(y,v) \in M$. The definition of M implies

$$\left\langle y, e^{1:i} \right\rangle = \sum_{j=1}^{i} y_j \geq 0, \quad i = 1, \ldots, p-1, \qquad \langle y, e \rangle = \sum_{i=1}^{p} y_i \geq \|v\|. \tag{8}$$

Let $(x,u) \in \mathcal{L}_{p,q}$ be arbitrary. The definition of $\mathcal{L}_{p,q}$ implies $x_1 - x_2 \geq 0, \ldots, x_{p-1} - x_p \geq 0$, and $x_p \geq \|u\|$, which together with (6) and (8) yield

$$\langle x, y \rangle = \sum_{i=1}^{p-1} (x_i - x_{i+1}) \sum_{j=1}^{i} y_j + x_p \sum_{i=1}^{p} y_i \geq \|u\| \|v\|.$$

Therefore, the last inequality and Cauchy's inequality imply

$$\langle (x,u), (y,v) \rangle = \langle x, y \rangle + \langle u, v \rangle \geq \|u\| \|v\| + \langle u, v \rangle \geq 0,$$

which proves the inclusion $M \subseteq \mathcal{L}^*_{p,q}$. To prove the second inclusion, take $(y,v) \in \mathcal{L}^*_{p,q}$. First note that $\left(e^{1:j}, 0\right) \in \mathcal{L}_{p,q}$. Thus, since $(y,v) \in \mathcal{L}^*_{p,q}$, we have $\left\langle \left(e^{1:j}, 0\right), (y,v) \right\rangle \geq 0$, for all $j = 1, 2, \ldots, p-1$, which implies

$$\left\langle y, e^{1:j} \right\rangle \geq 0, \qquad \forall\ j = 1, 2, \ldots, p-1. \tag{9}$$

To proceed, first assume $v = 0$. Since $(e, 0) \in \mathcal{L}_{p,q}$ and $(y, 0) \in \mathcal{L}^*_{p,q}$, we have

$$\langle y, e \rangle \geq 0 = \|v\|. \tag{10}$$

Now, assume $v \neq 0$. Since $(\|v\|e, -v) \in \mathcal{L}_{p,q}$ and $(y,v) \in \mathcal{L}^*_{p,q}$, we obtain that $\langle (\|v\|e, -v), (y,v) \rangle \geq 0$, which implies $\|v\| \langle y, e \rangle - \|v\|^2 \geq 0$. Thus, due to $v \neq 0$, we have $\langle y, e \rangle - \|v\| \geq 0$. Therefore, the last inequality together with (10) imply that

$$\langle y, e \rangle \geq \|v\|, \tag{11}$$

for all $(y,v) \in \mathcal{L}^*_{p,q}$. Hence, it follows from (9) and (11) that $(y,v) \in M$. Therefore, we conclude that $\mathcal{L}^*_{p,q} \subseteq M$. Since $M \subseteq \mathcal{L}^*_{p,q}$ and $\mathcal{L}^*_{p,q} \subseteq M$, we have $\mathcal{L}^*_{p,q} = M$. □

Remark 2. Letting $p = 1$ in (7), there are no inequalities, for $j = 1, \ldots, p-1$, because $p - 1 = 0$. Thus, the cone $\mathcal{L}^*_{p,q}$ becomes the Lorentz cone $\mathcal{L}_{1,p}$ (see also Remark 1).

4. The complementarity set

After finding the dual of the monotone extended second-order cone, we want to find the complementarity set of this cone. In order to find the complementarity set, we need two inequalities introduced in the next lemma.

Lemma 4.1. *Let $(x,u) \in \mathcal{L}_{p,q}$ and $(y,v) \in \mathcal{L}^*_{p,q}$. Then,*

$$\langle x, y \rangle \geq \|u\| \langle y, e \rangle \geq \|u\| \|v\|. \tag{12}$$

Proof. Since $(x,u) \in \mathcal{L}_{p,q}$, we have $x_1 \geq x_2 \geq \cdots \geq x_p \geq \|u\|$. Thus, letting $0 \in \mathbb{R}^q$, we have $(x - \|u\|e, 0) \in \mathcal{L}_{p,q}$. Considering that $(y,v) \in \mathcal{L}^*_{p,q}$, the definition of $\mathcal{L}^*_{p,q}$ yields

$$0 \leq \langle (x - \|u\|e, 0), (y,v) \rangle = \langle x, y \rangle - \|u\| \langle y, e \rangle.$$

which implies the first inequality in (12). Since $(y,v) \in \mathcal{L}^*_{p,q}$, we have $\langle y, e \rangle \geq \|v\|$, from where the second inequality in (12) follows. □

In the next proposition we presents some relationships of the monotone extended second-order cone with the monotone nonnegative cone. Since its proof is an immediate consequence of (5), (7), (3) and (4), it will be omitted.

Proposition 4.2. *Let $(x,u), (y,v) \in \mathbb{R}^p \times \mathbb{R}^q$. Then, there hold:*

(i) $(x,u) \in \mathcal{L}_{p,q}$ *if and only if* $x - \|u\|e \in \mathbb{R}^p_{\geq +}$.
(ii) $(y,v) \in \mathcal{L}^*_{p,q}$ *if and only if* $y - \|v\|e^p \in (\mathbb{R}^p_{\geq +})^*$.

By using Lemma 4.1 and Proposition 4.2, next we determine the complementarity set of $\mathcal{L}_{p,q}$.

Proposition 4.3. *Let $x, y \in \mathbb{R}^p$ and $u, v \in \mathbb{R}^q \setminus \{0\}$. Then $(x,u,y,v) := ((x,u),(y,v)) \in C(\mathcal{L}_{p,q})$ if and only if $x_p = \|u\|$, $\langle y, e \rangle = \|v\|$, $\langle u, v \rangle = -\|u\|\|v\|$, and $(x - \|u\|e, y - \|v\|e^p) \in C(\mathbb{R}^p_{\geq +})$.*

Proof. Take $(x,u,y,v) \in C(\mathcal{L}_{p,q})$. The definition of $C(\mathcal{L}_{p,q})$ implies $(x,u) \in \mathcal{L}_{p,q}$, $(y,v) \in \mathcal{L}^*_{p,q}$ and $\langle (x,u),(y,v) \rangle = 0$. Since $(x,u) \in \mathcal{L}_{p,q}$ and $(y,v) \in \mathcal{L}^*_{p,q}$, Proposition 4.2 implies that $x - \|u\|e \in \mathbb{R}^p_{\geq +}$ and $y - \|v\|e^p \in (\mathbb{R}^p_{\geq +})^*$. Furthermore, the condition $\langle (x,u),(y,v) \rangle = 0$, Lemma 4.1 and the Cauchy inequality imply that

$$0 = \langle x, y \rangle + \langle u, v \rangle \geq \|u\| \langle y, e \rangle + \langle u, v \rangle \geq \|u\|\|v\| + \langle u, v \rangle \geq 0.$$

Thus, $\langle x, y \rangle = \|u\| \langle y, e \rangle$, $\|u\| \langle y, e \rangle = \|u\|\|v\|$ and $\langle u, v \rangle = -\|u\|\|v\|$. Moreover, taking into account that $u \neq 0$, we also have $\langle y, e \rangle = \|v\|$. Hence, using (6), we conclude that

$$(\|u\| - x_p)\,\|v\| = (\|u\| - x_p)\,\langle y, e \rangle = \sum_{i=1}^{p-1} (x_i - x_{i+1}) \sum_{j=1}^{i} y_j.$$

Since $(x,u) \in \mathcal{L}_{p,q}$ and $(y,v) \in \mathcal{L}^*_{p,q}$, the left hand side and the right hand side of the last equality have opposite signs. Hence, they must be 0. In particular $(\|u\| - x_p)\,\|v\| = 0$. Thus, due to $v \neq 0$, we conclude that $x_p = \|u\|$. On the other hand,

$$\langle x - \|u\|e, y - \|v\|e^p \rangle = \langle x, y \rangle - \|u\| \langle y, e \rangle - x_p \|v\| + \|u\|\|v\|,$$

which taking into account that $\langle x, y \rangle = \|u\| \langle y, e \rangle$ and $x_p = \|u\|$, yields $\langle x - \|u\|e, y - \|v\|e^p \rangle = 0$. Hence, $(x - \|u\|e, y - \|v\|e^p) \in C(\mathbb{R}^p_{\geq +})$, which concludes the proof of necessity.

Reciprocally, assume that $x_p = \|u\|$, $\langle y, e \rangle = \|v\|$, $\langle u, v \rangle = -\|u\|\|v\|$ and $(x - \|u\|e, y - \|v\|e^p) \in C(\mathbb{R}^p_{\geq +})$. First note that $x - \|u\|e \in \mathbb{R}^p_{\geq +}$, $y - \|v\|e^p \in (\mathbb{R}^p_{\geq +})^*$ and $\langle x - \|u\|e, y - \|v\|e^p \rangle = 0$. Since $x - \|u\|e \in \mathbb{R}^p_{\geq +}$ and $y - \|v\|e^p \in (\mathbb{R}^p_{\geq +})^*$, Proposition 4.2 implies$(x,u) \in \mathcal{L}_{p,q}$ and $(y,v) \in \mathcal{L}^*_{p,q}$. On the other hand, the equality

$\langle x - \|u\|e, y - \|v\|e^p \rangle = 0$ implies that

$$\langle x, y \rangle - \|u\| \langle y, e \rangle - x_p \|v\| + \|u\| \|v\| = 0.$$

Thus, due to $x_p = \|u\|$, we conclude that $\langle x, y \rangle = \|u\| \langle y, e \rangle$. Hence, also using $\langle u, v \rangle = -\|u\| \|v\|$ and $\langle y, e \rangle = \|v\|$, we obtain

$$\langle (x, u), (y, v) \rangle = \langle x, y \rangle + \langle u, v \rangle = \|u\| \langle y, e \rangle - \|u\| \|v\| = \|u\| \left(\langle y, e \rangle - \|v\| \right) = 0.$$

Therefore, $(x, u, y, v) \in C(\mathcal{L}_{p,q})$. □

5. Projection onto monotone extended second-order cone

The aim of this section is to present the formulas for projecting onto the pair of mutually dual monotone extended second-order cone. For that we need a preliminary result.

Lemma 5.1. *Let $(z, w) \in \mathbb{R}^p \times \mathbb{R}^q$. If $P_{\mathcal{L}_{p,q}}(z, w) = (x, u)$ and $P_{\mathcal{L}^*_{p,q}}(-z, -w) = (y, v)$, then the following statements hold:*

(i) $\langle P_{(\mathbb{R}^p_{\geq +})^*}(-z), e \rangle \geq \|w\|$ *if and only if* $u = 0$*;*
(ii) $P_{\mathbb{R}^p_{\geq +}}(z)_p \geq \|w\|$ *if and only if* $v = 0$.
(iii) $\langle P_{(\mathbb{R}^p_{\geq +})^*}(-z), e \rangle < \|w\|$ *and* $P_{\mathbb{R}^p_{\geq +}}(z)_p < \|w\|$ *if and only if* $u \neq 0$ *and* $v \neq 0$.

Proof. To prove item (i), we first assume that $u = 0$. Considering that $P_{\mathcal{L}_{p,q}}(z, w) = (x, 0)$ and $P_{\mathcal{L}^*_{p,q}}(-z, -w) = (y, v)$, Theorem 2.1 for $\mathcal{L}_{p,q}$ implies that $(x, 0) \in \mathcal{L}_{p,q}$, $(y, v) \in \mathcal{L}^*_{p,q}$, $\langle (x, 0), (y, v) \rangle = 0$ and $(z, w) = (x, 0) - (y, v)$. Hence, we have $x \in \mathbb{R}^p_{\geq +}$, $y \in (\mathbb{R}^p_{\geq +})^*$, $\langle y, e \rangle \geq \|v\|$, $\langle x, y \rangle = 0$, $z = x - y$ and $w = -v$. Hence, by applying Theorem 2.1 for $\mathbb{R}^p_{\geq +}$, we obtain that $x = P_{\mathbb{R}^p_{\geq +}}(z)$ and $y = P_{(\mathbb{R}^p_{\geq +})^*}(-z)$. Since $w = -v$ and $\langle y, e \rangle \geq \|v\|$, we have that $\langle P_{(\mathbb{R}^p_{\geq +})^*}(-z), e \rangle \geq \|w\|$. Conversely, suppose that $\langle P_{(\mathbb{R}^p_{\geq +})^*}(-z), e \rangle \geq \|w\|$. First note that $(P_{\mathbb{R}^p_{\geq +}}(z), 0) \in \mathcal{L}_{p,q}$ and, using $\langle P_{(\mathbb{R}^p_{\geq +})^*}(-z), e \rangle \geq \|w\|$, we have $(P_{(\mathbb{R}^p_{\geq +})^*}(-z), -w) \in \mathcal{L}^*_{p,q}$. Moreover, we conclude that $(P_{\mathbb{R}^p_{\geq +}}(z), 0, P_{(\mathbb{R}^p_{\geq +})^*}(-z), -w) \in C(\mathcal{L}_{p,q})$ and $(z, w) = (P_{\mathbb{R}^p_{\geq +}}(z), 0) - (P_{(\mathbb{R}^p_{\geq +})^*}(-z), -w)$. Hence, by applying Theorem 2.1 for $\mathcal{L}_{p,q}$, we have $P_{\mathcal{L}_{p,q}}(z, w) = (P_{\mathbb{R}^p_{\geq +}}(z), 0)$ and $P_{\mathcal{L}^*_{p,q}}(-z, -w) = (P_{(\mathbb{R}^p_{\geq +})^*}(-z), -w)$. Therefore, $u = 0$.

To prove item (ii), we first assume that $v = 0$. Considering that $P_{\mathcal{L}_{p,q}}(z, w) = (x, u)$ and $P_{\mathcal{L}^*_{p,q}}(-z, -w) = (y, 0)$, Theorem 2.1 for $\mathcal{L}_{p,q}$ implies that $(x, u) \in \mathcal{L}_{p,q}$, $(y, 0) \in \mathcal{L}^*_{p,q}$, $\langle (x, u), (y, 0) \rangle = 0$ and $(z, w) = (x, u) - (y, 0)$. Hence, we have $x \in \mathbb{R}^p_{\geq +}$, $y \in (\mathbb{R}^p_{\geq +})^*$, $x_p \geq \|u\|$, $\langle x, y \rangle = 0$, $z = x - y$ and $w = u$. Thus, by applying Theorem 2.1 for $\mathbb{R}^p_{\geq +}$, we obtain that $x = P_{\mathbb{R}^p_{\geq +}}(z)$ and $y = P_{(\mathbb{R}^p_{\geq +})^*}(-z)$. Since $w = u$ and $x_p \geq \|u\|$, we have that $P_{\mathbb{R}^p_{\geq +}}(z)_p \geq \|w\|$. Conversely, assume that $P_{\mathbb{R}^p_{\geq +}}(z)_p \geq \|w\|$. Note that $(P_{(\mathbb{R}^p_{\geq +})^*}(-z), 0) \in \mathcal{L}^*_{p,q}$ and, using $P_{\mathbb{R}^p_{\geq +}}(z)_p \geq \|w\|$, we have $(P_{\mathbb{R}^p_{\geq +}}(z), w) \in \mathcal{L}_{p,q}$. Moreover, $(P_{\mathbb{R}^p_{\geq +}}(z), w, P_{(\mathbb{R}^p_{\geq +})^*}(-z), 0) \in C(\mathcal{L}_{p,q})$ and $(z, w) = (P_{\mathbb{R}^p_{\geq +}}(z), w) - (P_{(\mathbb{R}^p_{\geq +})^*}(-z), 0)$. Hence, by applying Theorem 2.1 for the cone $\mathcal{L}_{p,q}$, we have $P_{\mathcal{L}_{p,q}}(z, w) = (P_{\mathbb{R}^p_{\geq +}}(z), w)$ and $P_{\mathcal{L}^*_{p,q}}(-z, -w) = (P_{(\mathbb{R}^p_{\geq +})^*}(-z), 0)$. Therefore, $v = 0$.

Item (iii) is an immediate consequence of items *(i)* and *(ii)*. □

The next lemma is essential for reducing the projection onto the MESOC to isotonic regression.

Lemma 5.2. *Let $(z,w) \in \mathbb{R}^p \times \mathbb{R}^q$ such that $w \neq 0$. Assume that $P_{\mathcal{L}_{p,q}}(z,w) = (x, \beta w)$ for some $x \in \mathbb{R}^p$ and $\beta > 0$. Then,*

$$P_{\mathbb{R}^{p+1}_{\geq +}}(z, \|w\|) = (x, \beta \|w\|). \tag{13}$$

Proof. Suppose by contradiction that (13) does not hold. Hence, $P_{\mathbb{R}^{p+1}_{\geq +}}(z, \|w\|) = (y, \mu\|w\|)$ for some $y \in \mathbb{R}^p$ and $\mu \geq 0$ with $(y, \mu\|w\|) \neq (x, \beta\|w\|)$. Let $u := \beta w$ and $v := \mu w$. Then, we have $P_{\mathbb{R}^{p+1}_{\geq +}}(z, \|w\|) = (y, \|v\|)$ and consequently $(y, v) \in \mathcal{L}_{p,q}$. Hence, due to

$$\mathbb{R}^{p+1}_{\geq +} \ni (x, \|u\|) \neq (y, \|v\|) = P_{\mathbb{R}^{p+1}_{\geq +}}(z, \|w\|),$$

we obtain that

$$\|z - x\|^2 + (\|w\| - \|u\|)^2 > \|z - y\|^2 + (\|w\| - \|v\|)^2.$$

Because w, u and u are collinear vectors with the same orientation, the last inequality implies $\|z - x\|^2 + \|w - u\|^2 > \|z - y\|^2 + \|w - v\|^2$, or equivalently

$$\|(z,w) - (x,u)\|^2 > \|(z,w) - (y,v)\|^2,$$

which contradicts $P_{\mathcal{L}_{p,q}}(z,w) = (x,u)$, as $(y,v) \in \mathcal{L}_{p,q}$. □

In order to simplify the notations of our main result, for a fixed $z \in \mathbb{R}^p$ and $w \in \mathbb{R}^q$, we define

$$f(\lambda) := z - \frac{1}{1+\lambda}\|w\|e + \frac{\lambda}{1+\lambda}\|w\|e^p. \tag{14}$$

Theorem 5.3. *Let $(z,w) \in \mathbb{R}^p \times \mathbb{R}^q$, then the following statements hold:*

(1) *If $\langle P_{(\mathbb{R}^p_{\geq +})^*}(-z), e\rangle \geq \|w\|$, then*

$$P_{\mathcal{L}_{p,q}}(z,w) = (P_{\mathbb{R}^p_{\geq +}}(z), 0), \qquad P_{\mathcal{L}^*_{p,q}}(-z,-w) = (P_{(\mathbb{R}^p_{\geq +})^*}(-z), -w);$$

(2) *If $P_{\mathbb{R}^p_{\geq +}}(z)_p \geq \|w\|$, then*

$$P_{\mathcal{L}_{p,q}}(z,w) = (P_{\mathbb{R}^p_{\geq +}}(z), w), \qquad P_{\mathcal{L}^*_{p,q}}(-z,-w) = (P_{(\mathbb{R}^p_{\geq +})^*}(-z), 0);$$

(3) *If $\langle P_{(\mathbb{R}^p_{\geq +})^*}(-z), e\rangle < \|w\|$ and $P_{\mathbb{R}^p_{\geq +}}(z)_p < \|w\|$, then there holds*

$$P_{\mathcal{L}_{p,q}}(z,w) = \left(P_{\mathbb{R}^p_{\geq +}}(f(\lambda)) + \frac{1}{1+\lambda}\|w\|e, \frac{1}{1+\lambda}w\right), \tag{15}$$

$$P_{\mathcal{L}^*_{p,q}}(-z,-w) = \left(P_{(\mathbb{R}^p_{\geq +})^*}(-f(\lambda)) + \frac{\lambda}{1+\lambda}\|w\|e^p, -\frac{\lambda}{1+\lambda}w\right), \tag{16}$$

where $\lambda := \|w\|/\langle P_{\mathbb{R}^{p+1}_{\geq +}}(z, \|w\|), e^{p+1}\rangle - 1$.

Proof. Let $(z, w) \in \mathbb{R}^p \times \mathbb{R}^q$. Our task is to find $(x, u) \in \mathcal{L}_{p,q}$ and $(y, v) \in \mathcal{L}^*_{p,q}$ such that

$$P_{\mathcal{L}_{p,q}}(z, w) = (x, u), \qquad P_{\mathcal{L}^*_{p,q}}(-z, -w) = (y, v). \tag{17}$$

To prove item (1), assume that $\langle P_{(\mathbb{R}^p_{\geq +})^*}(-z), e\rangle \geq \|w\|$. Thus, by item (i) of Lemma 5.1 we must have $u = 0$. Since $P_{\mathcal{L}_{p,q}}(z, w) = (x, 0)$ and $P_{\mathcal{L}^*_{p,q}}(-z, -w) = (y, v)$, applying Theorem 2.1 for $\mathcal{L}_{p,q}$ we have $(x, 0) \in \mathcal{L}_{p,q}$, $(y, v) \in \mathcal{L}^*_{p,q}$, $\langle (x, 0), (y, v)\rangle = 0$ and $(z, w) = (x, 0) - (y, v)$. Thus, $x \in \mathbb{R}^p_{\geq +}$ and $y \in (\mathbb{R}^p_{\geq +})^*$, $\langle x, y\rangle = 0$, $z = x - y$ and $v = -w$. Now, applying Theorem 2.1 for $\mathbb{R}^p_{\geq +}$ we conclude that $x = P_{\mathbb{R}^p_{\geq +}}(z)$ and $y = P_{(\mathbb{R}^p_{\geq +})^*}(-z)$, which together with (17), $u = 0$ and $v = -w$ proves item (1).

We proceed to prove item (2). Since $P_{\mathbb{R}^p_{\geq +}}(z)_p \geq \|w\|$, the item (ii) of Lemma 5.1 implies $v = 0$. Considering that $P_{\mathcal{L}_{p,q}}(z, w) = (x, u)$ and $P_{\mathcal{L}^*_{p,q}}(-z, -w) = (y, 0)$, applying Theorem 2.1 for $\mathcal{L}_{p,q}$ we have $(x, u) \in \mathcal{L}_{p,q}$, $(y, 0) \in \mathcal{L}^*_{p,q}$, $\langle (x, u), (y, 0)\rangle = 0$ and $(z, w) = (x, u) - (y, 0)$. Hence, $x \in \mathbb{R}^p_{\geq +}$ and $y \in (\mathbb{R}^p_{\geq +})^*$, $\langle x, y\rangle = 0$, $z = x - y$ and $u = w$. Using Theorem 2.1 for $\mathbb{R}^p_{\geq +}$, we conclude that $x = P_{\mathbb{R}^p_{\geq +}}(z)$ and $y = P_{(\mathbb{R}^p_{\geq +})^*}(-z)$, which together with (17), $v = 0$ and $u = w$ yields item (2).

To prove item (3), we first note that conditions $\langle P_{(\mathbb{R}^p_{\geq +})^*}(-z), e\rangle < \|w\|$ and $P_{\mathbb{R}^p_{\geq +}}(z)_p < \|w\|$ together with item (iii) of Lemma 5.1 implies that $u \neq 0$ and $v \neq 0$. Moreover, it follows from Theorem 2.1 that (17) is equivalent to

$$(x, u, y, v) \in C(\mathcal{L}_{p,q}) \qquad (z, w) = (x, u) - (y, v). \tag{18}$$

Due to $u \neq 0$, $v \neq 0$ and (18), we apply Proposition 4.3 to obtain the following equivalent conditions

$$x_p = \|u\|, \qquad \langle y, e\rangle = \|v\|, \qquad \langle u, v\rangle = -\|u\|\|v\|, \qquad (x - \|u\|e, y - \|v\|e^p) \in C(\mathbb{R}^p_{\geq +}), \tag{19}$$

$$z = x - y, \qquad w = u - v. \tag{20}$$

Since $\langle u, v\rangle = -\|u\|\|v\|$, $u \neq 0$ and $v \neq 0$, there exists $\lambda > 0$ such that $v = -\lambda u$. Hence, it follows from the second equality in (20) that

$$u = \frac{1}{1+\lambda} w, \qquad v = -\frac{\lambda}{1+\lambda} w. \tag{21}$$

Meanwhile, the second equality in (19) gives $\langle y, e\rangle = \|v\|$. Thus we have that

$$\langle y, e\rangle = \frac{\lambda}{1+\lambda}\|w\|. \tag{22}$$

Since $(x - \|u\|e, y - \|v\|e^p) \in C(\mathbb{R}^p_{\geq +})$ by (19), applying Theorem 2.1 for $\mathbb{R}^p_{\geq +}$ we obtain

$$x - \|u\|e = P_{\mathbb{R}^p_{\geq +}}\left(x - \|u\|e - y + \|v\|e^p\right), \qquad y - \|v\|e^p = P_{(\mathbb{R}^p_{\geq +})^*}\left(-x + \|u\|e + y - \|v\|e^p\right).$$

Thus, by using the first equality in (20) and (21), we obtain after some calculations that

$$x = P_{\mathbb{R}^p_{\geq+}}\left(z - \frac{1}{1+\lambda}\|w\|e + \frac{\lambda}{1+\lambda}\|w\|e^p\right) + \frac{1}{1+\lambda}\|w\|e; \tag{23}$$

$$y = P_{(\mathbb{R}^p_{\geq+})^*}\left(-z + \frac{1}{1+\lambda}\|w\|e - \frac{\lambda}{1+\lambda}\|w\|e^p\right) + \frac{\lambda}{1+\lambda}\|w\|e^p. \tag{24}$$

Hence, combining (17) with (21), (23) and (24) and considering (14), we obtain (15) and (16). It remains to compute λ. For that, by using (15) we can apply Lemma 13 with

$$x = P_{\mathbb{R}^p_{\geq+}}(f(\lambda)) + \frac{1}{1+\lambda}\|w\|e, \qquad \beta = \frac{1}{1+\lambda},$$

to conclude that

$$P_{\mathbb{R}^{p+1}_{\geq+}}(z, \|w\|) = \left(P_{\mathbb{R}^p_{\geq+}}(f(\lambda)) + \frac{1}{1+\lambda}\|w\|e, \frac{1}{1+\lambda}\|w\|\right), \tag{25}$$

which gives $\langle P_{\mathbb{R}^{p+1}_{\geq+}}(z, \|w\|), e^{p+1}\rangle = \frac{1}{1+\lambda}\|w\|$. Therefore, $\lambda = \|w\|/\langle P_{\mathbb{R}^{p+1}_{\geq+}}(z, \|w\|), e^{p+1}\rangle - 1$, which concludes the proof. □

Remark 3. If $p = 1$, then the projection formulas in Theorem 5.3 become the projection onto the second-order cone (see Exercise 8.3 (c) in [22]).

The next theorem states that to compute a projection onto the cone $\mathbb{R}^p_{\geq+}$ it is sufficient to know how to compute a projection onto the cones $\mathbb{R}^p_{\geq}$ and $\mathbb{R}^p_{+}$, its proof can be found in [3]. For the sake of completeness we include its proof here.

Theorem 5.4. *For any $z \in \mathbb{R}^p$, there holds $P_{\mathbb{R}^p_{\geq+}}(z) = P_{\mathbb{R}^p_{\geq}}(z)^+ = P_{\mathbb{R}^p_{+}}(P_{\mathbb{R}^p_{\geq}}(z))$.*

Proof. To simplify the notations set $\mathcal{K} = \mathbb{R}^p_{\geq}$. Thus, Theorem 2.1 yields

$$z = P_{\mathcal{K}}(z) - P_{\mathcal{K}^*}(-z), \qquad \langle P_{\mathcal{K}}(z), P_{\mathcal{K}^*}(-z)\rangle = 0.$$

Moreover, as $P_{\mathcal{K}}(z) = P_{\mathcal{K}}(z)^+ - P_{\mathcal{K}}(z)^-$, the last inequality becomes

$$z = P_{\mathcal{K}}(z)^+ - P_{\mathcal{K}}(z)^- - P_{\mathcal{K}^*}(-z), \qquad \langle P_{\mathcal{K}}(z)^+ - P_{\mathcal{K}}(z)^-, P_{\mathcal{K}^*}(-z)\rangle = 0. \tag{26}$$

Note that $P_{\mathcal{K}}(z)^+ \in \mathcal{K}$ and $-P_{\mathcal{K}}(z)^- \in \mathcal{K}$. Indeed, due to $P_{\mathcal{K}}(z) \in \mathcal{K}$ and $-P_{\mathcal{K}}(z) \in \mathcal{K}$, we have from (1) that $P_{\mathcal{K}}(z)_1 \geq P_{\mathcal{K}}(z)_2 \geq \cdots \geq P_{\mathcal{K}}(z)_p$ and $-P_{\mathcal{K}}(z)_1 \geq -P_{\mathcal{K}}(z)_2 \geq \cdots \geq -P_{\mathcal{K}}(z)_p$. Hence, bearing in mind that the functions $\mathbb{R} \ni t \mapsto t^+$ and $\mathbb{R} \ni t \mapsto -t^-$ are monotone increasing, we also have $P_{\mathcal{K}}(z)_1^+ \geq P_{\mathcal{K}}(z)_2^+ \geq \cdots \geq P_{\mathcal{K}}(z)_p^+ \geq 0$ and $-P_{\mathcal{K}}(z)_1^- \geq -P_{\mathcal{K}}(z)_2^- \geq \cdots \geq -P_{\mathcal{K}}(z)_p^-$. Thus, $P_{\mathcal{K}}(z)^+ \in \mathbb{R}^p_{\geq+} \subset \mathcal{K}$ and $-P_{\mathcal{K}}(z)^- \in \mathcal{K}$. Therefore, the second equality in (26) yields

$$\langle P_{\mathcal{K}}(z)^+, P_{\mathcal{K}^*}(-z)\rangle = \langle P_{\mathcal{K}}(z)^-, P_{\mathcal{K}^*}(-z)\rangle = 0. \tag{27}$$

On the other hand, (2) and (4) implies $\mathcal{K}^* \subset (\mathbb{R}^p_{\geq+})^*$. Furthermore, due to $P_{\mathcal{K}}(z)^- \in$

$\mathbb{R}^p_+$ and $\mathbb{R}^p_+ \subset (\mathbb{R}^p_{\geq+})^*$, we conclude that

$$P_{\mathcal{K}}(z)^- + P_{\mathcal{K}^*}(-z) \in (\mathbb{R}^p_{\geq+})^*. \tag{28}$$

Considering (27) and $\langle P_{\mathcal{K}}(z)^+, P_{\mathcal{K}}(z)^-\rangle = 0$, we also have

$$\langle P_{\mathcal{K}}(z)^+, P_{\mathcal{K}}(z)^- + P_{\mathcal{K}^*}(-z)\rangle = 0. \tag{29}$$

Therefore, the reformulation $z = P_{\mathcal{K}}(z)^+ - (P_{\mathcal{K}}(z)^- + P_{\mathcal{K}^*}(-z))$ of $(26)_1$, together with the formulas $P_{\mathcal{K}}(z)^+ \in \mathbb{R}^p_{\geq+}$, (28), (29) and Theorem 2.1 imply that $P_{\mathcal{K}}(z)^+ = P_{\mathbb{R}^p_{\geq+}}(z)$, which is the desired result. □

We end this section by pointing out that efficient numerical methods to compute projection onto the cone $\mathbb{R}^p_{\geq}$ can be found by using the pool-adjacent-violators algorithm for isotonic regreession [2,4]. For projecting onto the cone $\mathbb{R}^p_+$, we only need to apply the formula of Theorem 5.4 to the output of the p-dimensional PAVA.

In the next section we present a conic optimisation problem with respect to the MESOC related to a portfolio optimisation problem. We note that this problem is an adaptation of Xiao's application in Chapter 4 of his PhD dissertation [23] (which is an improved version of the application in Section 3 of [24]) to the monotone case. Such problems can be solved by algorithms where the projection onto the intersection of MESOC with a hyperplane is important [25]. Our efficient projection method onto MESOC can be incorporated into Dykstra's alternating projection method [26] for the aforementioned intersection. One can also investigate a possible more direct adaptation of our method to such projections.

6. An application of the monotone extended second-order cone to portfolio optimisation

Markowitz developed the mean-variance (MV) model in [27], which is the classical method in investigating the problem of portfolio optimisation. Suppose we build portfolio by using n arbitrary assets. Let $w \in \mathbb{R}^n$ denote the weights of the assets, $r \in \mathbb{R}^n$ represent the return of assets and $\Sigma \in \mathbb{R}^n \times \mathbb{R}^n$ be the covariance matrix. Then, the two traditional and equivalent MV models could be given as:

$$\min_w \left\{ w^\top \Sigma w : r^\top w \geq \alpha,\ e^\top w = 1 \right\}$$

and

$$\max_w \left\{ r^\top w : w^\top \Sigma w \leq \beta,\ e^\top w = 1 \right\},$$

where α is the minimum profit that the investor demands and β is the minimum risk that the investor wants to tolerate. They are typical quadratic optimisation problems with higher computational complexity.

In order to reduce the complexity of solving the portfolio optimisation problem, based on the traditional mean-variance model, Konno and Yamazaki developed the mean-absolute deviation (MAD) model in [17], by replacing the risk measure from the covariance matrix to the absolute deviation. They demonstrated that the results obtained by using MAD model are similar with the results obtained by using the MV model when the return of assets are multivariate normally distributed. It has also been recognized that the MAD model has reduced the computational complexity significantly [28,29]. Before introducing the MAD model, we will give the definitions of some key parameters.

Denote the returns of assets be $\tilde{r} = (\tilde{r_1}, \ldots, \tilde{r_n})^\top \in \mathbb{R}^n$. Suppose that they are distributed over a finite sequence of points $R^j = \left(R_1^j, \ldots, R_n^j\right)^\top \in \mathbb{R}^n$, where $j = 1, \ldots, T$ and R^j denotes T different scenarios such that the behaviour of the assets are different in different scenarios. Meanwhile, denote f_j the probability distribution of the rates of returns of assets, that is

$$f_j = \text{Probability}\left\{(\tilde{r_1}, \ldots, \tilde{r_n})^\top = \left(R_1^j, \ldots, R_n^j\right)^\top\right\}, \qquad j = 1, \ldots, T.$$

The sequences $\{R^j\}_{j=1,\ldots,T}$ and $\{f_j\}_{j=1,\ldots,T}$ can be obtained by using the historical data of assets and some techniques for the future projection of these assets. Meanwhile, since $f_j \in [0,1]$, $j = 1, \ldots, T$ represent probabilities, we will have $e^\top f = 1$, where $f = (f_1, \ldots, f_T)$. In particular, we have

$$r = \mathbb{E}[\tilde{r}] = f_1 R^1 + \cdots + f_T R^T.$$

In order to measure the uncertainty of the returns of the assets for $j = 1, \ldots, T$, let us define $U = (U_1, \ldots, U_T)^\top$, where $U_j = R^j - r$. Let y_j denote the upper bound of disturbance of return at day j. Then, the traditional MAD model can be represented as the following linear programming problem:

$$\begin{aligned} \min_{y,w} \quad & c_0 f^\top y - r^\top w \\ \text{s.t.} \quad & y_j \geq |U_j^\top w|, \quad j = 1, \ldots, T, \\ & e^\top w = 1, \end{aligned}$$

where $c_0 > 0$ is the Arrow-Pratt absolute risk-aversion index defined in [30].

In reality, the uncertainty of the returns of the assets will increase with the increasing of the investment horizon. Thus, it is meaningful to optimize the MAD model to make it more in line with the real-world market behaviour. Meanwhile, by using Cauchy's inequality, we also have $|U_j^\top w| \leq \|U_j\| \|w\|$ for any j. Then, based on the current MAD model, we obtain the following related problem

$$\begin{aligned} \min_{y,w} \quad & c_0 f^\top y - r^\top w \\ \text{s.t.} \quad & y_T \geq y_{T-1} \geq \ldots \geq y_1 \geq \|U_{j^*}\| \|w\|, \\ & e^\top w = 1, \end{aligned}$$

where $j^* = \text{argmin}_j \, |U_j^\top w|$, for $j = 1, \ldots, T$. Note that the vector

$$\left(\frac{y_T}{\|U_{j^*}\|}, \frac{y_{T-1}}{\|U_{j^*}\|}, \ldots, \frac{y_1}{\|U_{j^*}\|}, w\right)^\top$$

belongs to the monotone extended second-order cone $\mathcal{L}_{T,n}$. Thus, the last problem is equivalent to the following conic optimization problem:

$$\begin{aligned} \min_{y,u} \quad & c_0 f^\top y - r^\top \frac{u}{\|U_{j^*}\|} \\ \text{s.t.} \quad & e^\top u = \|U_{j^*}\|, \\ & (y_T, y_{T-1}, \dots, y_1, u)^\top \in \mathcal{L}_{T,n}, \end{aligned}$$

where $u := w\|U_{j^*}\|$.

7. Final remarks

In this paper we have introduced the monotone extended second-order cone and its dual cone. We have reduced the projection onto the MESOC to two isotonic regressions in neighboring dimensions. The isotonic regression can be solved efficiently by pool-adjacent-violators algorithm [2,4]. We have also presented an application of the MESOC to portfolio optimization via a conic optimization problem related to the mean-absolute deviation model [17]. Knowing the projection onto the MESOC can be a useful "ingredient" of projection methods for the latter problem. We predict more direct applications of the projection onto MESOC to practical problems. These applications would be regressions with respect to a set of points whose distance (more generally a "cost") from a source point is expected to decrease and only the position of the point closest to the source is important. For example to capture a strong enough "signal" of a point from the source it is expected to put the better capturing devices further from the source. If one point is (significantly) closer to the source than the other ones, than its position becomes important, because any obstacle "between" this point and the source will have a dominant impact in comparison to the other points. In this probably a better device would be needed than one based on the distance from the source only. Similar types of problems can be imagined in case of a football (i.e., soccer) game where one would expect the defenders to be in general further from the opponents goal and the striker's position to be much more important.

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