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ARTICLE

Complete subgraphs in a multipartite graph

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Abstract

In 1975 Bollobás, Erdős, and Szemerédi asked the following question: given positive integers n, t, r with $2 \le t \le r-1$, what is the largest minimum degree $\delta(G)$ among all r-partite graphs G with parts of size n and which do not contain a copy of K_{t+1}? The $r = t+1$ case has attracted a lot of attention and was fully resolved by Haxell and Szabó, and Szabó and Tardos in 2006. In this article, we investigate the $r > t+1$ case of the problem, which has remained dormant for over 40 years. We resolve the problem exactly in the case when $r \equiv -1 \pmod t$, and up to an additive constant for many other cases, including when $r \ge (3t-1)(t-1)$. Our approach utilizes a connection to the related problem of determining the maximum of the minimum degrees among the family of balanced r-partite rn-vertex graphs of chromatic number at most t.

Keywords: Turán-type problems; multipartite graphs

2020 MSC Codes: Primary: 05C35, Secondary: 05C07, 05D15, 05C15

1. Introduction

The foundation stone of extremal graph theory is Turán's theorem from 1941 [14], which states that the Turán graph $T_t(n)$ (the complete t-partite graph on n vertices with parts of size $\lceil \frac{n}{t} \rceil$ or $\lfloor \frac{n}{t} \rfloor$) has the most edges among all K_{t+1}-free graphs on n vertices. Erdős [7] and Bollobás, Erdős, and Szemerédi [5] asked the following Turán-type problem for multipartite graphs.

Problem 1.1. Given integers n and $2 \le t \le r-1$, what is the largest minimum degree $\delta(G)$ among all r-partite graphs G with parts of size n and which do not contain a copy of K_{t+1}?

Let $f(n, r, t+1)$ denote the answer to Problem 1.1. At a meeting in 1972, Erdős conjectured that $f(n, r, r) = (r-2)n$, see [7, Problem 2, 353–354]. Graver gave a short and elegant proof for $r = 3$ but Seymour constructed counterexamples for $r \ge 4$, see [5]. The study of $f(n, r, r)$ (mostly in its complementary form concerning *independent transversals*) has been a central topic in Combinatorics (see, e.g., [1, 9–11, 13]) due to its applications in graph arboricity, list colouring, and strong chromatic numbers. The problem of determining $f(n, r, r)$ was finally settled by Haxell and Szabó [9] and Szabó and Tardos [13]; indeed, for every $n \in \mathbb{N}$ and even $r \ge 2$,

$$f(n, r+1, r+1) - n = f(n, r, r) = (r-1)n - \left\lceil \frac{rn}{2(r-1)} \right\rceil. \tag{1.1}$$

In contrast, little is known about the value of $f(n, r, t+1)$ for $r > t+1$. In 1975 Bollobás, Erdős, and Szemerédi [5] stated Problem 1.1 explicitly and noted that Turán's theorem easily implies that

$$f(n,r,t+1)=\left(r-\frac{r}{t}\right)n \quad \text{when } t \text{ divides } r. \tag{1.2}$$

Indeed, for any $r \geq t+1$, Turán's theorem implies that every K_{t+1}-free graph G on rn vertices has at most $(1-1/t)(rn)^2/2$ edges, and thus $\delta(G) \leq (1-1/t)rn$. On the other hand, we may let G be the complete t-partite graph on rn vertices with parts of size $\lceil \frac{r}{t} \rceil n$ or $\lfloor \frac{r}{t} \rfloor n$ (in other words, G is an n-vertex blow-up of the Turán graph $T_t(r)$). Then

$$\left(r-\left\lceil \frac{r}{t} \right\rceil\right)n \leq f(n,r,t+1) \leq \left(r-\frac{r}{t}\right)n. \tag{1.3}$$

Extending Graver's work on $f(n,3,3)$, Bollobás, Erdős, and Straus [4] answered Problem 1.1 for all (not necessarily balanced) r-partite graphs G when $t=2$. Their result implies that for every $n \in \mathbb{N}$ and $r \geq 3$,

$$f(n,r,3)=\lfloor r/2 \rfloor n.$$

The aim of this article is to rebuild momentum on Problem 1.1 for $r > t+1 \geq 4$. For any such choice of r and t, our results either resolve Problem 1.1 or provide a lower bound on $f(n,r,t+1)$ that improves that given in (1.3). In particular, in the case that $r \equiv -1 \pmod t$, our first result shows that the lower bound in (1.3) is tight.

Theorem 1.2. *Given integers $n \geq 1$, $m \geq 2$, and $t \geq 3$, let $r = mt-1$. Then $f(n,r,t+1) = (r - \lceil r/t \rceil)\, n$.*

It turns out that the lower bound in (1.3) is best possible only if $r \equiv 0, -1 \pmod t$; in all other cases we give constructions that improve on this lower bound (see Section 4). Moreover, in many such cases, including when $r \geq (3t-1)(t-1)$, we determine $f(n,r,t+1)$ up to an additive constant.

Theorem 1.3. *Given integers $n \geq 1, m \geq 2, t \geq 3$, let $r = mt-a$ with $2 \leq a \leq \min\{m, t-1\}$. Suppose*

(i) $r \geq a(3t-1)$ *or*

(ii) $\dfrac{r}{t(3t-1)(m-1)} - \dfrac{a}{t(m-1)} + \dfrac{a-1}{mt-2} \geq \dfrac{1}{n}$.

Then

$$(r-1)n-(m-1)\left\lceil \frac{(r-1)n}{mt-2} \right\rceil \leq f(n,r,t+1) \leq (r-1)n-\left\lceil \frac{(m-1)(r-1)n}{mt-2} \right\rceil.$$

Thus, up to an additive constant, (1.2), Theorems 1.2 and 1.3 together resolve Problem 1.1 whenever $r \geq (3t-1)(t-1)$. In particular, if $r \geq (3t-1)(t-1)$, $r \not\equiv 0 \pmod t$, and $\lceil r/t \rceil t - 2$ divides n, then

$$f(n,r,t+1)=(r-1)n-\frac{\lceil r/t \rceil - 1}{\lceil r/t \rceil t-2}(r-1)n.$$

Combining the last two theorems with (1.2), we essentially determine $f(n,r,4)$ for all $r \neq 7$.

Corollary 1.4. *Let $r \geq 5$ with $r \neq 7$. Suppose $n \geq 60$ if $r = 10$; $n \geq 22$ if $r = 13$, and $n \in \mathbb{N}$ otherwise. Then*

$$f(n,r,4)=\begin{cases} (r-\lfloor r/3 \rfloor)\,\frac{r-1}{r}n - c_r & \text{if } r \equiv 1 \pmod 3, \\ (r-\lceil r/3 \rceil)\, n & \text{otherwise,} \end{cases}$$

where $0 \leq c_r \leq r/3$.

The proof of Theorem 1.3 applies a result of Andrásfai, Erdős, and Sós [2]. In particular, this result allows us conclude that, if r is large compared to t, then $f(n,r,t+1)$ is equal to the

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maximum of the minimum degrees among the family of balanced r-partite rn-vertex graphs of chromatic number at most t. This approach is also utilized in the proof of Theorem 1.2 when $m \geq 3$. Interestingly, this approach breaks down when $m = 2$, so we require a separate direct argument in this case (which hinges on the fact that $r = 2t - 1$).

There are two extremal problems closely related to Problem 1.1. First, a multipartite Turán theorem has been known since the 1970s. Bollobás, Erdős, and Szemerédi [5] showed that the n-vertex blow-up of $T_t(r)$ has the most edges among all r-partite K_{t+1}-free graphs with n vertices in each part. In fact, this easily follows from Turán's theorem and indicates that for multipartite graphs, the minimum degree version, Problem 1.1, is indeed harder than the Turán problem.[1] Furthermore, Bollobás, Erdős, and Straus [4] determined the largest size of (not necessarily balanced) r-partite K_{t+1}-free graphs. Another related problem concerns finding the smallest d_r^t such that every r-partite graph whose parts have pairwise edge density greater than d_r^t contains a copy of K_t. Bondy, Shen, Thomassé, and Thomassen [6] showed that $d_3^3 = 0.618\ldots$, the golden ratio. Pfender [12] showed that $d_r^t = (t-2)/(t-1)$ for sufficiently large r; in particular, $d_r^3 = 1/2$ for $r \geq 13$.

1.1 Notation

Given a graph G and $x \in V(G)$, we write $N(x)$ for *the neighbourhood of* x in G and define $d(x) := |N(x)|$ as the *degree of* x in G. If $X \subseteq V(G)$ we write $N(x, X) := N(x) \cap X$ and $d(x, X) := |N(x, X)|$. We write $G[X]$ for the induced subgraph of G with vertex set X.

Let G be an r-partite graph with parts $V_1, \ldots, V_r$. A *transversal* is a subset $S \subset V(G)$ so that $|S \cap V_i| = 1$ for each part V_i of G. A set $S \subset V(G)$ is *crossing* if $|S \cap V_i| \leq 1$ for each part V_i of G. Thus, an independent transversal is simply a crossing independent set of size r. Define $K_r(n)$ to be the complete r-partite graph where each part has size n.

1.2 Organization of paper

In Section 2, we formally restate Problem 1.1 in its complementary form and prove Theorem 1.2 for $m = 2$. In Section 3, we introduce a related parameter $\delta(n, r, t)$ that is equal to $f(n, r, t+1)$ if r is large compared to t. In Section 4, we give constructions that improve on the lower bound in (1.3) whenever $r \not\equiv 0, -1 \pmod{t}$. We give an upper bound on $\delta(n, r, t)$ in Section 5 that allows us to easily deduce Theorems 1.2 and 1.3 in Section 6. We finish the paper with concluding remarks in Section 7.

2. The complementary problem and Proof of Theorem 1.2 for $m = 2$

In most papers on $f(n, r, r)$, the complementary form of $f(n, r, r)$ was considered, namely, the smallest $\Delta(G)$ among all r-partite graphs G with parts $V_1, \ldots, V_r$ of size n and without an independent transversal. For a couple of our proofs, it will also be easier to work in the corresponding complementary setting also.

Let $\Delta(n, r, t) := (r-1)n - f(n, r, t)$ denote the smallest maximum degree $\Delta(G)$ among all r-partite graphs G with parts $V_1, \ldots, V_r$ of size n and without a crossing independent set of size t. Note that (1.3) is equivalent to

$$\left(\frac{r}{t} - 1\right) n \leq \Delta(n, r, t+1) \leq \left(\left\lceil \frac{r}{t} \right\rceil - 1\right) n.$$

[1] In contrast, Turán's theorem easily implies that $\max \delta(G) = n - \lceil n/t \rceil$ among all K_{t+1}-free graphs on n vertices.

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We now prove the following lemma, which is the $m = 2$ case of Theorem 1.2.

Lemma 2.1. *For every $n \in \mathbb{N}$ and $t \geq 3$, $f(n, 2t-1, t+1) = (2t-3)n$.*

Proof. Let $r := 2t-1$. By (1.3), it suffices to prove that $f(n, 2t-1, t+1) \leq (2t-3)n$. Further, by considering the complementary problem, it suffices to prove that if G is an r-partite graph with vertex classes $V_1, \ldots, V_r$ of size n so that G does not contain a crossing independent set of size $t+1$, then $\Delta(G) \geq n$.

Suppose for a contradiction that $\Delta(G) < n$.

Claim 2.2. *For all $x \in V(G)$ and $i \in [r]$, we have $d(x, V_i) < n/(t-1)$.*

Proof of claim. Suppose the claim is false. Let $D := \max\{d(v, V_i)\colon v \in V(G), i \in [r]\} \geq n/(t-1)$. Without loss of generality, we may assume that there exists $x_1 \in V_1$ such that $d(x_1, V_2) = D$. Since $\Delta(G) < n$, there exists $x_2 \in V_2 \setminus N(x_1)$. Furthermore, since $\Delta(G) < n$ and $r = 2t-1$, we can greedily find $x_3, \ldots, x_{t-1}$ such that $S = \{x_1, \ldots, x_{t-1}\}$ is a crossing independent set.[2] Without loss of generality, assume that $x_i \in V_i$ for $i \in [t-1]$.

For each $i \in [t]$, let W_i consist of all the vertices of V_{r+1-i} that are not adjacent to any vertex in S. Set $n_i := |W_i|$ and without loss of generality, assume that $n_1 \geq n_2 \geq \ldots \geq n_t$. Let $\ell := \max\{i \in [t]\colon n_i > 0\}$. Note that

$$\begin{aligned} n_1 + \ldots + n_\ell = n_1 + \ldots + n_t &= \left| \bigcup_{t \leq i \leq r} V_i \setminus \bigcup_{j \in [t-1]} N(x_j) \right| \\ &\geq tn - ((t-1)\Delta(G) - d(x_1, V_2)) > n + D. \end{aligned}$$

By averaging, we have

$$n_1 + \ldots + n_{\ell-1} > (\ell-1)(n+D)/\ell. \tag{2.1}$$

Notice that the ℓ-partite subgraph of G induced by $W_1, \ldots, W_\ell$ must be complete (otherwise one can extend S into a crossing independent set of size $t+1$, a contradiction). Hence, there exists $y \in W_\ell$ such that $\bigcup_{i \in [\ell-1]} W_i \subseteq N(y)$. By the definition of D, we deduce that

$$n_1 + \ldots + n_{\ell-1} \leq \sum_{i \in [\ell-1]} d(y, V_{r+1-i}) \leq (\ell-1)D.$$

Together with (2.1), this implies that $D > n/(\ell-1)$ and so

$$\Delta(G) \geq d(y) \geq n_1 + \ldots + n_{\ell-1} > (\ell-1)(n+D)/\ell \geq n,$$

a contradiction. □

Given a crossing independent set S of G, let $\sigma(S) := \sum_{x \in S} d(x, V_S)$ where V_S is the union of all V_i that contain a vertex from S. As mentioned in Footnote 2, we can greedily construct a crossing independent set of size t. Let S be a crossing independent set of size t with $\sigma(S)$ maximal. Without loss of generality, $S \cap \bigcup_{i \in [t-1]} V_i = \emptyset$.

Consider any $(t-1)$-set $S' \subset S$. By Claim 2.2, there exists a vertex $y \in V_1$ that is not adjacent to any vertex in S'. Note that $S' \cup \{y\}$ is a crossing independent set of size t. Hence, $\sigma(S) \geq \sigma(S' \cup \{y\}) \geq \sigma(S') + \sum_{x \in S'} d(x, V_1)$. By summing over all $(t-1)$-sets $S' \subset S$, we obtain that

$$\begin{aligned} t\sigma(S) &\geq \sum_{S' \subset S\colon |S'| = t-1} \sigma(S') + (t-1) \sum_{x \in S} d(x, V_1) = (t-2)\sigma(S) + (t-1) \sum_{x \in S} d(x, V_1) \\ &\geq (t-2)\sigma(S) + (t-1)n, \end{aligned}$$

[2] We can actually find a crossing independent set of size t. However, considering a crossing independent set of size $t-1$ is crucial to the argument here.

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where the last inequality follows as every vertex in V_1 must be adjacent to at least one vertex in S (else there exists a crossing independent set of size $t+1$, a contradiction). Thus, $\sigma(S) \geq (t-1)n/2$.

As $S \cap \bigcup_{i\in[t-1]} V_i = \emptyset$, then $\bigcup_{i\in[t-1]} V_i \subseteq \bigcup_{x\in S} N(x)$ or else there exists a crossing independent set of size $t+1$, a contradiction. Therefore,

$$t\Delta(G) \geq \sum_{x\in S} d(x) \geq \sigma(S) + \left|\bigcup_{i\in[t-1]} V_i\right| \geq \frac{(t-1)n}{2} + (t-1)n \geq tn,$$

implying that $\Delta(G) \geq n$, a contradiction. □

3. A connection to the parameter $\delta(n, r, t)$

The following problem turns out to be closely related to Problem 1.1. Let $\mathcal{G}(n, r, t)$ be the family of all r-partite graphs G with parts of size n and with chromatic number $\chi(G) \leq t$. Let $\delta(n, r, t) := \max\{\delta(G)\colon G \in \mathcal{G}(n, r, t)\}$. An n-vertex blow-up of the Turán graph $T_t(r)$ is a member of $\mathcal{G}(n, r, t)$. Together with (1.3), this gives

$$\left(r - \left\lceil \frac{r}{t} \right\rceil\right) n \leq \delta(n, r, t) \leq f(n, r, t+1) \leq \left(r - \frac{r}{t}\right) n. \tag{3.1}$$

Therefore, when t divides r, we have $\delta(n, r, t) = f(n, r, t+1) = (r - r/t)n$.

When r is large compared to t, Corollary 3.3 below implies that $f(n, r, t+1) = \delta(n, r, t)$ as well. In fact, this is an easy consequence of the following result of Andrásfai, Erdős, and Sós [2].

Theorem 3.1. (Andrásfai, Erdős, and Sós [2].) *Let $t \geq 2$ and let G be a K_{t+1}-free graph on N vertices. If $\delta(G) > \frac{3t-4}{3t-1}N$, then $\chi(G) \leq t$.*

Corollary 3.2. *For $r > t \geq 2$, $f(n, r, t+1) \leq \max\left\{\frac{3t-4}{3t-1}(rn), \delta(n, r, t)\right\}$.*

Proof. Let G be a K_{t+1}-free r-partite graph with n vertices in each part. If $\delta(G) > \frac{3t-4}{3t-1}(rn)$, then $\chi(G) \leq t$ by Theorem 3.1. Thus $G \in \mathcal{G}(n, r, t)$ and $\delta(G) \leq \delta(n, r, t)$. □

By applying (3.1) together with Corollary 3.2, one can conclude that $f(n, r, t+1) = \delta(n, r, t)$ provided that $r \geq (t-1)(3t-1)$.

Corollary 3.3. *Let $r > t \geq 2$ and $0 \leq a \leq t-1$ so that $r \equiv -a \pmod t$. If $r \geq a(3t-1)$ then $f(n, r, t+1) = \delta(n, r, t)$.*

Proof. (1.2) covers the case when $a = 0$ so we assume that $a \in [t-1]$. By Corollary 3.2, it suffices to show that $\delta(n, r, t) \geq \frac{3t-4}{3t-1}rn$. By (3.1), we have

$$\delta(n, r, t) \geq \left(r - \left\lceil \frac{r}{t} \right\rceil\right) n = \left(r - \frac{r+a}{t}\right) n = \left(\frac{r}{t(3t-1)} + \frac{3t-4}{3t-1}r - \frac{a}{t}\right) n \geq \frac{3t-4}{3t-1}rn,$$

where we use the fact that $r \geq a(3t-1)$ in the last inequality. □

4. Lower bound constructions

In this section, we give constructions that improve on the lower bound in (1.3) whenever $r \not\equiv 0, -1 \pmod t$.

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Proposition 4.1. *Let $r \geq m, t \geq 2$ be such that $m(t-1) \leq r \leq mt-1$. Then there exists a graph $G \in \mathcal{G}(n,r,t)$ such that $\delta(G) = (r-1)n - (m-1)\left\lceil \frac{(r-1)n}{mt-2} \right\rceil$. In particular,*

$$f(n,r,t+1) \geq \delta(n,r,t) \geq (r-1)n - (m-1)\left\lceil \frac{(r-1)n}{mt-2} \right\rceil.$$

Proof. When $r = mt-1$, the desired bound follows from (3.1). We thus assume $r \leq mt-2$. Let $\ell := \lceil (r-1)n/(mt-2) \rceil \leq n$. Let $K := K_r(n)$ and let $V_1, \dots, V_r$ denote its parts. For $i \in [t-1]$, let $B_i := \{(i-1)m+1, \dots, im\}$. So $B_1, \dots, B_{t-1}$ form an equipartition of $[m(t-1)]$. For $i \in [t-1]$, let $W_i \subset \bigcup_{j \in B_i} V_j$ be such that $|W_i \cap V_j| = \ell$ for $j \in B_i$. Let $W_t := V(K) \setminus \bigcup_{i \in [t-1]} W_i$. Then

$$|W_1| = \cdots = |W_{t-1}| = \ell m \quad \text{and} \quad |W_t| = rn - m(t-1)\ell.$$

Let G' be the complete t-partite graph with parts $W_1, \dots, W_t$. We set $G := K \cap G'$; that is, G is the graph on $V(K)$ such that, for $x \in V_i \cap W_j$ and $x' \in V_{i'} \cap W_{j'}$, we have $xx' \in E(G)$ if and only if $i \neq i'$ and $j \neq j'$. Clearly $\chi(G) \leq \chi(G') \leq t$. If $x \in V_i$ with $i \notin [m(t-1)]$, then as x is only non-adjacent to the vertices in W_t, we have that $d(x) = rn - |W_t|$. If $x \in V_i$ with $i \in [m(t-1)]$, then

$$d(x) = \begin{cases} rn - (n + |W_1| - \ell) & \text{if } x \notin W_t, \\ rn - (n + |W_t| - (n-\ell)) & \text{if } x \in W_t. \end{cases}$$

By our choice of ℓ,

$$\begin{aligned} \delta(G) &= (r-1)n - \max\{(m-1)\ell, (r-1)n - (m(t-1)-1)\ell\} \\ &= (r-1)n - (m-1)\ell = (r-1)n - (m-1)\left\lceil \frac{(r-1)n}{mt-2} \right\rceil, \end{aligned}$$

as required. □

Note that the lower bound in Proposition 4.1 improves the lower bound from (1.3) in the case when $r = mt-a$ with $2 \leq a \leq \min\{m, t-1\}$. Indeed, in this case (1.3) gives a lower bound of $(r-m)n$ while

$$\begin{aligned} (r-1)n - (m-1)\left\lceil \frac{(r-1)n}{mt-2} \right\rceil &> (r-1)n - (m-1)\frac{(r-1)}{mt-2}n - (m-1) \\ &= (r-m)n + \frac{(a-1)(m-1)}{mt-2}n - (m-1). \end{aligned} \tag{4.1}$$

Thus, if $n \geq (mt-2)/(a-1)$, then the lower bound in Proposition 4.1 improves the lower bound from (1.3).

In the remaining case – when $m < a \leq t-1$ – the next result beats the lower bound from (1.3) when n is not too small.

Proposition 4.2. *Let $r > t \geq 3$ be such that $r = mt - a$ with $2 \leq m < a < t$. Then*

$$f(n,r,t+1) \geq \delta(n,r,t) \geq (r-1)n - (m-1)\left\lceil \frac{(m(t-1-a+m)-1)n}{m(t-a+m)-2} \right\rceil.$$

Proof. Let $t' := t-a+m$ and $r' := m(t'-1)$. By Proposition 4.1, there exists a graph $G' \in \mathcal{G}(n,r',t')$ such that

$$\delta(G') = (r'-1)n - (m-1)\left\lceil \frac{(r'-1)n}{mt'-2} \right\rceil \leq (r'-1)n - (m-2)n,$$

where the last inequality is due to fact that

$$(m-1)(r'-1) - (mt'-2)(m-2) = m(t'-m+2) - 3 = (t-a+2)m - 3 > 0.$$

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We now construct a graph $G \in \mathcal{G}(n,r,t)$ from G' as follows. Let $W_{a-m+1} := V(G')$. Let $W_1, \dots, W_{a-m}$ be vertex sets each of size $(m-1)n$ such that $W_1, \dots, W_{a-m+1}$ are disjoint. Let G be the resulting graph on $\bigcup_{i\in[a-m+1]} W_i$ obtained from G' by adding edges xx' for all $x \in W_i$ and $x' \in W_{i'}$ with $i \neq i'$. Note that $\chi(G) = \chi(G') + a - m \leq t$. Since each W_i with $i \in [a-m]$ can be partitioned into $m-1$ vertex classes of size n, we deduce that $G \in \mathcal{G}(n,r,t)$.

For $x \in V(G) \setminus V(G')$, x is non-adjacent to the vertices in the class W_i it lies in and so $d_G(x) = rn - (m-1)n$. For $x \in V(G')$, we have $d_G(x) = (a-m)(m-1)n + d_{G'}(x)$ and so the vertex $y \in V(G')$ with smallest degree $d_G(y)$ satisfies

$$d_G(y) = (a-m)(m-1)n + \delta(G') = (r-1)n - (m-1)\left\lceil \frac{(r'-1)n}{mt'-2} \right\rceil \leq (r-1)n - (m-2)n.$$

Hence,

$$\delta(G) = (r-1)n - (m-1)\left\lceil \frac{(r'-1)n}{mt'-2} \right\rceil,$$

as required. □

By applying Corollary 3.2 with Proposition 4.1, we can obtain the following result, which improves on Corollary 3.3 in most cases when n is not too small.

Corollary 4.3. *Given $m, t \geq 2$, let $r = mt - a$ where $2 \leq a \leq \min\{m, t-1\}$. If*

$$\frac{r}{t(3t-1)(m-1)} - \frac{a}{t(m-1)} + \frac{a-1}{tm-2} \geq \frac{1}{n},$$

then $f(n,r,t+1) = \delta(n,r,t)$.

Proof. By (3.1) and Corollary 3.2, it suffices to prove that

$$\delta(n,r,t) \geq \frac{3t-4}{3t-1} rn. \tag{4.2}$$

Proposition 4.1 and (4.1) together imply that

$$\delta(n,r,t) > (r-m)n + \frac{(a-1)(m-1)}{mt-2} n - (m-1).$$

On the other hand, $\frac{3t-4}{3t-1} r = r - m + \frac{a}{t} - \frac{r}{t(3t-1)}$. Thus, to prove (4.2), it suffices to have

$$\frac{(a-1)(m-1)}{mt-2} n - (m-1) \geq \frac{an}{t} - \frac{rn}{t(3t-1)}.$$

Indeed, this is equivalent to our assumption

$$\frac{r}{t(3t-1)(m-1)} - \frac{a}{t(m-1)} + \frac{a-1}{mt-2} \geq \frac{1}{n}.$$

□

5. An upper bound on $\delta(n,r,t)$

In this section, we prove the following upper bound on $\delta(n,r,t)$.

Proposition 5.1. *Let $r, n \in \mathbb{N}$ and $m, t \geq 2$ be such that $(m-1)t < r < mt$. Then,*

$$\delta(n,r,t) \leq (r-1)n - \left\lceil \frac{(m-1)(r-1)n}{mt-2} \right\rceil.$$

Proof. Let $\Delta^* := (m-1)(r-1)n/(mt-2)$. Since $\delta(n,r,t)$ is an integer, it suffices to show that $\delta(n,r,t) \leq (r-1)n - \Delta^*$. Suppose to the contrary that $\delta(n,r,t) > (r-1)n - \Delta^*$. Let

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$G \in \mathcal{G}(n,r,t)$ be such that $\delta(G) = \delta(n,r,t)$. As $G \in \mathcal{G}(n,r,t)$, $V(G)$ can be partitioned into r independent sets $V_1, \dots, V_r$ each of size n; as $\chi(G) \le t$, $V(G)$ can be partitioned into t colour classes $W_1, \dots, W_t$. For every $x \in V_i \cap W_j$, x is non-adjacent to those vertices in $V_i \cup W_j$ and so $d(x) \le rn - |V_i| - |W_j| + |V_i \cap W_j| = (r-1)n - |W_j| + |V_i \cap W_j|$.

For $i \in [r]$, let $C(i) := \{j \in [t] : |V_i \cap W_j| \neq 0\}$ be the set of colours present in V_i. For $j \in [t]$, let $\mathrm{supp}(j) := \{i \in [r] : |V_i \cap W_j| \neq 0\}$ be the set of parts that colour j is present. Let

$$\Delta_j := |W_j| - \min_{i \in \mathrm{supp}(j)} |V_i \cap W_j|.$$

Thus $\delta(G) \le \min_{j \in [t]} \{(r-1)n - \Delta_j\}$. Hence we have for all $j \in [t]$,

$$\Delta_j < \Delta^* = \frac{(m-1)(r-1)n}{mt-2}. \tag{5.1}$$

Claim 5.2. *For all $j \in [t]$, $|W_j| < m\Delta^*/(m-1)$.*

Proof of claim. Note that

$$\frac{m\Delta^*}{m-1} = \frac{m(r-1)n}{mt-2} \ge \frac{m(m-1)tn}{mt-2} > (m-1)n.$$

If $|\mathrm{supp}(j)| \le m-1$, then $|W_j| \le (m-1)n < \frac{m}{m-1}\Delta^*$ as desired. Hence, we may assume that $|\mathrm{supp}(j)| \ge m$. Using the definition of Δ_j and (5.1), we obtain that

$$\frac{m-1}{m}|W_j| \le \frac{|\mathrm{supp}(j)|-1}{|\mathrm{supp}(j)|}|W_j| \le \Delta_j < \Delta^*.$$

Hence the claim follows. □

Suppose that $|C(i)| = 1$ for all $i \in [r]$. Every V_i is a subset of some W_j and consequently, there exists $j \in [t]$ with $|W_j| \ge \lceil r/t \rceil n \ge mn$. It follows that

$$\Delta_j \ge |W_j| - n \ge (m-1)n \ge (m-1)\frac{r-1}{mt-2}n = \Delta^*,$$

a contradiction.

Without loss of generality, we assume that $|C(1)| = s \ge 2$. For every $j \in C(1)$, we know that $|W_j| \le \Delta_j + |V_1 \cap W_j|$ from the definition of Δ_j. Hence,

$$\sum_{j \in C(1)} |W_j| \le \sum_{j \in C(1)} \left(\Delta_j + |V_1 \cap W_j|\right) \overset{(5.1)}{<} s\Delta^* + n.$$

Together with Claim 5.2, this gives

$$rn = \sum_{j \in [t]} |W_j| < s\Delta^* + n + (t-s)\frac{m}{m-1}\Delta^* = \frac{mt-s}{m-1}\Delta^* + n \le \frac{mt-2}{m-1}\Delta^* + n = rn,$$

a contradiction. □

6. Proof of the main results

The proofs of Theorems 1.2 and 1.3 and Corollary 1.4 now follow easily from our auxiliary results.

Proof of Theorem 1.2. The $m = 2$ case of the theorem is precisely Lemma 2.1. For $m \ge 3$, we may apply Corollary 3.3 (with $a := 1$) to conclude that $f(n,r,t+1) = \delta(n,r,t)$. Then Proposition 5.1 implies $\delta(n,r,t) \le (r-m)n = (r - \lceil r/t \rceil)n$; together with the lower bound in (1.3) this completes the proof. □

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Proof of Theorem 1.3. Under Condition (i), we first apply Corollary 3.3 to obtain that $f(n, r, t+1) = \delta(n, r, t)$. Then Propositions 4.1 and 5.1 give the desired lower and upper bounds, respectively. Under Condition (ii), we apply Corollary 4.3 instead of Corollary 3.3. □

Proof of Corollary 1.4. The case when $r \equiv 0 \pmod 3$ follows from (1.2). The case when $r \equiv 2 \pmod 3$ follows immediately from Theorem 1.2. If $r \equiv 1 \pmod 3$ then $r = 3m - 2$ for some $m \geq 4$. Thus Condition (i) of Theorem 1.3 holds provided that $m \geq 6$. If $r = 10$ and $n \geq 60$ or $r = 13$ and $n \geq 22$, then it is easy to check that Condition (ii) of Theorem 1.3 holds. Thus, Theorem 1.3 yields the corollary in this case. □

7. Concluding remarks

In this article, we have resolved Problem 1.1 for many choices of r and t. For the remaining open cases, it would be interesting to establish when (if at all) a lower bound construction from Section 4 is extremal. One obvious case would be to determine $f(n, 7, 4)$, which is the only remaining case for $f(n, r, 4)$.

Our results show that $f(n, r, t+1) = \delta(n, r, t)$ when r is large compared to t. It would be interesting to determine all values of r and t for which this equality holds. Proposition 5.1 and (1.1) together show that $f(n, t+1, t+1) > \delta(n, t+1, t)$ when $t \geq 3$ is odd and n is sufficiently large. Indeed, Proposition 5.1 implies that $\delta(n, t+1, t) \leq tn - \lceil \frac{tn}{2t-2} \rceil$ for every $t \geq 2$. If t is odd, then by (1.1), we have

$$f(n, t+1, t+1) = tn - \left\lceil \frac{(t+1)n}{2t} \right\rceil > tn - \left\lceil \frac{tn}{2t-2} \right\rceil \geq \delta(n, t+1, t).$$

As mentioned in the Introduction, Bollobás, Erdős, and Straus [4] determined the largest $\delta(G)$ among all K_3-free (not necessarily balanced) r-partite graphs G for all r. It is natural to extend the results in the present paper to *unbalanced* multipartite graphs as well.

It is also natural to ask for the largest $\delta(G)$ among H-free multipartite graphs G for a fixed graph $H \neq K_t$. For example, Bollobás, Erdős, and Szemerédi [5] showed that if G is a tripartite graph with n vertices in each part and with $\delta(G) \geq n + \frac{1}{\sqrt{2}} n^{3/4}$, then G contains a copy of $K_3(2)$; they asked if $\delta(G) \geq n + Cn^{1/2}$ suffices. Furthermore, extending the aforementioned multipartite Turán theorem of Bollobás, Erdős, and Straus [4], there has been recent work on determining the largest $e(G)$ among all multipartite graphs G on n vertices that contain no multiple (disjoint) copies of K_t, see, e.g., [3, 8].

The following result might be useful for constructing extremal examples for the remaining open cases of Problem 1.1. It shows that given an upper bound on $\Delta(n_0, r_0, t_0)/n_0$ one can obtain an upper bound on $\Delta(n, r, t)/n$ for other triples (n, r, t).

Proposition 7.1. *Let $r_0, t_0 \in \mathbb{N}$ so that $2 \leq t_0 \leq r_0$. Let $\Delta_0 \geq 0$ be such that $\Delta(n_0, r_0, t_0)/n_0 \leq \Delta_0$ for all $n_0 \in \mathbb{N}$. Let $n, k \geq 2$ be integers. Set $r := r_0 k$ and $t := k + t_0$. Then there exists an r-partite graph G with parts of size n so that:*

- *G contains no crossing independent set of size t;*
- $\Delta(G) \leq (r_0 - 1) \left\lceil \dfrac{\Delta_0 + (k-1)r_0}{\Delta_0 + kr_0 - 1} \cdot n \right\rceil$.

That is,

$$\Delta(n, r, t) \leq (r_0 - 1) \left\lceil \frac{\Delta_0 + (k-1)r_0}{\Delta_0 + kr_0 - 1} \cdot n \right\rceil,$$

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or equivalently

$$f(n,r,t) \geq (r-1)n - (r_0-1)\left\lceil \frac{\Delta_0 + (k-1)r_0}{\Delta_0 + kr_0 - 1} \cdot n \right\rceil.$$

Proof. Let $\ell := \lfloor (r_0-1)n/(\Delta_0 + kr_0 - 1) \rfloor$. We construct an r-partite graph G with parts $V_1, \ldots, V_r$ of size n as follows. Partition each V_i into $L_i \cup S_i$ such that $|S_i| = \ell$ and $|L_i| = n - \ell$. We call the vertices in each L_i *large* and those in each S_i *small*. We partition $[r]$ into k blocks of size r_0 by assigning i and j to the same block if $\lceil i/r_0 \rceil = \lceil j/r_0 \rceil$. We form a complete bipartite graph between L_i and L_j (*joining L_i and L_j* for short) if and only if i and j are in the same block. We join S_i and S_j if i and j are not in the same block. By the definition of Δ_0, there exists an r_0-partite graph G_S with ℓ vertices in each part and $\Delta(G_S) \leq \Delta_0 \ell$ containing no crossing independent set of size t_0. We place a copy of G_S in each block so that the sets S_i in the block each form one of the vertex classes of G_S.

We claim that G contains no crossing independent set of size t. Indeed, a crossing independent set contains at most one large vertex from each block, and at most $t_0 - 1$ small vertices. Thus, the largest crossing independent set has size $k + t_0 - 1 = t - 1$.

Note that

$$\Delta(G) = \max\{(\Delta_0 + (k-1)r_0)\ell, (r_0-1)(n-\ell)\}.$$

The choice of ℓ ensures

$$\Delta(G) \leq (r_0-1)\left\lceil \frac{\Delta_0 + (k-1)r_0}{\Delta_0 + kr_0 - 1} \cdot n \right\rceil,$$

as desired. □

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