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Three-dimensional stochastic cubic nonlinear wave equation with almost space-time white noise

Tadahiro Oh^{1,2} · Yuzhao Wang³ · Younes Zine^{1,2}

This article is dedicated to István Gyöngy on the occasion of his 70th birthday.

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Abstract

We study the stochastic cubic nonlinear wave equation (SNLW) with an additive noise on the three-dimensional torus \mathbb{T}^3 . In particular, we prove local well-posedness of the (renormalized) SNLW when the noise is almost a space-time white noise. In recent years, the paracontrolled calculus has played a crucial role in the well-posedness study of singular SNLW on \mathbb{T}^3 by Gubinelli et al. (Paracontrolled approach to the three-dimensional stochastic nonlinear wave equation with quadratic nonlinearity, 2018, [arXiv:1811.07808](https://arxiv.org/abs/1811.07808) [math.AP]), Oh et al. (Focusing Φ_3^4 -model with a Hartree-type nonlinearity, 2020, [arXiv:2009.03251](https://arxiv.org/abs/2009.03251) [math.PR]), and Bringmann (Invariant Gibbs measures for the three-dimensional wave equation with a Hartree nonlinearity II: Dynamics, 2020, [arXiv:2009.04616](https://arxiv.org/abs/2009.04616) [math.AP]). Our approach, however, does not rely on the paracontrolled calculus. We instead proceed with the second order expansion and study the resulting equation for the residual term, using multilinear dispersive smoothing.

Keywords Stochastic nonlinear wave equation · Nonlinear wave equation · Pathwise well-posedness

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1 Introduction

1.1 Singular stochastic nonlinear wave equation

In this paper, we study the following Cauchy problem for the stochastic nonlinear wave equation (SNLW) with a cubic nonlinearity on the three dimensional torus $\mathbb{T}^3 = (\mathbb{R}/(2\pi\mathbb{Z}))^3$, driven by an additive noise:

$$\begin{cases} \partial_t^2 u + (1 - \Delta)u + u^3 = \phi\xi & (x, t) \in \mathbb{T}^3 \times \mathbb{R}, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \quad (1.1)$$

where $\xi(x, t)$ denotes a (Gaussian) space-time white noise on $\mathbb{T}^3 \times \mathbb{R}$ with the space-time covariance given by

$$\mathbb{E}[\xi(x_1, t_1)\xi(x_2, t_2)] = \delta(x_1 - x_2)\delta(t_1 - t_2)$$

and ϕ is a bounded operator on $L^2(\mathbb{T}^3)$. Our main goal is to present a concise proof of local well-posedness of (1.1), when ϕ is the Bessel potential of order α :

$$\phi = \langle \nabla \rangle^{-\alpha} = (1 - \Delta)^{-\frac{\alpha}{2}} \quad (1.2)$$

for any $\alpha > 0$. Namely, we consider (1.1) with an ‘‘almost’’ space-time white noise.

Given $\alpha \in \mathbb{R}$, let $\phi = \phi_\alpha$ be as in (1.2). Then, a standard computation shows that the stochastic convolution:

$$\dagger = \mathcal{I}(\langle \nabla \rangle^{-\alpha} \xi) \tag{1.3}$$

belongs almost surely to $C(\mathbb{R}; W^{s,\infty}(\mathbb{T}^3))$ for any $s < \alpha - \frac{1}{2}$. See Lemma 3.1 below. Here, we adopted Hairer’s convention to denote stochastic terms by trees; the vertex “ \cdot ” in \dagger corresponds to the random noise $\phi\xi = \langle \nabla \rangle^{-\alpha} \xi$, while the edge denotes the Duhamel integral operator:

$$\mathcal{I} = (\partial_t^2 + (1 - \Delta))^{-1}, \tag{1.4}$$

corresponding to the forward fundamental solution to the linear wave equation. Note that when $\alpha > \frac{1}{2}$, the stochastic convolution \dagger is a function of positive (spatial) regularity $\alpha - \frac{1}{2} - \varepsilon$.¹ Then, by proceeding with the first order expansion:

$$u = \dagger + v$$

and studying the equation for the residual term $v = u - \dagger$, we can show that (1.1) is locally well-posed, when $\alpha > \frac{1}{2}$. See [13,58] in the case of the deterministic cubic nonlinear wave equation (NLW):

$$\partial_t^2 u + (1 - \Delta)u + u^3 = 0 \tag{1.5}$$

with random initial data. Furthermore, by controlling the growth of the \mathcal{H}^1 -norm of the residual term v via a Gronwall-type argument, we can prove global well-posedness of (1.1), when $\alpha > \frac{1}{2}$.² See [13].

When $\alpha \leq \frac{1}{2}$, solutions to (1.1) are expected to be merely distributions of negative regularity $\alpha - \frac{1}{2} - \varepsilon$, inheriting the regularity of the stochastic convolution, and thus we need to consider the renormalized version of (1.1), which formally reads

$$\begin{cases} \partial_t^2 u + (1 - \Delta)u + u^3 - \infty \cdot u = \langle \nabla \rangle^{-\alpha} \xi \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \tag{1.6}$$

where the formal expression $u^3 - \infty \cdot u$ denotes the renormalization of the cubic power u^3 . In the range $\frac{1}{4} < \alpha \leq \frac{1}{2}$, a straightforward computation with the second order expansion:

$$u = \dagger - \Psi^* + v$$

¹ In this discussion, we only discuss spatial regularities. Moreover, we do not worry about the regularity of the initial data (u_0, u_1) .

² This globalization argument is the only place, where the defocusing nature of the nonlinearity plays a role. See also Remarks 1.4 and 1.6. In particular, all the local-in-time results, including Theorem 1.1, also hold in the focusing case.

yields local well-posedness of the renormalized SNLW (1.6) (in the sense of Theorem 1.1 below). Here, the second order process Ψ is defined by

$$\Psi = \mathcal{I}(\Psi),$$

where Ψ denotes the renormalized version of \uparrow^3 . See [51] for this argument in the context of the deterministic renormalized cubic NLW (1.5) with random initial data.

We state our main result.

Theorem 1.1 *Let $0 < \alpha \leq \frac{1}{2}$. Given $s > \frac{1}{2}$, let $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3) = H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$. Then, there exists a unique local-in-time solution to the renormalized cubic SNLW (1.6) with $(u, \partial_t u)|_{t=0} = (u_0, u_1)$.*

More precisely, given $N \in \mathbb{N}$, let $\xi_N = \pi_N \xi$, where π_N is the frequency projector onto the spatial frequencies $\{|n| \leq N\}$ defined in (1.13) below. Then, there exists a sequence of time-dependent constants $\{\sigma_N(t)\}_{N \in \mathbb{N}}$ tending to ∞ (see (1.16) below) such that, given small $\varepsilon = \varepsilon(s) > 0$, the solution u_N to the following truncated renormalized SNLW:

$$\begin{cases} \partial_t^2 u_N + (1 - \Delta)u_N + u_N^3 - 3\sigma_N u_N = \langle \nabla \rangle^{-\alpha} \xi_N \\ (u_N, \partial_t u_N)|_{t=0} = (u_0, u_1) \end{cases} \tag{1.7}$$

converges to a non-trivial³ stochastic process $u \in C([-T, T]; H^{\alpha-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))$ almost surely, where $T = T(\omega)$ is an almost surely positive stopping time.

Stochastic nonlinear wave equations have been studied extensively in various settings; see [15, Chapter 13] for the references therein. In particular, over the last few years, we have witnessed a rapid progress in the theoretical understanding of nonlinear wave equations with singular stochastic forcing and/or rough random initial data; see [12,19,20,25–27,45,47–51,53–57,66]. In [26], Gubinelli, Koch, and the first author studied the quadratic SNLW on \mathbb{T}^3 :

$$\partial_t^2 u + (1 - \Delta)u + u^2 = \xi. \tag{1.8}$$

By adapting the paracontrolled calculus [24], originally introduced by Gubinelli, Imkeller, and Perkowski in the study of stochastic parabolic PDEs, to the dispersive setting, the authors of [26] reduced (1.8) into a system of two unknowns. This system was then shown to be locally well-posed by exploiting the following two ingredients: (i) multilinear dispersive smoothing coming from a multilinear interaction of random waves (see also [12,45]) and (ii) novel random operators (the so-called paracontrolled operators) which incorporate the paracontrolled structure in their definition. These random operators are used to replace commutators which are standard in the parabolic paracontrolled approach [14,40].

³ Here, non-triviality means that the limiting process u is not zero or a linear solution. As we see below, the limiting process u admits a decomposition $u = \uparrow - \Psi + v$, where the residual term v satisfies the nonlinear equation (1.25). See Remark 1.4(ii) on a trivality result for the unrenormalized equation. See also [30,47,51,54] for related trivality results.

More recently, Okamoto, Tolomeo, and the first author [48] and Bringmann [12] independently studied the following SNLW with a cubic Hartree-type nonlinearity:⁴

$$\partial_t^2 u + (1 - \Delta)u + (V * u^2)u = \xi, \tag{1.9}$$

where V is the kernel of the Bessel potential $\langle \nabla \rangle^{-\beta}$ of order $\beta > 0$.⁵ In [48], the authors proved local well-posedness for $\beta > 1$ by viewing the nonlinearity as the nested bilinear interactions and utilizing the paracontrolled operators introduced in [26]. In [12], Bringmann went much further and proved local well-posedness of (1.9) for any $\beta > 0$. The main strategy in [12] is to extend the paracontrolled approach in [26] to the cubic setting. The main task is then to study regularity properties of various random operators and random distributions. This was done by an intricate combination of deterministic analysis, stochastic analysis, counting arguments, the random matrix/tensor approach by Bourgain [9,10] and Deng, Nahmod, and Yue [18], and the physical space approach via the (bilinear) Strichartz estimates due to Klainerman and Tataru [36], analogous to the random data Cauchy theory for the nonlinear Schrödinger equations on \mathbb{R}^d as in [2–4].

From the scaling point of view, the cubic SNLW (1.6) with a slightly smoothed space-time white noise (i.e. small $\alpha > 0$) is essentially the same as the Hartree SNLW (1.9) with small $\beta > 0$. Hence, Theorem 1.1 is expected to hold in view of Bringmann’s recent result [12]. The main point of this paper is that we present a concise proof of Theorem 1.1 *without* using the paracontrolled calculus. In the next subsection, we outline our strategy.

Due to the time reversibility of the equation, we only consider positive times in the remaining part of the paper.

Remark 1.2 The equations (1.1) and (1.6) indeed correspond to the stochastic nonlinear Klein–Gordon equations. The same results with inessential modifications also hold for the stochastic nonlinear wave equation, where we replace the linear part in (1.1) and (1.6) by $\partial_t^2 u - \Delta u$. In the following, we simply refer to (1.1) and (1.6) as the stochastic nonlinear wave equations.

Remark 1.3 Our argument also applies to the deterministic (renormalized) cubic NLW on \mathbb{T}^3 with random initial data of the form:

$$(u_0^\omega, u_1^\omega) = \left(\sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle^{1+\alpha}} e^{in \cdot x}, \sum_{n \in \mathbb{Z}^3} \frac{h_n(\omega)}{\langle n \rangle^\alpha} e^{in \cdot x} \right),$$

⁴ In [12], Bringmann studied the corresponding deterministic Hartree NLW with random initial data.

⁵ We point out that the scope of the papers [12,48] goes much further than what is described here. The main goal of [48] is to study the focusing problem, in particular the (non-)construction of the focusing Gibbs measure associated to the focusing Hartree SNLW. They identified the critical value $\beta = 2$ and proved sharp global well-posedness of the focusing problem (with a small coefficient in front of the nonlinearity when $\beta = 2$). On the other hand, the main goal in [12] is the construction of global-in-time dynamics in the defocusing case, where there was a significant difficulty in adapting Bourgain’s invariant measure argument [8,9]. This is due to (i) the singularity of the associated Gibbs measure with respect to the base Gaussian free field for $0 < \beta \leq \frac{1}{2}$ [11,48] and (ii) the paracontrolled structure imposed in the local theory, which must be propagated in the construction of global-in-time solutions. See the introductions of [12,48] for further discussion.

where the series $\{g_n\}_{n \in \mathbb{Z}^3}$ and $\{h_n\}_{n \in \mathbb{Z}^3}$ are two families of independent standard complex-valued Gaussian random variables conditioned that $g_n = \overline{g_{-n}}$, $h_n = \overline{h_{-n}}$, $n \in \mathbb{Z}^3$. In particular, Theorem 1.1 provides an improvement of the main result (almost sure local well-posedness) in [51] from $\alpha > \frac{1}{4}$ to $\alpha > 0$.

Remark 1.4 (i) The first part of the statement in Theorem 1.1 is merely a formal statement in view of the divergent behavior $\sigma_N(t) \rightarrow \infty$ for $t \neq 0$. In the next subsection, we provide a precise meaning to what it means to be a solution to (1.6) and also make the uniqueness statement more precise. See Remark 1.9.

(ii) In the case of the defocusing cubic SNLW with damping:

$$\partial_t^2 u + \partial_t u + (1 - \Delta)u + u^3 = \langle \nabla \rangle^{-\alpha} \xi,$$

a combination of our argument with that in [47] yield the following triviality result. Consider the following truncated (unrenormalized) SNLW with damping:

$$\begin{cases} \partial_t^2 u_N + \partial_t u_N + (1 - \Delta)u_N + u_N^3 = \langle \nabla \rangle^{-\alpha} \xi_N \\ (u_N, \partial_t u_N)|_{t=0} = (u_0, u_1), \end{cases}$$

where $\xi_N = \pi_N \xi$. As we remove the regularization (i.e. take $N \rightarrow \infty$), the solution u_N converges in probability to the trivial function $u_\infty \equiv 0$ for any (smooth) initial data (u_0, u_1) . See [47] for details.

Remark 1.5 (i) In our proof, we use the Fourier restriction norm method (i.e. the $X^{s,b}$ -spaces defined in (2.8)), following [12,57]. While it may be possible to give a proof of Theorem 1.1 based only on the physical-side spaces (such as the Strichartz spaces) as in [25–27], we do not pursue this direction since our main goal is to present a concise proof of Theorem 1.1 by adapting various estimates in [12] to our current setting. Note that the use of the physical-side spaces would allow us to take the initial data (u_0, u_1) in the critical space $\mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3)$ (for the cubic NLW on \mathbb{T}^3). See for example [25]. One may equally use the Fourier restriction norm method adapted to the space of functions of bounded p -variation and its pre-dual, introduced and developed by Tataru, Koch, and their collaborators [28,31,37], which would also allow us to take the initial data (u_0, u_1) in the critical space $\mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3)$. See for example [3,46] in the context of the nonlinear Schrödinger equations with random initial data. Since our main focus is to handle rough noises (and not about rough deterministic initial data), we do not pursue this direction.

(ii) On \mathbb{T}^3 , the Bessel potential $\phi_\alpha = \langle \nabla \rangle^{-\alpha}$ is Hilbert–Schmidt from $L^2(\mathbb{T}^3)$ to $H^s(\mathbb{T}^3)$ for $s < \alpha - \frac{3}{2}$. It would be of interest to extend Theorem 1.1 to a general Hilbert–Schmidt operator ϕ , say from $L^2(\mathbb{T}^3)$ to $H^{\alpha-\frac{3}{2}}(\mathbb{T}^3)$ as in [16,44,52].⁶ Note that our argument uses the independence of the Fourier coefficients of the stochastic convolution \dagger but that such independence will be lost for a general Hilbert–Schmidt operator ϕ .

⁶ Or a general γ -radonifying operator ϕ as in [21], where the authors proved local well-posedness of the one-dimensional stochastic cubic nonlinear Schrödinger equation with an almost space-time white noise.

Remark 1.6 (i) When $\alpha = 0$, SNLW (1.6) with damping

$$\partial_t^2 u + \partial_t u + (1 - \Delta)u + u^3 - \infty \cdot u = \xi \tag{1.10}$$

corresponds to the so-called canonical stochastic quantization equation⁷ for the Gibbs measure given by the Φ_3^4 -measure on u and the white noise measure on $\partial_t u$. See [60]. In this case (i.e. when $\alpha = 0$), our approach and the more sophisticated approach of Bringmann [12] for (1.9) with $\beta > 0$ completely break down. This is a very challenging problem, for which one would certainly need to use the paracontrolled approach in [12,26,48] and combine with the techniques in [18].

(ii) As mentioned above, when $\alpha > \frac{1}{2}$, the globalization argument by Burq and Tzvetkov [13] yields global well-posedness of SNLW (1.1) with ϕ as in (1.2). When $\alpha = 0$, we expect that (a suitable adaptation of) Bourgain’s invariant measure argument would yield almost sure global well-posedness once we could prove local well-posedness of (1.10) (but this is a very challenging problem). It would be of interest to investigate the issue of global well-posedness of (1.6) for $0 < \alpha \leq \frac{1}{2}$. See [27,66] for the global well-posedness results on SNLW with an additive space-time white noise in the two-dimensional case.

1.2 Outline of the proof

Let us now describe the strategy to prove Theorem 1.1. Let W denote a cylindrical Wiener process on $L^2(\mathbb{T}^3)$:⁸

$$W(t) = \sum_{n \in \mathbb{Z}^3} B_n(t) e_n,$$

where $e_n(x) = e^{in \cdot x}$ and $\{B_n\}_{n \in \mathbb{Z}^3}$ is defined by $B_n(t) = \langle \xi, \mathbf{1}_{[0,t]} \cdot e_n \rangle_{x,t}$. Here, $\langle \cdot, \cdot \rangle_{x,t}$ denotes the duality pairing on $\mathbb{T}^3 \times \mathbb{R}$. As a result, we see that $\{B_n\}_{n \in \mathbb{Z}^3}$ is a family of mutually independent complex-valued Brownian motions conditioned so that $B_{-n} = \overline{B_n}$, $n \in \mathbb{Z}^3$. In particular, B_0 is a standard real-valued Brownian motion. Note that we have, for any $n \in \mathbb{Z}^2$,

$$\text{Var}(B_n(t)) = \mathbb{E}[\langle \xi, \mathbf{1}_{[0,t]} \cdot e_n \rangle_{x,t} \overline{\langle \xi, \mathbf{1}_{[0,t]} \cdot e_n \rangle_{x,t}}] = \|\mathbf{1}_{[0,t]} \cdot e_n\|_{L^2_{x,t}}^2 = t.$$

With this notation, we can formally write the stochastic convolution $\uparrow = \mathcal{I}(\langle \nabla \rangle^{-\alpha} \xi)$ in (1.3) as

$$\uparrow = \int_0^t \frac{\sin((t-t')\langle \nabla \rangle)}{\langle \nabla \rangle^{1+\alpha}} dW(t') = \sum_{n \in \mathbb{Z}^3} e_n \int_0^t \frac{\sin((t-t')\langle n \rangle)}{\langle n \rangle^{1+\alpha}} dB_n(t'), \tag{1.11}$$

⁷ Namely, the Langevin equation with the momentum $v = \partial_t u$.

⁸ By convention, we endow \mathbb{T}^3 with the normalized Lebesgue measure $(2\pi)^{-3} dx$.

where $\langle \nabla \rangle = \sqrt{1 - \Delta}$ and $\langle n \rangle = \sqrt{1 + |n|^2}$. We indeed construct the stochastic convolution \uparrow in (1.11) as the limit of the truncated stochastic convolution \uparrow_N defined by

$$\uparrow_N = \mathcal{I}(\pi_N \langle \nabla \rangle^{-\alpha} \xi) = \sum_{\substack{n \in \mathbb{Z}^3 \\ |n| \leq N}} e_n \int_0^t \frac{\sin((t - t') \langle n \rangle)}{\langle n \rangle^{1+\alpha}} dB_n(t') \tag{1.12}$$

for $N \in \mathbb{N}$, where π_N denotes the (spatial) frequency projector defined by

$$\pi_N f = \sum_{|n| \leq N} \widehat{f}(n) e_n. \tag{1.13}$$

A standard computation shows that the sequence $\{\uparrow_N\}_{N \in \mathbb{N}}$ is almost surely Cauchy in $C([0, T]; W^{\alpha - \frac{1}{2}-, \infty}(\mathbb{T}^3))$ and thus converges almost surely to some limit, which we denote by \uparrow , in the same space. See Lemma 3.1 below.

We then define the Wick powers \mathbb{V}_N and \mathbb{V}_N by

$$\begin{aligned} \mathbb{V}_N(x, t) &= (\uparrow_N(x, t))^2 - \sigma_N(t), \\ \mathbb{V}_N(x, t) &= (\uparrow_N(x, t))^3 - 3\sigma_N(t) \cdot \uparrow_N(x, t), \end{aligned} \tag{1.14}$$

and the second order process \mathbb{V}_N by

$$\mathbb{V}_N = \mathcal{I}(\mathbb{V}_N), \tag{1.15}$$

where \mathcal{I} denotes the Duhamel integral operator in (1.4). Here, $\sigma_N(t)$ is defined by¹⁰

$$\begin{aligned} \sigma_N(t) &= \mathbb{E}[(\uparrow_N(x, t))^2] = \sum_{|n| \leq N} \int_0^t \left[\frac{\sin((t - t') \langle n \rangle)}{\langle n \rangle^{1+\alpha}} \right]^2 dt' \\ &= \sum_{|n| \leq N} \left\{ \frac{t}{2 \langle n \rangle^{2+2\alpha}} - \frac{\sin(2t \langle n \rangle)}{4 \langle n \rangle^{3+2\alpha}} \right\} \sim \begin{cases} t \log N, & \text{for } \alpha = \frac{1}{2}, \\ tN^{1-2\alpha}, & \text{for } 0 < \alpha < \frac{1}{2}. \end{cases} \end{aligned} \tag{1.16}$$

We point out that a standard argument shows that \mathbb{V}_N and \mathbb{V}_N converge almost surely to \mathbb{V} in $C([0, T]; W^{2\alpha-1-, \infty}(\mathbb{T}^3))$ and to \mathbb{V} in $C([0, T]; W^{3\alpha-\frac{3}{2}-, \infty}(\mathbb{T}^3))$, respectively, but that we do not need these regularity properties of the Wick powers \mathbb{V} and \mathbb{V} in this paper.

⁹ Hereafter, we use $a-$ (and $a+$) to denote $a - \varepsilon$ (and $a + \varepsilon$, respectively) for arbitrarily small $\varepsilon > 0$. If this notation appears in an estimate, then an implicit constant is allowed to depend on $\varepsilon > 0$ (and it usually diverges as $\varepsilon \rightarrow 0$).

¹⁰ In our spatially homogeneous setting, the variance $\sigma_N(t)$ is independent of $x \in \mathbb{T}^3$.

As for the second order process Ψ_N in (1.15), if we proceed with a “parabolic thinking”,¹¹ then we expect that Ψ_N has regularity¹² $3\alpha - \frac{1}{2}- = (3\alpha - \frac{3}{2}-) + 1$, which is negative for $\alpha \leq \frac{1}{6}$. In the dispersive setting, however, we can exhibit multilinear smoothing by exploiting multilinear dispersion coming from an interaction of (random) waves. In fact, by adapting the argument in [12] to our current problem, we can show an extra $\sim \frac{1}{2}$ -smoothing for Ψ_N , uniformly in $N \in \mathbb{N}$, and for the limit $\Psi = \mathcal{I}(\Psi) = \lim_{N \rightarrow \infty} \Psi_N$ and thus they have positive regularity. See Lemma 3.1. As in [12,26], such multilinear smoothing plays a fundamental role in our analysis.

Let us now start with the truncated renormalized SNLW (1.7) and obtain the limiting formulation of our problem. By proceeding with the second order expansion:

$$u_N = \mathfrak{I}_N - \Psi_N + v_N, \tag{1.17}$$

we rewrite (1.7) as

$$\begin{aligned} (\partial_t^2 + 1 - \Delta)v_N &= -(v_N + \mathfrak{I}_N - \Psi_N)^3 + 3\sigma_N(\mathfrak{I}_N - \Psi_N + v_N) + \Psi_N \\ &= -v_N^3 + 3(\Psi_N - \mathfrak{I}_N)v_N^2 - 3(\Psi_N^2 - 2\Psi_N\mathfrak{I}_N)v_N - 3\Psi_N v_N \\ &\quad + \Psi_N^3 - 3\Psi_N^2\mathfrak{I}_N + 3\Psi_N\Psi_N, \end{aligned} \tag{1.18}$$

where we used (1.14). The main problem in studying singular stochastic PDEs lies in making sense of various products. In this formal discussion, let us apply the following “rules”:

- A product of functions of regularities s_1 and s_2 is defined if $s_1 + s_2 > 0$. When $s_1 > 0$ and $s_1 \geq s_2$, the resulting product has regularity s_2 .
- A product of stochastic objects (not depending on the unknown) is always well defined, possibly with a renormalization. The product of stochastic objects of regularities s_1 and s_2 has regularity $\min(s_1, s_2, s_1 + s_2)$.

We postulate that the unknown v has regularity $\frac{1}{2}+$,¹³ which is subcritical with respect to the standard scaling heuristics for the three-dimensional cubic NLW. In order to close the Picard iteration argument, we need all the terms on the right-hand side of (1.18) to have regularity $-\frac{1}{2}+$. With the aforementioned regularities of the stochastic terms $\mathfrak{I}_N, \Psi_N,$ and Ψ_N and applying the rules above, we can handle the products on the right-hand side of (1.18), giving regularity $-\frac{1}{2}+$, *except* for the following terms (for small $\alpha > 0$):

$$\Psi_N\mathfrak{I}_N\Psi_N, \quad \Psi_N\Psi_N, \quad \text{and} \quad \Psi_N\Psi_N. \tag{1.19}$$

¹¹ Namely, if we only take into account the (uniformly bounded in N) regularity $3\alpha - \frac{3}{2}-$ of Ψ_N and one degree of smoothing from the Duhamel integral operator \mathcal{I} *without* taking into account the product structure and the oscillatory nature of the linear wave propagator.

¹² By “regularity”, we mean the spatial regularity s of Ψ_N as an element in $C([0, T]; W^{s,\infty}(\mathbb{T}^3))$, uniformly bounded in $N \in \mathbb{N}$.

¹³ As for the unknown v , we measure its regularity in (the local-in-time version of) the $X^{s,\frac{1}{2}+}$ -norm.

As for the first term $\Psi_N \uparrow_N v_N$, we first use stochastic analysis to make sense of $\Psi_N \uparrow_N$ with regularity $\alpha - \frac{1}{2}-$, uniformly in $N \in \mathbb{N}$, (see Lemma 3.3) and then interpret the product as

$$\Psi_N \uparrow_N v_N = (\Psi_N \uparrow_N) v_N.$$

Note that the right-hand side is well defined since the sum of the regularities is positive: $(\alpha - \frac{1}{2}-) + (\frac{1}{2}+) > 0$. The last product $\Psi_N \Psi_N$ in (1.19) makes sense but the resulting regularity is $2\alpha - 1-$, smaller than the required regularity $-\frac{1}{2}+$, when α is close to 0. As for the second term in (1.19), it depends on the unknown v_N and thus the product does not make sense (at this point) since the sum of regularities is negative (when $\alpha > 0$ is small).

As we see below, by studying the last two terms in (1.19) under the Duhamel integral operator \mathcal{I} , we can indeed give a meaning to them and exhibit extra $(\frac{1}{2}+)$ -smoothing with the resulting regularity $\frac{1}{2}+$ (under \mathcal{I}), which allows us to close the argument. By writing (1.18) with initial data (u_0, u_1) in the Duhamel formulation, we have

$$\begin{aligned} v_N &= S(t)(u_0, u_1) + \mathcal{I}(-v_N^3 + 3(\Psi_N - \uparrow_N)v_N^2 - 3\Psi_N^2 v_N) \\ &\quad + 6\mathcal{I}((\Psi_N \uparrow_N)v_N) - 3\mathcal{J}^{\Psi_N}(v_N) \\ &\quad + \mathcal{I}(\Psi_N^3 - 3\Psi_N^2 \uparrow_N) + 3\Psi_N, \end{aligned} \tag{1.20}$$

where $S(t)(u_0, u_1) = \cos(t\langle \nabla \rangle)u_0 + \frac{\sin(t\langle \nabla \rangle)}{\langle \nabla \rangle}u_1$ denotes the (deterministic) linear solution. Here, \mathcal{J}^{Ψ_N} denotes the random operator defined by

$$\mathcal{J}^{\Psi_N}(v) = \mathcal{I}(\Psi_N v) \tag{1.21}$$

and (as the notation suggests), the last term in (1.20) is defined by

$$\Psi_N = \mathcal{I}(\Psi_N \Psi_N) \tag{1.22}$$

(without a renormalization). By exploiting random multilinear dispersion, we show that

- the random operator \mathcal{J}^{Ψ_N} maps functions of regularity $\frac{1}{2}+$ to those of regularity $\frac{1}{2}+$ (measured in the $X^{s,b}$ -spaces) with the operator norm uniformly bounded in $N \in \mathbb{N}$ and \mathcal{J}^{Ψ_N} converges to some limit, denoted by \mathcal{J}^{Ψ} , as $N \rightarrow \infty$. We study the random operator \mathcal{J}^{Ψ_N} via the random matrix approach [9,10,12,18,59].¹⁴ See Lemma 3.5.

¹⁴ We also mention a recent preprint [61], where the random matrix approach is also used to prove probabilistic local well-posedness of the Zakharov–Yukawa system on the two-dimensional torus \mathbb{T}^2 .

- the third order process Ψ_N has regularity $\frac{1}{2}+$ (measured in the $X^{s,b}$ -spaces) with the norm uniformly bounded in $N \in \mathbb{N}$ and Ψ_N converges to some limit, denoted by Ψ , as $N \rightarrow \infty$. See Lemma 3.4.

We deduce these claims as corollaries to Bringmann’s work [12]. In [12], the smoothing coming from the potential $V = \langle \nabla \rangle^{-\beta}$ in the Hartree nonlinearity $(V * u^2)u$ played an important role. In our problem, this is replaced by the smoothing $\langle \nabla \rangle^{-\alpha}$ on the noise and we reduce our problem to that in [12], essentially by the following simple observation:

$$\prod_{j=1}^k \langle n_j \rangle^{-\gamma} \lesssim \langle n_1 + \dots + n_k \rangle^{-\gamma} \tag{1.23}$$

for any $\gamma \geq 0$.

Remark 1.7 In the following, we also set

$$\Psi_N^\dagger = \mathcal{I}(\Psi_N^2 \uparrow_N). \tag{1.24}$$

By carrying out analysis analogous to (but more involved than) that for $\Psi_N \uparrow_N$ studied in Lemma 3.3 below, we can show that $\{\Psi_N^\dagger\}_{N \in \mathbb{N}}$ forms a Cauchy sequence in $C([0, T]; W^{\alpha-\frac{1}{2}-, \infty}(\mathbb{T}^3))$ almost surely, thus converging to some limit Ψ^\dagger . In this paper, however, we proceed with space-time analysis as in [12]. Namely, we study Ψ_N^\dagger in the $X^{s,b}$ -spaces and show that it converges to some limit denoted by Ψ^\dagger . See Lemma 3.4.

Putting everything together, we can take $N \rightarrow \infty$ in (1.20) and obtain the following limiting equation for $v = u - \uparrow + \Psi^\dagger$:

$$\begin{aligned} v &= S(t)(u_0, u_1) + \mathcal{I}(-v^3 + 3(\Psi - \uparrow)v^2 - 3\Psi^2 v) \\ &\quad + 6\mathcal{I}((\Psi \uparrow)v) - 3\mathcal{J}^V(v) \\ &\quad + \mathcal{I}(\Psi^3) - 3\Psi^\dagger + 3\Psi. \end{aligned} \tag{1.25}$$

By the Fourier restriction norm method with the Strichartz estimates, we can then prove local well-posedness of (1.25) in the deterministic manner. Namely, given the following enhanced data set

$$\Xi = (u_0, u_1, \uparrow, \Psi, \Psi \uparrow, \Psi^\dagger, \Psi^\dagger, \mathcal{J}^V) \tag{1.26}$$

of appropriate regularities (depicted by stochastic analysis), there exists a unique local-in-time solution v to (1.25), continuously depending on the enhanced data set Ξ . See Proposition 3.7 for a precise statement.

This local well-posedness result together with the convergence of \uparrow_N and Ψ_N then yields the convergence of $u_N = \uparrow_N - \Psi_N + v_N$ in (1.17) to the limiting process

$$u = \uparrow - \Psi + v,$$

where v is the solution to (1.25).

Remark 1.8 In terms of regularity counting, the sum of the regularities in $\uparrow \cdot v^2$ is positive. In the parabolic setting, one may then proceed with a product estimate. In the current dispersive setting, however, integrability of functions plays an important role and thus we need to proceed with care. See Lemmas 2.7 and 3.6.

Remark 1.9 (i) By the use of stochastic analysis, the stochastic terms $\uparrow, \Psi, \Psi \uparrow, \Psi \Psi, \Psi \Psi \Psi,$ and \mathfrak{I}^Ψ in the enhanced data set are defined as the unique limits of their truncated versions. Furthermore, by deterministic analysis, we prove that a solution v to (1.25) is pathwise unique in an appropriate class. Therefore, under the decomposition $u = \uparrow - \Psi + v$, the uniqueness of u refers to (a) the uniqueness of \uparrow and Ψ as the limits of \uparrow_N and Ψ_N and (b) the uniqueness of v as a solution to (1.25).

(ii) In this paper, we work with the frequency projector π_N with a sharp cutoff function on the frequency side. It is also possible to work with smooth mollifiers $\eta_\delta(x) = \delta^{-3}\eta(\delta^{-1}x)$, where $\eta \in C^\infty(\mathbb{R}^3; [0, 1])$ is a smooth, non-negative, even function with $\int \eta dx = 1$ and $\text{supp } \eta \subset (-\pi, \pi]^3 \simeq \mathbb{T}^3$. In this case, working with

$$\begin{cases} \partial_t^2 u_\delta + (1 - \Delta)u_\delta + u_\delta^3 - 3\sigma_\delta u_\delta = \langle \nabla \rangle^{-\alpha} \eta_\delta * \xi \\ (u_\delta, \partial_t u_\delta)|_{t=0} = (u_0, u_1), \end{cases} \tag{1.27}$$

we can show that a solution u_δ to (1.27) converges in probability to some limit u in $C([-T_\omega, T_\omega]; H^{\alpha-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))$ as $\delta \rightarrow 0$. Furthermore, the limit u_δ is independent of the choice of a mollification kernel η and agrees with the limiting process u constructed in Theorem 1.1. This is the second meaning of the uniqueness of the limiting process u .

Remark 1.10 (i) From the “scaling” point of view, our problem for $0 < \alpha \ll 1$ is more difficult than the quadratic SNLW (1.8) considered in [26], where the paracontrolled calculus played an essential role. On the other hand, for the proof of Theorem 1.1, we do not need to use the paracontrolled ansatz for the remainder terms $v = u - \uparrow + \Psi$ thanks to the smoothing on the noise and the use of space-time estimates, which allows us to place v in the subcritical regularity $\frac{1}{2}+$.

Our approach to (1.6) and Bringmann’s approach in [12] crucially exploit various multilinear smoothing, gaining $\sim \frac{1}{2}$ -derivative. When $\alpha = 0$ (or $\beta = 0$ in the Hartree SNLW (1.9)), such multilinear smoothing seems to give (at best) $\frac{1}{2}$ -smoothing and thus the arguments in this paper and in [12] break down in the $\alpha = 0$ case.

- (ii) In [26], Gubinelli, Koch, and the first author studied the quadratic SNLW on \mathbb{T}^3 with an additive space-time white noise (i.e. $\alpha = 0$):

$$\partial_t^2 u + (1 - \Delta)u + u^2 = \xi. \tag{1.28}$$

With the Wick renormalization and the second order expansion $u = \uparrow - \Upsilon + v$, where $\Upsilon = \mathcal{I}(v)$, the remainder term $v = u - \uparrow + \Upsilon$ satisfies

$$(\partial_t^2 + 1 - \Delta)v = -(v - \Upsilon)^2 - 2\uparrow v + 2\uparrow\Upsilon. \tag{1.29}$$

As observed in [26], the main issue in studying (1.29) comes from the regularity $\frac{1}{2}-$ of v , which is inherited from the regularity $-\frac{1}{2}-$ of $\uparrow\Upsilon$. As a result, the product $\uparrow v$ in (1.29) is not well defined since the sum of the regularities of \uparrow and v is negative. As in (1.21), it is tempting to directly define the random operator $\mathfrak{J}^\uparrow(v) = \mathcal{I}(\uparrow v)$, using the random matrix estimates. However, there is an issue in handling the “high \times high \rightarrow low” interaction and thus the random matrix approach alone is not sufficient to close the argument. In [26], this issue was overcome by a paracontrolled ansatz and an iteration of the Duhamel formulation. We point out that the use of the paracontrolled ansatz in [26] led to the following paracontrolled operator $\mathfrak{J}_\otimes(v) = \mathcal{I}(v \otimes \uparrow)$, which avoids the undesirable high \times high \rightarrow low interaction. Instead of the paracontrolled calculus, one may use the random averaging operator from [17] together with an iteration of the Duhamel formulation. We, however, point out that due to the problematic high \times high interaction, the random averaging operator as introduced in [17] alone (without iterating the Duhamel formulation) does not seem to be sufficient to study the quadratic SNLW (1.28).

Organization of the paper In Sect. 2, we go over the basic definitions and lemmas from deterministic and stochastic analysis. In Sect. 3, we first state the almost sure regularity and convergence properties of (the truncated versions of) the stochastic objects in the enhanced data set Ξ in (1.26). Then, we present the proof of our main result (Theorem 1.1). In Sect. 4, we establish the almost sure regularity and convergence properties of the stochastic objects in the enhanced data set. In Appendix A, we recall the counting lemmas from [12] which play a crucial role in Sect. 4. In Appendices B and C, we provide the basic definitions and lemmas on multiple stochastic integrals and (random) tensors, respectively.

2 Notations and basic lemmas

We write $A \lesssim B$ to denote an estimate of the form $A \leq CB$. Similarly, we write $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$ and use $A \ll B$ when we have $A \leq cB$ for small $c > 0$. We also use $a+$ (and $a-$) to mean $a + \varepsilon$ (and $a - \varepsilon$, respectively) for arbitrarily small $\varepsilon > 0$.

When we work with space-time function spaces, we use short-hand notations such as $C_T H_x^s = C([0, T]; H^s(\mathbb{T}^3))$.

When there is no confusion, we simply use \widehat{u} or $\mathcal{F}(u)$ to denote the spatial, temporal, or space-time Fourier transform of u , depending on the context. We also use $\mathcal{F}_x, \mathcal{F}_t$, and $\mathcal{F}_{x,t}$ to denote the spatial, temporal, and space-time Fourier transforms, respectively.

We use the following short-hand notation: $n_{ij} = n_i + n_j$, etc. For example, $n_{123} = n_1 + n_2 + n_3$.

2.1 Sobolev spaces and Besov spaces

Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. We define the L^2 -based Sobolev space $H^s(\mathbb{T}^3)$ by the norm:

$$\|f\|_{H^s} = \|\langle n \rangle^s \widehat{f}(n)\|_{\ell_n^2}$$

and set $\mathcal{H}^s(\mathbb{T}^3)$ to be

$$\mathcal{H}^s(\mathbb{T}^3) = H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3).$$

We also define the L^p -based Sobolev space $W^{s,p}(\mathbb{T}^3)$ by the norm:

$$\|f\|_{W^{s,p}} = \|\mathcal{F}^{-1}(\langle n \rangle^s \widehat{f}(n))\|_{L^p}.$$

When $p = 2$, we have $H^s(\mathbb{T}^3) = W^{s,2}(\mathbb{T}^3)$.

Let $\phi : \mathbb{R} \rightarrow [0, 1]$ be a smooth bump function supported on $[-\frac{8}{5}, \frac{8}{5}]$ and $\phi \equiv 1$ on $[-\frac{5}{4}, \frac{5}{4}]$. For $\xi \in \mathbb{R}^3$, we set $\phi_0(\xi) = \phi(|\xi|)$ and

$$\phi_j(\xi) = \phi\left(\frac{|\xi|}{2^j}\right) - \phi\left(\frac{|\xi|}{2^{j-1}}\right)$$

for $j \in \mathbb{N}$. Note that we have

$$\sum_{j \in \mathbb{N}_0} \phi_j(\xi) = 1 \tag{2.1}$$

for any $\xi \in \mathbb{R}^3$. Then, for $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we define the Littlewood-Paley projector \mathbf{P}_j as the Fourier multiplier operator with a symbol ϕ_j . Thanks to (2.1), we have

$$f = \sum_{j=0}^{\infty} \mathbf{P}_j f. \tag{2.2}$$

Next, we recall the following paraproduct decomposition due to Bony [6]. See [1,24] for further details. Let f and g be functions on \mathbb{T}^3 of regularities s_1 and s_2 , respectively. Using (2.2), we write the product fg as

$$\begin{aligned} fg &= f \odot g + f \ominus g + f \circledast g \\ &:= \sum_{j < k-2} \mathbf{P}_j f \mathbf{P}_k g + \sum_{|j-k| \leq 2} \mathbf{P}_j f \mathbf{P}_k g + \sum_{k < j-2} \mathbf{P}_j f \mathbf{P}_k g. \end{aligned} \tag{2.3}$$

The first term $f \odot g$ (and the third term $f \ominus g$) is called the paraproduct of g by f (the paraproduct of f by g , respectively) and it is always well defined as a distribution of regularity $\min(s_2, s_1 + s_2)$. On the other hand, the resonant product $f \ominus g$ is well defined in general only if $s_1 + s_2 > 0$.

We briefly recall the basic properties of the Besov spaces $B_{p,q}^s(\mathbb{T}^3)$ defined by the norm:

$$\|u\|_{B_{p,q}^s} = \left\| 2^{sj} \|\mathbf{P}_j u\|_{L_x^p} \right\|_{\ell_j^q(\mathbb{N}_0)}.$$

Note that $H^s(\mathbb{T}^3) = B_{2,2}^s(\mathbb{T}^3)$.

Lemma 2.1 (i) (paraproduct and resonant product estimates) *Let $s_1, s_2 \in \mathbb{R}$ and $1 \leq p, p_1, p_2, q \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, we have*

$$\|f \odot g\|_{B_{p,q}^{s_2}} \lesssim \|f\|_{L^{p_1}} \|g\|_{B_{p_2,q}^{s_2}}. \tag{2.4}$$

When $s_1 < 0$, we have

$$\|f \odot g\|_{B_{p,q}^{s_1+s_2}} \lesssim \|f\|_{B_{p_1,q}^{s_1}} \|g\|_{B_{p_2,q}^{s_2}}. \tag{2.5}$$

When $s_1 + s_2 > 0$, we have

$$\|f \ominus g\|_{B_{p,q}^{s_1+s_2}} \lesssim \|f\|_{B_{p_1,q}^{s_1}} \|g\|_{B_{p_2,q}^{s_2}}. \tag{2.6}$$

(ii) *Let $s_1 < s_2$ and $1 \leq p, q \leq \infty$. Then, we have*

$$\|u\|_{B_{p,q}^{s_1}} \lesssim \|u\|_{W^{s_2,p}}. \tag{2.7}$$

The product estimates (2.4), (2.5), and (2.6) follow easily from the definition (2.3) of the paraproduct and the resonant product. See [1,39] for details of the proofs in the non-periodic case (which can be easily extended to the current periodic setting). The embedding (2.7) follows from the ℓ^q -summability of $\{2^{(s_1-s_2)j}\}_{j \in \mathbb{N}_0}$ for $s_1 < s_2$ and the uniform boundedness of the Littlewood-Paley projector \mathbf{P}_j .

We also recall the following product estimate from [25].

Lemma 2.2 *Let $0 \leq s \leq 1$. Let $1 < p, q, r < \infty$ such that $s \geq 3(\frac{1}{p} + \frac{1}{q} - \frac{1}{r})$. Then, we have*

$$\|\langle \nabla \rangle^{-s}(fg)\|_{L^r(\mathbb{T}^3)} \lesssim \|\langle \nabla \rangle^{-s} f\|_{L^p(\mathbb{T}^3)} \|\langle \nabla \rangle^s g\|_{L^q(\mathbb{T}^3)}.$$

Note that while Lemma 2.2 was shown only for $s = 3(\frac{1}{p} + \frac{1}{q} - \frac{1}{r})$ in [25], the general case $s \geq 3(\frac{1}{p} + \frac{1}{q} - \frac{1}{r})$ follows the embedding $L^{r_1}(\mathbb{T}^3) \subset L^{r_2}(\mathbb{T}^3)$, $r_1 \geq r_2$.

2.2 Fourier restriction norm method and Strichartz estimates

We first recall the so-called $X^{s,b}$ -spaces, also known as the hyperbolic Sobolev spaces, due to Klainerman-Machedon [34] and Bourgain [7], defined by the norm:

$$\|u\|_{X^{s,b}(\mathbb{T}^3 \times \mathbb{R})} = \| \langle n \rangle^s \langle |\tau| - \langle n \rangle \rangle^b \widehat{u}(n, \tau) \|_{\ell_n^2 L_\tau^2(\mathbb{Z}^3 \times \mathbb{R})}. \tag{2.8}$$

For $b > \frac{1}{2}$, we have $X^{s,b} \subset C(\mathbb{R}; H^s(\mathbb{T}^3))$. Given an interval $I \subset \mathbb{R}$, we define the local-in-time version $X^{s,b}(I)$ as a restriction norm:

$$\|u\|_{X^{s,b}(I)} = \inf \{ \|v\|_{X^{s,b}(\mathbb{T}^3 \times \mathbb{R})} : v|_I = u \}. \tag{2.9}$$

When $I = [0, T]$, we set $X_T^{s,b} = X^{s,b}(I)$.

Next, we recall the Strichartz estimates for the linear wave/Klein–Gordon equation. Given $0 \leq s \leq 1$, we say that a pair (q, r) is s -admissible if $2 < q \leq \infty$, $2 \leq r < \infty$,

$$\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - s \quad \text{and} \quad \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}.$$

Then, we have the following Strichartz estimates.

Lemma 2.3 *Given $0 \leq s \leq 1$, let (q, r) be s -admissible. Then, we have*

$$\|S(t)(\phi_0, \phi_1)\|_{L_T^q L_x^r(\mathbb{T}^3)} \lesssim \|(\phi_0, \phi_1)\|_{\mathcal{H}^s(\mathbb{T}^3)} \tag{2.10}$$

for any $0 < T \leq 1$.

See Ginibre–Velo [23], Lindblad–Sogge [38], and Keel–Tao [32] for the Strichartz estimates on \mathbb{R}^d . See also [33]. The Strichartz estimates (2.10) on \mathbb{T}^3 in Lemma 2.3 follows from those on \mathbb{R}^3 and the finite speed of propagation.

When $b > \frac{1}{2}$, the $X^{s,b}$ -spaces enjoy the transference principle. In particular, as a corollary to Lemma 2.3, we obtain the following space-time estimate. See [35,64] for the proof.

Lemma 2.4 *Let $0 < T \leq 1$. Given $0 \leq s \leq 1$, let (q, r) be s -admissible. Then, for $b > \frac{1}{2}$, we have*

$$\|u\|_{L_T^q L_x^r} \lesssim \|u\|_{X_T^{s,b}}.$$

We also state the nonhomogeneous linear estimate. See [22].

Lemma 2.5 *Let $-\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1$. Then, for $0 < T \leq 1$, we have*

$$\|\mathcal{I}(F)\|_{X_T^{s,b}} = \left\| \int_0^t \frac{\sin((t-t')\langle \nabla \rangle)}{\langle \nabla \rangle} F(t') dt' \right\|_{X_T^{s,b}} \lesssim T^{1-b+b'} \|F\|_{X_T^{s-1,b'}}.$$

In the following, we briefly go over the main trilinear estimate for the basic local well-posedness of the cubic NLW (1.5) in $\mathcal{H}^{\frac{1}{2}+\varepsilon}(\mathbb{T}^3)$.

Lemma 2.6 *Fix small $\delta_1, \delta_2 > 0$ with $4\delta_2 \leq \delta_1$. Then, we have*

$$\|\mathcal{I}(u_1 u_2 u_3)\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \lesssim T^{\delta_2} \prod_{j=1}^3 \|u_j\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \tag{2.11}$$

for any $0 < T \leq 1$.

Proof Recall that $(q, r) = (4, 4)$ is $\frac{1}{2}$ -admissible. Then, in view of Lemma 2.4, interpolating

$$\|u\|_{L_{T,x}^4} \lesssim \|u\|_{X_T^{\frac{1}{2}, \frac{1}{2}+\delta_0}} \quad \text{and} \quad \|u\|_{L_{T,x}^2} = \|u\|_{X_T^{0,0}} \tag{2.12}$$

with small $\delta_0 > 0$, we obtain

$$\|u\|_{L_{T,x}^{\frac{4}{1+2\delta_1}}} \lesssim \|u\|_{X_T^{\frac{1}{2}-\delta_1, \frac{1}{2}-\frac{1}{2}\delta_1}}. \tag{2.13}$$

Moreover, noting that $(\frac{12}{3-2\delta_1}, \frac{12}{3-2\delta_1})$ is $(\frac{1}{2} + \frac{2}{3}\delta_1)$ -admissible, we obtain from Lemma 2.4 that

$$\|u\|_{L_{T,x}^{\frac{12}{3-2\delta_1}}} \leq C_{\delta_1, \delta_2} \|u\|_{X_T^{\frac{1}{2}+\frac{2}{3}\delta_1, \frac{1}{2}+\delta_2}} \tag{2.14}$$

for any $\delta_2 > 0$.

Hence, from Lemma 2.5, duality, Hölder’s inequality, (2.13), and (2.14), we obtain

$$\begin{aligned} \|\mathcal{I}(u_1 u_2 u_3)\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} &\lesssim T^{\delta_2} \|u_1 u_2 u_3\|_{X^{-\frac{1}{2}+\delta_1, -\frac{1}{2}+2\delta_2}} \\ &= T^{\delta_2} \sup_{\|w\|_{X^{\frac{1}{2}-\delta_1, \frac{1}{2}-2\delta_2}}=1} \left| \int_0^T \int_{\mathbb{T}^3} u_1 u_2 u_3 w dx dt \right| \\ &\leq T^{\delta_2} \sup_{\|w\|_{X^{\frac{1}{2}-\delta_1, \frac{1}{2}-2\delta_2}}=1} \left(\prod_{j=1}^3 \|u_j\|_{L_{T,x}^{\frac{12}{3-2\delta_1}}} \right) \|w\|_{L_{T,x}^{\frac{4}{1+2\delta_1}}} \\ &\lesssim T^{\delta_2} \prod_{j=1}^3 \|u_j\|_{X_T^{\frac{1}{2}+\frac{2}{3}\delta_1, \frac{1}{2}+\delta_2}}, \end{aligned}$$

provided that $0 < 4\delta_2 \leq \delta_1 \ll 1$. This proves (2.11). □

We conclude this part by establishing the following trilinear estimate, which will be used to control the term $\uparrow v^2$ in (1.25). See Proposition 8.6 in [12] for an analogous trilinear estimate.

Lemma 2.7 *Let $\delta_1, \delta_2 > 0$ be sufficiently small such that $8\delta_2 \leq \delta_1$. Then, we have*

$$\|u_1 u_2 u_3\|_{X_T^{-\frac{1}{2}+\delta_1, -\frac{1}{2}+2\delta_2}} \lesssim \|u_1\|_{L_T^\infty W_x^{-\frac{1}{2}+2\delta_1, \infty}} \|u_2\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \|u_3\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \quad (2.15)$$

for any $0 < T \leq 1$.

Proof By applying the Littlewood-Paley decompositions, we have

$$\begin{aligned} & \text{LHS of (2.15)} \\ & \leq \sum_{j_1, j_{23}, j_{123}=0}^{\infty} \|\mathbf{P}_{j_{123}}(\mathbf{P}_{j_1} u_1 \mathbf{P}_{j_{23}}(u_2 u_3))\|_{X_T^{-\frac{1}{2}+\delta_1, -\frac{1}{2}+2\delta_2}}. \end{aligned}$$

For simplicity of notation, we set $N_1 = 2^{j_1}$, $N_{23} = 2^{j_{23}}$, and $N_{123} = 2^{j_{123}}$, denoting the dyadic frequency sizes of n_1 (for u_1), n_{23} (for $u_2 u_3$), and n_{123} (for $u_1 u_2 u_3$), respectively. We set $v_k = \mathbf{P}_{j_k} u_k$. In view of $n_{123} = n_1 + n_{23}$, we separately estimate the contributions from (i) $N_{123} \sim \max(N_1, N_{23})$ and (ii) $N_{123} \ll \max(N_1, N_{23})$.

Case 1: $N_{123} \sim \max(N_1, N_{23})$.

By Hölder’s inequality and the L^4 -Strichartz estimate (2.12), we have

$$\begin{aligned} & \|\mathbf{P}_{j_{123}}(v_1 \mathbf{P}_{j_{23}}(u_2 u_3))\|_{X_T^{-\frac{1}{2}+\delta_1, -\frac{1}{2}+2\delta_2}} \lesssim N_{123}^{-\frac{1}{2}+\delta_1} \|v_1 \mathbf{P}_{j_{23}}(u_2 u_3)\|_{L_{T,x}^2} \\ & \lesssim N_{123}^{-\delta_1} \|u_1\|_{L_T^\infty W_x^{-\frac{1}{2}+2\delta_1, \infty}} \prod_{j=2}^3 \|u_j\|_{L_{T,x}^4} \\ & \lesssim N_{123}^{-\delta_1} \|u_1\|_{L_T^\infty W_x^{-\frac{1}{2}+2\delta_1, \infty}} \prod_{j=2}^3 \|u_j\|_{X_T^{\frac{1}{2}, \frac{1}{2}+\delta_2}}. \end{aligned}$$

This is summable in dyadic $N_1, N_{23}, N_{123} \geq 1$, yielding (2.15) in this case.

Case 2: $N_{123} \ll \max(N_1, N_{23})$.

In this case, we further apply the Littlewood-Paley decompositions for u_2 and u_3 and write

$$u_2 u_3 = \sum_{j_2, j_3=0}^{\infty} (\mathbf{P}_{j_2} u_2)(\mathbf{P}_{j_3} u_3).$$

Without loss of generality, assume $N_3 \geq N_2$, where $N_k = 2^{j_k}$, $k = 2, 3$. Then, we have

$$N_{123} \lesssim N_1 \sim N_{23} \lesssim N_3. \quad (2.16)$$

By duality and (2.13) (with $\delta_1 = 4\delta_2$), we have

$$\begin{aligned} \|\mathbf{P}_j u\|_{X_T^{0, -\frac{1}{2} + 2\delta_2}} &= \sup_{\|v\|_{X^{0, \frac{1}{2} - 2\delta_2}} = 1} \left| \int_0^T \int_{\mathbb{T}^3} (\mathbf{P}_j u) (\mathbf{P}_{j-1} + \mathbf{P}_j + \mathbf{P}_{j+1}) v dx dt \right| \\ &\lesssim 2^{(\frac{1}{2} - 4\delta_2)j} \|\mathbf{P}_j u\|_{L_{T,x}^{\frac{4}{3-8\delta_2}}}. \end{aligned} \tag{2.17}$$

Then, from (2.17), (2.14), and (2.16) with $8\delta_2 \leq \delta_1$, we have

$$\begin{aligned} \|\mathbf{P}_{j_{123}}(v_1 \mathbf{P}_{j_{23}}(v_2 v_3))\|_{X_T^{-\frac{1}{2} + \delta_1, -\frac{1}{2} + 2\delta_2}} &\lesssim N_{123}^{-\frac{1}{2} + \delta_1} N_{123}^{\frac{1}{2} - 4\delta_2} \|v_1 \mathbf{P}_{j_{23}}(v_2 v_3)\|_{L_{T,x}^{\frac{4}{3-8\delta_2}}} \\ &\lesssim N_{123}^{\delta_1 - 4\delta_2} N_1^{\frac{1}{2} - 2\delta_1} \|v_1\|_{L_T^\infty W_x^{-\frac{1}{2} + 2\delta_1, \infty}} \|v_2\|_{L_{T,x}^{\frac{4}{1-8\delta_2}}} \|v_3\|_{L_{T,x}^2} \\ &\lesssim N_{123}^{\delta_1 - 4\delta_2} N_1^{\frac{1}{2} - 2\delta_1} N_3^{-\frac{1}{2} - \delta_1} \\ &\quad \times \|u_1\|_{L_T^\infty W_x^{-\frac{1}{2} + 2\delta_1, \infty}} \|u_2\|_{X_T^{\frac{1}{2} + 8\delta_2, \frac{1}{2} + \delta_2}} \|u_3\|_{X_T^{\frac{1}{2} + \delta_1, 0}} \\ &\lesssim N_{123}^{-4\delta_2} N_1^{-\delta_1} N_3^{-\delta_1} \|u_1\|_{L_T^\infty W_x^{-\frac{1}{2} + 2\delta_1, \infty}} \prod_{j=2}^3 \|u_j\|_{X_T^{\frac{1}{2} + \delta_1, \frac{1}{2} + \delta_2}}. \end{aligned}$$

This is summable in dyadic $N_1, N_2, N_3, N_{23}, N_{123} \geq 1$, yielding (2.15) in this case. □

2.3 On discrete convolutions

Next, we recall the following basic lemma on a discrete convolution.

Lemma 2.8 (i) *Let $d \geq 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy*

$$\alpha + \beta > d \quad \text{and} \quad \alpha, \beta < d.$$

Then, we have

$$\sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta} \lesssim \langle n \rangle^{d-\alpha-\beta}$$

for any $n \in \mathbb{Z}^d$.

(ii) *Let $d \geq 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy $\alpha + \beta > d$. Then, we have*

$$\sum_{\substack{n=n_1+n_2 \\ |n_1| \sim |n_2|}} \frac{1}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta} \lesssim \langle n \rangle^{d-\alpha-\beta}$$

for any $n \in \mathbb{Z}^d$.

Namely, in the resonant case (ii), we do not have the restriction $\alpha, \beta < d$. Lemma 2.8 follows from elementary computations. See, for example, Lemmas 4.1 and 4.2 in [41] for the proof.

2.4 Tools from stochastic analysis

We conclude this section by recalling useful lemmas from stochastic analysis. See [5,43,62] for basic definitions. See also Appendix B for basic definitions and properties for multiple stochastic integrals.

Let (H, B, μ) be an abstract Wiener space. Namely, μ is a Gaussian measure on a separable Banach space B with $H \subset B$ as its Cameron-Martin space. Given a complete orthonormal system $\{e_j\}_{j \in \mathbb{N}} \subset B^*$ of $H^* = H$, we define a polynomial chaos of order k to be an element of the form $\prod_{j=1}^{\infty} H_{k_j}(\langle x, e_j \rangle)$, where $x \in B, k_j \neq 0$ for only finitely many j 's, $k = \sum_{j=1}^{\infty} k_j, H_{k_j}$ is the Hermite polynomial of degree k_j , and $\langle \cdot, \cdot \rangle = {}_B \langle \cdot, \cdot \rangle_{B^*}$ denotes the B - B^* duality pairing. We then denote the closure of polynomial chaoses of order k under $L^2(B, \mu)$ by \mathcal{H}_k . The elements in \mathcal{H}_k are called homogeneous Wiener chaoses of order k . We also set

$$\mathcal{H}_{\leq k} = \bigoplus_{j=0}^k \mathcal{H}_j$$

for $k \in \mathbb{N}$.

Let $L = \Delta - x \cdot \nabla$ be the Ornstein-Uhlenbeck operator.¹⁵ Then, it is known that any element in \mathcal{H}_k is an eigenfunction of L with eigenvalue $-k$. Then, as a consequence of the hypercontractivity of the Ornstein-Uhlenbeck semigroup $U(t) = e^{tL}$ due to Nelson [42], we have the following Wiener chaos estimate [63, Theorem I.22]. See also [65, Proposition 2.4].

Lemma 2.9 *Let $k \in \mathbb{N}$. Then, we have*

$$\|X\|_{L^p(\Omega)} \leq (p - 1)^{\frac{k}{2}} \|X\|_{L^2(\Omega)}$$

for any $p \geq 2$ and any $X \in \mathcal{H}_{\leq k}$.

The following lemma will be used in studying regularities of stochastic objects. We say that a stochastic process $X : \mathbb{R}_+ \rightarrow \mathcal{D}'(\mathbb{T}^d)$ is spatially homogeneous if $\{X(\cdot, t)\}_{t \in \mathbb{R}_+}$ and $\{X(x_0 + \cdot, t)\}_{t \in \mathbb{R}_+}$ have the same law for any $x_0 \in \mathbb{T}^d$. Given $h \in \mathbb{R}$, we define the difference operator δ_h by setting

$$\delta_h X(t) = X(t + h) - X(t).$$

Lemma 2.10 *Let $\{X_N\}_{N \in \mathbb{N}}$ and X be spatially homogeneous stochastic processes $:\mathbb{R}_+ \rightarrow \mathcal{D}'(\mathbb{T}^d)$. Suppose that there exists $k \in \mathbb{N}$ such that $X_N(t)$ and $X(t)$ belong to $\mathcal{H}_{\leq k}$ for each $t \in \mathbb{R}_+$.*

¹⁵ For simplicity, we write the definition of the Ornstein-Uhlenbeck operator L when $B = \mathbb{R}^d$.

(i) Let $t \in \mathbb{R}_+$. If there exists $s_0 \in \mathbb{R}$ such that

$$\mathbb{E}[|\widehat{X}(n, t)|^2] \lesssim \langle n \rangle^{-d-2s_0} \tag{2.18}$$

for any $n \in \mathbb{Z}^d$, then we have $X(t) \in W^{s, \infty}(\mathbb{T}^d)$, $s < s_0$, almost surely.

(ii) Suppose that X_N , $N \in \mathbb{N}$, satisfies (2.18). Furthermore, if there exists $\gamma > 0$ such that

$$\mathbb{E}[|\widehat{X}_N(n, t) - \widehat{X}_M(n, t)|^2] \lesssim N^{-\gamma} \langle n \rangle^{-d-2s_0}$$

for any $n \in \mathbb{Z}^d$ and $M \geq N \geq 1$, then $X_N(t)$ is a Cauchy sequence in $W^{s, \infty}(\mathbb{T}^d)$, $s < s_0$, almost surely, thus converging to some limit in $W^{s, \infty}(\mathbb{T}^d)$.

(iii) Let $T > 0$ and suppose that (i) holds on $[0, T]$. If there exists $\sigma \in (0, 1)$ such that

$$\mathbb{E}[|\delta_h \widehat{X}(n, t)|^2] \lesssim \langle n \rangle^{-d-2s_0+\sigma} |h|^\sigma$$

for any $n \in \mathbb{Z}^d$, $t \in [0, T]$, and $h \in [-1, 1]$,¹⁶ then we have $X \in C([0, T]; W^{s, \infty}(\mathbb{T}^d))$, $s < s_0 - \frac{\sigma}{2}$, almost surely.

(iv) Let $T > 0$ and suppose that (ii) holds on $[0, T]$. Furthermore, if there exists $\gamma > 0$ such that

$$\mathbb{E}[|\delta_h \widehat{X}_N(n, t) - \delta_h \widehat{X}_M(n, t)|^2] \lesssim N^{-\gamma} \langle n \rangle^{-d-2s_0+\sigma} |h|^\sigma$$

for any $n \in \mathbb{Z}^d$, $t \in [0, T]$, $h \in [-1, 1]$, and $M \geq N \geq 1$, then X_N is a Cauchy sequence in $C([0, T]; W^{s, \infty}(\mathbb{T}^d))$, $s < s_0 - \frac{\sigma}{2}$, almost surely, thus converging to some process in $C([0, T]; W^{s, \infty}(\mathbb{T}^d))$.

Lemma 2.10 follows from a straightforward application of the Wiener chaos estimate (Lemma 2.9). For the proof, see Proposition 3.6 in [41] and Appendix in [50]. As compared to Proposition 3.6 in [41], we made small adjustments. In studying the time regularity, we made the following modifications: $\langle n \rangle^{-d-2s_0+2\sigma} \mapsto \langle n \rangle^{-d-2s_0+\sigma}$ and $s < s_0 - \sigma \mapsto s < s_0 - \frac{\sigma}{2}$ so that it is suitable for studying the wave equation. Moreover, while the result in [41] is stated in terms of the Besov-Hölder space $C^s(\mathbb{T}^d) = B_{\infty, \infty}^s(\mathbb{T}^d)$, Lemma 2.10 handles the L^∞ -based Sobolev space $W^{s, \infty}(\mathbb{T}^3)$. Note that the required modification of the proof is straightforward since $W^{s, \infty}(\mathbb{T}^d)$ and $B_{\infty, \infty}^s(\mathbb{T}^d)$ differ only logarithmically:

$$\|f\|_{W^{s, \infty}} \leq \sum_{j=0}^{\infty} \|\mathbf{P}_j f\|_{W^{s, \infty}} \lesssim \|f\|_{B_{\infty, \infty}^{s+\varepsilon}} \tag{2.19}$$

for any $\varepsilon > 0$. For the proof of the almost sure convergence claims, see [50].

¹⁶ We impose $h \geq -t$ such that $t + h \geq 0$.

3 Local well-posedness of SNLW, $\alpha > 0$

In this section, we present the proof of local well-posedness of (1.25) (Theorem 1.1). In Sect. 3.1, we first state the regularity and convergence properties of the stochastic objects in the enhanced data set Ξ in (1.26). In Sect. 3.2, we then present a deterministic local well-posedness result by viewing elements in the enhanced data set as given (deterministic) distributions and a given (deterministic) operator with prescribed regularity properties.

3.1 On the stochastic terms

In this subsection, we state the regularity and convergence properties of the stochastic objects in (1.26) whose proofs are presented in Sect. 4.

Lemma 3.1 *Let $\alpha > 0$ and $T > 0$.*

- (i) *For any $s < \alpha - \frac{1}{2}$, $\{\uparrow_N\}_{N \in \mathbb{N}}$ defined in (1.12) is a Cauchy sequence in $C([0, T]; W^{s, \infty}(\mathbb{T}^3))$, almost surely. In particular, denoting the limit by \uparrow (formally given by (1.11)), we have*

$$\uparrow \in C([0, T]; W^{\alpha - \frac{1}{2} - \varepsilon, \infty}(\mathbb{T}^3))$$

for any $\varepsilon > 0$, almost surely.

- (ii) *Let $0 < \alpha \leq \frac{1}{2}$. Then, for any $s < \alpha$, $\{\Psi_N\}_{N \in \mathbb{N}}$ defined in (1.15) is a Cauchy sequence in $C([0, T]; W^{s, \infty}(\mathbb{T}^3))$ almost surely. In particular, denoting the limit by Ψ , we have*

$$\Psi \in C([0, T]; W^{\alpha - \varepsilon, \infty}(\mathbb{T}^3))$$

for any $\varepsilon > 0$, almost surely.

Remark 3.2 (i) As mentioned in Sect. 1, a parabolic thinking gives regularity $3\alpha - \frac{1}{2}$ for Ψ . Lemma 3.1 (ii) states that, when $\alpha > 0$ is small, we indeed gain about $\frac{1}{2}$ -regularity by exploiting multilinear dispersion as in the quadratic case studied in [26]. We point out that our proof is based on an adaptation of Bringmann’s analysis on the corresponding term in the Hartree case [12] and thus the regularities we obtain in Lemma 3.1 (ii) as well as Lemmas 3.3, 3.4, and 3.5 may not be sharp (especially for large $\alpha > 0$; see, for example, a crude bound (4.9)). They are, however, sufficient for our purpose.

- (ii) In this section, we only state almost sure convergence but the same argument also yields convergence in $L^p(\Omega)$ with an exponential tail estimate (as in [12, 27, 48]). Our goal is, however, to prove local well-posedness and thus the almost sure convergence suffices for our purpose.

Lemma 3.3 *Let $0 < \alpha \leq \frac{1}{2}$ and $T > 0$. Let $\{\uparrow_N\}_{N \in \mathbb{N}}$ and $\{\Psi_N\}_{N \in \mathbb{N}}$ be as in (1.12) and (1.15). Then, for any $s < \alpha - \frac{1}{2}$, $\{\Psi_N \uparrow_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $C([0, T]; W^{s, \infty}(\mathbb{T}^3))$ almost surely. In particular, denoting the limit by $\Psi \uparrow$, we have*

$$\Psi \uparrow \in C([0, T]; W^{\alpha - \frac{1}{2} - \varepsilon, \infty}(\mathbb{T}^3))$$

for any $\varepsilon > 0$, almost surely.

Lemma 3.4 *Let $\alpha > 0$, $T > 0$, and $b > \frac{1}{2}$ be sufficiently close to $\frac{1}{2}$.*

- (i) *For any $s < \alpha + \frac{1}{2}$, $\{\Psi_N\}_{N \in \mathbb{N}}$ defined in (1.22) is a Cauchy sequence in $X^{s, b}([0, T])$. In particular, denoting the limit by Ψ , we have*

$$\Psi \in X^{\alpha + \frac{1}{2} - \varepsilon, b}([0, T]),$$

for any $\varepsilon > 0$, almost surely.

- (ii) *For any $s < \alpha + \frac{1}{2}$, $\{\Psi_N^{\bullet}\}_{N \in \mathbb{N}}$ defined in (1.24) is a Cauchy sequence in $X^{s, b}([0, T])$. In particular, denoting the limit by Ψ^{\bullet} , we have*

$$\Psi^{\bullet} \in X^{\alpha + \frac{1}{2} - \varepsilon, b}([0, T]),$$

for any $\varepsilon > 0$, almost surely.

Given Banach spaces B_1 and B_2 , we use $\mathcal{L}(B_1; B_2)$ to denote the space of bounded linear operators from B_1 to B_2 . We also set

$$\mathcal{L}_{T_0}^{s_1, s_2, b} = \bigcap_{0 < T < T_0} \mathcal{L}(X^{s_1, b}([0, T]); X^{s_2, b}([0, T])) \tag{3.1}$$

endowed with the norm given by

$$\|S\|_{\mathcal{L}_{T_0}^{s_1, s_2, b}} = \sup_{0 < T < T_0} T^{-\theta} \|S\|_{\mathcal{L}(X_T^{s_1, b}; X_T^{s_2, b})} \tag{3.2}$$

for some small $\theta > 0$.

Lemma 3.5 *Let $\alpha > 0$ and $T_0 > 0$. Then, given sufficiently small $\delta_1, \delta_2 > 0$, the sequence of the random operators $\{\mathfrak{J}^{\Psi_N}\}_{N \in \mathbb{N}}$ defined in (1.21) is a Cauchy sequence in the class $\mathcal{L}_{T_0}^{\frac{1}{2} + \delta_1, \frac{1}{2} + \delta_1, \frac{1}{2} + \delta_2}$, almost surely. In particular, denoting the limit by \mathfrak{J}^{Ψ} , we have*

$$\mathfrak{J}^{\Psi} \in \mathcal{L}_{T_0}^{\frac{1}{2} + \delta_1, \frac{1}{2} + \delta_1, \frac{1}{2} + \delta_2},$$

almost surely.

The following trilinear estimate is an immediate consequence of Lemma 2.7.

Lemma 3.6 *Let $\alpha > 0$. Let $\delta_1, \delta_2, \varepsilon > 0$ be sufficiently small such that $2\delta_1 + \varepsilon \leq \alpha$. Then, we have*

$$\|\uparrow v_1 v_2\|_{X_T^{-\frac{1}{2}+\delta_1, -\frac{1}{2}+2\delta_2}} \lesssim \|\uparrow\|_{L_T^\infty W_x^{\alpha-\frac{1}{2}-\varepsilon, \infty}} \|v_1\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \|v_2\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}}$$

for any $0 < T \leq 1$.

3.2 Proof of Theorem 1.1

In this section, we prove the following proposition. Theorem 1.1 then follows from this proposition and Lemmas 3.1 - 3.5.

Proposition 3.7 *Let $\alpha > 0, s > \frac{1}{2}$, and $T_0 > 0$. Then, there exists small $\varepsilon = \varepsilon(\alpha, s)$, $\delta_1 = \delta_1(\alpha, s), \delta_2 = \delta_2(\alpha, s) > 0$ such that if*

- \uparrow is a distribution-valued function belonging to $C([0, T_0]; W^{\alpha-\frac{1}{2}-\varepsilon, \infty}(\mathbb{T}^3))$,
- Ψ is a distribution-valued function belonging to $C([0, T_0]; W^{\alpha-\varepsilon, \infty}(\mathbb{T}^3))$,
- $\Psi\uparrow$ is a distribution-valued function belonging to $C([0, T_0]; W^{\alpha-\frac{1}{2}-\varepsilon, \infty}(\mathbb{T}^3))$,
- Ψ is a function belonging to $X^{\alpha+\frac{1}{2}-\varepsilon, \frac{1}{2}+\delta_2}([0, T_0])$,
- $\Psi\uparrow$ is a function belonging to $X^{\alpha+\frac{1}{2}-\varepsilon, \frac{1}{2}+\delta_2}([0, T_0])$,
- the operator \mathcal{J}^V belongs to the class $\mathcal{L}_{T_0}^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}$ defined in (3.1),

then the Eq. (1.25) is locally well-posed in $\mathcal{H}^s(\mathbb{T}^3)$. More precisely, given any $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$, there exist $0 < T \leq T_0$ and a unique solution v to the cubic SNLW (1.25) on $[0, T]$ in the class

$$X^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}([0, T]) \subset C([0, T]; H^{\frac{1}{2}+\delta_1}(\mathbb{T}^3)).$$

Furthermore, the solution v depends continuously on the enhanced data set

$$\Xi = (u_0, u_1, \uparrow, \Psi, \Psi\uparrow, \Psi, \Psi\uparrow, \mathcal{J}^V) \tag{3.3}$$

in the class

$$\begin{aligned} \mathcal{X}_T^{s, \alpha, \varepsilon} &= \mathcal{H}^s(\mathbb{T}^3) \times C([0, T]; W^{\alpha-\frac{1}{2}-\varepsilon, \infty}(\mathbb{T}^3)) \\ &\times C([0, T]; W^{\alpha-\varepsilon, \infty}(\mathbb{T}^3)) \times C([0, T]; W^{\alpha-\frac{1}{2}-\varepsilon, \infty}(\mathbb{T}^3)) \\ &\times X^{\alpha+\frac{1}{2}-\varepsilon, \frac{1}{2}+\delta_2}([0, T]) \times X^{\alpha+\frac{1}{2}-\varepsilon, \frac{1}{2}+\delta_2}([0, T]) \\ &\times \mathcal{L}(X^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}([0, T]); X^{\frac{1}{2}+\delta_2, \frac{1}{2}+\delta_2}([0, T])). \end{aligned}$$

Proof Given $\alpha > 0$ and $s > \frac{1}{2}$, fix small $\varepsilon > 0$ such that $\varepsilon < \min(\alpha, s - \frac{1}{2})$. Given an enhanced data set Ξ as in (3.3), we set

$$\Xi(\xi) = (\mathfrak{r}, \Psi, \Psi\mathfrak{r}, \Psi^2\mathfrak{r}, \Psi, \mathfrak{J}^V)$$

and

$$\begin{aligned} \|\Xi(\xi)\|_{\mathcal{Y}_{T_0}^{\alpha,\varepsilon}} &= \|\mathfrak{r}\|_{C_{T_0}W_x^{\alpha-\frac{1}{2}-\varepsilon,\infty}} + \|\Psi\|_{C_{T_0}W_x^{\alpha-\varepsilon,\infty}} + \|\Psi\mathfrak{r}\|_{C_{T_0}W_x^{\alpha-\frac{1}{2}-\varepsilon,\infty}} \\ &+ \|\Psi\|_{X_{T_0}^{\alpha+\frac{1}{2}-\varepsilon,\frac{1}{2}+\delta_2}} + \|\Psi\mathfrak{r}\|_{X_{T_0}^{\alpha+\frac{1}{2}-\varepsilon,\frac{1}{2}+\delta_2}} \\ &+ \|\mathfrak{J}^V\|_{\mathcal{L}_{T_0}^{\frac{1}{2}+\delta_1,\frac{1}{2}+\delta_1,\frac{1}{2}+\delta_2}}, \end{aligned}$$

where $\mathcal{L}_{T_0}^{\frac{1}{2}+\delta_1,\frac{1}{2}+\delta_1,\frac{1}{2}+\delta_2}$ is as in (3.2). In the following, we assume that

$$\|\Xi(\xi)\|_{\mathcal{Y}_{T_0}^{\alpha,\varepsilon}} \leq K \tag{3.4}$$

for some $K \geq 1$.

Given the enhanced data set Ξ in (3.3), define a map Γ_Ξ by

$$\begin{aligned} \Gamma_\Xi(v) &= S(t)(u_0, u_1) + \mathcal{I}(-v^3 + 3(\Psi - \mathfrak{r})v^2 - 3\Psi^2v) \\ &+ 6\mathcal{I}((\Psi\mathfrak{r})v) - 3\mathfrak{J}^V(v) \\ &+ \mathcal{I}(\Psi^3) - 3\Psi\mathfrak{r} + 3\Psi. \end{aligned}$$

Fix $0 < T \leq T_0$. From Lemmas 2.5 and 2.4 with (3.4), we have

$$\begin{aligned} \|\mathcal{I}(\Psi v^2)\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}+\delta_2}} &\lesssim T^{\delta_2} \|\Psi v^2\|_{X_T^{-\frac{1}{2}+\delta_1,-\frac{1}{2}+2\delta_2}} \leq T^\theta \|\Psi\|_{L_{T,x}^\infty} \|v\|_{L_{T,x}^4}^2 \\ &\lesssim T^\theta K \|v\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}+\delta_2}}^2 \end{aligned} \tag{3.5}$$

and

$$\|\mathcal{I}(\Psi^2 v)\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}+\delta_2}} \leq T^\theta \|\Psi\|_{L_{T,x}^\infty}^2 \|v\|_{L_{T,x}^2} \leq T^\theta K^2 \|v\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}+\delta_2}} \tag{3.6}$$

for some $\theta > 0$. Similarly, we have

$$\|\mathcal{I}(\Psi^3)\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \lesssim T^{\delta_2} \|\Psi^3\|_{X_T^{-\frac{1}{2}+\delta_1, -\frac{1}{2}+2\delta_2}} \leq T^\theta \|\Psi\|_{L_{T,x}^\infty}^3 \leq T^\theta K^3. \tag{3.7}$$

From Lemma 2.5 and Lemma 2.2 with (3.4), we have

$$\begin{aligned} \|\mathcal{I}((\Psi^\dagger)v)\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} &\lesssim T^{\delta_2} \|(\Psi^\dagger)v\|_{X_T^{-\frac{1}{2}+\delta_1, -\frac{1}{2}+2\delta_2}} \lesssim T^{\delta_2} \|(\Psi^\dagger)v\|_{L_T^2 H_x^{-\frac{1}{2}+\delta_1}} \\ &\lesssim T^\theta \|\Psi^\dagger\|_{L_T^\infty W_x^{-\frac{1}{2}+\delta_1, \infty}} \|v\|_{L_T^2 H_x^{\frac{1}{2}-\delta_1}} \\ &\lesssim T^\theta K \|v\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}}, \end{aligned} \tag{3.8}$$

provided that $\delta_1 + \varepsilon \leq \alpha$. From (3.2) and (3.4), we have

$$\|\mathcal{J}^\vee(v)\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \leq T^\theta \|\mathcal{J}^\vee\|_{\mathcal{L}_{T_0}^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \|v\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \leq T^\theta K \|v\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}}. \tag{3.9}$$

Hence, by applying Lemmas 2.3 and 2.5, then Lemma 2.6, (3.5), Lemma 3.6, (3.6), (3.8), (3.9), (3.7), and Lemma 3.4 with (3.4), we have

$$\|\Gamma_\Xi(v)\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + T^\theta \left(\|v\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}}^3 + K^3 \right) + K.$$

An analogous computation yields a difference estimate on $\Gamma_\Xi(v_1) - \Gamma_\Xi(v_2)$. Therefore, Proposition 3.7 follows from a standard contraction argument. \square

4 Regularities of the stochastic terms

In this section, we present the proof of Lemmas 3.1 - 3.5, which are basic tools in applying Proposition 3.7 to finally prove Theorem 1.1. In view of the local well-posedness result in [51], we assume that $0 < \alpha \leq \frac{1}{4}$ in the following. Without loss of generality, we assume that $T \leq 1$. The main tools in this section are the counting estimates from [12, Section 4] and the random matrix estimate (see Lemma C.3 below) from [18], which capture the multilinear dispersive effect of the wave equation. For readers' convenience, we collect the relevant counting estimates in Appendix A and the relevant definitions and estimates for random matrices and tensors in Appendix C. We show in details how to reduce the relevant stochastic estimates to some basic counting and (random) matrix/tensor estimates studied in [12, Section 4] and [18].

In the remaining part of this section, we assume $0 < T < T_0 \leq 1$.

4.1 Basic stochastic terms

We first present the proof of Lemma 3.1.

Proof (i) Let $t \geq 0$. From (1.16), we have

$$\mathbb{E} [|\widehat{\mathfrak{I}}_N(n, t)|^2] \leq C(t) \langle n \rangle^{-2-2\alpha} \tag{4.1}$$

for any $n \in \mathbb{Z}^3$ and $N \geq 1$. Also, by the mean value theorem and an interpolation argument as in [26], we have

$$\mathbb{E} [|\widehat{\mathfrak{I}}_N(n, t_1) - \widehat{\mathfrak{I}}_N(n, t_2)|^2] \lesssim_T \langle n \rangle^{-2(1+\alpha)+\theta} |t_1 - t_2|^\theta$$

for any $\theta \in [0, 1]$, $n \in \mathbb{Z}^3$, and $0 \leq t_2 \leq t_1 \leq T$ with $t_1 - t_2 \leq 1$, uniformly in $N \in \mathbb{N}$. Hence, from Lemma 2.10, we conclude that $\mathfrak{I}_N \in C([0, T]; W^{\alpha-\frac{1}{2}-\varepsilon, \infty}(\mathbb{T}^3))$ for any $\varepsilon > 0$, almost surely. Moreover, a slight modification of the argument, using Lemma 2.10, yields that $\{\mathfrak{I}_N\}_{N \in \mathbb{N}}$ is almost surely a Cauchy sequence in $C([0, T]; W^{\alpha-\frac{1}{2}-\varepsilon, \infty}(\mathbb{T}^3))$, thus converging to some limit \mathfrak{I} . Since the required modification is exactly the same as in [26], we omit the details here.

Remark 4.1 In the remaining part of this section, we establish uniform (in N) regularity bounds on the truncated stochastic terms (such as Ψ_N) but may omit the convergence part of the argument. Furthermore, as for $\Psi_N \mathfrak{I}_N$ studied in Lemma 3.3, we only establish a uniform (in N) regularity bound on $\Psi_N \mathfrak{I}_N(t)$ for each fixed $0 < t \leq T \leq 1$. A slight modification as above yields continuity in time but we omit details.

(ii) It is possible to prove this part by proceeding as in [26,45] (i.e. without the use of the $X^{s,b}$ -spaces). In the following, however, we follow Bringmann’s approach [12], adapted to the stochastic PDE setting. More precisely, we show that given any $\delta_1 > 0$ and sufficiently small $\delta_2 > 0$, the sequence $\{\Psi_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $X^{\alpha-1-\delta_1, -\frac{1}{2}+\delta_2}([0, T])$, almost surely, and thus converges almost surely to Ψ in the same space, where Ψ is the almost sure limit of $\{\Psi_N\}_{N \in \mathbb{N}}$ in $C([0, T]; W^{3\alpha-\frac{3}{2}-\infty}(\mathbb{T}^3))$ discussed in Sect. 1.

Our first goal is to prove the following bound; given any $\delta_1 > 0$ and sufficiently small $\delta_2 > 0$, there exists $\theta > 0$ such that

$$\left\| \|\Psi_N\|_{X_T^{\alpha-1-\delta_1, -\frac{1}{2}+\delta_2}} \right\|_{L^p(\Omega)} \lesssim p^{\frac{3}{2}} T^\theta \tag{4.2}$$

for any $p \geq 1$ and $0 < T \leq 1$, uniformly in $N \in \mathbb{N}$.

Let us first compute the space-time Fourier transform of Ψ_N (with a time cutoff function). From (1.14) with (1.12), we can write the spatial Fourier transform $\widehat{\Psi}_N(n, t)$

as the following multiple Wiener–Ito integral (as in [41]):

$$\widehat{\Psi}_N(n, t) = \sum_{\substack{n=n_1+n_2+n_3 \\ |n_j| \leq N}} \int_0^t \int_0^t \int_0^t \prod_{j=1}^3 \frac{\sin((t - t_j)\langle n_j \rangle)}{\langle n_j \rangle^{1+\alpha}} dB_{n_3}(t_3) dB_{n_2}(t_2) dB_{n_1}(t_1). \tag{4.3}$$

We emphasize that the renormalization in (1.14) is embedded in the definition of the multiple Wiener–Ito integral.

We now compute the space-time Fourier transform of $\mathbf{1}_{[0, T]} \Psi$, where $\mathbf{1}_{[0, T]}$ denotes the sharp cutoff function on the time interval $[0, T]$. From (4.3) and the stochastic Fubini theorem ([15, Theorem 4.33]; see also Lemma B.2), we have

$$\begin{aligned} \widehat{\mathbf{1}_{[0, T]} \Psi}_N(n, \tau) &= \frac{1}{\sqrt{2\pi}} \sum_{\substack{n=n_1+n_2+n_3 \\ |n_j| \leq N}} \int_{\mathbb{R}} \mathbf{1}_{[0, T]} e^{-it\tau} \int_0^t \int_0^t \int_0^t \\ &\quad \prod_{j=1}^3 \frac{\sin((t - t_j)\langle n_j \rangle)}{\langle n_j \rangle^{1+\alpha}} dB_{n_3}(t_3) dB_{n_2}(t_2) dB_{n_1}(t_1) dt \\ &= \frac{1}{\sqrt{2\pi}} \sum_{\substack{n=n_1+n_2+n_3 \\ |n_j| \leq N}} \int_0^T \int_0^T \int_0^T F_{n_1, n_2, n_3}(t_1, t_2, t_3) dB_{n_3}(t_3) dB_{n_2}(t_2) dB_{n_1}(t_1), \end{aligned} \tag{4.4}$$

where $F_{n_1, n_2, n_3}(t_1, t_2, t_3, \tau)$ is defined by

$$F_{n_1, n_2, n_3}(t_1, t_2, t_3, \tau) = \int_0^T e^{-it\tau} \prod_{j=1}^3 \frac{\sin((t - t_j)\langle n_j \rangle)}{\langle n_j \rangle^{1+\alpha}} \mathbf{1}_{[0, t]}(t_j) dt. \tag{4.5}$$

Note that $F_{n_1, n_2, n_3}(t_1, t_2, t_3, \tau)$ is symmetric in t_1, t_2, t_3 .

Given dyadic $N_j \geq 1, j = 1, 2, 3$, let us denote by A_{N_1, N_2, N_3}^N the contribution to $\mathbf{1}_{[0, T]} \Psi^N$ from $|n_j| \sim N_j, j = 1, 2, 3$, in (4.4). We first compute the $X^{s-1, b}$ -norm of A_{N_1, N_2, N_3}^N with $b = -\frac{1}{2} - \delta$ for $\delta > 0$. We then interpolate it with the trivial $X^{0, 0}$ -bound. Recall the trivial bound:

$$\begin{aligned} \|u\|_{X^{s, b}} &= \|\langle n \rangle^s \langle |\tau| - \langle n \rangle \rangle^b \widehat{u}(n, \tau)\|_{\ell_n^2 L_\tau^2} \\ &\leq \sum_{\varepsilon_0 \in \{-1, 1\}} \|\langle n \rangle^s \langle \tau + \varepsilon_0 \langle n \rangle \rangle^b \widehat{u}(n, \tau)\|_{\ell_n^2 L_\tau^2} \\ &= \sum_{\varepsilon_0 \in \{-1, 1\}} \|\langle n \rangle^s \langle \tau \rangle^b \widehat{u}(n, \tau - \varepsilon_0 \langle n \rangle)\|_{\ell_n^2 L_\tau^2} \end{aligned} \tag{4.6}$$

for any $s, b \in \mathbb{R}$. Then, defining $\kappa(\bar{n}) = \kappa_{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3}(n_1, n_2, n_3)$ by

$$\kappa(\bar{n}) = \varepsilon_0 \langle n_{123} \rangle + \varepsilon_1 \langle n_1 \rangle + \varepsilon_2 \langle n_2 \rangle + \varepsilon_3 \langle n_3 \rangle, \tag{4.7}$$

with $\varepsilon_j \in \{-1, 1\}$ for $j = 0, 1, 2, 3$, it follows from (4.6), (4.4), Fubini’s theorem, Ito’s isometry, and expanding the sine functions in (4.5) in terms of the complex exponentials that

$$\begin{aligned}
 & \left\| \|A_{N_1, N_2, N_3}^N\|_{X_T^{s-1, -\frac{1}{2}-\delta}} \right\|_{L^2(\Omega)}^2 \\
 & \lesssim \sum_{\varepsilon_0 \in \{-1, 1\}} \sum_{n \in \mathbb{Z}^3} \int_{\mathbb{R}} \langle n \rangle^{2(s-1)} \langle \tau \rangle^{-1-2\delta} \\
 & \quad \times \left\{ \sum_{\substack{n=n_1+n_2+n_3 \\ |n_j| \leq N \\ |n_j| \sim N_j}} \int_{[0, T]^3} |F_{n_1, n_2, n_3}(t_1, t_2, t_3, \tau - \varepsilon_0(n))|^2 dt_3 dt_2 dt_1 \right\} d\tau \\
 & \lesssim \sum_{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}} \sum_{n \in \mathbb{Z}^3} \int_{\mathbb{R}} \langle n \rangle^{2(s-1)} \langle \tau \rangle^{-1-2\delta} \left\{ \sum_{\substack{n=n_1+n_2+n_3 \\ |n_j| \leq N \\ |n_j| \sim N_j}} \prod_{j=1}^3 \frac{1}{\langle n_j \rangle^{2(1+\alpha)}} \right. \\
 & \quad \times \left. \int_{[0, T]^3} \left| \int_{\max(t_1, t_2, t_3)}^T e^{-it(\tau - \kappa(\bar{n}))} dt \right|^2 dt_3 dt_2 dt_1 \right\} d\tau \\
 & \lesssim \sum_{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}} \sum_{n \in \mathbb{Z}^3} \sum_{\substack{n=n_1+n_2+n_3 \\ |n_j| \sim N_j}} \frac{\langle n \rangle^{2(s-1)}}{\prod_{j=1}^3 \langle n_j \rangle^{2(1+\alpha)}} \int_{\mathbb{R}} \frac{1}{\langle \tau \rangle^{1+2\delta} \langle \tau - \kappa(\bar{n}) \rangle^2} d\tau \\
 & \lesssim \sum_{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}} \sum_{n \in \mathbb{Z}^3} \sum_{\substack{n=n_1+n_2+n_3 \\ |n_j| \sim N_j}} \frac{\langle n \rangle^{2(s-1)}}{\prod_{j=1}^3 \langle n_j \rangle^{2(1+\alpha)}} \langle \kappa(\bar{n}) \rangle^{-1-2\delta} \\
 & \lesssim \sum_{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}} \sup_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}^3} \sum_{\substack{n=n_1+n_2+n_3 \\ |n_j| \sim N_j}} \frac{\langle n \rangle^{2(s-1)}}{\prod_{j=1}^3 \langle n_j \rangle^{2(1+\alpha)}} \cdot \mathbf{1}_{\{|\kappa(\bar{n})-m| \leq 1\}} \quad (4.8)
 \end{aligned}$$

for any $\delta > 0$, uniformly in dyadic $N_j \geq 1, j = 1, 2, 3$. By noting

$$\prod_{j=1}^3 \langle n_j \rangle^{-2\alpha} \lesssim \langle n_{12} \rangle^{-2\alpha}, \tag{4.9}$$

we can reduce the right-hand side of (4.8) to the setting of the Hartree nonlinearity studied in [12]. In particular, from (4.8) with (4.9) and the cubic sum estimate (Lemma A.1), we obtain

$$\left\| \|A_{N_1, N_2, N_3}^N\|_{X_T^{s-1, -\frac{1}{2}-\delta}} \right\|_{L^2(\Omega)} \lesssim N_{\max}^{s-\alpha}, \tag{4.10}$$

where $N_{\max} = \max(N_1, N_2, N_3)$. This provides an estimate for $s < \alpha$ and $b = -\frac{1}{2} - \delta < -\frac{1}{2}$.

On the other hand, using (4.4), we have

$$\begin{aligned} & \left\| \|A_{N_1, N_2, N_3}^N\|_{X_T^{0,0}} \right\|_{L^2(\Omega)}^2 = \left\| \|A_{N_1, N_2, N_3}^N\|_{L^2_{T,x}} \right\|_{L^2(\Omega)}^2 \\ & \lesssim T^\theta \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^3 \\ |n_j| \sim N_j}} \prod_{j=1}^3 \langle n_j \rangle^{-2(1+\alpha)} \\ & \lesssim T^\theta N_{\max}^{3-6\alpha} \end{aligned} \tag{4.11}$$

for some $\theta > 0$. Hence, it follows from interpolating (4.10) and (4.11) and then applying the Wiener chaos estimate (Lemma 2.9) that given $s < \alpha$, there exist small $\delta_2 > 0$ and $\varepsilon > 0$ such that

$$\left\| \|A_{N_1, N_2, N_3}^N\|_{X_T^{s-1, -\frac{1}{2}+\delta_2}} \right\|_{L^p(\Omega)} \lesssim p^{\frac{3}{2}} T^\theta N_{\max}^{-\varepsilon}$$

for any $p \geq 1$, uniformly in dyadic $N_j \geq 1, j = 1, 2, 3$. By summing over dyadic blocks $N_j \geq 1, j = 1, 2, 3$, we obtain the bound (4.2) (with $b = -\frac{1}{2} + \delta_2 > -\frac{1}{2}$).

As for the convergence of Ψ_N to Ψ in $X^{\alpha-1-\delta_1, -\frac{1}{2}+\delta_2}([0, T])$, we can simply repeat the computation above to estimate the difference $\mathbf{1}_{[0,T]}\Psi_M - \mathbf{1}_{[0,T]}\Psi_N$ for $M \geq N \geq 1$. Fix $s < \alpha$. Then, in (4.8), we replace the restriction $|n_j| \leq N$ in the summation of $n_j, j = 1, 2, 3$, by $N \leq \max(|n_1|, |n_2|, |n_3|) \leq M$, which allows us to gain a small negative power of N . As a result, in place of (4.10), we obtain

$$\left\| \|A_{N_1, N_2, N_3}^M - A_{N_1, N_2, N_3}^N\|_{X_T^{s-1, -\frac{1}{2}-\delta}} \right\|_{L^2(\Omega)} \lesssim N^{-\varepsilon} N_{\max}^{s-\alpha+\varepsilon}$$

for any small $\varepsilon > 0$ and $M \geq N \geq 1$. Then, the interpolation argument with (4.11) as above yields that given $s < \alpha$, there exist small $\delta_2 > 0$ and $\varepsilon > 0$ such that

$$\left\| \left\| \mathbf{1}_{[0,T]}\Psi_M - \mathbf{1}_{[0,T]}\Psi_N \right\|_{X_T^{s-1, -\frac{1}{2}+\delta_2}} \right\|_{L^p(\Omega)} \lesssim p^{\frac{3}{2}} T^\theta N^{-\varepsilon} \tag{4.12}$$

for any $p \geq 1$ and $M \geq N \geq 1$. Then, by applying Chebyshev’s inequality and the Borel–Cantelli lemma, we conclude the almost sure convergence of Ψ_N . See [51].

Finally, fix $s < \alpha$. Given $N \in \mathbb{N}$, let $H_N = \mathcal{I}(\mathbf{1}_{[0,T]}(\Psi_N - \Psi))$. Then, we have

$$\Psi_N(t) - \Psi(t) = H_N(t) \tag{4.13}$$

for $t \in [0, T]$. Note that from (4.4), we have $\widehat{H}_N(n, t) \in \mathcal{H}_3$ and, furthermore, by the independence of $\{B_n\}_{n \in \mathbb{Z}^3}$ (modulo $B_{-n} = \overline{B_n}$), we have

$$\mathbb{E}[\widehat{H}_N(n, t_1)\widehat{H}_N(m, t_2)] = \mathbf{1}_{n+m=0} \mathbb{E}[\widehat{H}_N(n, t_1)\overline{\widehat{H}_N(n, t_2)}] \tag{4.14}$$

for any $t_1, t_2 \in \mathbb{R}$. Then, by (4.13), Sobolev’s inequality (with finite $r \gg 1$ such that $r\delta_0 > 3$ for some small $\delta_0 > 0$), Minkowski’s integral inequality, the Wiener chaos estimate (Lemma 2.9) with (4.14), Hausdorff–Young’s inequality (in time), we have, for any $p \geq \max(q, r) \gg 1$,

$$\begin{aligned} \|\Psi_N - \Psi\|_{L_T^\infty W_x^{s,\infty}} \|_{L^p(\Omega)} &\lesssim \|\|H_N\|_{W_t^{\delta_0,r} W_x^{s+\delta_0,r}}\|_{L^p(\Omega)} \\ &\leq \left\| \left\| \sum_{n \in \mathbb{Z}^3} \langle \nabla_t \rangle^{\delta_0} \langle n \rangle^{s+\delta_0} \widehat{H}_N(n, t) e_n(x) \right\|_{L^p(\Omega)} \right\|_{L_x^r L_t^r} \\ &\lesssim p^{\frac{3}{2}} \left\| \left\| \sum_{n \in \mathbb{Z}^3} \langle \nabla_t \rangle^{\delta_0} \langle n \rangle^{s+\delta_0} \widehat{H}_N(n, t) e_n(x) \right\|_{L^2(\Omega)} \right\|_{L_x^r L_t^r} \\ &\lesssim p^{\frac{3}{2}} \|\langle \tau \rangle^{\delta_0} \langle n \rangle^{s+\delta_0} \widehat{H}_N(n, \tau)\|_{L^2(\Omega; \ell_n^2 L_{\tau'}^r)}. \end{aligned}$$

Now, by the triangle inequality: $\langle \tau \rangle^{\delta_0} \lesssim (|\tau| - \langle n \rangle)^{\delta_0} \langle n \rangle^{\delta_0}$, Hölder’s inequality (in τ), followed by the nonhomogeneous linear estimate (Lemma 2.5) and (4.12) (with $p = 2, M = \infty$, and s replaced by $s + 2\delta_0 < \alpha$), we obtain

$$\begin{aligned} \|\Psi_N - \Psi\|_{L_T^\infty W_x^{s,\infty}} \|_{L^p(\Omega)} &\lesssim p^{\frac{3}{2}} \|\|H_N\|_{X^{s+2\delta_0, \frac{1}{2}+\delta_0}}\|_{L^2(\Omega)} \\ &\lesssim p^{\frac{3}{2}} T^\theta N^{-\varepsilon} \end{aligned}$$

by choosing $\delta_0 > 0$ sufficiently small. Then, the regularity and convergence claim for $\{\Psi_N\}_{N \in \mathbb{N}}$ follows from applying Chebyshev’s inequality and the Borel–Cantelli lemma as before. □

Remark 4.2 Given a function $f \in L^2((\mathbb{Z}^3 \times \mathbb{R}_+)^k)$, define the multiple stochastic integral $I_k[f]$ by

$$I_k[f] = \sum_{n_1, \dots, n_k \in \mathbb{Z}^3} \int_{[0, \infty)^k} f(n_1, t_1, \dots, n_k, t_k) dB_{n_1}(t_1) \cdots dB_{n_k}(t_k).$$

See Appendix B for the basic definitions and properties of multiple stochastic integrals. In terms of multiple stochastic integrals, we can express (4.3) as

$$\widehat{\Psi}_N(n, t) = I_3[f_{n,t}],$$

where $f_{n,t}$ is defined by

$$f_{n,t}(n_1, t_1, n_2, t_2, n_3, t_3) = \mathbf{1}_{n=n_{123}} \cdot \left(\prod_{j=1}^3 \frac{\sin((t - t_j)\langle n_j \rangle)}{\langle n_j \rangle^{1+\alpha}} \cdot \mathbf{1}_{|n_j| \leq N} \cdot \mathbf{1}_{[0,t]}(t_j) \right)$$

for $(n_1, t_1, n_2, t_2, n_3, t_3) \in (\mathbb{Z}^3 \times \mathbb{R})^3$. Then, by Fubini’s theorem for multiple stochastic integrals (Lemma B.2), we have

$$\widehat{\mathbf{1}}_{[0,T]} \Psi_N(n, \tau) = I_3 [\mathcal{F}_t(\mathbf{1}_{[0,T]} f_{n,\cdot})(\tau)],$$

where \mathcal{F}_t denotes the Fourier transform in time. With this notation, it follows from Lemma B.1 that we can write the second moment of the $X^{s,b}$ -norm of A_{N_1, N_2, N_3}^N , appearing in (4.8) and (4.11), in a concise manner:

$$\begin{aligned} & \left\| \|A_{N_1, N_2, N_3}^N\|_{X_T^{s,b}} \right\|_{L^2(\Omega)}^2 \\ &= 3! \sum_{n \in \mathbb{Z}^3} \int_{\mathbb{R}} \langle n \rangle^{2s} \langle |\tau| - \langle n \rangle \rangle^{2b} \|\mathcal{F}_t(\mathbf{1}_{[0,T]} f_{n,\cdot}^{\bar{N}})(\tau)\|_{\ell_{n_1, n_2, n_3}^2 L_{t_1, t_2, t_3}^2}^2 d\tau, \end{aligned}$$

where $f_{n,t}^{\bar{N}}$ is given by

$$f_{n,t}^{\bar{N}} = f_{n,t} \cdot \prod_{j=1}^3 \mathbf{1}_{|n_j| \sim N_j}.$$

In the following, for conciseness of the presentation, we express various stochastic objects as multiple stochastic integrals on $(\mathbb{Z}^3 \times \mathbb{R}_+)^k$ and carry out analysis. For this purpose, we set

$$z_j = (n_j, t_j) \in \mathbb{Z}^3 \times \mathbb{R}_+ \tag{4.15}$$

and use the following short-hand notation:

$$\|f(z_j)\|_{L_{z_j}^p} = \|f(n_j, t_j)\|_{\ell_{n_j}^p L_{t_j}^p}. \tag{4.16}$$

Note, however, that one may also carry out equivalent analysis at the level of multiple Wiener–Ito integrals as in the proof of Lemma 3.1 presented above.

Next, we briefly discuss the proof of Lemma 3.3.

Proof of Lemma 3.3 By the paraproduct decomposition (2.3), we have

$$\Psi_N \uparrow_N = \Psi_N \otimes \uparrow_N + \Psi_N \ominus \uparrow_N + \Psi_N \otimes \uparrow_N.$$

In view of Lemma 2.1 with (2.19), the paraproducts $\Psi_N \otimes \uparrow_N$ and $\Psi_N \ominus \uparrow_N$ belong to $C([0, T]; W^{\alpha-\frac{1}{2}-\varepsilon, \infty}(\mathbb{T}^3))$ for any $\varepsilon > 0$, almost surely. Hence, it remains to study the resonant product $\Psi_N := \Psi_N \ominus \uparrow_N$. We only study the regularity of the resonant product for a fixed time since the continuity in time and the convergence follow from a systematic modification. In the following, we show

$$\mathbb{E} [|\widehat{\Psi}_N(n, t)|^2] \lesssim \langle n \rangle^{-2-4\alpha} \tag{4.17}$$

for any $n \in \mathbb{Z}^3$ and $N \geq 1$. Note the bound (4.17) together with Lemma 2.10 shows that the resonant product Ψ_N is smoother and has (spatial) regularity $2\alpha - \frac{1}{2} - = (\alpha -) + (\alpha - \frac{1}{2} -)$.

As in [41], by decomposing $\widehat{\Psi}_N(n, t)$ into components in the homogeneous Wiener chaoses $\mathcal{H}_k, k = 2, 4$, we have

$$\widehat{\Psi}_N(n, t) = \widehat{\Psi}_N^{(4)}(n, t) + \widehat{\Psi}_N^{(2)}(n, t),$$

where $\widehat{\Psi}_N^{(4)}(n, t) \in \mathcal{H}_4$ and $\widehat{\Psi}_N^{(2)}(n, t) \in \mathcal{H}_2$. See, for example, [43, Proposition 1.1.2] and Lemma B.4 on the product formula for multiple Wiener–Ito integrals (and it also follows from Ito’s lemma as explained in [41]). From the orthogonality of \mathcal{H}_4 and \mathcal{H}_2 , we have

$$\mathbb{E}\left[|\widehat{\Psi}_N(n, t)|^2\right] = \mathbb{E}\left[|\widehat{\Psi}_N^{(4)}(n, t)|^2\right] + \mathbb{E}\left[|\widehat{\Psi}_N^{(2)}(n, t)|^2\right].$$

Hence, it suffices to prove (4.17) for $\Psi_N^{(j)}, j = 2, 4$.

From a slight modification¹⁷ of (4.8) with Lemma A.2, we have

$$\mathbb{E}\left[|\widehat{\Psi}_N(n, t)|^2\right] \leq C(t)\langle n \rangle^{-3-2\alpha} \tag{4.18}$$

for any $n \in \mathbb{Z}^3$ and $N \geq 1$. Then, from Jensen’s inequality (see (B.2)),¹⁸ (4.1), (4.18), and Lemma 2.8, we have

$$\begin{aligned} \mathbb{E}\left[|\widehat{\Psi}_N^{(4)}(n, t)|^2\right] &\lesssim \sum_{\substack{n \in \mathbb{Z}^3 \\ |n_1| \sim |n-n_1|}} \mathbb{E}\left[|\widehat{\Psi}_N(n_1, t)|^2\right] \mathbb{E}\left[|\widehat{\Psi}_N(n-n_1, t)|^2\right] \\ &\leq C(t) \sum_{\substack{n \in \mathbb{Z}^3 \\ |n_1| \sim |n-n_1|}} \frac{1}{\langle n_1 \rangle^{3+2\alpha} \langle n-n_1 \rangle^{2+2\alpha}} \\ &\leq C(t)\langle n \rangle^{-2-4\alpha} \end{aligned} \tag{4.19}$$

for any $n \in \mathbb{Z}^3$ and $N \geq 1$, where $|n_1| \sim |n - n_1|$ signifies the resonant product \ominus . This yields (4.17) for $\widehat{\Psi}_N^{(4)}$.

¹⁷ Namely, with $s = 0$ and dropping the summation over n in (4.8).

¹⁸ See the discussion on Ψ in Section 4 of [41]. See also Section 10 in [29].

From Ito’s lemma (see also the product formula, Lemma B.4), (1.12), and (4.3) with (4.15), we have

$$\widehat{\Psi}_N^{(2)}(n, t) = 3 \int_0^t I_2 [g_{n,t,t'}(z_2, z_3)] dt',$$

where $g_{n,t,t'}$ is defined by

$$\begin{aligned} g_{n,t,t'}(z_2, z_3) &= \sum_{\substack{|n_1| \leq N \\ |n_1| \sim |n_{123}|}} \mathbf{1}_{n=n_{23}} \cdot \mathbf{1}_{|n_2| \leq N} \cdot \mathbf{1}_{|n_3| \leq N} \int_0^{t'} \frac{\sin((t-t')\langle n_{123} \rangle)}{\langle n_{123} \rangle} \\ &\times \left(\prod_{j=1}^3 \frac{\sin((t'-t_j)\langle n_j \rangle)}{\langle n_j \rangle^{1+\alpha}} \cdot \mathbf{1}_{[0,t']}(t_j) \right) \frac{\sin((t-t_1)\langle n_1 \rangle)}{\langle n_1 \rangle^{1+\alpha}} dt_1. \end{aligned} \tag{4.20}$$

Note that $g_{n,t,t'}(z_2, z_3)$ is symmetric (in z_2 and z_3). From Fubini’s theorem (Lemma B.2), we have

$$\widehat{\Psi}_N^{(2)}(n, t) = 3I_2 \left[\int_0^t g_{n,t,t'}(z_2, z_3) dt' \right]. \tag{4.21}$$

We now apply Lemma B.1 to compute the second moment of(4.21). Then, with $\kappa(\bar{n})$ as in (4.7), it follows from expanding the sine functions in (4.20) in terms of the complex exponentials and switching the order of integration in t' and t_1 that

$$\begin{aligned} \mathbb{E} [|\widehat{\Psi}_N^{(2)}(n, t)|^2] &\sim \left\| \int_0^t g_{n,t,t'}(n_2, t_2, n_3, t_3) dt' \right\|_{\ell_{n_2, n_3}^2 L_{t_2, t_3}^2}^2 \\ &\lesssim \sum_{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}} \sum_{\substack{n=n_2+n_3 \\ |n_j| \leq N}} \frac{1}{\langle n_2 \rangle^{2+2\alpha} \langle n_3 \rangle^{2+2\alpha}} \\ &\times \left(\sum_{|n_1| \leq N} \frac{1}{\langle \kappa(\bar{n}) \rangle \langle n_{123} \rangle \langle n_1 \rangle^{2+2\alpha}} \right)^2 \\ &\lesssim \sum_{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}} \sum_{\substack{n=n_2+n_3 \\ |n_j| \leq N}} \frac{1}{\langle n_2 \rangle^{2+2\alpha} \langle n_3 \rangle^{2+2\alpha}} \\ &\times \left(\sum_{m \in \mathbb{Z}} \sum_{|n_1| \leq N} \frac{1}{\langle m \rangle \langle n_{123} \rangle \langle n_1 \rangle^{2+2\alpha}} \cdot \mathbf{1}_{\{|\kappa(\bar{n})-m| \leq 1\}} \right)^2. \end{aligned}$$

Under the condition $|n_1| \sim |n_{123}|$ and $n = n_2 + n_3$, we have $|n_1| \gtrsim |n|$. Then, by applying the basic resonant estimate (Lemma A.3) and Lemma 2.8, we obtain

$$\begin{aligned} \mathbb{E}[|\Psi_N^{(2)}(n, t)|^2] &\lesssim \frac{1}{\langle n \rangle^2} \sum_{\substack{|n| \lesssim N_1 \lesssim N \\ \text{dyadic}}} \frac{\log^2(2 + N_1)}{N_1^{4\alpha}} \sum_{\substack{n=n_2+n_3 \\ |n_j| \leq N_j}} \frac{1}{\langle n_2 \rangle^{2+2\alpha} \langle n_3 \rangle^{2+2\alpha}} \\ &\quad \times \\ &\lesssim \frac{1}{\langle n \rangle^{2+4\alpha-}} \sum_{\substack{n=n_2+n_3 \\ |n_j| \leq N_j}} \frac{1}{\langle n_2 \rangle^{2+2\alpha} \langle n_3 \rangle^{2+2\alpha}} \lesssim \langle n \rangle^{-3-8\alpha+}. \end{aligned} \tag{4.22}$$

This computation with Lemma 2.10 shows that $\Psi_N^{(2)}$ is even smoother and has (spatial) regularity $4\alpha -$.

Therefore, putting (4.19) and (4.22) together, we obtain the desired bound (4.17). □

4.2 Quintic stochastic term

In this subsection, we present the proof of Lemma 3.4(i) on the quintic stochastic process Ψ_N defined in (1.22). In view of Lemma 2.5, we prove the following bound; given any $\varepsilon > 0$ and sufficiently small $\delta_2 > 0$, there exists $\theta > 0$ such that

$$\left\| \|\Psi_N\|_{X_T^{\alpha-\frac{1}{2}-\varepsilon, -\frac{1}{2}+\delta_2}} \right\|_{L^p(\Omega)} \lesssim p^{\frac{5}{2}} T^\theta \tag{4.23}$$

for any $p \geq 1$ and $0 < T \leq 1$, uniformly in $N \in \mathbb{N}$.

We start by computing the space-time Fourier transform of Ψ_N with a time cutoff. As shown in (1.22), the quintic stochastic objects Ψ_N is a convolution of $\hat{\Psi}_N$ in (1.15) and \hat{v}_N in (1.14):

$$\hat{\Psi}_N(n, t) = \sum_{n=n_{123}+n_{45}} \hat{\Psi}_N(n_{123}, t) \hat{v}_N(n_{45}, t). \tag{4.24}$$

Using Lemma B.2, we can write $\hat{\Psi}_N$ and \hat{v}_N as multiple stochastic integrals:

$$\begin{aligned} \hat{\Psi}_N(n, t) &= \int_0^t I_3[f_{n,t,t'}(z_1, z_2, z_3)] dt' = I_3 \left[\int_0^t f_{n,t,t'}(z_1, z_2, z_3) dt' \right], \\ \hat{v}_N(n, t) &= I_2[g_{n,t}], \end{aligned} \tag{4.25}$$

where $f_{n,t,t'}$ and $g_{n,t}$ are defined by

$$\begin{aligned}
 f_{n,t,t'}(z_1, z_2, z_3) &= \mathbf{1}_{n=n_{123}} \cdot \frac{\sin((t-t')\langle n_{123} \rangle)}{\langle n_{123} \rangle} \\
 &\quad \times \left(\prod_{j=1}^3 \frac{\sin((t'-t_j)\langle n_j \rangle)}{\langle n_j \rangle^{1+\alpha}} \cdot \mathbf{1}_{|n_j| \leq N} \cdot \mathbf{1}_{[0,t']}(t_j) \right), \tag{4.26} \\
 g_{n,t}(z_1, z_2) &= \mathbf{1}_{n=n_{12}} \cdot \left(\prod_{j=1}^2 \frac{\sin((t-t_j)\langle n_j \rangle)}{\langle n_j \rangle^{1+\alpha}} \cdot \mathbf{1}_{|n_j| \leq N} \cdot \mathbf{1}_{[0,t]}(t_j) \right).
 \end{aligned}$$

By the product formula (Lemma B.4) to (4.24), we can decompose $\widehat{\Psi}_N$ into the components in the homogeneous Wiener chaoses $\mathcal{H}_k, k = 1, 3, 5$:

$$\widehat{\Psi}_N(n, t) = \widehat{\Psi}_N^{(5)}(n, t) + \widehat{\Psi}_N^{(3)}(n, t) + \widehat{\Psi}_N^{(1)}(n, t), \tag{4.27}$$

where $\widehat{\Psi}_N^{(5)} \in \mathcal{H}_5, \widehat{\Psi}_N^{(3)} \in \mathcal{H}_3,$ and $\widehat{\Psi}_N^{(1)} \in \mathcal{H}_1.$ By taking the Fourier transforms in time, the relation (4.27) still holds. Then, by using the orthogonality of $\mathcal{H}_5, \mathcal{H}_3,$ and $\mathcal{H}_1,$ we have

$$\mathbb{E} \left[|\widehat{\Psi}_N(n, t)|^2 \right] = \sum_{j \in \{1,3,5\}} \mathbb{E} \left[|\widehat{\Psi}_N^{(j)}(n, t)|^2 \right].$$

Hence, it suffices to prove (4.23) for each $\widehat{\Psi}_N^{(j)}, j = 1, 3, 5.$

Case (i): Non-resonant term $\widehat{\Psi}_N^{(5)}$. From (4.25) and (4.26), we have

$$\widehat{\Psi}_N^{(5)}(n, t) = I_5[f_{n,t}^{(5)}],$$

where $f_{n,t}^{(5)}$ is defined by

$$\begin{aligned}
 f_{n,t}^{(5)}(z_1, z_2, z_3, z_4, z_5) &= \mathbf{1}_{n=n_{12345}} \cdot \int_0^t \frac{\sin((t-t')\langle n_{123} \rangle)}{\langle n_{123} \rangle} \\
 &\quad \times \left(\prod_{j=1}^3 \frac{\sin((t'-t_j)\langle n_j \rangle)}{\langle n_j \rangle^{1+\alpha}} \cdot \mathbf{1}_{|n_j| \leq N} \cdot \mathbf{1}_{[0,t']}(t_j) \right) dt' \\
 &\quad \times \left(\prod_{j=4}^5 \frac{\sin((t-t_j)\langle n_j \rangle)}{\langle n_j \rangle^{1+\alpha}} \cdot \mathbf{1}_{|n_j| \leq N} \cdot \mathbf{1}_{[0,t]}(t_j) \right). \tag{4.28}
 \end{aligned}$$

Let $\text{Sym}(f_{n,t}^{(5)})$ be the symmetrization of $f_{n,t}^{(5)}$ defined in (B.1). Then, from Lemma B.1 (ii), we have

$$\widehat{\Psi}_N^{(5)}(n, t) = I_5[\text{Sym}(f_{n,t}^{(5)})].$$

Then, by taking the temporal Fourier transform and applying Fubini’s theorem (Lemma B.2), we have

$$\mathcal{F}_t(\mathbf{1}_{[0,T]}\widehat{\Psi}_N^{(5)})(n, \tau) = I_5[\mathcal{F}_t(\mathbf{1}_{[0,T]}\text{Sym}(f_{n,\cdot}^{(5)})(\tau))] = I_5[\text{Sym}(\mathcal{F}_t(\mathbf{1}_{[0,T]}f_{n,\cdot}^{(5)})(\tau))].$$

Then, by (4.6), Fubini’s theorem, and Lemma B.1 (iii) with (4.15) and (4.16), we have

$$\begin{aligned} & \left\| \mathbf{1}_{[0,T]}\widehat{\Psi}_N^{(5)} \right\|_{X_T^{s,b}} \Big|_{L^2(\Omega)}^2 \\ & \lesssim \sum_{\varepsilon_0 \in \{-1,1\}} \sum_{n \in \mathbb{Z}^3} \int_{\mathbb{R}} \langle n \rangle^{2s} \langle \tau \rangle^{2b} \left\| \text{Sym}(\mathcal{F}_t(\mathbf{1}_{[0,T]}f_{n,\cdot}^{(5)})(\bar{z}))(\tau - \varepsilon_0 \langle n \rangle) \right\|_{L^2_{z_1, \dots, z_5}}^2 d\tau, \end{aligned} \tag{4.29}$$

where $\bar{z} = (z_1, \dots, z_5)$.

By expanding the sine functions in (4.28) in terms of the complex exponentials, we have

$$\begin{aligned} f_{n,t}^{(5)}(z_1, z_2, z_3, z_4, z_5) &= c \cdot \mathbf{1}_{n=n_{12345}} \sum_{\mathcal{E}} \widehat{\varepsilon} \cdot \frac{e^{it\kappa_1(\bar{n})}}{\langle n_{123} \rangle} \int_{\max(t_1, t_2, t_3)}^t e^{-it'\kappa_2(\bar{n})} dt' \\ & \times \left(\prod_{j=1}^5 \frac{1}{\langle n_j \rangle^{1+\alpha}} \cdot \mathbf{1}_{|n_j| \leq N} \right) \left(\prod_{j=4}^5 \mathbf{1}_{[0,t]}(t_j) \right) F_1(z_1, \dots, z_5), \end{aligned} \tag{4.30}$$

where $F_1(z_1, \dots, z_5)$ is independent of t and t' with $|F_1| \leq 1$. Here, $\mathcal{E}, \widehat{\varepsilon}, \kappa_1(\bar{n})$, and $\kappa_2(\bar{n})$ are defined by

$$\begin{aligned} \mathcal{E} &= \{ \varepsilon_1, \dots, \varepsilon_5, \varepsilon_{123} \in \{-1, 1\} \}, \quad \widehat{\varepsilon} = \varepsilon_{123} \prod_{j=1}^5 \varepsilon_j, \\ \kappa_1(\bar{n}) &= \varepsilon_{123} \langle n_{123} \rangle + \varepsilon_4 \langle n_4 \rangle + \varepsilon_5 \langle n_5 \rangle, \\ \kappa_2(\bar{n}) &= \varepsilon_{123} \langle n_{123} \rangle - \varepsilon_1 \langle n_1 \rangle - \varepsilon_2 \langle n_2 \rangle - \varepsilon_3 \langle n_3 \rangle. \end{aligned} \tag{4.31}$$

By integrating in t' , we have

$$\int_{\max(t_1, t_2, t_3)}^t e^{-it'\kappa_2(\bar{n})} dt' = \frac{e^{-it\kappa_2(\bar{n})} - e^{-it_{123}^* \kappa_2(\bar{n})}}{-i\kappa_2(\bar{n})}, \tag{4.32}$$

where $t_{123}^* = \max(t_1, t_2, t_3)$. Then, from (4.30) and (4.32), we have

$$\begin{aligned}
 |\mathcal{F}_t(\mathbf{1}_{[0,T]} f_{n,\cdot}^{(5)}(\bar{z}))(\tau - \varepsilon_0 \langle n \rangle)| &\lesssim \mathbf{1}_{n=n_{12345}} \frac{1}{\langle \kappa_2(\bar{n}) \rangle \langle \min(|\tau - \kappa_3(\bar{n})|, |\tau - \kappa_4(\bar{n})|) \rangle} \\
 &\times \frac{1}{\langle n_{123} \rangle} \left(\prod_{j=1}^5 \frac{1}{\langle n_j \rangle^{1+\alpha}} \cdot \mathbf{1}_{|n_j| \leq N} \cdot \mathbf{1}_{[0,T]}(t_j) \right),
 \end{aligned}
 \tag{4.33}$$

where $\kappa_3(\bar{n})$ and $\kappa_4(\bar{n})$ are defined by

$$\begin{aligned}
 \kappa_3(\bar{n}) &= \varepsilon_0 \langle n_{12345} \rangle + \varepsilon_{123} \langle n_{123} \rangle + \varepsilon_4 \langle n_4 \rangle + \varepsilon_5 \langle n_5 \rangle, \\
 \kappa_4(\bar{n}) &= \varepsilon_0 \langle n_{12345} \rangle + \sum_{j=1}^5 \varepsilon_j \langle n_j \rangle.
 \end{aligned}
 \tag{4.34}$$

Given dyadic $N_j \geq 1, j = 1, 2, 3, 4, 5$, we denote by B_{N_1, \dots, N_5}^N the contribution to $\mathbf{1}_{[0,T]} \mathfrak{V}_N^{(5)}$ from $|n_j| \sim N_j$ in (4.33). Let $\mathcal{E}_0 = \mathcal{E} \cup \{\varepsilon_0 \in \{-1, 1\}\}$ and $N_{\max} = \max(N_1, \dots, N_5)$. Then, from (4.29), Jensen’s inequality (B.2), and (4.33) with (1.23), we have

$$\begin{aligned}
 &\left\| \mathbf{1}_{[0,T]} B_{N_1, \dots, N_5}^N \right\|_{X_T^{s-1, -\frac{1}{2}-\delta}}^2 \\
 &\lesssim T^\theta \sum_{\mathcal{E}_0} \sum_{n \in \mathbb{Z}^3} \sum_{\substack{n=n_{12345} \\ |n_j| \sim N_j}} \frac{\langle n \rangle^{2(s-1)}}{\langle n_{123} \rangle^2} \frac{1}{\langle \kappa_2(\bar{n}) \rangle^2} \left(\prod_{j=1}^5 \frac{1}{\langle n_j \rangle^{2+2\alpha}} \right) \\
 &\quad \times \int_{\mathbb{R}} \frac{1}{\langle \tau \rangle^{1+2\delta} \langle \min(|\tau - \kappa_3(\bar{n})|, |\tau - \kappa_4(\bar{n})|) \rangle^2} d\tau \\
 &\lesssim T^\theta \sum_{\mathcal{E}_0} \sup_{m, m' \in \mathbb{Z}} \sum_{\substack{n_1, \dots, n_5 \in \mathbb{Z}^3 \\ |n_j| \sim N_j}} \frac{\langle n_{12345} \rangle^{2(s-\alpha+\frac{1}{2}\varepsilon-1)}}{\langle n_{1234} \rangle^{\frac{1}{2}\varepsilon} \langle n_{12} \rangle^{\frac{1}{2}\varepsilon} \langle n_{123} \rangle^2 \prod_{j=1}^5 \langle n_j \rangle^2} \\
 &\quad \times \mathbf{1}_{\{|\kappa_2(\bar{n})-m|\leq 1\}} \left(\mathbf{1}_{\{|\kappa_3(\bar{n})-m'|\leq 1\}} + \mathbf{1}_{\{|\kappa_4(\bar{n})-m'|\leq 1\}} \right)
 \end{aligned}
 \tag{4.35}$$

for some $\theta > 0$, provided that $\delta > 0$. In the last step, we used the following bound:

$$\begin{aligned}
 &\int_{\mathbb{R}} \frac{1}{\langle \tau \rangle^{1+2\delta} \langle \min(|\tau - \kappa_3(\bar{n})|, |\tau - \kappa_4(\bar{n})|) \rangle^2} d\tau \\
 &\leq \int_{\mathbb{R}} \frac{1}{\langle \tau \rangle^{1+2\delta} \langle \tau - \kappa_3(\bar{n}) \rangle^2} d\tau + \int_{\mathbb{R}} \frac{1}{\langle \tau \rangle^{1+2\delta} \langle \tau - \kappa_4(\bar{n}) \rangle^2} d\tau \\
 &\lesssim \langle \kappa_3(\bar{n}) \rangle^{-1-2\delta} + \langle \kappa_4(\bar{n}) \rangle^{-1-2\delta} \\
 &\lesssim \sum_{m' \in \mathbb{Z}} \frac{1}{\langle m' \rangle^{1+2\delta}} \left(\mathbf{1}_{\{|\kappa_3(\bar{n})-m'|\leq 1\}} + \mathbf{1}_{\{|\kappa_4(\bar{n})-m'|\leq 1\}} \right)
 \end{aligned}$$

for $\delta > 0$. Then, by applying Lemma A.4 to (4.35), we obtain

$$\left\| \mathbf{1}_{[0,T]} B_{N_1, \dots, N_5}^N \right\|_{X_T^{\alpha - \frac{1}{2} - \varepsilon, -\frac{1}{2} - \delta}}^2 \Big\|_{L^2(\Omega)}^2 \lesssim T^\theta N_{\max}^{-\delta_0} \tag{4.36}$$

for some $\delta_0 > 0$, provided that $\varepsilon, \delta > 0$. Using (4.29) and (4.33), a crude bound shows

$$\left\| \mathbf{1}_{[0,T]} B_{N_1, \dots, N_5}^N \right\|_{X^{0,0}}^2 \Big\|_{L^2(\Omega)}^2 \lesssim T^\theta N_{\max}^K \tag{4.37}$$

for some (possibly large) $K > 0$. By interpolating (4.36) and (4.37), applying the Winner chaos estimate (Lemma 2.9), and then summing over dyadic $N_j, j = 1, \dots, 5$, we obtain

$$\left\| \left\| \mathbf{1}_{[0,T]} \Psi_N^{(5)} \right\|_{X^{\alpha - \frac{1}{2} - \varepsilon, -\frac{1}{2} + \delta_2}([0,T])} \right\|_{L^p(\Omega)} \lesssim p^{\frac{5}{2}} T^\theta$$

for some $\theta > 0$, uniformly in $N \in \mathbb{N}$. Proceeding as in the end of the proof of Lemma 3.1 (ii) on Ψ_N , a slight modification of the argument above yields convergence of $\Psi_N^{(5)}$ to $\Psi^{(5)}$. Since the required modification is straightforward, we omit details.

A similar comment applies to $\Psi_N^{(3)}$ and $\Psi_N^{(1)}$ studied below.

Case (ii): Single-resonance term $\Psi_N^{(3)}$. In view of the product formula (Lemma B.4)¹⁹ and Definition B.3 together with (4.25) and (4.26), we have

$$\widehat{\Psi}_N^{(3)}(n, t) = I_3[f_{n,t}^{(3)}],$$

where $f_{n,t}^{(3)}$ is defined by

$$\begin{aligned} f_{n,t}^{(3)}(z_1, z_2, z_4) &= \sum_{n_3 \in \mathbb{Z}^3} \mathbf{1}_{n=n_{124}} \cdot \left(\prod_{j=1}^4 \mathbf{1}_{|n_j| \leq N} \right) \frac{\sin((t - t_j)\langle n_4 \rangle)}{\langle n_4 \rangle^{1+\alpha}} \cdot \mathbf{1}_{[0,t]}(t_4) \\ &\times \int_0^t \frac{\sin((t - t')\langle n_{123} \rangle)}{\langle n_{123} \rangle} \left(\prod_{j=1}^2 \frac{\sin((t' - t_j)\langle n_j \rangle)}{\langle n_j \rangle^{1+\alpha}} \cdot \mathbf{1}_{[0,t']}(t_j) \right) \\ &\times \left(\int_0^{t'} \frac{\sin((t - t_3)\langle n_3 \rangle) \sin((t' - t_3)\langle n_3 \rangle)}{\langle n_3 \rangle^{2+2\alpha}} dt_3 \right) dt'. \end{aligned}$$

¹⁹ Note that both $f_{n,t,t'}$ and $g_{n,t}$ in (4.26) are symmetric in their arguments.

By the Wiener chaos estimate (Lemma 2.9) and Hölder’s inequality, we have

$$\begin{aligned} \left\| \|\Psi_N^{(3)}\|_{X_T^{\alpha-\frac{1}{2}-\varepsilon, -\frac{1}{2}+\delta_2}} \right\|_{L^p(\Omega)} &\lesssim \left\| \|\Psi_N^{(3)}\|_{L_T^2 H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^p(\Omega)} \\ &\lesssim T^{\frac{1}{2}} p^{\frac{3}{2}} \sup_{t \in [0, T]} \left\| \|\Psi_N^{(3)}(t)\|_{H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^2(\Omega)} \end{aligned} \tag{4.38}$$

for small $\delta_2 > 0$. Hence, (4.23) follows once we prove

$$\sup_{t \in [0, T]} \left\| \|\Psi_N^{(3)}(t)\|_{H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^2(\Omega)} < \infty \tag{4.39}$$

for $\varepsilon > 0$, uniformly in $N \in \mathbb{N}$.

With the symmetrization $\text{Sym}(f_{n,t}^{(3)})$ defined in (B.1), it follows from Lemma B.1 and Jensen’s inequality (B.2) that

$$\begin{aligned} \left\| \|\Psi_N^{(3)}(t)\|_{H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^2(\Omega)}^2 &= \sum_{n \in \mathbb{Z}^3} \langle n \rangle^{2\alpha-1-2\varepsilon} \|I_3[\text{Sym}(f_{n,t}^{(3)})]\|_{L^2(\Omega)}^2 \\ &\lesssim \sum_{n \in \mathbb{Z}^3} \sum_{\substack{n=n_{124} \\ |n_j| \leq N}} \langle n \rangle^{2\alpha-1-2\varepsilon} \left(\prod_{j \in \{1,2,4\}} \frac{1}{\langle n_j \rangle^{2+2\alpha}} \right) \int_{[0,t]^3} |\mathcal{J}^{(3)}(z_1, z_2, t_4)|^2 dt_1 dt_2 dt_4, \end{aligned} \tag{4.40}$$

where $\mathcal{J}^{(3)}(z_1, z_2, t_4)$ is defined by

$$\begin{aligned} \mathcal{J}^{(3)}(z_1, z_2, t_4) &= \sum_{|n_3| \leq N} \frac{1}{\langle n_{123} \rangle \langle n_3 \rangle^{2+2\alpha}} \int_{\max(t_1, t_2)}^t \sin((t-t') \langle n_{123} \rangle) \\ &\times \left(\prod_{j=1}^2 \sin((t'-t_j) \langle n_j \rangle) \right) \\ &\times \int_0^{t'} \sin((t-t_3) \langle n_3 \rangle) \sin((t'-t_3) \langle n_3 \rangle) dt_3 dt'. \end{aligned} \tag{4.41}$$

By switching the order of the integrals in (4.41) (with $a = \max(t_1, t_2)$):

$$\int_a^t \int_0^{t'} f dt_3 dt' = \int_0^a \int_a^t f dt' dt_3 + \int_a^t \int_{t_3}^t f dt' dt_3$$

and integrating in t' first, we have

$$|\mathcal{J}^{(3)}(z_1, z_2, t_4)| \lesssim \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_{123} \in \{-1, 1\}} \sum_{|n_3| \leq N} \frac{1}{\langle n_{123} \rangle \langle n_3 \rangle^{2+2\alpha} \langle \kappa_2(\bar{n}) \rangle}, \tag{4.42}$$

where $\kappa_2(\bar{n})$ is as in (4.31). Hence, from (4.40), (4.42), and Lemma A.3, we obtain

$$\begin{aligned} \left\| \|\Psi_N^{(3)}\|_{H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^2(\Omega)}^2 &\lesssim \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_{123} \in \{-1, 1\}} \sum_{n \in \mathbb{Z}^3} \sum_{\substack{n=n_{124} \\ |n_j| \leq N}} \langle n \rangle^{2\alpha-1-2\varepsilon} \left(\prod_{j \in \{1, 2, 4\}} \frac{1}{\langle n_j \rangle^{2+2\alpha}} \right) \\ &\quad \times \left| \sum_{\substack{1 \leq N_3 \leq N \\ \text{dyadic}}} \sum_{m \in \mathbb{Z}} \sum_{|n_3| \sim N_3} \frac{\mathbf{1}_{\{|\kappa_2(\bar{n})-m| \leq 1\}}}{\langle n_{123} \rangle \langle n_3 \rangle^{2+2\alpha} \langle m \rangle} \right|^2 \\ &\lesssim \sum_{n \in \mathbb{Z}^3} \sum_{\substack{n=n_{124} \\ |n_j| \leq N}} \frac{1}{\langle n \rangle^{1-2\alpha+2\varepsilon} \langle n_{12} \rangle^2} \left(\prod_{j \in \{1, 2, 4\}} \frac{1}{\langle n_j \rangle^{2+2\alpha}} \right) \\ &= \sum_{|n_1| \leq N} \frac{1}{\langle n_1 \rangle^{2+2\alpha}} \left\{ \sum_{|n_2| \leq N} \frac{1}{\langle n_{12} \rangle^2 \langle n_2 \rangle^{2+2\alpha}} \right. \\ &\quad \left. \times \left(\sum_{|n_4| \leq N} \frac{1}{\langle n_{124} \rangle^{1-2\alpha+2\varepsilon} \langle n_4 \rangle^{2+2\alpha}} \right) \right\}. \end{aligned}$$

By applying Lemma 2.8 iteratively, we then obtain

$$\left\| \|\Psi_N^{(3)}\|_{H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^2(\Omega)}^2 \lesssim 1,$$

provided that $\delta_1 > 0$. This yields (4.39).

Case (iii): Double-resonance term $\widehat{\Psi}_N^{(1)}$. As in Case (ii), from the product formula (Lemma B.4) and Definition B.3 together with (4.25) and (4.26), we have

$$\widehat{\Psi}_N^{(1)}(n, t) = I_1[f_{n,t}^{(1)}],$$

where $f_{n,t}^{(1)}$ is defined by

$$\begin{aligned} f_{n,t}^{(1)}(z_1) &= \sum_{n_2, n_3 \in \mathbb{Z}^3} \mathbf{1}_{n=n_1} \cdot \left(\prod_{j=1}^3 \mathbf{1}_{|n_j| \leq N} \right) \\ &\quad \times \int_0^t \frac{\sin((t-t')\langle n_{123} \rangle)}{\langle n_{123} \rangle} \frac{\sin((t'-t_1)\langle n_1 \rangle)}{\langle n_1 \rangle^{1+\alpha}} \cdot \mathbf{1}_{[0,t']}(t_1) \\ &\quad \times \left(\int_0^{t'} \int_0^{t'} \prod_{j=2}^3 \frac{\sin((t-t_j)\langle n_j \rangle)}{\langle n_j \rangle^{2+2\alpha}} \frac{\sin((t'-t_j)\langle n_j \rangle)}{\langle n_j \rangle^{2+2\alpha}} dt_2 dt_3 \right) dt'. \end{aligned}$$

Arguing as in (4.38), it suffices to show

$$\sup_{t \in [0, T]} \left\| \|\Psi_N^{(1)}(t)\|_{H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^2(\Omega)} < \infty \tag{4.43}$$

for $\varepsilon > 0$, uniformly in $N \in \mathbb{N}$.

With the symmetrization $\text{Sym}(f_{n,t}^{(1)})$ defined in (B.1), it follows from Lemma B.1 and Jensen’s inequality (B.2) that

$$\begin{aligned} \left\| \left\| \Downarrow_N^{(1)}(t) \right\|_{H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^2(\Omega)}^2 &= \sum_{n \in \mathbb{Z}^3} \langle n \rangle^{2\alpha-1-2\varepsilon} \left\| I_1 \left[\text{Sym}(f_{n,t}^{(1)}) \right] \right\|_{L^2(\Omega)}^2 \\ &\lesssim \sum_{|n_1| \leq N} \langle n_1 \rangle^{-3-2\varepsilon} \int_{[0,t]} |\mathcal{J}^{(1)}(z_1)|^2 dt_1, \end{aligned} \tag{4.44}$$

where $\mathcal{J}^{(1)}(z_1)$ is defined by

$$\begin{aligned} \mathcal{J}^{(1)}(z_1) &= \sum_{|n_2|, |n_3| \leq N} \frac{1}{\langle n_{123} \rangle \langle n_2 \rangle^{2+2\alpha} \langle n_3 \rangle^{2+2\alpha}} \\ &\quad \times \int_{t_1}^t \sin((t-t') \langle n_{123} \rangle) \sin((t'-t_1) \langle n_1 \rangle) \\ &\quad \times \int_0^{t'} \int_0^{t'} \prod_{j=2}^3 \sin((t-t_j) \langle n_j \rangle) \sin((t'-t_j) \langle n_j \rangle) dt_2 dt_3 dt'. \end{aligned} \tag{4.45}$$

By switching the order of the integrals in (4.45) and integrating in t' first, we have

$$|\mathcal{J}^{(1)}(z_1)| \lesssim \sum_{|n_2|, |n_3| \leq N} \frac{1}{\langle n_{123} \rangle \langle n_2 \rangle^{2+2\alpha} \langle n_3 \rangle^{2+2\alpha} \langle \kappa_2(\bar{n}) \rangle}, \tag{4.46}$$

where $\kappa_2(\bar{n})$ is as in (4.31). Hence, from (4.44) and (4.46), we obtain

$$\begin{aligned} \left\| \left\| \Downarrow_N^{(1)} \right\|_{H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^2(\Omega)}^2 &\lesssim \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_{123} \in \{-1, 1\}} \sum_{|n_1| \leq N} \langle n_1 \rangle^{-3-2\varepsilon} \\ &\quad \times \left| \sum_{m \in \mathbb{Z}} \sum_{|n_2|, |n_3| \leq N} \frac{\mathbf{1}_{\{|\kappa_2(\bar{n})-m| \leq 1\}}}{\langle n_{123} \rangle \langle n_2 \rangle^{2+2\alpha} \langle n_3 \rangle^{2+2\alpha} \langle m \rangle} \right|^2. \end{aligned}$$

Now, apply the dyadic decompositions $|n_j| \sim N_j, j = 1, 2, 3$. By noting that $\langle n_{12} \rangle^\alpha \lesssim N_1^\alpha N_2^\alpha$ and that $|\kappa_2(\bar{n}) - m| \leq 1$ implies $|m| \lesssim N_{\max} = \max(N_1, N_2, N_3)$, it follows from Lemma A.5 that

$$\begin{aligned}
 \left\| \left\| \mathbb{V}_N^{(1)} \right\|_{H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^2(\Omega)}^2 &\lesssim \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_{123} \in \{-1, 1\}} \sum_{\substack{1 \leq N_1, N_2, N_3 \lesssim N \\ \text{dyadic}}} N_{\max}^\gamma \frac{N_1^{2\alpha}}{N_2^{2\alpha} N_3^{4\alpha}} \\
 &\times \sum_{|n_1| \sim N_1} \langle n_1 \rangle^{-3-2\varepsilon} \left| \sup_{m \in \mathbb{Z}} \sum_{\substack{|n_2| \sim N_2 \\ |n_3| \sim N_3}} \frac{\mathbf{1}_{\{|\kappa_2(\bar{n})-m| \leq 1\}}}{\langle n_{123} \rangle \langle n_{12} \rangle^\alpha \langle n_2 \rangle^2 \langle n_3 \rangle^2} \right|^2 \\
 &\lesssim \sum_{\substack{1 \leq N_1, N_2, N_3 \lesssim N \\ \text{dyadic}}} N_{\max}^\gamma \frac{N_1^{2\alpha-2\varepsilon}}{N_2^{2\alpha} N_3^{4\alpha}} \max(N_1, N_2)^{-2\alpha+2\gamma} \\
 &\lesssim 1,
 \end{aligned}$$

provided that $\varepsilon > 0$, where $\gamma = \gamma(\varepsilon, \alpha) > 0$ is sufficiently small. This yields (4.43). This concludes the proof of Lemma 3.4(i).

4.3 Septic stochastic term

In this subsection, we present the proof of Lemma 3.4(ii) on the septic stochastic term \mathbb{V}_N defined in (1.24). Proceeding as in (4.38), it suffices to show

$$\sup_{t \in [0, T]} \left\| \left\| \mathbb{V}_N(t) \right\|_{H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^2(\Omega)} \lesssim 1 \tag{4.47}$$

for $\varepsilon > 0$, uniformly in $N \in \mathbb{N}$. As in the previous subsections, we decompose $\widehat{\mathbb{V}}_N(n, t)$ into the components in the homogeneous Wiener chaoses \mathcal{H}_k , $k = 1, 3, 5, 7$:

$$\widehat{\mathbb{V}}_N(n, t) = \sum_{j=0}^3 \widehat{\mathbb{V}}_N^{(2j+1)}(n, t), \tag{4.48}$$

where $\widehat{\mathbb{V}}_N^{(2j+1)} \in \mathcal{H}_{2j+1}$. From the orthogonality of \mathcal{H}_k , we have

$$\mathbb{E} \left[\left| \widehat{\mathbb{V}}_N(n, t) \right|^2 \right] = \sum_{j=0}^3 \mathbb{E} \left[\left| \widehat{\mathbb{V}}_N^{(2j+1)}(n, t) \right|^2 \right].$$

Hence, it suffices to prove (4.47) for $\widehat{\mathbb{V}}_N^{(2j+1)}$, $j = 0, 1, 2, 3$.

Case (i): *Non-resonant septic term* We first study the non-resonant term $\widehat{\Psi_N^{(7)}} \in \mathcal{H}_7$. From (1.12) and (4.25) with (4.26) and (4.15), we have

$$\widehat{\Psi_N^{(7)}}(n, t) = I_7[f_{n,t}^{(7)}], \tag{4.49}$$

where $f_{n,t}^{(7)}$ is defined by

$$\begin{aligned} f_{n,t}^{(7)}(z_1, \dots, z_7) &= \mathbf{1}_{n=n_{1234567}} \cdot \left(\prod_{j=1}^7 \mathbf{1}_{|n_j| \leq N} \right) \\ &\times \int_0^t \frac{\sin((t-t')\langle n_{123} \rangle)}{\langle n_{123} \rangle} \left(\prod_{j=1}^3 \frac{\sin((t'-t_j)\langle n_j \rangle)}{\langle n_j \rangle^{1+\alpha}} \mathbf{1}_{[t_j, t]}(t') \right) dt' \\ &\times \int_0^t \frac{\sin((t-t'')\langle n_{456} \rangle)}{\langle n_{456} \rangle} \left(\prod_{j=4}^6 \frac{\sin((t''-t_j)\langle n_j \rangle)}{\langle n_j \rangle^{1+\alpha}} \mathbf{1}_{[t_j, t]}(t'') \right) dt'' \\ &\times \frac{\sin((t-t_7)\langle n_7 \rangle)}{\langle n_7 \rangle^{1+\alpha}} \mathbf{1}_{[0, t]}(t_7). \end{aligned} \tag{4.50}$$

By defining the amplitude Φ by

$$\Phi(t, z_1, z_2, z_3) = \int_{\max(t_1, t_2, t_3)}^t \frac{\sin((t-t')\langle n_{123} \rangle)}{\langle n_{123} \rangle} \prod_{j=1}^3 \frac{\sin((t'-t_j)\langle n_j \rangle)}{\langle n_j \rangle^{1+\alpha}} dt', \tag{4.51}$$

we have

$$f_{n,t}^{(7)}(z_1, \dots, z_7) = \Phi(t, z_1, z_2, z_3) \Phi(t, z_4, z_5, z_6) \frac{\sin((t-t_7)\langle n_7 \rangle)}{\langle n_7 \rangle^{1+\alpha}}.$$

Let $\kappa_2(\bar{n})$ be as in (4.31). Then, from (4.51), we have

$$\sup_{t \in [0, T]} |\Phi(t, z_1, z_2, z_3)| \lesssim K(n_1, n_2, n_3) \prod_{j=1}^3 \langle n_j \rangle^{-\alpha},$$

where $K(n_1, n_2, n_3)$ is defined by

$$K(n_1, n_2, n_3) = \frac{1}{\langle n_{123} \rangle \langle \kappa_2(\bar{n}) \rangle} \prod_{j=1}^3 \frac{1}{\langle n_j \rangle}. \tag{4.52}$$

Note that from Lemma A.1, we have

$$\sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^3 \\ |n_j| \sim N_j}} K^2(n_1, n_2, n_3) \lesssim \max(N_1, N_2, N_3)^\gamma \tag{4.53}$$

for any $\gamma > 0$. In view of (4.52) and (4.31), $K(n_1, n_2, n_3)$ depends on $\varepsilon_{123}, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}$. In the following, however, we drop the dependence on $\varepsilon_{123}, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}$ since (4.53) uniformly in $\varepsilon_{123}, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}$. The same comment applies to (4.54) below.

With the symmetrization $\text{Sym}(f_{n,t}^{(7)})$ defined in (B.1), it follows from Lemma B.1, Jensen’s inequality (B.2), and Lemma 2.8 (to sum over n_7) that

$$\begin{aligned} & \left\| \left\| \mathfrak{V}_N^{(7)}(t) \right\|_{H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^2(\Omega)}^2 \\ & \sim \sum_{n \in \mathbb{Z}^3} \langle n \rangle^{2\alpha-1-2\varepsilon} \sum_{\substack{n=n_{1234567} \\ |n_j| \leq N}} \int_{[0,t]} |\text{Sym}(f_{n,t}^{(7)})|^2 dt_1 \cdots dt_7 \\ & \lesssim T^\theta \sum_{\substack{n_1, \dots, n_7 \in \mathbb{Z}^3 \\ |n_j| \leq N}} \frac{\langle n_{1234567} \rangle^{2\alpha-1-2\varepsilon}}{\langle n_7 \rangle^{2+2\alpha}} \left(\prod_{j=1}^6 \frac{1}{\langle n_j \rangle^{2\alpha}} \right) K^2(n_1, n_2, n_3) K^2(n_4, n_5, n_6) \\ & \lesssim T^\theta \sum_{\substack{n_1, \dots, n_6 \in \mathbb{Z}^3 \\ |n_j| \leq N}} \frac{1}{\langle n_{123456} \rangle^{2\varepsilon}} \left(\prod_{j=1}^6 \frac{1}{\langle n_j \rangle^{2\alpha}} \right) K^2(n_1, n_2, n_3) K^2(n_4, n_5, n_6) \end{aligned}$$

for some $\theta > 0$, provided that $\delta_1 > 0$. By applying the dyadic decomposition $|n_j| \sim N_j, j = 1, \dots, 7$, and then applying (4.53), we then obtain

$$\left\| \left\| \mathfrak{V}_N^{(7)}(t) \right\|_{H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^2(\Omega)}^2 \sim \sum_{\substack{1 \leq N_1, \dots, N_6 \lesssim N \\ \text{dyadic}}} \left(\prod_{j=1}^6 \frac{1}{N_j^{2\alpha-\gamma}} \right) \lesssim 1,$$

as long as $\gamma < 2\alpha$. This proves (4.47).

Case (ii): *General septic terms* As we saw in the previous subsections, all other terms in (4.48) come from the contractions of the product of $\mathfrak{V}_N \cdot \mathfrak{V}_N \cdot \mathfrak{I}_N$. In order to fully describe these terms, we recall the notion of a pairing from [12, Definition 4.30] to describe the structure of the contractions.

Definition 4.3 (pairing) Let $J \geq 1$. We call a relation $\mathcal{P} \subset \{1, \dots, J\}^2$ a pairing if

- (i) \mathcal{P} is reflexive, i.e. $(j, j) \notin \mathcal{P}$ for all $1 \leq j \leq J$,
- (ii) \mathcal{P} is symmetric, i.e. $(i, j) \in \mathcal{P}$ if and only if $(j, i) \in \mathcal{P}$,
- (iii) \mathcal{P} is univalent, i.e. for each $1 \leq i \leq J, (i, j) \in \mathcal{P}$ for at most one $1 \leq j \leq J$.

If $(i, j) \in \mathcal{P}$, the tuple (i, j) is called a pair. If $1 \leq j \leq J$ is contained in a pair, we say that j is paired. With a slight abuse of notation, we also write $j \in \mathcal{P}$ if j is paired. If j is not paired, we also say that j is unpaired and write $j \notin \mathcal{P}$. Furthermore, given a partition $\mathcal{A} = \{A_\ell\}_{\ell=1}^L$ of $\{1, \dots, J\}$, we say that \mathcal{P} respects \mathcal{A} if $i, j \in A_\ell$ for some $1 \leq \ell \leq L$ implies that $(i, j) \notin \mathcal{P}$. Namely, \mathcal{P} does not pair elements of the same set $A_\ell \in \mathcal{A}$. We say that $(n_1, \dots, n_J) \in (\mathbb{Z}^3)^J$ is admissible if $(i, j) \in \mathcal{P}$ implies that $n_i + n_j = 0$.

In order to represent $\widehat{\mathfrak{V}}_N^{(k)}(n, t)$, $k = 1, 3, 5$, as multiple stochastic integrals as in (4.49), we start with (4.50) and perform a contraction over the variables $z_j = (n_j, t_j)$, namely, we consider a (non-trivial)²⁰ pairing on $\{1, \dots, 7\}$. Then, by integrating in t' and t'' first in (4.50) after a contraction, a computation analogous to that in Case (i) yields

$$\begin{aligned} & \left\| \left\| \widehat{\mathfrak{V}}_N^{(k)}(t) \right\|_{H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^2(\Omega)} \\ & \lesssim \sum_{\varepsilon_{123}, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}} \sum_{\mathcal{P} \in \Pi_k} \sum_{\{n_j\}_{j \notin \mathcal{P}}} \langle n_{nr} \rangle^{2\alpha-1-2\varepsilon} \\ & \quad \times \left(\sum_{\{n_j\}_{j \in \mathcal{P}}} \mathbf{1}_{\substack{(n_1, \dots, n_7) \\ \text{admissible}}} \cdot \frac{K(n_1, n_2, n_3)K(n_4, n_5, n_6)}{\langle n_7 \rangle^{1+\alpha}} \prod_{j=1}^6 \frac{1}{\langle n_j \rangle^\alpha} \right)^2, \end{aligned} \tag{4.54}$$

where K is as in (4.52) and the non-resonant frequency n_{nr} is defined by

$$n_{nr} = \sum_{j \notin \mathcal{P}} n_j. \tag{4.55}$$

Here, Π_k denotes the collection of pairings \mathcal{P} on $\{1, \dots, 7\}$ such that (i) \mathcal{P} respects the partition $\mathcal{A} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7\}\}$ and (ii) $|\mathcal{P}| = 7 - k$ (when we view \mathcal{P} as a subset of $\{1, \dots, 7\}$). Note that the estimate on $\widehat{\mathfrak{V}}_N^{(7)}$ discussed in Case (i) is a special case of (4.54) with $\mathcal{P} = \emptyset$. By applying Lemma A.6 (with (1.23)), we then obtain

$$\left\| \left\| \widehat{\mathfrak{V}}_N^{(k)}(t) \right\|_{H_x^{\alpha-\frac{1}{2}-\varepsilon}} \right\|_{L^2(\Omega)} \lesssim 1,$$

provided that $\varepsilon > 0$. This concludes the proof of Lemma 3.4 (ii).

4.4 Random operator

In this subsection, we present the proof of Lemma 3.5 on the random operator $\mathfrak{J}^{\vee N}$ defined in (1.21).

In view of (3.1) and (3.2) in the definition of $\mathcal{L}_{T_0}^{s_1, s_2, b}$, (1.21), and the nonhomogeneous linear estimate (Lemma 2.5), it suffices to show the following bound:

$$\left\| \sup_{T \in [0, 1]} \sup_{\|v\|_{X_T^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \leq 1} \|\mathfrak{V}_N v\|_{X_T^{-\frac{1}{2}+\delta_1, -\frac{1}{2}+2\delta_2}} \right\|_{L^p(\Omega)} \lesssim p \tag{4.56}$$

²⁰ Namely, $\mathcal{P} = \emptyset$.

for some small $\delta_1, \delta_2 > 0$ and any $p \geq 1$, uniformly in $N \in \mathbb{N}$. From (2.9), we see that (4.56) follows once we prove

$$\left\| \sup_{T \in [0,1]} \sup_{\|v\|_{X^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \leq 1} \|\mathbf{v}_N v\|_{X_T^{-\frac{1}{2}+\delta_1, -\frac{1}{2}+2\delta_2}} \right\|_{L^p(\Omega)} \lesssim p. \tag{4.57}$$

Furthermore, by inserting a sharp time-cutoff function on $[0, 1]$, we may drop the supremum in T and reduce the bound (4.57) to proving

$$\left\| \sup_{\|v\|_{X^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \leq 1} \|\mathbf{1}_{[0,1]}(t) \cdot \mathbf{v}_N v\|_{X^{-\frac{1}{2}+\delta_1, -\frac{1}{2}+2\delta_2}} \right\|_{L^p(\Omega)} \lesssim p. \tag{4.58}$$

As in the proof of Lemma 3.1 (ii), we first prove

$$\left\| \sup_{\|v\|_{X^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \leq 1} \|\mathbf{1}_{[0,1]}(t) \cdot \mathbf{v}_N v\|_{X^{-\frac{1}{2}+\delta_1, -\frac{1}{2}-\delta}} \right\|_{L^p(\Omega)} \lesssim p, \tag{4.59}$$

namely with $b = -\frac{1}{2} - \delta < -\frac{1}{2}$ on the $X^{s,b}$ -norm of $\mathbf{1}_{[0,1]}(t) \cdot \mathbf{v}_N v$ for $\delta > 0$. In fact, we prove a frequency-localized version of (4.59) (see (4.72) below) and interpolate it with a trivial $X^{0,0}$ estimate (see (4.73) below), as in the proof of Lemma 3.1 (ii) and Lemma 3.4 (i), to establish (4.58) with $b = -\frac{1}{2} + 2\delta_2 > -\frac{1}{2}$

We start by computing the space-time Fourier transform of $\mathbf{1}_{[0,1]}(t) \cdot \mathbf{v}_N v$. From (4.25) and (4.26), we have

$$\mathcal{F}_x(\mathbf{v}_N \cdot v)(n, t) = \sum_{n_3 \in \mathbb{Z}^3} \widehat{v}(n_3, t) I_2[g_{n-n_3, t}],$$

where $g_{n-n_3, t}(z_1, z_2)$ is as in (4.26). Now, write $v = v_1 + v_{-1}$, where

$$\widehat{v}_1(n, \tau) = \mathbf{1}_{[0, \infty)}(\tau) \cdot \widehat{v}(n, \tau) \quad \text{and} \quad \widehat{v}_{-1}(n, \tau) = \mathbf{1}_{(-\infty, 0)}(\tau) \cdot \widehat{v}(n, \tau).$$

Then, by noting $|\widehat{v}(n, \tau)|^2 = |\widehat{v}_1(n, \tau)|^2 + |\widehat{v}_{-1}(n, \tau)|^2$, we have

$$\begin{aligned} \|v\|_{X^{s,b}}^2 &= \sum_{\varepsilon_3 \in \{-1, 1\}} \|v_{\varepsilon_3}\|_{X^{s,b}}^2 \\ &= \sum_{\varepsilon_3 \in \{-1, 1\}} \|\langle n \rangle^s \langle \tau \rangle^b \widehat{v}_{\varepsilon_3}(n, \tau + \varepsilon_3 \langle n \rangle)\|_{\ell_n^2 L_\tau^2}^2. \end{aligned} \tag{4.60}$$

With this in mind, we write

$$\begin{aligned} &\mathcal{F}_{x,t}(\mathbf{1}_{[0,1]}(t) \cdot \mathbf{v}_N v_{\varepsilon_3})(n, \tau - \varepsilon_0 \langle n \rangle) \\ &= \langle n \rangle^{\frac{1}{2}-\delta_1} \sum_{|n_3| \leq N} \langle n_3 \rangle^{\frac{1}{2}+\delta_1} \int_{\mathbb{R}} \widehat{v}_{\varepsilon_3}(n_3, \tau_3 + \varepsilon_3 \langle n_3 \rangle) H(n, n_3, \tau, \tau_3) d\tau_3, \end{aligned} \tag{4.61}$$

where $\varepsilon_0, \varepsilon_3 \in \{-1, 1\}$ and the kernel $H = H^{\varepsilon_0, \varepsilon_3}$ is given by

$$H(n, n_3, \tau, \tau_3) = \langle n \rangle^{-\frac{1}{2}+\delta_1} \langle n_3 \rangle^{-\frac{1}{2}-\delta_1} \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-it(\tau-\tau_3-\varepsilon_0\langle n \rangle-\varepsilon_3\langle n_3 \rangle)} I_2[g_{n-n_3, t}] dt.$$

By Fubini’s theorem (Lemma B.2), we can write H as

$$H(n, n_3, \tau, \tau_3) = \langle n \rangle^{-\frac{1}{2}+\delta_1} \langle n_3 \rangle^{-\frac{1}{2}-\delta_1} I_2[h_{n, n_3, \tau, \tau_3}], \tag{4.62}$$

where h_{n, n_3, τ, τ_3} is given by

$$h_{n, n_3, \tau, \tau_3}(z_1, z_2) = \mathbf{1}_{n-n_3=n_{12}} \cdot \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-it(\tau-\tau_3-\varepsilon_0\langle n \rangle-\varepsilon_3\langle n_3 \rangle)} \times \left(\prod_{j=1}^2 \frac{\sin((t-t_j)\langle n_j \rangle)}{\langle n_j \rangle^{1+\alpha}} \cdot \mathbf{1}_{|n_j| \leq N} \cdot \mathbf{1}_{[0, t]}(t_j) \right) dt. \tag{4.63}$$

Then, by (4.6), (4.61), Cauchy–Schwarz’s inequality, and (4.60), we have

$$\begin{aligned} & \left\| \sup_{\|v\|_{X^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \leq 1} \|\mathbf{1}_{[0,1]}(t) \cdot \mathbf{v}_N v\|_{X^{-\frac{1}{2}+\delta_1, -\frac{1}{2}-\delta}} \right\|_{L^p(\Omega)} \\ & \leq \sum_{\varepsilon_3 \in \{-1, 1\}} \left\| \sup_{\|v\|_{X^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \leq 1} \|\mathbf{1}_{[0,1]}(t) \cdot \mathbf{v}_N v_{\varepsilon_3}\|_{X^{-\frac{1}{2}+\delta_1, -\frac{1}{2}-\delta}} \right\|_{L^p(\Omega)} \\ & \lesssim \sup_{\varepsilon_0, \varepsilon_3 \in \{-1, 1\}} \left\| \sup_{\|v\|_{X^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \leq 1} \left\| \langle \tau \rangle^{-\frac{1}{2}-\delta} \sum_{|n_3| \leq N} \langle n_3 \rangle^{\frac{1}{2}+\delta_1} \right. \right. \\ & \quad \times \left. \left\| \int_{\mathbb{R}} \widehat{v}_{\varepsilon_3}(n_3, \tau_3 + \varepsilon_3 \langle n_3 \rangle) H(n, n_3, \tau, \tau_3) d\tau_3 \right\| \right\|_{L^p(\Omega; L_n^2)} \\ & \lesssim \sup_{\varepsilon_0, \varepsilon_3 \in \{-1, 1\}} \left\| \left\| \langle \tau \rangle^{-\frac{1}{2}-\delta} \langle \tau_3 \rangle^{-\frac{1}{2}-\delta_2} \|H(n, n_3, \tau, \tau_3)\|_{\ell_{n_3}^2 \rightarrow \ell_n^2} \right\| \right\|_{L^p(\Omega; L_{\tau, \tau_3}^2)} \\ & \lesssim \sup_{\varepsilon_0, \varepsilon_3} \sup_{\tau, \tau_3 \in \mathbb{R}} \left\| \|H(n, n_3, \tau, \tau_3)\|_{\ell_{n_3}^2 \rightarrow \ell_n^2} \right\|_{L^p(\Omega)}, \end{aligned}$$

as long as $\delta, \delta_2 > 0$, where, in the last step, we used Minkowski’s integral inequality followed by Hölder’s inequality (in τ and τ_3). Here, we viewed $H(n, n_3, \tau, \tau_3)$ (for fixed $\tau, \tau_3 \in \mathbb{R}$) as an infinite dimensional matrix operator mapping from $\ell_{n_3}^2$ into ℓ_n^2 . Hence, the estimate (4.59) is reduced to proving

$$\sup_{\varepsilon_0, \varepsilon_3} \sup_{\tau, \tau_3 \in \mathbb{R}} \left\| \|H(n, n_3, \tau, \tau_3)\|_{\ell_{n_3}^2 \rightarrow \ell_n^2} \right\|_{L^p(\Omega)} \lesssim p. \tag{4.64}$$

As mentioned above, we instead establish a frequency-localized version of (4.64):

$$\sup_{\varepsilon_0, \varepsilon_3} \sup_{\tau, \tau_3 \in \mathbb{R}} \left\| \| H_{N_1, N_2, N_3}(n, n_3, \tau, \tau_3) \|_{\ell_{n_3}^2 \rightarrow \ell_n^2} \right\|_{L^p(\Omega)} \lesssim p N_{\max}^{-\delta_0}, \tag{4.65}$$

for some small $\delta_0 > 0$, uniformly in dyadic $N_1, N_2, N_3 \geq 1$, where $N_{\max} = \max(N_1, N_2, N_3)$ and H_{N_1, N_2, N_3} is defined by (4.62) and (4.63) with extra frequency localizations $\mathbf{1}_{|n_j| \sim N_j}$, $j = 1, 2, 3$. Namely, we have

$$H_{N_1, N_2, N_3}(n, n_3, \tau, \tau_3) = \langle n \rangle^{-\frac{1}{2} + \delta_1} \langle n_3 \rangle^{-\frac{1}{2} - \delta_1} I_2[h_{n, n_3, \tau, \tau_3}^{N_1, N_2, N_3}], \tag{4.66}$$

where $h_{n, n_3, \tau, \tau_3}^{N_1, N_2, N_3}$ is given by

$$\begin{aligned} h_{n, n_3, \tau, \tau_3}^{N_1, N_2, N_3}(z_1, z_2) &= \sum_{\varepsilon_1, \varepsilon_2 \in \{-1, 1\}} c_{\varepsilon_1, \varepsilon_2} \mathbf{1}_{n=n_3} \cdot \mathbf{1}_{|n_3| \sim N_3} \cdot \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-it(\tau - \tau_3 - \kappa(\bar{n}))} \\ &\times \left(\prod_{j=1}^2 \frac{e^{-it_j \varepsilon_j \langle n_j \rangle}}{\langle n_j \rangle^{1+\alpha}} \cdot \mathbf{1}_{|n_j| \sim N_j} \cdot \mathbf{1}_{|n_j| \leq N} \right) dt \end{aligned} \tag{4.67}$$

with $\kappa(\bar{n})$ as in (4.7).

For $m \in \mathbb{Z}$, define the tensor \mathfrak{h}^m by

$$\begin{aligned} \mathfrak{h}_{nn_1n_2n_3}^m &= c_{\varepsilon_1, \varepsilon_2} \mathbf{1}_{n=n_{123}} \cdot \mathbf{1}_{|n_3| \sim N_3} \left(\prod_{j=1}^2 \mathbf{1}_{|n_j| \sim N_j} \right) \\ &\times \mathbf{1}_{\{|\kappa(\bar{n}) - m| \leq 1\}} \frac{\langle n \rangle^{-\frac{1}{2} + \delta_1}}{\langle n_1 \rangle^{1+\alpha} \langle n_2 \rangle^{1+\alpha} \langle n_3 \rangle^{\frac{1}{2} + \delta_1}}. \end{aligned} \tag{4.68}$$

Then, from (4.66), (4.67), and (4.68), we have

$$\begin{aligned} H_{N_1, N_2, N_3}(n, n_3, \tau, \tau_3) &= \sum_{\varepsilon_1, \varepsilon_2 \in \{-1, 1\}} \sum_{m \in \mathbb{Z}} H^m(n, n_3, \tau, \tau_3) \\ &:= \sum_{\varepsilon_1, \varepsilon_2 \in \{-1, 1\}} \sum_{m \in \mathbb{Z}} I_2[\mathfrak{h}_{nn_1n_2n_3}^m \mathfrak{H}_{n_3, \tau, \tau_3}^m], \end{aligned} \tag{4.69}$$

where $\mathfrak{H}_{n_3, \tau, \tau_3}^m$ is given by

$$\begin{aligned} \mathfrak{H}_{n_3, \tau, \tau_3}^m(z_1, z_2) &= \frac{1}{\sqrt{2\pi}} \int_0^1 \mathbf{1}_{\{|\kappa(\bar{n}) - m| \leq 1\}} e^{-it(\tau - \tau_3 - \kappa(\bar{n}))} \left(\prod_{j=1}^2 e^{-it_j \varepsilon_j \langle n_j \rangle} \cdot \mathbf{1}_{[0, t]}(t_j) \right) dt. \end{aligned}$$

Performing t -integration, we have

$$\|\mathfrak{H}_{n_3, \tau, \tau_3}^m(z_1, z_2)\|_{\ell_{n_1, n_2}^\infty L_{t_1, t_2}^2([0,1]^2)} \lesssim \langle \tau - \tau_3 - m \rangle^{-1}. \tag{4.70}$$

Then from Lemma C.3, (4.70), and Lemma C.2 (with (1.23)), there exists $\delta_3 > 0$ such that

$$\begin{aligned} & \left\| \|H^m(n, n_3, \tau, \tau_3)\|_{\ell_{n_3}^2 \rightarrow \ell_n^2} \right\|_{L^p(\Omega)} \\ & \lesssim pN_{\max}^\varepsilon \langle \tau - \tau_3 - m \rangle^{-1} \\ & \quad \times \max \left(\|\mathfrak{h}^m\|_{n_1 n_2 n_3 \rightarrow n}, \|\mathfrak{h}^m\|_{n_3 \rightarrow n n_1 n_2}, \|\mathfrak{h}^m\|_{n_1 n_3 \rightarrow n n_2}, \|\mathfrak{h}^m\|_{n_2 n_3 \rightarrow n n_1} \right) \\ & \lesssim pN_{\max}^{\varepsilon - \delta_3} \langle \tau - \tau_3 - m \rangle^{-1}. \end{aligned} \tag{4.71}$$

for any $\varepsilon > 0$, provided that $\delta_1 < \alpha$, which is needed to apply Lemma C.2. Hence, by noting that the condition $|\kappa(\bar{n}) - m| \leq 1$ implies $|m| \lesssim N_{\max}$ and summing over $m \in \mathbb{Z}$, the bound (4.65) follows from (4.69) and (4.71) (by taking $\varepsilon > 0$ sufficiently small), which in turn implies

$$\left\| \sup_{\|v\|_{X^{\frac{1}{2} + \delta_1, \frac{1}{2} + \delta_2}} \leq 1} \|\mathbf{1}_{[0,1]}(t) \cdot \mathbf{v}_N^{N_1, N_2} v_{N_3}\|_{X^{-\frac{1}{2} + \delta_1, -\frac{1}{2} - \delta}} \right\|_{L^p(\Omega)} \lesssim pN_{\max}^{-\delta_0} \tag{4.72}$$

for some $\delta_0 > 0$, where $v_{N_3} = \mathcal{F}_x^{-1}(\mathbf{1}_{|n| \sim N_3} \widehat{v}(n))$ and

$$\widehat{V}_N^{N_1, N_2}(n, t) = I_2 \left[\mathbf{1}_{n=n_{12}} \cdot \left(\prod_{j=1}^2 \frac{\sin(t - t_j) \langle n_j \rangle}{\langle n_j \rangle^{1+\alpha}} \cdot \mathbf{1}_{|n_j| \sim N_j} \cdot \mathbf{1}_{[0,t]}(t_j) \right) \right].$$

Namely, the frequencies n_1, n_2 , and n_3 are localized to the dyadic blocks $\{|n_j| \sim N_j\}$, $j = 1, 2, 3$.

On the other hand, a crude bound shows

$$\left\| \sup_{\|v\|_{X^{\frac{1}{2} + \delta_1, \frac{1}{2} + \delta_2}} \leq 1} \|\mathbf{1}_{[0,1]}(t) \cdot \mathbf{v}_N^{N_1, N_2} v_{N_3}\|_{X^{0,0}} \right\|_{L^p(\Omega)} \lesssim pN_{\max}^K \tag{4.73}$$

for some (possibly large) $K > 0$. By interpolating (4.72) and (4.73) and then summing over dyadic $N_j, j = 1, \dots, 3$, we obtain (4.58) for some small $\delta_2 > 0$.

Lastly, as for the convergence of \mathfrak{I}^{V_N} to \mathfrak{I}^V , we can simply repeat the computation above to estimate the difference $\mathbf{1}_{[0,1]} \mathfrak{V}_M v - \mathbf{1}_{[0,1]} \mathfrak{V}_N v$ for $M \geq N \geq 1$. In considering the difference of the tensors \mathfrak{h}^m in (4.68), we then obtain a new restriction $\max(|n_1|, |n_2|) \gtrsim N$, which allows us to gain a small negative power of N . As a result,

we obtain

$$\left\| \sup_{\|v\|_{X^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \leq 1} \left\| \mathbf{1}_{[0,1]}(t) \cdot \left(\mathbf{v}_N^{N_1, N_2} - \mathbf{v}_M^{N_1, N_2} \right) v_{N_3} \right\|_{X^{-\frac{1}{2}+\delta_1, -\frac{1}{2}-\delta}} \right\|_{L^p(\Omega)} \lesssim p N^{-\varepsilon} N_{\max}^{-\delta'_0}$$

for some small $\varepsilon, \delta'_0 > 0$. Then, interpolating this with (4.73) and summing over dyadic blocks, we then obtain

$$\left\| \left\| \mathfrak{J}^{\mathbf{V}_M} - \mathfrak{J}^{\mathbf{V}_N} \right\|_{\mathcal{L}^{\frac{1}{2}+\delta_1, \frac{1}{2}+\delta_1, \frac{1}{2}+\delta_2}} \right\|_{L^p(\Omega)} \lesssim p N^{-\varepsilon},$$

for any $p \geq 1$ and $M \geq N \geq 1$. Then, by applying Chebyshev’s inequality, summing over $N \in \mathbb{N}$, and applying the Borel–Cantelli lemma, we conclude the almost sure convergence of $\mathfrak{J}^{\mathbf{V}_N}$. This concludes the proof of Lemma 3.5.

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Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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Appendix A: Counting estimates

In this section, we state the counting estimates used in Sect. 4 to study the regularities of the stochastic terms. These lemmas are taken from Bringmann [12]. Note that some statements are given in a slightly simplified form. The same comment applies to Lemma C.2.

Lemma A.1 (Proposition 4.20 in [12]) *Let $0 < s \leq \frac{1}{2}$ and $0 \leq \beta \leq \frac{1}{2}$. Given $\varepsilon_j \in \{-1, 1\}$ for $j = 0, 1, 2, 3$, let $\kappa(\bar{n}) = \kappa_{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3}(n_1, n_2, n_3)$ be as in (4.7). Then, we have*

$$\sup_{m \in \mathbb{Z}} \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^3 \\ |n_j| \sim N_j}} \langle n_{123} \rangle^{2(s-1)} \frac{\mathbf{1}_{\{|\kappa(\bar{n})-m| \leq 1\}}}{\langle n_{12} \rangle^{2\beta} \prod_{j=1}^3 \langle n_j \rangle^2} \lesssim N_{\max}^{2(s-\beta)},$$

uniformly in dyadic $N_1, N_2, N_3 \geq 1$ and $\varepsilon_j \in \{-1, 1\}$ for $j = 0, 1, 2, 3$, where $N_{\max} = \max(N_1, N_2, N_3)$.

Lemma A.2 (Lemma 4.22(i) in [12]) *Given $\varepsilon_j \in \{-1, 1\}$ for $j = 0, 1, 2, 3$, let $\kappa(\bar{n}) = \kappa_{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3}(n_1, n_2, n_3)$ be as in (4.7). Then, we have*

$$\sup_{m \in \mathbb{Z}} \sup_{n \in \mathbb{Z}^3} \#\left\{ (n_1, n_2, n_3) \in \mathbb{Z}^3 : |n_j| \sim N_j, j = 1, 2, 3, n = n_{123}, \{|\kappa(\bar{n}) - m| \leq 1\} \right\} \lesssim \text{med}(N_1, N_2, N_3)^3 \min(N_1, N_2, N_3)^2,$$

uniformly in dyadic $N_1, N_2, N_3 \geq 1$ and $\varepsilon_j \in \{-1, 1\}$ for $j = 0, 1, 2, 3$.

Next, we recall the basic resonant estimate.

Lemma A.3 (Lemma 4.25 in [12]) *Given $\varepsilon_j \in \{-1, 1\}$ for $j = 0, 1, 2, 3$, let $\kappa(\bar{n}) = \kappa_{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3}(n_1, n_2, n_3)$ be as in (4.7). Then, we have*

$$\sum_{m \in \mathbb{Z}} \sum_{\substack{n_1 \in \mathbb{Z}^3 \\ |n_1| \sim N_1}} \frac{\mathbf{1}_{\{|\kappa(\bar{n}) - m| \leq 1\}}}{\langle m \rangle \langle n_{123} \rangle \langle n_1 \rangle^2} \lesssim \frac{\log(2 + N_1)}{\langle n_{23} \rangle},$$

uniformly in dyadic $N_1 \geq 1$ and $\varepsilon_j \in \{-1, 1\}$ for $j = 0, 1, 2, 3$.

The next two lemmas (and Lemma A.3 above) are used for estimating the quintic stochastic term.

Lemma A.4 *Let $s \leq \frac{1}{2} - \eta$ and $\beta > 0$ for some $\eta > 0$. Given $\varepsilon_{123}, \varepsilon_j \in \{-1, 1\}$ for $j = 0, \dots, 5$, let $\kappa_2(\bar{n}), \kappa_3(\bar{n}),$ and $\kappa_4(\bar{n})$ be as in (4.31) and (4.34). Then, we have*

$$\begin{aligned} & \sup_{m, m' \in \mathbb{Z}} \sum_{\substack{n_1, \dots, n_5 \in \mathbb{Z}^3 \\ |n_j| \sim N_j}} \frac{\langle n_{12345} \rangle^{2(s-1)}}{\langle n_{1234} \rangle^{2\beta} \langle n_{12} \rangle^{2\beta} \langle n_{123} \rangle^2 \prod_{j=1}^5 \langle n_j \rangle^2} \\ & \times \mathbf{1}_{\{|\kappa_2(\bar{n}) - m| \leq 1\}} \left(\mathbf{1}_{\{|\kappa_3(\bar{n}) - m'| \leq 1\}} + \mathbf{1}_{\{|\kappa_4(\bar{n}) - m'| \leq 1\}} \right) \\ & \lesssim \max(N_1, N_2, N_3, N_4)^{-2\beta + \varepsilon} N_5^{-\eta} \end{aligned} \tag{A.1}$$

for any $\varepsilon > 0$, uniformly in dyadic $N_1, \dots, N_5 \geq 1$ and $\varepsilon_{123}, \varepsilon_j \in \{-1, 1\}$ for $j = 0, \dots, 5$, where $N_{\max} = \max(N_1, \dots, N_5)$.

Lemma A.4 is essentially Lemma 4.27 in [12], where the condition $|\kappa_4(\bar{n}) - m'| \leq 1$ in (A.1) is replaced by $|\kappa_4(\bar{n}) + \varepsilon_{123} \langle n_{123} \rangle - m'| \leq 1$. We point out that this modification does not make any difference in the proof. In our notation, the first step of the proof of Lemma 4.27 in [12] is to sum over n_5 , using [12, Lemma 4.17], for which the conditions $|\kappa_4(\bar{n}) - m'| \leq 1$ in (A.1) and $|\kappa_4(\bar{n}) + \varepsilon_{123} \langle n_{123} \rangle - m'| \leq 1$ do not make any difference since the extra term $\varepsilon_{123} \langle n_{123} \rangle$ is fixed in summing over n_5 .

Lemma A.5 (Lemma 4.29 in [12]) *Let $\beta > 0$. Given $\varepsilon_{123}, \varepsilon_j \in \{-1, 1\}$ for $j = 1, 2, 3$, let $\kappa_2(\bar{n})$ be as in (4.31). Then, we have*

$$\sup_{m \in \mathbb{Z}^3} \sup_{|n_1| \sim N_1} \sum_{\substack{|n_2| \sim N_2 \\ |n_3| \sim N_3}} \frac{\mathbf{1}_{\{|\kappa_2(\bar{n})-m| \leq 1\}}}{\langle n_{123} \rangle \langle n_{12} \rangle^\beta \langle n_2 \rangle^2 \langle n_3 \rangle^2} \lesssim \max(N_1, N_2)^{-\beta + \varepsilon}$$

for any $\varepsilon > 0$, uniformly in dyadic $N_1, N_2, N_3 \geq 1$ and $\varepsilon_{123}, \varepsilon_j \in \{-1, 1\}$ for $j = 1, 2, 3$.

Lastly, we state the septic counting estimate. See Definition 4.3 in Sect. 4.3 for the definition of a pairing.

Lemma A.6 (Lemma 4.31 in [12])

Let $\frac{1}{2} < s < 1$ and $\beta > 0$. Given $\varepsilon_{123}, \varepsilon_j \in \{-1, 1\}$ for $j = 1, 2, 3$, let $\kappa_2(\bar{n})$ be as in (4.31) and set

$$\mathcal{K}(n_1, n_2, n_3) = \sum_{m \in \mathbb{Z}} \frac{\mathbf{1}_{\{|\kappa_2(\bar{n})-m| \leq 1\}}}{\langle m \rangle \langle n_{123} \rangle \langle n_{12} \rangle^\beta} \prod_{j=1}^3 \frac{1}{\langle n_j \rangle}.$$

Let \mathcal{P} be a pairing on $\{1, \dots, 7\}$ which respects the partition $\{\{1, 2, 3\}, \{4, 5, 6\}, \{7\}\}$. Then, we have

$$\begin{aligned} & \sum_{\{n_j\}_{j \notin \mathcal{P}}} \langle n_{nr} \rangle^{2(s-1)} \left(\sum_{\{n_j\}_{j \in \mathcal{P}}} \mathbf{1}_{|n_{1234567}| \sim N_{1234567}} \cdot \mathbf{1}_{|n_{1237}| \sim N_{1237}} \cdot \mathbf{1}_{|n_{456}| \sim N_{456}} \cdot \mathbf{1}_{|n_7| \sim N_7} \right. \\ & \quad \times \mathbf{1}_{\substack{(n_1, \dots, n_7) \\ \text{admissible}}} \cdot \left. \frac{\mathcal{K}(n_1, n_2, n_3) \mathcal{K}(n_4, n_5, n_6)}{\langle n_7 \rangle} \right)^2 \\ & \lesssim N_{\max}^{2s-1+\varepsilon} \end{aligned}$$

for any $\varepsilon > 0$, uniformly in dyadic $N_{1234567}, N_{1237}, N_{456}, N_7 \geq 1$ and $\varepsilon_{123}, \varepsilon_j \in \{-1, 1\}$ for $j = 1, 2, 3$, where $N_{\max} = \max(N_1, \dots, N_7)$ and n_{nr} is as in (4.55).

Appendix B: Multiple stochastic integrals

In this section, we go over the basic definitions and properties of multiple stochastic integrals. See [43] and also [12, Section 4] for further discussion.

Let λ be the measure on $Z := \mathbb{Z}^3 \times \mathbb{R}_+$ defined by

$$d\lambda = dn dt,$$

where dn is the counting measure on \mathbb{Z}^3 . Given $k \in \mathbb{N}$, we set $\lambda_k = \otimes_{j=1}^k \lambda$ and $L^2(Z^k) = L^2((\mathbb{Z}^3 \times \mathbb{R}_+)^k, \lambda_k)$. Given a function $f \in L^2(Z^k)$, we can adapt the

discussion in [43, Section 1.1] (in particular, [43, Example 1.1.2]) to the complex-valued setting and define the multiple stochastic integral $I_k[f]$ by

$$I_k[f] = \sum_{n_1, \dots, n_k \in \mathbb{Z}^3} \int_{[0, \infty)^k} f(n_1, t_1, \dots, n_k, t_k) dB_{n_1}(t_1) \cdots dB_{n_k}(t_k).$$

Given a function $f \in L^2(Z^k)$, we define its symmetrization $\text{Sym}(f)$ by

$$\text{Sym}(f)(z_1, \dots, z_k) = \frac{1}{k!} \sum_{\sigma \in S_k} f(z_{\sigma(1)}, \dots, z_{\sigma(k)}), \tag{B.1}$$

where $z_j = (n_j, t_j)$ as in (4.15) and S_k denotes the symmetric group on $\{1, \dots, k\}$. Note that by Jensen’s equality, we have

$$|\text{Sym}(f)(z_1, \dots, z_k)|^p \leq \frac{1}{k!} \sum_{\sigma \in S_k} |f(z_{\sigma(1)}, \dots, z_{\sigma(k)})|^p \tag{B.2}$$

for any $p \geq 1$. We say that f is symmetric if $\text{Sym}(f) = f$. We now recall some basic properties of multiple stochastic integrals.

Lemma B.1 *Let $k, \ell \in \mathbb{N}$. The following statements hold for any $f \in L^2(Z^k)$ and $g \in L^2(Z^\ell)$:*

- (i) $I_k : L^2(Z^k) \rightarrow \mathcal{H}_k \subset L^2(\Omega)$ is a linear operator, where \mathcal{H}_k denotes the k th Wiener chaos.
- (ii) $I_k[\text{Sym}(f)] = I_k[f]$.
- (iii) Ito isometry:

$$\mathbb{E}[I_k[f] \overline{I_\ell[g]}] = \mathbf{1}_{k=\ell} \cdot k! \int_{(\mathbb{Z}^3 \times \mathbb{R})^k} \text{Sym}(f) \overline{\text{Sym}(g)} d\lambda_k.$$

- (iv) Furthermore, suppose that f is symmetric. Then, we have

$$I_k[f] = k! \sum_{n_1, \dots, n_k \in \mathbb{Z}^3} \int_0^\infty \int_0^{t_1} \int_0^{t_{k-1}} f(n_1, t_1, \dots, n_k, t_k) dB_{n_k}(t_k) \cdots dB_{n_1}(t_1),$$

where the iterated integral on the right-hand side is understood as an iterated Ito integral.

We state a version of Fubini’s theorem for multiple stochastic integrals that is convenient for our purpose. See, for example, [15, Theorem 4.33] for a version of the stochastic Fubini theorem.

Lemma B.2 *Let $k \geq 1$. Given finite $T > 0$, let $f \in L^2((\mathbb{Z}^3 \times [0, T])^k \times [0, T], d\lambda_k \otimes dt)$. (In particular, we assume that the temporal support (for the variables t_1, \dots, t_k, t)*

of f is contained in $[0, T]^{k+1}$ for any (n_1, \dots, n_k) .) Then, we have

$$\int_0^T I_k[f(\cdot, t)]dt = I_k\left[\int_0^T f(\cdot, t)dt\right] \tag{B.3}$$

in $L^2(\Omega)$.

Proof From Lemma B.1 (ii), we may assume that $f(z_1, \dots, z_k, t)$ is symmetric in $z_j = (n_j, t_j)$, $j = 1, \dots, k$. Let $\mathbf{n} = (n_1, \dots, n_k)$ and $\mathbf{t} = (t_1, \dots, t_k)$. From Minkowski’s integral inequality, Lemma B.1 (iii), and Cauchy–Schwarz’s inequality, we have

$$\begin{aligned} \left\| \int_0^T I_k[(f - \varphi)(\cdot, t)]dt \right\|_{L^2(\Omega)} &\lesssim \int_0^T \|(f - \varphi)(\cdot, t)\|_{\ell_{\mathbf{n}}^2(\mathbb{Z}^3)^k; L_{\mathbf{t}}^2([0, T]^k)} dt \\ &\leq T^{\frac{1}{2}} \|f - \varphi\|_{\ell_{\mathbf{n}}^2(\mathbb{Z}^3)^k; L_{\mathbf{t}, \mathbf{t}}^2([0, T]^{k+1})}. \end{aligned} \tag{B.4}$$

On the other hand, by Lemma B.1 (iii) and Cauchy–Schwarz’s inequality, we have

$$\begin{aligned} \left\| I_k\left[\int_0^T (f - \varphi)(\cdot, t)dt\right] \right\|_{L^2(\Omega)} &\sim \left\| \int_0^T (f - \varphi)(\cdot, t)dt \right\|_{\ell_{\mathbf{n}}^2(\mathbb{Z}^3)^k; L_{\mathbf{t}}^2(\mathbb{R}_+^k)} \\ &\leq T^{\frac{1}{2}} \|f - \varphi\|_{\ell_{\mathbf{n}}^2(\mathbb{Z}^3)^k; L_{\mathbf{t}, \mathbf{t}}^2([0, T]^{k+1})}. \end{aligned} \tag{B.5}$$

Hence, it follows from (B.4), (B.5) and the density²¹ of $\ell_{\mathbf{n}}^2(\mathbb{Z}^3)^k; C_{\mathbf{t}, \mathbf{t}}^\infty([0, T]^{k+1})$ in $\ell_{\mathbf{n}}^2(\mathbb{Z}^3)^k; L_{\mathbf{t}, \mathbf{t}}^2([0, T]^{k+1})$ that we may assume that f is symmetric and belongs to $\ell_{\mathbf{n}}^2(\mathbb{Z}^3)^k; C_{\mathbf{t}, \mathbf{t}}^\infty([0, T]^{k+1})$. Furthermore, we may assume that f has a compact support in \mathbf{n} . Namely, there exists $K > 0$ such that if $\max(|n_1|, \dots, |n_k|) > K$, then $f(n_1, t_1, \dots, n_k, t_k, t) = 0$ for any $t_1, \dots, t_k, t \in [0, T]$. Then, together with Lemma B.1 (iv), we have

$$\begin{aligned} &\int_0^T I_k[f(\cdot, t)]dt \\ &= k! \int_0^T \sum_{\substack{n_1, \dots, n_k \in \mathbb{Z}^3 \\ \max(|n_1|, \dots, |n_k|) \leq K}} \int_0^T \int_0^{t_1} \cdots \int_0^{t_{k-1}} f(z_1, \dots, z_k, t) dB_{n_k}(t_k) \cdots dB_{n_1}(t_1) dt \\ &= k! \sum_{\substack{n_1, \dots, n_k \in \mathbb{Z}^3 \\ \max(|n_1|, \dots, |n_k|) \leq K}} \int_0^T \int_0^T \int_0^{t_1} \cdots \int_0^{t_{k-1}} f(z_1, \dots, z_k, t) dB_{n_k}(t_k) \cdots dB_{n_1}(t_1) dt, \end{aligned} \tag{B.6}$$

²¹ By identifying a function $f \in \ell_{\mathbf{n}}^2(\mathbb{Z}^3)^k; L_{\mathbf{t}, \mathbf{t}}^2([0, T]^{k+1})$ with a sequence $\{f_{\mathbf{n}}\}_{\mathbf{n} \in (\mathbb{Z}^3)^k} \subset L_{\mathbf{t}, \mathbf{t}}^2([0, T]^{k+1})$, we can approximate each $f_{\mathbf{n}}$ by a smooth function $\varphi_{\mathbf{n}}$ such that $\|f_{\mathbf{n}} - \varphi_{\mathbf{n}}\|_{L_{\mathbf{t}, \mathbf{t}}^2([0, T]^{k+1})} < \varepsilon_{\mathbf{n}}$ such that $\varepsilon_{\mathbf{n}}$ is symmetric in \mathbf{n} and $\sum_{\mathbf{n} \in (\mathbb{Z}^3)^k} \varepsilon_{\mathbf{n}} = \varepsilon$. Then, the function $\varphi \cong \{\varphi_{\mathbf{n}}\}_{\mathbf{n} \in (\mathbb{Z}^3)^k}$ approximates f within distance ε in $\ell_{\mathbf{n}}^2(\mathbb{Z}^3)^k; L_{\mathbf{t}, \mathbf{t}}^2([0, T]^{k+1})$. Since f is symmetric, we can choose φ to be symmetric.

since the summation is over a finite set of indices $\mathbf{n} = (n_1, \dots, n_k)$ and f is symmetric. Hence, it remains to justifying the t -integration with the stochastic integrals for each fixed $\mathbf{n} = (n_1, \dots, n_k)$. For this reason, we suppress the dependence of f on $\mathbf{n} = (n_1, \dots, n_k)$ in the following.

When $k = 1$, we can exploit the smoothness of f and have

$$\begin{aligned} \int_0^T \int_0^T f(t_1, t) dB_{n_1}(t_1) dt &= \int_0^T f(T, t) B_{n_1}(T) dt - \int_0^T \int_0^T B_{n_1}(t_1) \partial_{t_1} f(t_1, t) dt_1 dt \\ &= B_{n_1}(T) \int_0^T f(T, t) dt - \int_0^T B_{n_1}(t_1) \partial_{t_1} \left(\int_0^T f(t_1, t) dt \right) dt_1 \\ &= \int_0^T \int_0^T f(t_1, t) dt dB_{n_1}(t_1), \end{aligned}$$

where, at the second equality, we used the standard Fubini’s theorem in view of the almost sure boundedness of B_{n_1} on $[0, T]$. This proves (B.3) when $k = 1$.

For the general case, let us first consider the innermost integral in (B.6). For notational simplicity, let us suppress all the variables of f except for t_k and t . Let $\Delta_m = \{0 \leq \tau_0 < \tau_1 < \dots < \tau_m \leq T\}$ be a partition of $[0, T]$ and define a step function $f_m(\cdot, t)$ by setting $f_m(\tau, t) = f(\tau_{j-1}, t)$ for $\tau_{j-1} < \tau \leq \tau_j$. Then, by defining J_m by

$$J_m(t) := \int_0^{t_{k-1}} f_m(t_k, t) dB_{n_k}(t_k) = \sum_{j=1}^m (\mathbf{1}_{[0, t_{k-1}]}) f(\tau_{j-1}, t) (B_{n_k}(\tau_j) - B_{n_k}(\tau_{j-1})), \tag{B.7}$$

it follows from the definition of the Wiener integral that

$$J_m(t) \longrightarrow \int_0^{t_{k-1}} f(t_k, t) dB_{n_k}(t_k) \quad \text{in } L^2(\Omega), \tag{B.8}$$

as $m \rightarrow \infty$ (such that $|\Delta_m| \rightarrow 0$). By integrating (B.7) in t , we have

$$\int_0^T J_m(t) dt = \sum_{j=1}^m \left(\int_0^T (\mathbf{1}_{[0, t_{k-1}]}) f(\tau_{j-1}, t) dt \right) (B_{n_k}(\tau_j) - B_{n_k}(\tau_{j-1})). \tag{B.9}$$

By the definition of the Wiener integral once again, we have

$$\text{RHS of (B.9)} \longrightarrow \int_0^{t_{k-1}} \int_0^T f(t_k, t) dt dB_{n_k}(t_k) \quad \text{in } L^2(\Omega), \tag{B.10}$$

while from Minkowski’s integral inequality, (B.8), and the bounded convergence theorem (recall that f is smooth), we have

$$\begin{aligned} & \left\| \int_0^T J_m(t)dt - \int_0^T \int_0^{t_{k-1}} f(t_k, t)dB_{n_k}(t_k)dt \right\|_{L^2(\Omega)} \\ & \leq \int_0^T \left\| J_m(t) - \int_0^{t_{k-1}} f(t_k, t)dB_{n_k}(t_k) \right\|_{L^2(\Omega)} dt \longrightarrow 0, \end{aligned} \tag{B.11}$$

as $m \rightarrow \infty$. Hence, from (B.9), (B.10), and (B.11), we conclude that

$$\int_0^T \int_0^{t_{k-1}} f(t_k, t)dB_{n_k}(t_k)dt = \int_0^{t_{k-1}} \int_0^T f(t_k, t)dt dB_{n_k}(t_k) \text{ in } L^2(\Omega). \tag{B.12}$$

Next, we consider

$$\begin{aligned} & \int_0^{t_{k-2}} \int_0^T F(t_{k-1}, t)dt dB_{n_{k-1}}(t_{k-1}) \\ & := \int_0^{t_{k-2}} \int_0^T \left(\int_0^{t_{k-1}} f(t_{k-1}, t_k, t)dB_{n_k}(t_k) \right) dt dB_{n_{k-1}}(t_{k-1}). \end{aligned} \tag{B.13}$$

Given the partition Δ_m of $[0, T]$ as above, we define an adaptive step function $F_m(\cdot, t)$ by setting $F_m(\tau, t; \omega) = F(\tau_{j-1}, t; \omega)$ for $\tau_{j-1} < \tau \leq \tau_j$. Then, we can simply repeat the previous computation (but with Ito integrals instead of Wiener integrals) and obtain

$$\int_0^T \int_0^{t_{k-2}} F(t_{k-1}, t)dB_{n_{k-1}}(t_{k-1})dt = \int_0^{t_{k-2}} \int_0^T F(t_{k-1}, t)dt dB_{n_{k-1}}(t_{k-1}) \tag{B.14}$$

in $L^2(\Omega)$. Combining (B.13) and (B.14) with (B.12), we then obtain

$$\begin{aligned} & \int_0^T \int_0^{t_{k-2}} \int_0^{t_{k-1}} f(t_{k-1}, t_k, t)dB_{n_k}(t_k)dB_{n_{k-1}}(t_{k-1})dt \\ & = \int_0^{t_{k-2}} \int_0^{t_{k-1}} \int_0^T f(t_{k-1}, t_k, t)dt dB_{n_k}(t_k)dB_{n_{k-1}}(t_{k-1}) \end{aligned}$$

in $L^2(\Omega)$. By iterating this process, we conclude

$$\begin{aligned} & \int_0^T \int_0^T \int_0^{t_1} \cdots \int_0^{t_{k-1}} f(t_1, \dots, t_k, t)dB_{n_k}(t_k) \cdots dB_{n_1}(t_1)dt \\ & = \int_0^T \int_0^{t_1} \cdots \int_0^{t_{k-1}} \int_0^T f(t_1, \dots, t_k, t)dt dB_{n_k}(t_k) \cdots dB_{n_1}(t_1) \end{aligned}$$

in $L^2(\Omega)$. Together with (B.6), this proves (B.3). □

We conclude this section by stating the product formula (Lemma B.4). Before doing so, we first recall the contraction of two functions.

Definition B.3 Let $k, \ell \in \mathbb{N}$. Given an integer $0 \leq r \leq \min(k, \ell)$, we define the contraction $f \otimes_r g$ of r indices of $f \in L^2(Z^k)$ and $g \in L^2(Z^\ell)$ by

$$(f \otimes_r g)(z_1, \dots, z_{k+\ell-2r}) = \sum_{m_1, \dots, m_r \in \mathbb{Z}^3} \int_{\mathbb{R}_+^r} f(z_1, \dots, z_{k-r}, \zeta_1, \dots, \zeta_r) \times g(z_{k+1-r}, \dots, z_{k+\ell-2r}, \tilde{\zeta}_1, \dots, \tilde{\zeta}_r) ds_1 \cdots ds_r,$$

where $\zeta_j = (m_j, s_j)$ and $\tilde{\zeta}_j = (-m_j, s_j)$.

Note that even if f and g are symmetric, their contraction $f \otimes_r g$ is not symmetric in general. We now state the product formula. See [43, Proposition 1.1.3].

Lemma B.4 (product formula) Let $k, \ell \in \mathbb{N}$. Let $f \in L^2(Z^k)$ and $g \in L^2(Z^\ell)$ be symmetric functions. Then, we have

$$I_k[f] \cdot I_\ell[g] = \sum_{r=0}^{\min(k, \ell)} r! \binom{k}{r} \binom{\ell}{r} I_{k+\ell-2r}[f \otimes_r g].$$

Appendix C: Random tensors

In this section, we provide the basic definition and some lemmas on (random) tensors from [12, 18]. See [18, Sections 2 and 4] and [12, Section 4] for further discussion.

Definition C.1 Let A be a finite index set. We denote by n_A the tuple $(n_j : j \in A)$. A tensor $h = h_{n_A}$ is a function: $(\mathbb{Z}^3)^A \rightarrow \mathbb{C}$ with the input variables n_A . Note that the tensor h may also depend on $\omega \in \Omega$. The support of a tensor h is the set of n_A such that $h_{n_A} \neq 0$.

Given a finite index set A , let (B, C) be a partition of A . We define the norms $\|\cdot\|_{n_A}$ and $\|\cdot\|_{n_B \rightarrow n_C}$ by

$$\|h\|_{n_A} = \|h\|_{\ell_{n_A}^2} = \left(\sum_{n_A} |h_{n_A}|^2 \right)^{\frac{1}{2}}$$

and

$$\|h\|_{n_B \rightarrow n_C}^2 = \sup \left\{ \left| \sum_{n_C} \left| \sum_{n_B} h_{n_A} f_{n_B} \right|^2 \right. \right. : \|f\|_{\ell_{n_B}^2} = 1 \left. \right\}, \tag{C.1}$$

where we used the short-hand notation \sum_{n_Z} for $\sum_{n_Z \in (\mathbb{Z}^3)^Z}$ for a finite index set Z . Note that, by duality, we have $\|h\|_{n_B \rightarrow n_C} = \|h\|_{n_C \rightarrow n_B} = \|\bar{h}\|_{n_B \rightarrow n_C}$ for any tensor $h = h_{n_A}$. If $B = \emptyset$ or $C = \emptyset$, then we have $\|h\|_{n_B \rightarrow n_C} = \|h\|_{n_A}$.

For example, when $A = \{1, 2\}$, the norm $\|h\|_{n_1 \rightarrow n_2}$ denotes the usual operator norm $\|h\|_{\ell_{n_1}^2 \rightarrow \ell_{n_2}^2}$ for an infinite dimensional matrix operator $\{h_{n_1 n_2}\}_{n_1, n_2 \in \mathbb{Z}^3}$. By bounding the matrix operator norm by the Hilbert–Schmidt norm (= the Frobenius norm), we have

$$\|h\|_{\ell_{n_1}^2 \rightarrow \ell_{n_2}^2} \leq \|h\|_{\ell_{n_1, n_2}^2} \tag{C.2}$$

Let (B, C) be a partition of A . Then, by duality, we can write (C.1) as

$$\|h\|_{n_B \rightarrow n_C} = \sup \left\{ \sum_{n_C} \left| \sum_{n_B} h_{n_A} f_{n_B} g_{n_C} \right| : \|f\|_{\ell_{n_B}^2} = \|g\|_{\ell_{n_C}^2} = 1 \right\},$$

from which we obtain

$$\sup_{n_A} |h_{n_A}| = \sup_{n_B, n_C} |h_{n_B n_C}| \leq \|h\|_{n_B \rightarrow n_C}. \tag{C.3}$$

Next, we recall a key deterministic tensor bound in the study of the random cubic NLW from [12].

Lemma C.2 (Lemma 4.33 in [12])

Let $s < \frac{1}{2} + \beta$ for some $\beta > 0$. Given $\varepsilon_j \in \{-1, 1\}$ for $j = 0, 1, 2, 3$, let $\kappa(\bar{n})$ be as in (4.7). For $m \in \mathbb{Z}$, define the tensor h^m by

$$h_{nn_1 n_2 n_3}^m = \left(\prod_{j=1}^3 \mathbf{1}_{\substack{|n_j| \sim N_j \\ |n_j| \leq N}} \right) \mathbf{1}_{\{|\kappa(\bar{n}) - m| \leq 1\}} \frac{\langle n \rangle^{s-1}}{\langle n_{12} \rangle^\beta \langle n_1 \rangle \langle n_2 \rangle \langle n_3 \rangle^{\frac{1}{2}}}.$$

Then, there exists $\delta_0 > 0$ such that

$$\begin{aligned} & \max \left(\|h^m\|_{n_1 n_2 n_3 \rightarrow n}, \|h^m\|_{n_3 \rightarrow n n_1 n_2}, \|h^m\|_{n_1 n_3 \rightarrow n n_2}, \|h^m\|_{n_2 n_3 \rightarrow n n_1} \right) \\ & \lesssim \max(N_1, N_2, N_3)^{-\delta_0}, \end{aligned}$$

uniformly in $N \geq 1$, $m \in \mathbb{Z}$, dyadic $N_1, N_2, N_3 \geq 1$, and $\varepsilon_j \in \{-1, 1\}$ for $j = 0, 1, 2, 3$.

We conclude this section with the following random matrix estimate. This lemma is essentially Propositions 2.8 and 4.14 in [18]; see also Proposition 4.50 in [12]. In our stochastic PDE setting, however, we need a slightly different formulation (in particular, adapted to multiple stochastic integrals with general integrands) and thus for readers’ convenience, we present its proof.

Let A be a finite index set. As in (4.15) and (4.16), we set $z_A = (k_A, t_A)$ for $(k_A, t_A) \in (\mathbb{Z}^3)^A \times \mathbb{R}^A$ and write $f_{z_A} = f(z_A) = f(n_A, t_A)$.

Lemma C.3 *Let A be a finite index set with $k = |A| \geq 1$. Let $h = h_{bcn_A}$ be a tensor such that $n_j \in \mathbb{Z}^3$ for each $j \in A$ and $(b, c) \in (\mathbb{Z}^3)^d$ for some integer $d \geq 2$. Given $N \geq 1$, assume that*

$$\text{supp } h \subset \{|b|, |c|, |n_j| \lesssim N \text{ for each } j \in A\}. \tag{C.4}$$

Given a (deterministic) tensor $h_{bcn_A} \in \ell^2_{bcn_A}$, define the tensor $H = H_{bc}$ by

$$H_{bc} = I_k[h_{bcn_A} f_{z_A}] \tag{C.5}$$

for $f \in \ell^\infty_{n_A}((\mathbb{Z}^3)^A; L^2_{t_A}(\mathbb{R}^A_+))$, where I_k denotes the multiple stochastic integral defined in Appendix B. Then, for any $\theta > 0$, we have

$$\| \|H_{bc}\|_{b \rightarrow c} \|_{L^p(\Omega)} \lesssim p^{\frac{k}{2}} N^\theta \left(\max_{(B,C)} \|h\|_{bn_B \rightarrow cn_C} \right) \|f(n_A, t_A)\|_{\ell^\infty_{n_A} L^2_{t_A}}, \tag{C.6}$$

where the maximum is taken over all partitions (B, C) of A .

Remark C.4 (i) The assumption that $h_{bcn_A} \in \ell^2_{bcn_A}$ and $f \in \ell^\infty_{n_A}((\mathbb{Z}^3)^A; L^2_{t_A}(\mathbb{R}^A_+))$ ensures that the multiple stochastic integral $I_k[h_{bcn_A} f_{z_A}]$ in (C.5) is well defined. Note that if for instance we have a stronger condition $f \in \ell^2((\mathbb{Z}^3)^A; L^2(\mathbb{R}^A_+))$, then the conclusion (C.6) trivially holds without any loss in N . We also note that even if the tensor h is random, Lemma C.3 holds with the same proof as long as h is independent of the Brownian motions $\{B_{n_A}\}$ defining multiple stochastic integrals.

(ii) By translation invariance, we may replace the condition (C.4) in Lemma C.3 by

$$\text{supp } h \subset \{|b - b_*|, |c - c_*|, |n_j - n_{j,*}| \lesssim N \text{ for each } j \in A\}$$

for some $(b_*, c_*) \in (\mathbb{Z}^3)^d$ and $n_{j,*} \in \mathbb{Z}^3, j \in A$.

Proof of Lemma C.3 We follow the proof of Proposition 4.14 in [18] and use a higher order version of Bourgain’s TT^* -argument [9]. Let $T : \ell^2_c \rightarrow \ell^2_b$ be the linear operator whose kernel is H_{bc} . Namely, T is defined by

$$(Tg)_b = \sum_c H_{bc} g_c, \quad g \in \ell^2_c. \tag{C.7}$$

For $j \in \mathbb{N}$, we define the operator T_j by $T_j = (TT^*)^m$ if $j = 2m$, and $T_j = (TT^*)^m T$ if $j = 2m + 1$. We claim that T_j has a kernel which is given by a linear combination of terms \mathcal{T}_j of the form

$$\mathcal{T}_j = \begin{cases} I_\ell[y_{bb'}(z_D)], & \text{when } j \text{ is even,} \\ I_\ell[y_{bc}(z_D)], & \text{when } j \text{ is odd,} \end{cases} \tag{C.8}$$

for some finite index set D and $\ell = |D| \leq kj$, where $y_{bb'}(z_D)$ (or $y_{bc}(z_D)$) satisfies the following bound:

$$\begin{aligned} & \|y_{bb'}(z_D)\|_{\ell^2_{bcn_D} L^2_{t_D}} \quad (\text{or } \|y_{bc}(z_D)\|_{\ell^2_{bcn_D} L^2_{t_D}}) \\ & \lesssim \left(\max_{(B,C)} \|h\|_{bn_B \rightarrow cn_C} \right)^{j-1} \|h_{bcn_A}\|_{\ell^2_{bcn_A}} \|f(n_A, t_A)\|_{\ell^\infty_{n_A} L^2_{t_A}}^j. \end{aligned} \tag{C.9}$$

where the maximum is taken over all partitions (B, C) of A . Here, the implicit constant depends on k, ℓ , and j . While it grows with j (and ℓ), this does not cause an issue since for a given small $\theta > 0$ in (C.6), we fix $j = j(\theta) \gg 1$.

Let $j = 1$. In this case, comparing (C.8) with (C.7) and (C.5) and using Lemma B.1 (ii), we have $y_{bc}(z_D) = \text{Sym}(h_{bcn_A} f(z_A))$ with $D = A$ and thus the bound (C.9) follows from Hölder’s inequality. Note that, in this case, it follows from Lemma B.1 (iii) that

$$\|y_{bc}(z_A)\|_{\ell^2_{bcn_A} L^2_{t_A}} = (k!)^{-1} \| \|H_{bc}\|_{\ell^2_{bc}} \|_{L^2(\Omega)},$$

where the right-hand side is the second moment of the Hilbert–Schmidt norm of the operator T . By taking higher powers T_j , we control the operator norm of T .

Now, assume that the claim with (C.8) and (C.9) hold true for $j - 1$. We assume that j is odd. The proof for even j is analogous. Noting that $T_j = T_{j-1}T$, it follows from the inductive hypothesis (C.8) with (C.5) and Lemma B.1 (ii) that the kernel for T_j is given by a linear combination of terms \mathcal{T}_j of the form

$$\begin{aligned} (\mathcal{T}_j)_{bc} &= \sum_{b'} (\mathcal{T}_{j-1})_{bb'} H_{b'c} \\ &= \sum_{b'} I_\ell [y_{bb'}(z_D)] \cdot I_k [h_{b'cn_A} f(z_A)] \\ &= \sum_{b'} I_\ell [\text{Sym}(y_{bb'}(z_D))] \cdot I_k [\text{Sym}(h_{b'cn_A} f(z_A))]. \end{aligned}$$

Then, from the product formula (Lemma B.4), we have

$$(\mathcal{T}_j)_{bc} = \sum_{r=0}^{\min(k,\ell)} r! \binom{k}{r} \binom{\ell}{r} I_{k+\ell-2r} \left[\sum_{b'} (\text{Sym}(y_{bb'}) \otimes_r \text{Sym}(h_{b'c} f)) \right].$$

Hence, it suffices to show that $\sum_{b'} (\text{Sym}(y_{bb'}) \otimes_r \text{Sym}(h_{b'c} f))$ satisfies (C.9) for each $0 \leq r \leq \min(k, \ell)$. For notational simplicity, we drop Sym in $\text{Sym}(y_{bb'})$ and $\text{Sym}(h_{b'c} f)$ in the following. Note that this does not cause any issue since, in taking the $L^2(\Omega)$ -norm, we can remove Sym by Jensen’s inequality (B.2) as in Sect. 4.

Fix $0 \leq r \leq \min(k, \ell)$. From Definition B.3 on the contraction, we have

$$\begin{aligned} & (y_{bb'} \otimes_r h_{b'c} f)(z_B) \\ &= \sum_{n_C} \int_{\mathbb{R}_+^r} y_{bb'}(z_{B_1}, z_C) \cdot (h_{b'c} f)(z_{B_2}, \tilde{z}_C) dt_C, \end{aligned} \tag{C.10}$$

where $\tilde{z}_C = (-n_C, t_C)$ for given $z_C = (n_C, t_C)$. Here, B_1, B_2 , and C are pairwise disjoint sets such that $|B_1| = \ell - r, |B_2| = k - r, |C| = r, B = B_1 \cup B_2$, and (by suitable relabeling of indices)

$$B_1 \cup C = D \quad \text{and} \quad B_2 \cup C = A. \tag{C.11}$$

Then, from (C.10), Cauchy–Schwarz’s inequality (in t_C), Minkowski’s integral inequality (with $L_{t_B}^2 = L_{t_{B_1}}^2 L_{t_{B_2}}^2$, (C.1), and the identification in (C.11), we have

$$\begin{aligned} & \left\| \sum_{b'} (y_{bb'} \otimes_r h_{b'c} f)(n_B, t_B) \right\|_{\ell_{bcn_B}^2 L_{t_B}^2} \\ & \leq \left\| \sum_{b', n_C} \|y_{bb'}(n_{B_1}, t_{B_1}, n_C, t_C)\|_{L_{t_{B_1} t_C}^2} \cdot \|(h_{b'c} f)(n_{B_2}, t_{B_2}, -n_C, t_C)\|_{L_{t_{B_2} t_C}^2} \right\|_{\ell_{bcn_B}^2} \\ & \leq \|y_{bb'}(n_{B_1}, t_{B_1}, n_C, t_C)\|_{\ell_{bb'n_{B_1} n_C}^2 L_{t_{B_1} t_C}^2} \left\| \|(h_{b'c} f)(z_{B_2}, z_C)\|_{L_{t_{B_2} t_C}^2} \right\|_{b'n_C \rightarrow cn_{B_2}} \\ & = \|y_{bb'}(z_D)\|_{\ell_{bb'n_D}^2 L_{t_D}^2} \|h_{b'cn_A} \| f(z_A)\|_{L_{t_A}^2} \Big\|_{b'n_C \rightarrow cn_{A \setminus C}}. \end{aligned} \tag{C.12}$$

Moreover, from (C.1), we have

$$\begin{aligned} \|h_{b'cn_A} \| f(z_A)\|_{L_{t_A}^2} \Big\|_{b'n_C \rightarrow cn_{A \setminus C}} & \leq \|h_{b'cn_A} \|_{b'n_C \rightarrow cn_{A \setminus C}} \|f(n_A, t_A)\|_{\ell_{n_A}^\infty L_{t_A}^2} \\ & \leq \left(\max_{(A_1, A_2)} \|h_{bcn_A} \|_{bn_{A_1} \rightarrow cn_{A_2}} \right) \|f(n_A, t_A)\|_{\ell_{n_A}^\infty L_{t_A}^2}, \end{aligned} \tag{C.13}$$

where the maximum is taken over all partitions (A_1, A_2) of A . Hence, from (C.12), (C.13), and the inductive hypothesis (C.9) (with $j - 1$ in place of j), we obtain (C.9) for j . Therefore, by induction, the claim holds for any $j \in \mathbb{N}$.

We are now ready to prove (C.6). Consider the product $T_{2m} = (TT^*)^m$ for $m \geq 1$. Let us denote by \mathcal{R}_{2m} the kernel of T_{2m} , which consists of terms \mathcal{T}_j , satisfying (C.8) and (C.9). Namely, we have

$$(\mathcal{R}_{2m})_{bb'} = \sum_{j=1}^J I_{2k\ell_j} [y_{bb'}^{(j)}(z_{D^{(j)}})] \tag{C.14}$$

for some $J \geq 1, 0 \leq \ell_j \leq m$, and $y_{bb'}^{(j)}$, satisfying (C.9). Note that we have $\mathcal{R}_{2m} \in \mathcal{H}_{\leq 2mk}$. Then, by the standard TT^* argument, (C.2), Minkowski’s integral inequality,

(C.14), the Wiener chaos estimate (Lemma 2.9), Lemma B.1 (iii), and (C.9), we obtain

$$\begin{aligned}
 \| \|H_{bc} \|_{b \rightarrow c} \|_{L^p(\Omega)} &= \| \|H_{bc} \|_{b \rightarrow c}^{2m} \|_{L^{\frac{p}{2m}}(\Omega)}^{\frac{1}{2m}} = \| \|T \|_{\ell_c^2 \rightarrow \ell_b^2}^{2m} \|_{L^{\frac{p}{2m}}(\Omega)}^{\frac{1}{2m}} \\
 &= \| \| (TT^*)^m \|_{\ell_{b'}^2 \rightarrow \ell_b^2} \|_{L^{\frac{p}{2m}}(\Omega)}^{\frac{1}{2m}} \leq \| \| (\mathcal{R}_{2m})_{bb'} \|_{\ell_{bb'}^2} \|_{L^{\frac{p}{2m}}(\Omega)}^{\frac{1}{2m}} \\
 &\leq \| \| (\mathcal{R}_{2m})_{bb'} \|_{L^{\frac{p}{2m}}(\Omega)} \|_{\ell_{bb'}^2}^{\frac{1}{2m}} \\
 &\leq p^{\frac{k}{2}} \left(\sum_{j=1}^J \| \| I_{2k} \ell_j [y_{bb'}^{(j)}(z_{D(j)})] \|_{L^2(\Omega)} \|_{\ell_{bb'}^2} \right)^{\frac{1}{2m}} \\
 &\lesssim p^{\frac{k}{2}} \left(\sum_{j=1}^J \| y_{bb'}^{(j)}(z_{D(j)}) \|_{\ell_{bcn_{D(j)}}^2} L_{D(j)}^2 \right)^{\frac{1}{2m}} \\
 &\lesssim p^{\frac{k}{2}} \left(\max_{(B,C)} \| h \|_{bn_B \rightarrow cn_C} \right)^{1 - \frac{1}{2m}} \| h \|_{\ell_{bcn_A}^2}^{\frac{1}{2m}} \| f(n_A, t_A) \|_{\ell_{n_A}^\infty L_{t_A}^2}
 \end{aligned}
 \tag{C.15}$$

for any $p \geq 4m$. Moreover, from (C.4) and (C.3), we have

$$\| h \|_{\ell_{bcn_A}^2} \leq N^{\frac{3}{2}(d+k)} \sup_{b,c,n_A} |h_{bcn_A}| \leq N^{\frac{3}{2}(d+k)} \max_{(B,C)} \| h \|_{bn_B \rightarrow cn_C}. \tag{C.16}$$

Therefore, by combining (C.15) and (C.16) and taking m sufficiently large, we obtain the desired bound (C.6). □

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