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CYCLE DECOMPOSITIONS IN 3-UNIFORM HYPERGRAPHS

SIMÓN PIGA AND NICOLÁS SANHUEZA-MATAMALA

Abstract. We show that 3-graphs on n vertices whose codegree is at least $(2/3 + o(1))n$ can be decomposed into tight cycles and admit Euler tours, subject to the trivial necessary divisibility conditions. We also provide a construction showing that our bounds are best possible up to the $o(1)$ term. All together, our results answer in the negative some recent questions of Glock, Joos, Kühn, and Osthus.

§1. Introduction

1.1. **Cycle decompositions.** Given a k-uniform hypergraph H, a *decomposition of* H is a collection of subgraphs of H such that every edge of H is covered exactly once. When these subgraphs are all isomorphic copies of a single hypergraph F we say that it is an F*-decomposition*, and that H is F*-decomposable*. Finding decompositions of hypergraphs is one of the oldest problems in combinatorics. For instance, the well-known problem of the existence of designs and Steiner systems can be cast as the problem of decomposing a complete hypergraph into smaller complete hypergraphs of a fixed size. Thanks to the recent breakthroughs of Keevash [17] and Glock, Kühn, Lo, and Osthus [11] our knowledge about hypergraph decompositions has increased substantially; but many open questions remain. We refer the reader to the survey of Glock, Kühn, and Osthus [12] for an overview of the state of the art.

Here we focus in decompositions in which the subgraphs are all cycles. For $k \geqslant 2$ and $\ell \geqslant k+1$, the *k-uniform tight cycle of length ℓ* is the k-graph C^k_ℓ whose vertices are $\{v_1, v_2, \dots, v_\ell\}$ and whose edges are all k-sets of consecutive vertices of the form $\{v_i, v_{i+1}, \dots, v_{i+k-1}\}$ for $1 \leqslant i \leqslant \ell$, where the indices are understood modulo ℓ. Since no other kind of hypergraph cycles will be considered, we will refer to tight cycles as *cycles*. If k is clear from the context, we will just write C_ℓ instead of C^k_ℓ.

Given a vertex x in H the *degree of* x, $\deg_H(x)$, is the number of edges that contain x. For a positive integer k, when the degree of every vertex of a hypergraph H is divisible by k we say that H is *k-vertex-divisible*. Note that in a k-uniform cycle every vertex has degree exactly k. This implies that, for any $\ell \geqslant k+1$, any C^k_ℓ-decomposable k-graph H must necessarily be k-vertex-divisible. Another obvious necessary condition to find C_ℓ-decompositions in H is that the total number of edges of H must be divisible by ℓ. If H satisfies these two conditions, we say that H is C_ℓ*-divisible*.

However, not every C_ℓ-divisible k-graph is C_ℓ-decomposable. For instance, a cycle $C_{2\ell}$ is C_ℓ-divisible, but clearly does not have a C_ℓ-decomposition. This motivates the search of easily-checkable sufficient conditions which, together with the necessary C_ℓ-divisibility, already force the existence of C_ℓ-decompositions. A natural choice is to consider degree conditions, which in hypergraphs can be expressed in terms of *codegree*. For k-uniform graphs and a set S of $(k-1)$ vertices, we define the *codegree of* S, $\deg_H(S)$, as the number of edges of H that contain all of S. We denote the minimum (resp. maximum) codegree of a hypergraph H over all S by $\delta_{k-1}(H)$ (resp. $\Delta_{k-1}(H)$). The C^k_ℓ*-decomposition threshold* $\delta_{C^k_\ell}(n)$ is the minimum d such that every

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C_ℓ^k-divisible k-graph H on n vertices with $\delta_{k-1}(H) \geqslant d$ is C_ℓ^k-decomposable. Moreover, it is convenient to define $\delta_{C_\ell^k} = \limsup_{n\to\infty} \delta_{C_\ell^k}(n)/n$. Again, we may omit k from the notation and write $\delta_{C_\ell}(n)$ and δ_{C_ℓ}. The very general results of [11] imply that $\delta_{C_\ell^k} < 1$ for all $k \geqslant 2$ and $\ell > k$, but no precise values are known when $k \geqslant 3$.

In our main result, we find the value of $\delta_{C_\ell^3}$ for all but finitely many values of ℓ.

Theorem 1.1. *Suppose ℓ satisfies one of the following:* (i) *ℓ is divisible by* 3 *and at least* 9*, or* (ii) $\ell \geqslant 10^7$*. Then $\delta_{C_\ell^3} = 2/3$.*

Theorem 1.1 implies an interesting contrast with respect to what is known for C_ℓ^2-decomposition thresholds, which we now recall. In graphs (i.e. 2-uniform hypergraphs), the codegree conditions default to conditions on minimum degree. Barber, Kühn, Lo, and Osthus [3] introduced the technique of *iterative absorption* to study F-decompositions in graphs —this technique is also crucial to our present work, and will be reviewed in detail in Section 4. In particular, for cycle decompositions in graphs, their work implies that $\delta_{C_\ell}(n) \leqslant \delta^*_{C_\ell}(n) + o(n)$. Here, $\delta^*_{C_\ell}(n)$ is the minimum degree which guarantees the existence of 'fractional C_ℓ-decompositions' in n-vertex graphs. This notion corresponds to the natural fractional relaxation of decompositions (we define and discuss this in Section 7.2). Let $\delta^*_{C_\ell} = \limsup_{n\to\infty} \delta^*_{C_\ell}(n)/n$.

The famous Nash-Williams conjecture [18] says that $\delta_{C_3}(n) \leqslant 3n/4$. This is still open, with the current best upper bound given by $\delta^*_{C_3} \leqslant d \approx 0.827$ due to Delcourt and Postle [5]. Very recently, Joos and Kühn [15] proved that $\delta^*_{C_\ell}$ tends to $1/2$ whenever ℓ goes to infinity. Together with the best known lower bounds [3, 2], we now know that for all odd $\ell \geqslant 3$,

$$\frac{1}{2} + \frac{1}{2(\ell-1)} \leqslant \delta_{C_\ell} \leqslant \delta^*_{C_\ell} \leqslant \frac{1}{2} + O\left(\frac{\log \ell}{\ell}\right).$$

On the other hand, cycles of even length are bipartite, and Glock, Kühn, Lo, Montgomery, and Osthus [10] were able to characterise the 'decomposition thresholds' for all bipartite graphs. In particular, $\delta_{C_4} = 2/3$ and $\delta_{C_\ell} = 1/2$ for all even $\ell \geqslant 6$. Remarkably, Taylor [20] showed exact results for large n, by proving $\delta_{C_4}(n) = 2n/3 - 1$ and $\delta_{C_\ell}(n) = n/2$ for all even $\ell \geqslant 8$.

To summarise, for large ℓ the values of $\delta_{C_\ell^2}$ have a strong dependence on the parity of ℓ, being $\delta_{C_\ell^2} > 1/2$ if ℓ is odd, and $\delta_{C_\ell^2} = 1/2$ otherwise. In contrast, Theorem 1.1 implies that for $k = 3$ and large ℓ the behaviour is different: $\delta_{C_\ell^3} = 2/3$ for all ℓ sufficiently large, regardless of whether the cycle is tripartite or not.

The following simple corollary can be deduced from our main theorem. Say a k-graph has a *cycle decomposition* if it admits a decomposition into cycles. That is, there are edge-disjoint cycles —not necessarily of the same length— which cover every edge exactly once. This notion is weaker than that of having a C_ℓ-decomposition for a fixed ℓ. It is easy to see that any 3-graph having a cycle decomposition must be 3-vertex-divisible. As a corollary of Theorem 1.1, we obtain an upper bound on the minimum codegree sufficient to force a cycle decomposition.

Corollary 1.2. *Any* 3*-vertex-divisible* 3*-graph H with $\delta_2(H) \geqslant (2/3 + o(1))|H|$ has a cycle decomposition.*

1.2. **Euler tours.** Our focus in decompositions into cycles is partly motivated by its close connections with the celebrated problem of finding *Euler tours*. Given a k-graph H, a *tour* is a sequence of non-necessarily distinct vertices $v_1, \dots, v_\ell$ such that, for every $1 \leqslant i \leqslant \ell$ the k consecutive vertices $\{v_i, v_{i+1}, \dots, v_{i+k-1}\}$ induce an edge (understanding the indices modulo ℓ), and moreover all of these edges are distinct. If a hypergraph H contains a tour that covers each edge exactly once, we call it *Euler tour* and we say that H is *Eulerian*.

Famously, Euler [8] proved that every Eulerian graph must be 2-vertex-divisible, and stated (later proved by Hierholzer and Wiener [13]) that connected and 2-vertex-divisible graphs are Eulerian. Analogously, for $k \geqslant 3$, it is an easy observation that every Eulerian k-graph must be k-vertex-divisible. However, the characterisation of Eulerian k-graphs is not as simple as for

$k = 2$. In fact, until recently, it was not even known if complete k-vertex-divisible k-graphs were Eulerian. It was conjectured by Chung, Diaconis, and Graham [4] that indeed that should be the case, at least for sufficiently large complete k-graphs. This was proven to be true by Glock, Joos, Kühn, and Osthus [9], which deduced this from a more general result which finds Euler tours in k-graphs with certain quasirandom conditions (which are satisfied by complete graphs).

From this more general result, they also deduced a 'minimum codegree' version of their theorem: there exists $c > 0$ such that any sufficiently large 3-vertex-divisible hypergraph H with $\delta_2(H) \geqslant (1-c)|H|$ is Eulerian. The constant c which they obtained is fairly small (by inspecting their proof, we estimate $\log_2(c) \leqslant -10^{12}$) and therefore improving the minimum codegree condition becomes a natural problem. Their proof is based fundamentally on a reduction to the problem of finding a cycle decomposition. In the same fashion, we can use Theorem 1.1 to improve the minimum codegree condition.

Corollary 1.3. *Any 3-vertex-divisible 3-graph H with $\delta_2(H) \geqslant (2/3 + o(1))|H|$ is Eulerian.*

1.3. **Lower bounds and counterexamples.** Theorem 1.1, Corollary 1.2 and Corollary 1.3 hold for 3-graphs H satisfying $\delta_2(H) \geqslant (2/3 + o(1))|H|$. Glock, Kühn, and Osthus [12, Conjecture 5.6] conjectured that Corollary 1.2 should hold already for any H with $\delta_2(H) \geqslant (1/2 + o(1))|H|$. Similarly, in the setting of Corollary 1.3, Glock, Joos, Kühn, and Osthus [9, Conjecture 3] conjectured (reiterated in [12, Conjecture 5.4]) that a minimum codegree of $(1/2+o(1))|H|$ should be enough to guarantee the existence of Euler tours.

However, it turns out that the '2/3' in our statements cannot be lowered. We prove this by constructing a family of counterexamples which are able to cover all of the previous settings (C_ℓ-decompositions, cycle decompositions, and Euler tours) in a unified way.

A *tour decomposition* of H is a collection of edge-disjoint tours in H which, together, cover all edges of H. Note that a cycle is precisely a tour which does not repeat vertices. Thus we have that both C_ℓ-decompositions and cycle decompositions are particular instances of tour decompositions, and moreover Eulerian graphs are graphs which admit a tour decomposition consisting of a single tour. Thus the following result shows that Theorem 1.1, Corollary 1.2, and Corollary 1.3 are asymptotically tight for the minimum codegree condition.

Theorem 1.4. *Let $\ell \geqslant 4$ and $n \geqslant 3(\ell+3)$ be divisible by 18. Then there exists a C_ℓ-divisible 3-graph H on n vertices which satisfies $\delta_2(H) \geqslant (2n-15)/3$, but does not admit a tour decomposition.*

1.4. **Organisation of the paper.** In Section 2 we prove the lower bound of Theorem 1.4. In Section 3 we give short proofs of Corollaries 1.2 and 1.3 assuming Theorem 1.1.

In Section 4 we show Theorem 1.1 by using the technique of *iterative absorption*, which we review there. The technique relies on three main lemmata, the Vortex Lemma, Cover-Down Lemma and Absorbing Lemma. After some useful tools (Section 5), these three lemmata are proved in Sections 6, 7 and 8, respectively. We finish in Section 9 with some remarks and questions.

1.5. **Notation.** Since isolated vertices make no difference in our context, we usually do not distinguish from a hypergraph $H = (V, E)$ and its set of edges E. We will suppress brackets and commas to refer to pairs and triples of vertices when they are considered as edges of a hypergraph. For instance, for $x, y, z \in V(H)$, $xyz \in H$ means that the edge $\{x, y, z\}$ is in $E(H)$. For a vertex $x \in V(H)$, the *link graph of x* is the 2-graph $H(x)$ with edge set $\{yz \in \binom{V}{2} : xyz \in E(H)\}$. Moreover, given a set of vertices $U \subseteq V$ we denote the *restricted link graph* by $H(v, U) = H(v) \cap \binom{U}{2}$. The *degrees* $\deg_H(x)$ and $\deg_H(x, U)$ correspond to $|H(x)|$ and $|H(x, U)|$ respectively. For a pair of vertices xy in $V(H)$, the *neighbourhood of xy* $N_H(xy)$ is the set of vertices $z \in V(H)$ such that $xyz \in H$, given $U \subseteq V(H)$ then $N_H(xy, U) = N_H(xy) \cap U$. The *codegrees* $\deg_H(xy)$ and $\deg(xy, U)$ correspond to $|N_H(xy)|$ and $|N_H(xy, U)|$ respectively. We suppress H from the degrees, codegrees, and neighbourhoods if it can be deduced from context. The *shadow* ∂H of a

3-graph H is $\{uv \in \binom{V(H)}{2} : \deg(uv) > 0\}$. If $\mathcal{C} = \{C_1, \dots, C_r\}$ is a collection of subgraphs of H, sometimes we will let $E(\mathcal{C})$ be the hypergraph whose edges are $\bigcup_{1 \leqslant i \leqslant r} E(C_i)$.

We will use hierarchies in our statements. The phrase "$a \ll b$" means "for every $b > 0$, there exists $a_0 > 0$, such that for all $0 < a \leqslant a_0$ the following statements hold". We implicitly assume all constants in such hierarchies are positive, and if $1/a$ appears we assume a is an integer.

A *walk* in a 3-graph H is a sequence $W = (v_1, \dots, v_\ell)$ of vertices of H such that every 3 consecutive vertices form an edge of H. A *trail* is a walk in which no edge appears more than once, and a *path* is a trail in which no vertex appears more than once. A *closed walk* is a walk in which every cyclic shift is still a walk of H (thus tours are trails which are closed walks). Given a walk $W = (v_1, v_2, \dots, v_\ell)$, we define its *start* $s(W)$ and *terminus* $t(P)$ as $\{v_1, v_2\}$ and $\{v_{\ell-1}, v_\ell\}$ respectively, and we say W *goes from* (v_1, v_2) to $(v_{\ell-1}, v_\ell)$ and also that W is a $(v_1, v_2, v_{\ell-1}, v_\ell)$-*path*. We will use the simpler notation $W = v_1 v_2 \cdots v_\ell$ for walks, and, when useful, we will identify such walks with subgraphs of H (so we can say e.g. $e \in E(W)$).

§2. Lower bounds

In this section we prove Theorem 1.4. The following lemma captures divisibility constraints that tours in 3-graphs must satisfy, and it will be the basis of our constructions. For a 3-graph H, a subgraph $W \subseteq H$ and vertex sets X, Y, Z in $V(H)$, let $W[X, Y, Z]$ be the set of edges xyz in $E(W)$ such that $x \in X$, $y \in Y$, and $z \in Z$.

Lemma 2.1. *Let H be a 3-graph with a vertex partition $\{U_0, U_1, U_2\}$, and $H[U_0, U_1, U_2] = \varnothing$. Let W be a tour in H. Then $|W[U_1, U_1, U_2]| \equiv |W[U_1, U_2, U_2]| \bmod 3$.*

Proof. Let $W = w_1 w_2 \cdots w_r$, in cyclic order, and let $P = \sigma_1 \cdots \sigma_r$ be a cyclic word over the symbols $\{0, 1, 2\}$, where $\sigma_i = j$ if and only if $w_i \in U_j$. Since W is a tour, it does not repeat edges. Thus we have that $|W[U_1, U_1, U_2]|$ is exactly the same as the number of appearances of the patterns $F_1 = \{112, 121, 211\}$ formed by three consecutive symbols in P. Similarly, $|W[U_1, U_2, U_2]|$ is exactly counted by the number of appearances of $F_2 = \{122, 212, 221\}$ consecutively in P. In both cases we count the cyclic appearances of the patterns, i.e. we also consider the patterns formed by $\sigma_{r-1}\sigma_r\sigma_1$ and $\sigma_r\sigma_1\sigma_2$.

Define $\Phi(P)$ as follows. Scan the triples of consecutive symbols of P one by one, and if they belong to $F_1 \cup F_2$, we add the sum of the values of their symbols to $\Phi(P)$. More formally, let $I \subseteq [r]$ be such that $i \in I$ if and only if $\sigma_i\sigma_{i+1}\sigma_{i+2} \in F_1 \cup F_2$ (where the indices are always understood modulo r, i.e. $\sigma_{r+1} = \sigma_1$ and $\sigma_{r+2} = \sigma_2$), and then

$$\Phi(P) = \sum_{i \in I} (\sigma_i + \sigma_{i+1} + \sigma_{i+2}).$$

We aim to show that $\Phi(P) \equiv 0 \bmod 3$. If $I = \varnothing$, this is obvious, and if $I = [r]$ then $\Phi(P)$ sums every symbol of P three times, and thus also $\Phi(P) \equiv 0$. Thus we can assume $I \notin \{\varnothing, [r]\}$. We write I as a disjoint union of intervals of consecutive indices, minimising the number of intervals. Thus, without loss of generality (after shifting W and P cyclically) we can assume $I = I_1 \cup \cdots \cup I_k$, so each I_j is of the form $\{a_j, a_j + 1, \dots, b_j\}$ for some $a_j \leqslant b_j$ and further we have $a_1 = 1$, $b_j \leqslant a_{j+1} - 2$ for all $1 \leqslant j < k$ and $b_k \leqslant r - 1$. Setting $\Phi_j = \sum_{i \in I_j} (\sigma_i + \sigma_{i+1} + \sigma_{i+2})$ we have $\Phi(P) = \sum_{1 \leqslant j \leqslant k} \Phi_j$, so it is enough to show that $\Phi_j \equiv 0 \bmod 3$ for each j.

Let $1 \leqslant j \leqslant k$ be arbitrary, for brevity write $a = a_j$ and $b = b_j$. Let $P_j = \sigma_a\sigma_{a+1} \cdots \sigma_{b+1}\sigma_{b+2}$. We claim that P_j begins with two repeated symbols. Since $I_k \subseteq I$, we have $\sigma_a\sigma_{a+1}\sigma_{a+2} \in F_1 \cup F_2$, thus in particular σ_a and σ_{a+1} must be in $\{1, 2\}$. If $\sigma_a \neq \sigma_{a+1}$, then we would have $\sigma_a\sigma_{a+1} = 12$ or $\sigma_a\sigma_{a+1} = 21$. In any case, it cannot happen that $\sigma_{a-1} \in \{1, 2\}$, since then that would imply that $a - 1 \in I$, contradicting the choice of I_k. Thus $\sigma_{a-1} = 0$, and therefore $\sigma_{a-1}\sigma_a\sigma_{a+1} = 012$ or $\sigma_{a-1}\sigma_a\sigma_{a+1} = 021$. But this implies that W contains an edge in $H[U_0, U_1, U_2]$, a contradiction.

Thus P_j begins with two repeated symbols, and an analogous argument implies that P_j also ends with two repeated symbols.

If $a = b$, then we would have $\sigma_a\sigma_{a+1}\sigma_{a+2} = 111$ or $\sigma_a\sigma_{a+1}\sigma_{a+2} = 222$, then implying $a \notin I$, a contradiction. Thus $a < b$, and therefore P_j must have the form $P_j = xxQ_jyy$, where $x, y \in \{1, 2\}$ and Q_j is a (possibly empty) word. Thus we have

$$\Phi_j = \sum_{a \leqslant i \leqslant b} (\sigma_i + \sigma_{i+1} + \sigma_{i+2}) = x + 2x + 3\left(\sum_{a+2 \leqslant i \leqslant b} \sigma_i\right) + 2y + y \equiv 0 \bmod 3,$$

and this implies $\Phi(P) \equiv 0 \bmod 3$, as discussed before.

Finally, note that, for $j \in \{1, 2\}$, if $\sigma_i\sigma_{i+1}\sigma_{i+2} \in F_j$, then $\sigma_i + \sigma_{i+1} + \sigma_{i+2} \equiv j \bmod 3$. Thus $\Phi(P) \equiv |W[U_1, U_1, U_2]| + 2|W[U_1, U_2, U_2]| \bmod 3$. But since $\Phi(P) \equiv 0 \bmod 3$ and $2 \equiv -1 \bmod 3$, we deduce $|W[U_1, U_1, U_2]| \equiv |W[U_1, U_2, U_2]| \bmod 3$, as desired. ■

To prove Theorem 1.4, we will consider alterations of the following 3-graph.

Definition 2.2. *Let n be divisible by 18 and write $n = 18k$. Consider the 3-graph H_n on n vertices, whose vertex set is partitioned into three clusters V_0, V_1, V_2 whose sizes are n_0, n_1, n_2 respectively, and are defined by*

$$n_0 = 6k, \qquad n_1 = 6k - 2, \qquad \text{and} \qquad n_2 = 6k + 2. \tag{2.1}$$

Given a vertex $x \in V(H_n)$, the label *$l(x)$ of x is i if and only if $x \in V_i$. The edge set of H_n is*

$$E(H_n) = \{xyz : l(x) + l(y) + l(z) \not\equiv 0 \bmod 3\}.$$

In words, every 3-set is present as an edge in H_n, except for those which are entirely contained in one of the clusters V_i, or have non-empty intersection with all three clusters. Usually n will always be clear from context, and for a cleaner notation we will just write $H = H_n$ in the remainder of this section.

We begin our analysis by noting the 3-graph H has large minimum codegree.

Lemma 2.3. *Let $n \in 18\mathbb{N}$. Then $\delta_2(H) \geqslant (2n - 12)/3$.*

Proof. Let $x, y \in V(H)$, and set $p = l(x) + l(y)$. By the definition of H, a vertex z will form an edge together with xy whenever $p + l(z) \not\equiv 0 \bmod 3$. This is equivalent to $l(z) \equiv 1 - p \bmod 3$ or $l(z) \equiv 2 - p \bmod 3$. Thus, if $i, j \in \{0, 1, 2\}$ are such that $i \equiv 1 - p \bmod 3$ and $j \equiv 2 - p \bmod 3$, then $N(xy) = (V_i \cup V_j) \setminus \{x, y\}$. A quick case analysis reveals that $|N(xy)|$ is minimised whenever $x \in V_0$, $y \in V_1$, and in such a case $\deg_H(xy) = n_0 + n_1 - 2 = 12k - 4$. Thus $\delta_2(H) = 12k - 4 = (2n - 12)/3$, as required. ■

We note that equations (2.1) imply that, for $n = 18k$, all n_0, n_1, n_2 are even, and for all $i \in \{0, 1, 2\}$ we have

$$n_i \equiv i \pmod 3, \tag{2.2}$$

Given $(i, j, k) \in \{0, 1, 2\}^3$, write $H_{ijk} = H[V_i, V_j, V_k]$.

Lemma 2.4. *Let $n \in 18\mathbb{N}$. Then*

(M1) *for every $x \in V(H)$, $\deg_H(x) \equiv 1 \bmod 3$ and*
(M2) $|H_{112}| \not\equiv |H_{122}| \bmod 3$.

Proof. We begin by noting that $\binom{m}{2} \equiv 2m(m-1) \bmod 3$ holds for all integers m. Thus $\binom{m}{2} \equiv 1 \bmod 3$ if $m \equiv 2 \bmod 3$, and $\binom{m}{2} \equiv 0 \bmod 3$ otherwise.

Now let $x \in V_0$. Then the pairs yz such that $xyz \in H$ are those such that

(1) $y \in V_0 \setminus \{x\}$ and $z \in V_1 \cup V_2$, of which there are $(n_0 - 1)(n_1 + n_2)$ many,
(2) $yz \subseteq V_1$, of which there are $\binom{n_1}{2}$ many, and
(3) $yz \subseteq V_2$, of which there are $\binom{n_2}{2}$ many.

Thus we have $\deg_H(x) = (n_0 - 1)(n_1 + n_2) + \binom{n_1}{2} + \binom{n_2}{2}$. Together with (2.2), we have that $\deg_H(x) \equiv 0 + 0 + 1 \equiv 1 \bmod 3$. Analogous calculations show that

$$\deg_H(y) \equiv 0 + 0 + 1 \equiv 1 \bmod 3 \text{ for } y \in V_1 \text{ and}$$
$$\deg_H(z) \equiv 1 + 0 + 0 \equiv 1 \bmod 3 \text{ for } z \in V_2,$$

thus (M1) holds.

Finally, the sizes of $|H_{112}|$ and $|H_{122}|$ are $\binom{n_1}{2}n_2$ and $\binom{n_2}{2}n_1$ respectively, which then are easily seen to be equivalent to 0 and 1 modulo 3, respectively, which implies (M2). ■

Since H is not quite 3-vertex-divisible, our counterexample will consist actually of a slight alteration of H obtained by removing some sparse subgraph, which we define now.

Lemma 2.5. *Let $n \in 18\mathbb{N}$. Then there exists a perfect matching $F \subseteq H \setminus (H_{112} \cup H_{122})$.*

Proof. Let k be such that $n = 18k$. Let a, b be two distinct vertices in V_2, and let $V_1' = V_1 \cup \{a, b\}$ and $V_2' = V_2 \setminus \{a, b\}$. Note that $|V_0| = |V_1'| = |V_2'| = 6k$. Let $V_0 = \{x_1, \dots, x_{6k}\}$, $V_1' = \{y_1, \dots, y_{6k}\}$ and $V_2' = \{z_1, \dots, z_{6k}\}$, with $y_1 = a$ and $y_2 = b$. Then

$$F = \{y_{2i-1}y_{2i}x_{2i-1} : 1 \leqslant i \leqslant 3k\} \cup \{z_{2i-1}z_{2i}x_{2i} : 1 \leqslant i \leqslant 3k\}$$

is a perfect matching in which every edge intersects V_0 in exactly one vertex. Thus F has no edge in $H_{112} \cup H_{122}$, as required. ■

We are now ready to show Theorem 1.4.

Proof of Theorem 1.4. Consider the 3-graph $H = H_n$ given in Definition 2.2, and consider the perfect matching $F \subseteq H \setminus (H_{112} \cup H_{122})$ given by Lemma 2.5. Let $\ell' \in \{4, \dots, \ell + 3\}$ be such that $|E(H - F)| + \ell' \equiv 0 \bmod \ell$. Since $n = 18k \geqslant 3(\ell + 3)$, we have $|V_0| = 6k \geqslant \ell + 3 \geqslant \ell'$. To $H - F$, we add a cycle C of length ℓ', edge-disjoint from $H - F$, which is entirely contained in V_0. We claim $H' = (H \setminus F) \cup C$ has all of the desired properties.

We first check H' is C_ℓ-divisible. We start by checking H' is 3-vertex-divisible. Indeed, let $x \in V(H')$ be arbitrary. We have $\deg_H(x) \equiv 1 \bmod 3$ by Lemma 2.4(M1), we have $\deg_F(x) = 1$ since F is a perfect matching, and $\deg_C(x) \equiv 0 \bmod 3$ since C is a cycle on $\ell' \geqslant 4$ vertices. Thus $\deg_{H'}(x) \equiv 1 - 1 + 0 \equiv 0 \bmod 3$ for all $x \in V(H')$, as required. Also, the number of edges of H' is $|E(H')| = |E(H - F)| + \ell'$, which was chosen to be divisible by ℓ, so indeed H' is C_ℓ-divisible.

Now we check H' has large codegree. It suffices to show $H - F$ has large codegree. Removing a perfect matching from H decreases the codegree of every pair at most by 1, thus by Lemma 2.3, we have $\delta_2(H - F) \geqslant \delta_2(H) - 1 \geqslant (2n - 12)/3 - 1 = (2n - 15)/3$.

Now we prove H' does not have a tour decomposition. First, since $F \subseteq H \setminus (H_{112} \cup H_{122})$, we have $H'[V_1, V_1, V_2] = H_{112}$ and $H'[V_1, V_2, V_2] = H_{122}$. For a contradiction, suppose that $W^1, \dots, W^r$ are tours forming a tour decomposition in H'. For a walk W, let $W_{112} = H_{112} \cap E(W)$, and let $W_{122} = H_{122} \cap E(W)$. Since the tours are edge-disjoint and cover all edges of H', we have $\sum_{1 \leqslant i \leqslant r} |W^i_{112}| = |H_{112}|$ and $\sum_{1 \leqslant i \leqslant r} |W^i_{122}| = |H_{122}|$. Since $H_{012} = \varnothing$, Lemma 2.1 implies that $|W^i_{112}| \equiv |W^i_{122}| \bmod 3$ for each $1 \leqslant i \leqslant r$. We deduce $|H_{112}| \equiv |H_{122}| \bmod 3$, but this contradicts Lemma 2.4(M2). ■

Remark 2.6. For sufficiently large values of n, we can make our example vertex-regular instead of C_ℓ-divisible. This is needed, for instance, when we are looking at decompositions into spanning vertex-disjoint collections of cycles, such as Hamilton cycles.

Start from $H = H_n$, and remove F as before to get to $H' = H - F$ which is 3-vertex-divisible. Every vertex in V_i has the same degree d_i, for all $i \in \{0, 1, 2\}$, and a calculation reveals that $d_1 = d_0 - 9$ and $d_2 = d_0 - 3$. Then, adding 3 edge-disjoint Hamilton cycles to $H[V_1]$ and one Hamilton cycle to $H[V_2]$ leaves a 3-graph H^* in which every vertex has degree d_0, and it can be similarly proved that H^* does not admit any tour decomposition.

§3. Proof of Corollaries 1.2 and and 1.3

In this short section we deduce Corollaries 1.2 and 1.3 from Theorem 1.1.

Proof of Corollary 1.2. Let m be the number of edges of H, and write it as $m = 9q + r$ for some $q \geqslant 1$ and $0 \leqslant r < 9$. Find a cycle C of length $9 + r$ in H: this can be done greedily (see Section 5.1 for details). Then, $H' = H - C$ is a 3-divisible graph, its minimum codegree is $\delta_2(H') \geqslant \delta_2(H) - 2 \geqslant (2/3 + \varepsilon/2)n$, and its number of edges is $m - (9 + r) = 9(q - 2)$, which is divisible by 9. By Theorem 1.1, H' has a C_9-decomposition, together with C this is a cycle decomposition of H. ■

For the proof of Corollary 1.3 we use the strategy of Glock, Joos, Kühn, and Osthus [9]. Crucial part of their argument is (using our terminology) to first find a trail W which is *spanning* (i.e. every 2-tuple of distinct vertices of H is contained as a sequence of consecutive vertices of W) but at the same time is sparse (it satisfies $\Delta_2(W) = o(n)$).

We state their relevant lemma only in the particular case $k = 3$. A 3-graph H on n vertices is *α-connected* if for all distinct $v_1, v_2, v_4, v_5 \in V(H)$, there exist at least αn vertices $v_3 \in V(H)$ such that $v_1v_2v_3v_4v_5$ is a walk in H.

Lemma 3.1 ([9, Lemma 5])**.** *Suppose $n \in \mathbb{N}$ is sufficiently large in terms of α. Suppose H is an α-connected 3-graph on n vertices. Then H contains a spanning trail W satisfying $\Delta_2(W) \leqslant \log^3 n$.*

Proof of Corollary 1.3. Take n_0 such that $1/n_0 \ll \varepsilon$. Since H satisfies $\delta_2(H) \geqslant (2/3 + \varepsilon)n$, it is ε-connected. By Lemma 3.1 there exists a spanning trail $W = w_1 \cdots w_r$ satisfying $\Delta_2(W) \leqslant \log^3 n$. Use the ε-connected property of H to close W to a tour, using three extra vertices, while avoiding edges previously used by W (using that $\Delta_2(W) \leqslant \log^3 n$). The resulting $W' = w_1 \cdots w_{r+3}$ is a spanning tour which satisfies $\Delta_2(W') \leqslant 2\log^3 n$. Let $H' = H - W'$. Since W' is a tour and H is 3-vertex-divisible, W' is 3-vertex-divisible as well. Since $\Delta_2(W') \leqslant 2\log^3 n \leqslant \varepsilon n/2$ and $\delta_2(H) \geqslant (2/3 + \varepsilon)n$, we deduce $\delta_2(H') \geqslant (2/3 + \varepsilon/2)n$. Since n is sufficiently large, Corollary 1.2 implies that H' has a cycle decomposition. Fix one of those cycles $C = v_1v_2 \cdots v_m$ and note that the ordered pair (v_1, v_2) must appear consecutively in some part of W' (since W' is spanning). We may write $W' = W'_1v_1v_2W'_2$ and extend W' by taking $W'_1v_1v_2 \cdots v_mv_1v_2W'_2$, which is still an spanning tour, but now uses the edges of C in addition to those of W'. Attaching the cycles of the decomposition one by one to W', we obtain the desired Euler tour. ■

§4. Iterative absorption: proof of Theorem 1.1

Our proof of Theorem 1.1 follows the strategy of *iterative absorption* introduced by Barber, Kühn, Lo, and Osthus [3] and further developed by Glock, Kühn, Lo, Montgomery, and Osthus [10] to study decomposition thresholds in graphs. We base our outline in the exposition of Barber, Glock, Kühn, Lo, Montgomery, and Osthus [2].

The method of iterative absorption rests around three main lemmata, originally called the the Vortex Lemma, Absorbing Lemma, and the Cover-Down Lemma. We will introduce these lemmata first while explaining the global strategy, then we will use them to prove Theorem 1.1. The proof of these lemmata will take up the rest of the paper.

A sequence of nested subsets of vertices $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ is called a *(δ, ξ, m)-vortex in H* if satisfies the following properties.

(V1) $U_0 = V(H)$,
(V2) for each $1 \leqslant i \leqslant \ell$, $|U_i| = \lfloor \xi |U_{i-1}| \rfloor$,
(V3) $|U_\ell| = m$, and
(V4) $\deg(x, U_i) \geqslant \delta\binom{|U_i|}{2}$ for each $1 \leqslant i \leqslant \ell$ and $x \in U_{i-1}$, and
(V5) $\deg(xy, U_i) \geqslant \delta|U_i|$ for each $1 \leqslant i \leqslant \ell$ and $xy \in \binom{U_{i-1}}{2}$.

The existence of vortices for suitable parameters δ, ξ, and m is stated in the Vortex Lemma.

Lemma 4.1 (Vortex Lemma). *Let $\xi, \delta > 0$ and $m' \in \mathbb{N}$ be such that $1/m' \ll \xi$. Let H be a 3-graph on $n \geqslant m'$ vertices with $\delta_2(H) \geqslant \delta$. Then it has a $(\delta - \xi, \xi, m)$-vortex, for some $\lfloor \xi m' \rfloor \leqslant m \leqslant m'$.*

The main idea is to use the properties of the vortex to find a suitable *C_ℓ-packing*, i.e. a collection of edge-disjoint $C_\ell \subseteq H$. We will find a packing covering most edges of H, and moreover the non-covered edges will lie entirely in U_ℓ. The Absorbing Lemma will provide us with a small structure that we put aside at the beginning, and that will be used to deal with the small remainder left by our C_ℓ-packing. If $R \subseteq H$ is a subgraph of H, a *C_ℓ-absorber for R* is a subgraph $A \subseteq H$, edge-disjoint from R, such that both A and $A \cup R$ are C_ℓ-decomposable.

Lemma 4.2 (Absorbing Lemma). *Let $\ell \geqslant 7$, $\varepsilon > 0$, and $n, m \in \mathbb{N}$ such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be a 3-graph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Let $R \subseteq H$ be C_ℓ-divisible on at most m vertices. Then there exists a C_ℓ-absorber for R in H with at most $(4m\ell)^9$ edges.*

Finally, we construct the desired C_ℓ-packing step by step through the nested sets of the vortex. More precisely, suppose $U_i \supseteq U_{i+1}$ are two consecutive sets in a vortex of H. The Cover-Down Lemma will be applied to find a C_ℓ-packing which covers every edge of $H[U_i]$, except maybe for some in $H[U_{i+1}]$. Thus the packing will be found via reiterated applications.

Lemma 4.3 (Cover-Down Lemma). *Let $\ell \geqslant 9$ be divisible by 3 or at least 10^7, and $\varepsilon, \mu > 0$ and $n \in \mathbb{N}$ with $1/n \ll \mu, \varepsilon \ll 1/\ell$. Suppose H is a 3-graph on n vertices, and $U \subseteq V(H)$ with $|U| = \lfloor \varepsilon n \rfloor$, which satisfy*

- (C1) $\delta_2(H) \geqslant (2/3 + 2\varepsilon)n$,
- (C2) $\deg_H(x, U) \geqslant (2/3 + \varepsilon)\binom{|U|}{2}$ *for each* $x \in V(H)$,
- (C3) $\deg_H(xy, U) \geqslant (2/3 + \varepsilon)|U|$ *for each* $xy \in \binom{V(H)}{2}$, *and*
- (C4) $\deg_H(x)$ *is divisible by 3 for each* $x \in V(H) \smallsetminus U$.

Then H has a C_ℓ-decomposable subgraph F such that $H - H[U] \subseteq F$, and $\Delta_2(F[U]) \leqslant \mu n$.

Assuming lemmata 4.2–4.3, we prove Theorem 1.1 holds (cf. [2, Section 3.4]).

Proof of Theorem 1.1. It is enough to show that, for every $\varepsilon > 0$, there exists n_0 such that every C_ℓ-divisible 3-graph H on $n \geqslant n_0$ vertices with $\delta_2(H) \geqslant (2/3 + 8\varepsilon)n$ admits a C_ℓ-decomposition. Given ε and ℓ, we fix m', n_0 such that

$$1/n_0 \ll 1/m' \ll \varepsilon, 1/\ell. \tag{4.1}$$

Let H on $n \geqslant n_0$ vertices as before, we are done if we show H has a C_ℓ-decomposition.

Step 1: Setting the vortex and absorbers. By Lemma 4.1, H has a $(2/3 + 7\varepsilon, \varepsilon, m)$-vortex $U_0 \supseteq \cdots \supseteq U_\ell$, for some m such that $\lfloor \varepsilon m' \rfloor \leqslant m \leqslant m'$.

Let $\mathscr{L}$ be the family of all C_ℓ-divisible 3-graphs which are subgraphs of $H[U_\ell]$. Since $|U_\ell| = m$, clearly $|\mathscr{L}| \leqslant 2^{\binom{m}{3}}$. Let $L \in \mathscr{L}$ be arbitrary. Since $m \leqslant m'$ and (4.1), a suitable application of Lemma 4.2 yields a C_ℓ-absorber $A_L \subseteq H \smallsetminus H[U_1]$ of L with at most $(4m\ell)^9$ edges. Since $1/n \ll 1/m, \varepsilon, 1/\ell$, removing the edges of A_L only barely affects the codegree of H, thus we can repeat the argument to obtain an absorber $A_{L'} \subseteq H \smallsetminus H[U_1]$ for some $L' \neq L$, edge-disjoint from A_L. Since the total number of $L \in \mathscr{L}$ is tiny with respect to n, we can iterate this argument to obtain edge-disjoint C_ℓ-absorbers $A_L \subseteq H \smallsetminus H[U_1]$, one for each $L \in \mathscr{L}$. Moreover, each A_L contains at most $(4m\ell)^9$ edges, and hence, the union $A = \bigcup_{L \in \mathscr{L}} A_L \subseteq H \smallsetminus H[U_1]$ contains at most $|\mathscr{L}|(4m\ell)^9 \leqslant 2^{\binom{m}{3}}(4m\ell)^9 \leqslant \varepsilon n$ edges. By construction, we have A is C_ℓ-decomposable and for each $L \in \mathscr{L}$, $L \cup A$ is C_ℓ-decomposable.

Let $H' = H \smallsetminus A$ and observe that $\delta_2(H') \geqslant (2/3 + 7\varepsilon)n$ and $U_0 \supseteq \cdots \supseteq U_\ell$ is a $(2/3 + 6\varepsilon, \varepsilon, m)$-vortex for H' (for this, it is crucial that $A \subseteq H \smallsetminus H[U_1]$). Notice that since A and H are C_ℓ-divisible, we get that H' is C_ℓ-divisible.

Step 2: The cover-down. Now we aim to find a C_ℓ-packing in H' using every edge of $H' \smallsetminus H'[U_\ell]$. Let $U_{\ell+1} = \varnothing$. For each $0 \leqslant i \leqslant \ell$ we wish to find $H_i \subseteq H'[U_i]$ such that

(a_i) $H' - H_i$ has a C_ℓ-decomposition,
(b_i) $\delta_2(H_i) \geqslant (2/3 + 4\varepsilon)|U_i|$,
(c_i) $\deg_{H_i}(x, U_{i+1}) \geqslant (2/3 + 5\varepsilon)\binom{|U_{i+1}|}{2}$ for all $x \in U_i$,
(d_i) $\deg_{H_i}(xy, U_{i+1}) \geqslant (2/3 + 5\varepsilon)|U_{i+1}|$ for all $x, y \in U_i$, and
(e_i) $H_i[U_{i+1}] = H'[U_{i+1}]$.

For $i = 0$ this can be done by setting $H_0 = H'$. Now suppose H_i satisfying (a_i)–(e_i) is given for some $0 \leqslant i < \ell$, we wish to construct H_{i+1} satisfying (a_{i+1})–(e_{i+1}). By (a_i), H_i is C_ℓ-divisible. Let $H_i' = H_i \smallsetminus H_i[U_{i+2}]$. By ($b_i$)–($d_i$) and $|U_{i+2}| \leqslant \varepsilon|U_{i+1}| \leqslant \varepsilon^2|U_i|$, we have

(C1) $\delta_2(H_i') \geqslant \delta_2(H_i) - |U_{i+2}| \geqslant (2/3 + 3\varepsilon)|U_i|$,
(C2) $\deg_{H_i'}(x, U_{i+1}) \geqslant \deg_{H_i}(x, U_{i+1}) - |U_{i+2}|(|U_{i+1}| - 1) \geqslant (2/3 + 3\varepsilon)\binom{|U_{i+1}|}{2}$, for each $x \in U_i$,
(C3) $\deg_{H_i'}(xy, U_{i+1}) \geqslant \deg_{H_i'}(xy, U_{i+1}) - |U_{i+2}| \geqslant (2/3 + 4\varepsilon)|U_{i+1}|$ for each $x, y \in U_i$, and
(C4) $\deg_{H_i'}(x)$ is divisible by 3 for each $x \in U_i \smallsetminus U_{i+1}$.

This allows us to apply Lemma 4.3 with $\varepsilon, \varepsilon^4, |U_i|, H_i', U_{i+1}$ playing the rôles of $\varepsilon, \mu, n, H, U$. We obtain a C_ℓ-decomposable subgraph $F_i \subseteq H_i'$ such that $H_i' \smallsetminus H_i'[U_{i+1}] \subseteq F_i$ and that $\Delta_2(F_i[U_{i+1}]) \leqslant \varepsilon^4|U_i|$. Let $H_{i+1} = H_i[U_{i+1}] \smallsetminus F_i$, we prove it satisfies the required properties.

Clearly F_i is C_ℓ-divisible and $F_i \subseteq H_i' \subseteq H_i$, so ($a_i$) implies that $H' - H_{i+1} = (H' - H_i) \cup F_i$ has a C_ℓ-decomposition, thus (a_{i+1}) holds. From (d_i) and $\Delta_2(F_i[U_{i+1}]) \leqslant \varepsilon^4|U_i| \leqslant \varepsilon^2|U_{i+1}|$, we have $\delta_2(H_{i+1}) \geqslant (2/3 + 5\varepsilon)|U_{i+1}| - \varepsilon^2|U_{i+1}| \geqslant (2/3 + 4\varepsilon)|U_{i+1}|$, proving ($b_{i+1}$).

By the properties of $(2/3+6\varepsilon, \varepsilon, m)$-vortices, we have $\deg_{H'}(x, U_{i+2}) \geqslant (2/3+6\varepsilon)\binom{|U_i|}{2}$ for each $x \in U_{i+1}$, together with $\Delta_2(F_i[U_{i+1}]) \leqslant \varepsilon^2|U_{i+1}|$ and (e_i) we deduce (c_{i+1}) holds, and (d_{i+1}) can be verified similarly. Finally, since $F_i \subseteq H_i' = H_i \smallsetminus H_i[U_{i+1}]$, we have $F_i[U_{i+2}]$ is empty therefore $H_{i+1}[U_{i+2}] = H_i[U_{i+2}] = H'[U_{i+2}]$, which verifies ($e_{i+1}$).

Now $H_\ell \subseteq H'[U_\ell]$ is such that $H' \smallsetminus H_\ell$ has a C_ℓ-decomposition.

Step 3: Finish. Since both H' and $H' \smallsetminus H_\ell$ are C_ℓ-divisible, we deduce $H_\ell \subseteq H'[U_\ell]$ is C_ℓ-divisible. Therefore, $H_\ell \in \mathscr{L}$ and by construction of A we know that $H_\ell \cup A$ is C_ℓ-decomposable. Since H is the edge-disjoint union of $H' \smallsetminus H_\ell$ and $H_\ell \cup A$, and both of them have C_ℓ-decompositions, we deduce H has a C_ℓ-decomposition, as desired. ■

§5. Useful tools

We collect various results to be used during the proof of Lemmatas 4.2–4.3.

5.1. **Counting path extensions.** The following lemma find short trails between prescribed pairs of vertices. For a 3-graph H, a set of vertices $U \subseteq V(H)$, and a set of pairs $G \subseteq \binom{V(H)}{2}$ let $\delta_2^{(3)}(H; U, G)$ be the minimum of $|N(e_1) \cap N(e_2) \cap N(e_3) \cap U|$ over all possible choices of $e_1, e_2, e_3 \in G$. This is the size of the minimum joint neighbourhood in U of three distinct pairs in G. Also, let $\delta_2^{(3)}(H; U) = \delta_2^{(3)}(H, U, \binom{V(H)}{2})$ and $\delta_2^{(3)}(H) = \delta_2^{(3)}(H; V(H))$.

Lemma 5.1. *Let $\varepsilon > 0$ and $n, \ell \in \mathbb{N}$ be such that $\ell \geqslant 5$ and $1/n \ll \varepsilon, 1/\ell$. Let H be a 3-graph on n vertices, $U \subseteq V(H)$ and $G \subseteq \binom{V(H)}{2}$ such that $\{uv \in \binom{V(H)}{2} : u \in U\} \subseteq G$. Suppose $\delta_2^{(3)}(H; U, G) \geqslant 2\varepsilon n$. Then, for every two disjoint pairs v_1v_2 and $v_{\ell-1}v_\ell$ in G there exist at least $(\varepsilon n)^{\ell-4}$ many $(v_1, v_2, v_{\ell-1}, v_\ell)$-paths on ℓ vertices, whose internal vertices are in U.*

Proof. Every pair of vertices in G has at least $2\varepsilon n$ neighbours in U. For each $1 \leqslant i \leqslant \ell - 3$, since $\{uv \in \binom{V(H)}{2} : u \in U\} \subseteq G$ we can build a path $v_1v_2 \cdots v_i$ such that $\{v_{i-1}, v_i\} \in G$ by choosing vertices in U greedily. The path is then finished by choosing $v_{\ell-2}$ as a common neighbour in U of the pairs $v_{\ell-4}v_{\ell-3}$, $v_{\ell-3}v_{\ell-1}$ and $v_{\ell-1}v_\ell$, all of which belong to G. At any step we only need to avoid choosing one of vertices already chosen so far, which are at most $\ell \leqslant \varepsilon n$. Thus in each step there are at least εn possible choices, which gives the desired bound. ■

In the particular for a 3-graph H with $\delta_2(H) \geqslant (2/3+\varepsilon)n$ a simple application of Lemma 5.1 with $U = V(H)$ and $G = \binom{V(H)}{2}$ implies the existence of many trails of length $\ell \geqslant 5$ between arbitrary pairs of vertices.

Sometimes we want find many paths which also avoid a small prescribed set of vertices or edges, for instance to extend paths into cycles. This is accomplished as follows.

Lemma 5.2. *Let $\varepsilon, \mu > 0$ and $n, \ell \in \mathbb{N}$ be such that $\ell \geqslant 5$ and $1/n \ll \mu \ll \varepsilon, 1/\ell$. Suppose that $v_1, v_2, v_{\ell-1}, v_\ell \in V(H)$ and there are at least $2\varepsilon n^{\ell-4}$ many $(v_1, v_2, v_{\ell-1}, v_\ell)$-paths on ℓ vertices in H. Let $F \subseteq H$ with $\Delta_2(F) \leqslant \mu n$. Then there are at least $\varepsilon n^{\ell-4}$ many $(v_1, v_2, v_{\ell-1}, v_\ell)$-paths on ℓ vertices in $H \smallsetminus F$.*

Proof. The number of $(v_1, v_2, v_{\ell-1}, v_\ell)$-paths on ℓ vertices such that $v_1v_2v_3 \in F$ is at most $\deg_F(v_1v_2)n^{\ell-5} \leqslant \Delta_2(F)n^{\ell-5} \leqslant \mu n^{\ell-4}$. Similar bound are obtained for the paths of the same form such that $v_{\ell-2}v_{\ell-1}v_\ell \in F$, $v_3v_4v_5 \in F$, or $v_{\ell-3}v_{\ell-2}v_{\ell-1} \in F$. Finally, the paths such that $v_jv_{j+1}v_{j+2} \in F$ for some $3 \leqslant j \leqslant \ell-4$ is at most $|E(F)|n^{\ell-7} \leqslant \mu n^{\ell-4}$. All together, the number of paths destroyed by passing from H to $H \smallsetminus F$ is at most $(\ell-2)\mu n^{\ell-4} \leqslant \varepsilon n^{\ell-4}$, where the last inequality uses $\mu \ll \varepsilon$. ■

The following is an immediate corollary of Lemma 5.1 and Lemma 5.2.

Corollary 5.3. *Let $\varepsilon > 0$ and $n, \ell, \ell' \in \mathbb{N}$ be such that $1/n \ll \mu \ll \varepsilon \ll \varepsilon', 1/\ell, 1/\ell'$ and $\ell \geqslant \ell'+1$. Let H be a 3-graph on n vertices, $U \subseteq V(H)$ and $G \subseteq \binom{V(H)}{2}$ such that $\{uv \in \binom{V(H)}{2} : u \in U\} \subseteq G$. Suppose $\delta_2^{(3)}(H; U, G) \geqslant 2\varepsilon' n$. Let P be a path on ℓ' vertices in H, whose two endpoints are in G. Then there are at least $\varepsilon n^{\ell-\ell'}$ many cycles C on ℓ vertices which contain P, and $V(C) \smallsetminus V(P) \subseteq U$.*

Observe that for a 3-graph H with $\delta_2(H) \geqslant (2/3+\varepsilon)n$ and a set of vertices $W \subseteq V(H)$ with $|W| < \varepsilon n/2$, a simple application of Corollary 5.3 with $U = V(H) \smallsetminus W$ and $G = \binom{V(H)}{2}$ yields the existence of many cycles containing one fix path P and avoiding the set of vertices W.

5.2. **Probabilistic tools.** We shall use the following concentration inequalities [14, Corollary 2.3, Corollary 2.4, Remark 2.5, Theorem 2.10].

Theorem 5.4. *Let X be a random variable which is a sum of n independent $\{0,1\}$-random variables, or hypergeometric with parameters n, N, M.*

(i) *If $x \geqslant 7\mathbf{E}[X]$, then $\mathbf{P}[X \geqslant x] \leqslant \exp(-x)$,*
(ii) *$\mathbf{P}[|X - \mathbf{E}[X]| \geqslant t] \leqslant 2\exp(-2t^2/n)$, and*
(iii) *$\mathbf{P}[|X - \mathbf{E}[X]| \geqslant t] \leqslant 2\exp(-t^2/(3\mathbf{E}[X]))$.*

The following lemma allows us to bound the tail probabilities of sums of sequentially-dependent $\{0,1\}$-random variables by comparing them with binomial random variables. We use the probability-theoretic notion of conditioning in a sequence of random variables, which in our application will take the following form. If $X_1, \dots, X_i$ are random variables, $\mathbf{P}[X_i = 1 | X_1, \dots, X_{i-1}] \leqslant p_i$ means that the probability of $X_i = 1$ is always at most p_i, even after conditioning on any possible output of $X_1, \dots, X_{i-1}$.

Theorem 5.5. *Let $X_1, \dots, X_t$ be Bernoulli random variables (not necessarily independent) such that for each $1 \leqslant i \leqslant t$ we have $\mathbf{P}[X_i = 1 | X_1, \dots, X_{i-1}] \leqslant p_i$. Let $Y_1, \dots, Y_t$ be independent Bernoulli random variables such that $\mathbf{P}[Y_i = 1] = p_i$ for all $1 \leqslant i \leqslant t$. If $X = \sum_{i=1}^t X_i$ and $Y = \sum_{i=1}^t Y_i$, then $\mathbf{P}[X \geqslant k] \leqslant \mathbf{P}[Y \geqslant k]$ for all $k \in \{0, 1, \dots, t\}$.*

The proof of Theorem 5.5 was given by Jain [19, Lemma 7] in the particular case where $p_i = p$ for all $1 \leqslant i \leqslant t$. The slightly more general statement of Theorem 5.5 follows by mimicking that proof (which goes by induction on t), so we omit it.

§6. Vortex Lemma

We prove Lemma 4.1 by selecting random subsets (cf. [2, Lemma 3.7]).

Proof of Lemma 4.1. Let $n_0 = n$ and $n_i = \lfloor \xi n_{i-1} \rfloor$ for all $i \geqslant 1$. In particular, note $n_i \leqslant \xi^i n$. Let ℓ be the largest i such that $n_i \geqslant m'$ and let $m = n_{\ell+1}$. Note that $\lfloor \xi m' \rfloor \leqslant m \leqslant m'$.

Let $\xi_0 = 0$ and, for all $i \geqslant 1$, define $\xi_i = \xi_{i-1} + 2(\xi^i n)^{-1/3}$. Thus we have

$$\xi_{\ell+1} = 2n^{-1/3} \sum_{i=1}^{\ell} (\xi^{-1/3})^i \leqslant 2n^{-1/3} \sum_{i=1}^{\infty} (\xi^{-1/3})^i \leqslant \frac{2(n\xi)^{-1/3}}{1 - \xi^{-1/3}} \leqslant \xi,$$

where in the last inequality we used $1/m' \ll \xi$ and $n \geqslant m'$.

Note that taking $U_0 = V(H)$ yields a $(\delta - \xi_0, \xi, n_0)$-vortex in H. Suppose we have already found a $(\delta - \xi_{i-1}, \xi, n_{i-1})$-vortex $U_0 \supseteq \cdots \supseteq U_{i-1}$ in H for some $i \leqslant \ell + 1$. In particular, $\delta_2(H[U_{i-1}]) \geqslant (\delta - \xi_{i-1})|U_{i-1}|$. Let $U_i \subseteq U_{i-1}$ be a random subset of size n_i. By Theorem 5.4, with positive probability we have, for all $x, y \in U_{i-1}$, $\deg(xy, U_i) \geqslant (\delta - \xi_{i-1} - n_i^{-1/3})|U_i|$ and $\deg(x, U_i) \geqslant (\delta - \xi_{i-1} - n_i^{-1/3})\binom{|U_i|}{2}$. Since $\xi_{i-1} + n_i^{-1/3} \leqslant \xi_i$, we have found a $(\delta - \xi_i, \xi, n_i)$-vortex for H. In the end, we will have found a $(\delta - \xi_{\ell+1}, \xi, n_{\ell+1})$-vortex for H. Since we have $m = n_{\ell+1}$ and we have established $\xi_{\ell+1} \leqslant \xi$, we are done. ∎

§7. Cover-Down Lemma

7.1. **Extending paths into cycles.** More than once during our proof, we will be faced with the following situation: we have a family of (not too many) edge-disjoint tight paths, and we want to extend each of these paths into a tight cycle of a given length, such that all of the obtained cycles are edge-disjoint. In this subsection we will prove a lemma which will find such extensions for us.

Given a path P we say that a path or a cycle C is an *extension of* P if $P \subseteq C$. Let H be a 3-graph, for a path $P \subseteq H$ and a pair of vertices $e \in \binom{V(H)}{2}$ we say that P is of *type* r for e, where $r = \max\{e \cap s(P), e \cap t(P)\}$. The only possible types are 0, 1, or 2.

We say that a collection of edge-disjoint paths $\mathcal{P}$ in H is γ*-sparse* if, for each $e \in \binom{V(H)}{2}$ and each $r \in \{0, 1, 2\}$, $\mathcal{P}$ has at most γn^{3-r} paths P of type r for e.

Lemma 7.1 (Extending Lemma). *Let $\varepsilon, \mu, \gamma > 0$ and $n, \ell, \ell' \in \mathbb{N}$ such that $\ell' \geqslant 4$, $\ell \geqslant \ell' + 2$ and $1/n \ll \gamma \ll \mu \ll \varepsilon, 1/\ell$. Let H_1, H_2 be two edge-disjoint 3-graphs on the same vertex set V of size n. Let P be the 3-uniform tight path on ℓ' vertices, and let $\mathcal{P} = \{P_1, \ldots, P_t\}$ be an edge-disjoint collection of copies of P in H_1 such that*

- (F1) *$\mathcal{P}$ is γ-sparse, and*
- (F2) *for each $P_i \in \mathcal{P}$, there exists at least $2\varepsilon n^{\ell-\ell'}$ copies of C_ℓ in $H_1 \cup H_2$ which extend P_i using extra edges of H_2 only.*

Then, there exists a C_ℓ-decomposable subgraph $F \subseteq H_1 \cup H_2$, such that

- (C1) *$E(\mathcal{P}) \subseteq F$, and*
- (C2) *$\Delta_2(F \setminus E(\mathcal{P})) \leqslant \mu n$.*

Proof. The idea is to pick, sequentially, an extension C_i of P_i into an ℓ-cycle, chosen uniformly at random among all the extensions which do not use edges already used by $C_1, \ldots, C_{i-1}$. Since $\mathcal{P}$ is γ-sparse and there are plenty of choices for C_i in each step, we expect that in each step the random choices do not affect the codegree of the graph formed by the unused edges in H_2 by much. This will ensure that, even after removing the edges used by $C_1, \ldots, C_{i-1}$, there are still many extensions available for P_i. If all goes well, then we can continue the process until the end, thus achieving (C1) and (C2) by setting $F = \bigcup_{1 \leqslant i \leqslant t} E(C_i)$.

To formalise the above plan, we begin by noting that the removal of a sufficiently sparse 3-graph from H_2, there are still many extensions available for each P_i. Given $G \subseteq H_2$ and $1 \leqslant i \leqslant t$, let $\mathcal{C}_i(G)$ be the set of *G-avoiding cycle-extensions of* P_i, that is, the copies of C_ℓ in $H_1 \cup H_2$ which extend P_i and use extra edges from $H_2 \setminus G$ only. By assumption, $|\mathcal{C}_i(\varnothing)| \geqslant 2\varepsilon n^{\ell-\ell'}$, thus Lemma 5.2 implies that

$$\text{if } G \subseteq H_2 \text{ is such that } \Delta_2(G) \leqslant \mu n, \text{ then } |\mathcal{C}_i(G)| \geqslant \varepsilon n^{\ell-\ell'}. \tag{7.1}$$

We now describe the random process which outputs edge-disjoint extensions C_i of P_i for each $1 \leqslant i \leqslant t$. In the case of success each C_i will be an ℓ-cycle extending P_i. To account for the case of failure, in our description we will allow the degenerate case in which $C_i \setminus P_i$ is empty.

For each $1 \leqslant i \leqslant t$, assume we have already chosen $C_1, C_2, \dots, C_{i-1} \subseteq H_1 \cup H_2$ edge-disjoint graphs, and we describe the choice of C_i. Let $G_{i-1} = \bigcup_{1 \leqslant j < i} E(C_j) \setminus E(P_j)$ correspond to the edges of H_2 used by the previous choices of C_j, which we need to avoid when choosing C_i (note that G_0 is empty). If $\Delta_2(G_{i-1}) \leqslant \mu n$, then by (7.1) we have $|\mathcal{C}_i(G_{i-1})| \geqslant \varepsilon n^{\ell-\ell'}$, and we take $C_i \in \mathcal{C}_i(G_{i-1})$ uniformly at random. Otherwise, if $\Delta_2(G_{i-1}) > \mu n$, let $C_i = P_i$.

In any case, the process outputs a collection $C_1, \dots, C_t$ of edge-disjoint cycle or paths which extend P_i. Our task now is to show that with positive probability, there is a choice of $C_1, \dots, C_t$ such that $\Delta_2(G_t) \leqslant \mu n$. This would imply also that each C_i was an ℓ-cycle. Formally, for each $1 \leqslant i \leqslant t$, let $\mathcal{S}_i$ be the event that $\Delta_2(G_i) \leqslant \mu n$. Thus it is enough to show $\mathbf{P}[\mathcal{S}_t] > 0$.

Fix $e \in \binom{V}{2}$. For each $1 \leqslant i \leqslant t$, let $X_i(e)$ be the random variable which takes the value 1 precisely if e belongs to an edge of $C_i \setminus P_i$, and 0 otherwise. Equivalently, $X_i(e) = 1$ if and only if e belong to the shadow $\partial(C_i \setminus P_i)$. Since $\Delta_2(C_i) \leqslant 2$ for each $1 \leqslant i \leqslant t$, we have

$$\deg_{G_i}(e) \leqslant 2 \sum_{j=1}^{i} X_j(e). \tag{7.2}$$

For each $1 \leqslant i \leqslant t$, define

$$p_i^*(e) := \min\left\{1, \frac{c}{n^{2-r}}\right\},$$

where $r \in \{0, 1, 2\}$ is such that P_i is of type r for e, and $c := 4\ell\varepsilon^{-1}$.

Claim 7.2. *For each $e \in \binom{V}{2}$ and $1 \leqslant i \leqslant t$,*

$$\mathbf{P}[X_i(e) = 1 | X_1(e), X_2(e), \dots, X_{i-1}(e)] \leqslant p_i^*(e),$$

Proof of the claim. Using conditional probabilities, we separate our analysis depending on whether $\mathcal{S}_{i-1}$ holds or not. Assume first that $\mathcal{S}_{i-1}$ fails. Then the process declares $C_i = P_i$, thus $C_i \setminus P_i$ is empty. Therefore $X_i(e) = 0$ regardless of the values of $X_1(e), \dots, X_{i-1}(e)$, and we have

$$\mathbf{P}[X_i(e) = 1 | X_1(e), X_2(e), \dots, X_{i-1}(e), \mathcal{S}_{i-1}^c] = 0 \leqslant p_i^*(e).$$

Now assume that $\mathcal{S}_{i-1}$ holds. Then the set G_{i-1} of edges to be avoided while constructing C_i satisfies $\Delta_2(G_{i-1}) \leqslant \mu n$. By (7.1), C_i will be an ℓ-cycle extending P_i selected uniformly at random from the set $\mathcal{C}_i(G_{i-1})$, which has size at least $\varepsilon n^{\ell-\ell'}$; and this will happen no matter the values of $X_1(e), \dots, X_{i-1}(e)$.

If P_i is of type 2 for e, then we are required to bound a probability by $p_i^*(e) = 1$, which holds trivially. Suppose now that P_i is of type 0 for e, and suppose $P_i = v_1 v_2 \cdots v_{\ell'}$. For $C_i \in \mathcal{C}_i(G_{i-1})$, $C_i \setminus P_i$ is a path of the form $v_{\ell'-1} v_{\ell'} u_1 u_2 \cdots u_{\ell-\ell'} v_1 v_2$. We wish to estimate the number of such paths where $e \in \partial(C_i \setminus P_i)$. Since P_i is of type 0 for e, then $e \in \partial(C_i \setminus P_i)$ can only happen if $e = u_j u_k$ for $|j - k| \leqslant 2$. There are $(\ell - \ell' - 1) - (\ell - \ell' - 2) \leqslant 2\ell$ choices for j, k. Having fixed those, there are two 2 possibilities for assigning e to $\{u_j, u_k\}$, and having fixed those, there are at most n possibilities for each other u_p with $p \notin \{j, k\}$. All together, the number of C_i which extend P_i and such that $e \in \partial(C_i \setminus P_i)$ is certainly at most $4\ell n^{\ell-\ell'-2}$. Thus we have

$$\mathbf{P}[X_i(e) = 1 | X_1(e), X_2(e), \dots, X_{i-1}(e), \mathcal{S}_{i-1}] \leqslant \frac{4\ell n^{\ell'-\ell-2}}{|\mathcal{C}_i(G_{i-1})|} \leqslant \frac{4\ell}{\varepsilon n^2} = \frac{c}{n^2} = p_i^*(e),$$

as required. Finally, if P_i is of type 1 for e, then similar (but simpler) calculations show that $\mathbf{P}[X_i(e) = 1 | X_1(e), X_2(e), \ldots, X_{i-1}(e), \mathcal{S}_{i-1}] \leqslant \frac{6n^{\ell'-\ell-1}}{|\mathcal{C}_i(G_{i-1})|} \leqslant \frac{c}{n} = p_i^*(e)$, and we are done. □

Now, we use that $\mathcal{P}$ is γ-sparse to argue $\sum_{i=1}^t p_i^*(e)$ is suitably small. Indeed, for each $r \in \{0,1,2\}$, let t_r be the number of $i \in \{1,\ldots,t\}$ such that P_i is of type r for e. Since $\mathcal{P}$ is γ-sparse, we have $t_r \leqslant \gamma n^{3-r}$ for each $r \in \{0,1,2\}$. Therefore, we have

$$\sum_{i=1}^{t} p_i^*(e) = t_0 \frac{c}{n^2} + t_1 \frac{c}{n} + t_2 \leqslant \gamma c n + \gamma c n + \gamma n \leqslant \frac{\mu}{30} n, \tag{7.3}$$

where the last inequality follows from the choice of c and $\gamma \ll \mu, \varepsilon$.

We now claim that

$$\mathbf{P}\left[\sum_{i=1}^{t} X_i(e) \geqslant \frac{\mu}{3} n\right] \leqslant \exp\left(-\frac{\mu}{3} n\right). \tag{7.4}$$

Indeed, inequality (7.3) implies that $7\sum_{i=1}^t p_i^*(e) \leqslant \mu n/3$, so the bound follows from Theorem 5.5 combined with Theorem 5.4.

For each $e \in \binom{V(H)}{2}$, let $X_e := \sum_{i=1}^t X_i(e)$. Let $\mathcal{E}$ be the event that $\max_e X_e \leqslant \mu n/3$. By using an union bound over all the (at most n^2) possible choices of e and using (7.4), we deduce that $\mathcal{E}$ holds with probability at least $1 - o(1)$.

Now we can show that $\mathcal{S}_t$ holds with positive probability. We shall prove that $\mathbf{P}[\mathcal{S}_t | \mathcal{E}] = 1$, which then will imply $\mathbf{P}[\mathcal{S}_t] \geqslant \mathbf{P}[\mathcal{S}_t|\mathcal{E}]\mathbf{P}[\mathcal{E}] \geqslant 1 - o(1)$. So assume $\mathcal{E}$ holds, that is, $\max_e X_e \leqslant \mu n/3$. Note that $\mathcal{S}_0$ holds deterministically, and suppose $1 \leqslant i \leqslant t$ is the minimum such that $\mathcal{S}_i$ fails to hold. Since $\mathcal{S}_{i-1}$ holds, using (7.2) we deduce

$$\begin{aligned}\Delta_2(G_i) \leqslant 2 + \Delta_2(G_{i-1}) = 2 + \max_e \deg_{G_{i-1}}(e) &\leqslant 2\left(1 + \max_e \sum_{j=1}^{i-1} X_i(e)\right) \\ &\leqslant 2\left(1 + \max_e X_e\right) \leqslant 2\left(1 + \frac{\mu}{3}n\right) \leqslant \mu n,\end{aligned}$$

where in the second to last inequality we used $\mathcal{E}$, and in the last inequality we used $1/n \ll \mu$. Thus $\mathcal{S}_i$ holds, a contradiction. ■

7.2. **Well-behaved approximate cycle decompositions.** In this section we show the existence of approximate cycle decomposition which are 'well-behaved', meaning that the subgraph left by the uncovered edges has small codegree. The argument is different depending on the two setting considered by Theorem 1.1, and we start with the former.

When ℓ is divisible by 3, the tight cycle C_ℓ is 3-partite. By a well-known theorem from Erdős [7, Theorem 1], we know that the Turán number of C_ℓ is degenerate, i.e. edge-maximal C_ℓ-free 3-graphs on n vertices have at most $o(n^3)$ edges. This allows us to find an approximate decomposition of any 3-graph H with copies of C_ℓ if ℓ is divisible by 3, simply by removing copies of C_ℓ greedily until $o(n^3)$ edges remain. This argument alone does not provide us with the 'well-behavedness' condition we alluded to earlier, but it is, however, possible to modify such a packing locally to guarantee such a property holds.

Lemma 7.3 (Well-behaved approximate cycle decompositions, version 1)**.** *Let $\varepsilon, \gamma > 0$ and $n, \ell \in \mathbb{N}$ be such that $\ell \geqslant 9$ is divisible by 3 and $1/n \ll \varepsilon, \gamma, 1/\ell$. Let H be a 3-graph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Then H has a C_ℓ-packing $\mathcal{C}$ such that $\Delta_2(H \setminus E(\mathcal{C})) \leqslant \gamma n$.*

Results in a similar spirit were proven in [3]. The proof is not difficult but somewhat long and repetitive, thus we defer it to Appendix A.

Now we consider the second range of ℓ, where it $\ell \geqslant 10^7$, in which we can show the following.

Lemma 7.4 (Well-behaved approximate cycle decomposition, version 2)**.** *Let $\varepsilon, \gamma > 0$ and $n, \ell \in \mathbb{N}$ be such that $\ell \geqslant 10^7$ and $1/n \ll \varepsilon, \gamma, 1/\ell$. Let H be a 3-graph on n vertices with $\delta_2(H) \geqslant (2/3+\varepsilon)n$. Then H has a C_ℓ-packing $\mathcal{C}$ such that $\Delta_2(H \smallsetminus E(\mathcal{C})) \leqslant \gamma n$.*

In this range we exploit the connection of *fractional graph decompositions* with their integral counterparts. Given a 3-graph H, let $\mathcal{C}_\ell(H)$ be the family of all ℓ-cycles in H, and given $X \in E(H)$ let $\mathcal{C}_\ell(H, X) \subseteq \mathcal{C}_\ell(H)$ be those cycles which use the edge X. A *fractional C_ℓ-decomposition* of a 3-graph H is a function $\omega : \mathcal{C}_\ell(H) \to [0,1]$ such that for every edge $X \in H$ we have $\sum_{C \in \mathcal{C}_\ell(H,X)} \omega(C) = 1$. Joos and Kühn [15] proved the existence of fractional C^k_ℓ-decompositions under general conditions. We state their results only in the particular case $k = 3$. A 3-graph H on n vertices is *(α, ℓ)-connected* if for every two ordered edges (s_1, s_2, s_3), $(t_1, t_2, t_3) \in V(H)^3$, there are at least $\alpha n^{\ell-1}/(3!|E(H)|)$ walks with ℓ edges starting at (s_1, s_2, s_3), ending at (t_1, t_2, t_3).

Theorem 7.5 (Joos and Kühn [15])**.** *For all $\alpha \in (0,1)$, $\mu \in (0, 1/3)$ and $\ell \geqslant 2$, there is n_0 such that the following holds for all $n \geqslant n_0$. Suppose H is an (α, ℓ_0)-connected 3-graph on n vertices with $540\frac{\ell_0}{\alpha} \log \frac{\ell_0}{\alpha} \log \frac{1}{\mu} \leqslant \ell$. Then there is a fractional C_ℓ-decomposition ω of H with*

$$(1-\mu)\frac{2|E(H)|}{\Delta(H)^\ell} \leqslant \omega(C) \leqslant (1+\mu)\frac{2|E(H)|}{\delta(H)^\ell}$$

for all ℓ-cycles C in H.

To use this theorem, we show that 3-graphs with $\delta_2(H) \geqslant 2n/3$ are (α, ℓ_0)-connected for some suitable α, ℓ_0. The following argument is due to Reiher [15, Lemma 2.3]. We include it for completeness and since for $k = 3$ one can give a better value of α, which in turn increases the range of ℓ in which one can apply Theorem 7.5.

Lemma 7.6. *For each $d \geqslant 1/2$, every 3-graph H on n vertices and such that $\delta_2(H) \geqslant (d+o(1))n$ is $(d^2(2d-1)^4, 8)$-connected.*

Proof. Let $V = V(H)$ and $(s_1, s_2, s_3), (t_1, t_2, t_3) \in V^3$ be two arbitrary ordered edges of H. For $z \in V(H)$, let the function $I_z : V^2 \to \{0,1\}$ be such that $I_z(x_1, x_2) = 1$ if and only if $s_2s_3x_1x_2t_1t_2$ is a path in the link-graph of z in H. Let $N = N_H(s_2s_3) \cap N_H(t_1t_2)$ and note that $|N| > (2d-1)n$. Note that if $z_1, z_2 \in N$ (possibly equal) and $(x_1, x_2) \in V^2$ are such that $I_{z_1}(x_1, x_2) = I_{z_2}(x_1, x_2) = 1$, then $s_1s_2s_3z_1x_1x_2z_2t_1t_2t_3$ is a walk from (s_1, s_2, s_3) to (t_1, t_2, t_3) using 8 edges, call such walks *standard*.

First, note that having fixed $z \in N$, the number of $(x_1, x_2) \in V^2$ such that $I_z(x_1, x_2) = 1$ can be bounded as follows: choose $x_1 \in N_H(s_3z)$ arbitrarily (there are at least dn choices) and then $x_2 \in N_H(zx_1) \cap N_H(zt_1)$ (of which there are at least $(2d-1)n$ choices). Thus we have $\sum_{(x_1,x_2) \in V^2} I_z(x_1, x_2) \geqslant d(2d-1)n^2$ for all $z \in N$.

On the other hand, note that for a fixed (x_1, x_2) with $x_1 \neq x_2$, the number of standard walks which use (x_1, x_2) is exactly $(\sum_{z\in N} I_z(x_1, x_2))^2$. Thus the number of standard walks is at least (using Jensen's inequality in the first inequality, and $|N| \geqslant (2d-1)n$ in the third inequality)

$$\begin{aligned}\sum_{(x_1,x_2)\in V^2} \left(\sum_{z\in N} I_z(x_1,x_2)\right)^2 &\geqslant n^2 \left(\frac{1}{n^2}\sum_{z\in N}\sum_{(x_1,x_2)\in V^2} I_z(x)\right)^2 \\ &\geqslant n^2\left(\frac{1}{n^2}\sum_{z\in N} d(2d-1)n^2\right)^2 \geqslant d^2(2d-1)^4n^4,\end{aligned}$$

as required. ■

To prove Lemma 7.4, we combine the fractional matching of Theorem 7.5 with a nibble-type matching argument. We use a result of Alon and Yuster [1] (but see also Kahn [16] and Ehard, Glock and Joos [6] for variations and extensions).

Proof of Lemma 7.4. Let $\alpha = 4 \times 3^{-6}$ (as in Lemma 7.6 for $d = 2/3$) and $\ell_0 = 8$. By Lemma 7.6, H is (α, ℓ_0)-connected. A numerical calculation shows that we can fix $\mu \in (0, 1/3)$ such that $540\frac{\ell_0}{\alpha}\log\frac{\ell_0}{\alpha}\log\frac{1}{\mu} \leqslant 10^7 \leqslant \ell$. Thus Theorem 7.5 informs us that there exists a fractional C_ℓ-decomposition ω of H with

$$\omega(C) \leqslant (1+\mu)\frac{2|E(H)|}{\delta_2(H)^\ell} \leqslant 4\frac{|E(H)|}{\delta_2(H)^\ell} \leqslant \frac{4n^3}{\delta_2(H)^\ell} \leqslant \frac{4 \times 3^\ell}{n^{\ell-3}}$$

for all $C \in \mathcal{C}_\ell(H)$.

Consider the auxiliary ℓ-uniform hypergraph F with vertex set $E(H)$, and an edge for each cycle in $\mathcal{C}_\ell(H)$ corresponding to its set of ℓ edges. Define a random subgraph $F' \subseteq F$ by keeping each edge C with probability $p_C := n^{1/2}\omega(C)$. By the bounds on $\omega(C)$ and $1/n \ll 1/\ell$ we have $p_C \leqslant 1$ for all $C \in \mathcal{C}_\ell(H)$. For each edge $e \in E(H)$, we have $\mathbf{E}[\deg_{F'}(e)] = n^{1/2}\sum_{C \in \mathcal{C}_\ell(H,e)}\omega(C) = n^{1/2}$. Two distinct edges $e, f \in E(H)$ can participate together in at most $O(n^{\ell-4})$ ℓ-cycles in H, thus we have $\mathbf{E}[\deg_{F'}(e,f)] = O(n^{-1/2})$. Standard concentration inequalities (Theorem 5.4(i) and (iii)), imply that with very high probability F' satisfies $\deg_{F'}(e) = (1+o(1))n^{1/2}$ for each $e \in V(F')$, and thus $\delta_1(F') \geqslant (1-o(1))\Delta_1(F')$; and moreover $\Delta_2(F') = o(n^{1/2})$.

For each 2-set uv of vertices of H, let $H_{uv} \subseteq V(F)$ correspond to the edges in H which contain uv. There are at most n^2 such sets and each has size at least $2n/3$. Thus, the Alon–Yuster theorem [1] implies the existence of a matching M in F' such that at most γn vertices in $V(F')$ are uncovered in each H_{uv}. The matching M in $F' \subseteq F$ translates to a C_ℓ-packing $\mathcal{C}$ in H, and the latter condition implies $\Delta_2(H \setminus E(\mathcal{C})) \leqslant \gamma n$, as desired. ■

7.3. **Proof of the Cover-Down Lemma.** As a final tool, we borrow the following theorem of Thomassen [21] about path-decompositions of graphs.

Theorem 7.7 ([21]). *Any 171-edge-connected graph G such that $|E(G)|$ is divisible by 3 has a P_3-decomposition.*

Proof of Lemma 4.3. Let $\gamma_1, p_1, p_2 > 0$ such that $\gamma_1 \ll p_1 \ll p_2 \ll \mu, \varepsilon$. For $i \in \{0,1,2,3\}$, say an edge e of H is of *type* i if $|e \cap U| = i$, and let $H_i \subseteq H$ be the edges of H which are of type i. For $i \in \{1,2\}$, let $R_i \subseteq H_i$ be defined by choosing edges independently at random from H_i with probability $3p_i/2$. By assumption, $\delta_2^{(3)}(H;U) \geqslant 3\varepsilon|U|$ (see definition at the beginning of Section 5.1).

By Theorem 5.4 we get that, for $i \in \{1,2\}$, with non-zero probability, that

$$\Delta_2(R_i) \leqslant 2p_i n, \tag{7.5}$$

$$\delta_2^{(3)}(R_1 \cup R_2 \cup H[U]; U) \geqslant 2\varepsilon p_1|U|, \text{ and} \tag{7.6}$$

$$\delta_2^{(3)}(R_2 \cup H[U]; U, G) \geqslant 2\varepsilon p_2|U|, \tag{7.7}$$

where $G \subseteq \binom{V(H)}{2}$ corresponds to the pairs e such that $e \cap U \neq \varnothing$. From now on we assume R_1, R_2 are fixed with those properties.

Let $H' = H - H[U] - R_1 - R_2$. Recall that, by assumption, $\delta_2(H) \geqslant (2/3+2\varepsilon)n$ and $|U| = \lfloor \varepsilon n \rfloor$. By our choice of $p_1, p_2 \ll \varepsilon, \mu$ and (7.5), we deduce that $\delta_2(H') \geqslant (2/3+\varepsilon/2)n$.

We consider two possible cases depending on the value of ℓ. If $\ell \geqslant 9$ is divisible by 3, then we apply Lemma 7.3, otherwise by assumption $\ell \geqslant 10^7$, and we can apply Lemma 7.4. In any case, the output is a C_ℓ-packing $\mathcal{C}$ in H' such that $\Delta_2(H' \setminus E(\mathcal{C})) \leqslant \gamma_1 n$. Let $J = H' \setminus E(\mathcal{C})$ be the edges in H' not covered by $\mathcal{C}$, and for each $i \in \{0,1,2\}$ let J_i be the edges of type i in J. We shall cover the edges in J with cycles of length ℓ and for that we will proceed in three steps, covering the edges of J_0, J_1, and J_2 in order.

Consider each edge in J_0 as a path on three vertices $v_1v_2v_3$, assigning to each edge an arbitrary order. Let $\mathcal{P}_0$ be the collection of those paths. The inequalities $\Delta_2(J_0) \leqslant \Delta_2(J) \leqslant \gamma_1 n$ show that

$\mathcal{P}_0$ is γ_1-sparse. Let $\mu_1, \varepsilon_1 > 0$ satisfy $\gamma_1 \ll \mu_1 \ll \varepsilon_1 \ll p_1, \varepsilon$. Equation (7.6) and Corollary 5.3 imply that each $P \in \mathcal{P}_0$ can be extended to at least $2\varepsilon_1 n^{\ell-3}$ cycles C, such that $C \smallsetminus P \subseteq R_1 \cup R_2 \cup H[U]$ and $V(C) \smallsetminus V(P) \subseteq U$. Then an application of Lemma 7.1 with $\varepsilon_1, \mu_1, 3, J_0, R_1 \cup R_2 \cup H[U], \mathcal{P}_0$ in place of $\varepsilon, \mu, \ell', H_1, H_2, \mathcal{P}$ respectively, implies that there is a C_ℓ-decomposable subgraph F_0 such that $F_0 \supseteq J_0$, and

$$\Delta_2(F_0 \smallsetminus J_0) \leqslant \mu_1 n. \tag{7.8}$$

By construction, F_0 is edge-disjoint with the cycles in $\mathcal{C}$, and then $F_0' = E(\mathcal{C}) \cup F_0$ is C_ℓ-descomposable. Note that all edges not covered by F_0' lie in $(J_1 \cup J_2) \cup (R_1 \cup R_2) \cup H[U]$.

Let $J_1' = (J_1 \cup R_1) \smallsetminus F_0'$ and $R_2' = (R_2 \cup H[U]) \smallsetminus F_0'$. Let $\gamma_2, \mu_2, \varepsilon_2 > 0$ be such that $p_1 \ll \gamma_2 \ll \mu_2 \ll \varepsilon_2 \ll p_2, \varepsilon$. Since $J_1' \subseteq J_1 \cup R_1 \subseteq J \cup R_1$, we have

$$\Delta_2(J_1') \leqslant \Delta_2(J) + \Delta_2(R_1) \leqslant \gamma n + 2p_1 n \leqslant \gamma_2 n.$$

Since each edge in J_1' is of type 1 in H, we can consider each edge in J_1' as a path $P = v_1v_2v_3$ where $v_2 \in U$ and $v_1, v_3 \notin U$; and let $\mathcal{P}_1$ be the collection of those paths. Then $\Delta_2(J_1') \leqslant \gamma_2 n$ implies $\mathcal{P}_1$ is γ_2-sparse. By (7.7) and (7.8), together with Corollary 5.3, we deduce that each $P \in \mathcal{P}_1$ can be extended to at least $2\varepsilon_2 n^{\ell-3}$ cycles C, such that $C \smallsetminus P \subseteq R_2'$ and $V(C) \smallsetminus V(P) \subseteq U$. Apply Lemma 7.1 with $\varepsilon_2, \mu_2, \gamma_2, 3, J_1', R_2', \mathcal{P}_1$ in place of $\varepsilon, \mu, \gamma, \ell', H_1, H_2, \mathcal{P}$ to obtain a C_ℓ-decomposable subgraph F_1 such that $F_1 \supseteq J_1'$, and

$$\Delta_2(F_1 \smallsetminus J_1) \leqslant \mu_2 n. \tag{7.9}$$

By construction, F_1 and F_0' are edge-disjoint, and then $F_1' = F_1 \cup F_0'$ is C_ℓ-decomposable. Note that the edges not covered by F_1' lie in $J_2 \cup R_2 \cup H[U]$.

Let $J_2' = (J_2 \cup R_2) \smallsetminus F_1'$. Note that each edge in J_2' is of type 2. For each $v \in V(H) \smallsetminus U$, let $G_v = J_2'(v, U)$, that is, G_v is the link graph of v in J_2' restricted to U. Fix $v \in V(H) \smallsetminus U$. Given $x, y \in U$, the equations (7.7) and (7.9) imply that x and y have at least $2\varepsilon p_2 |U| - 2\mu_2 n \geqslant 171$ common neighbours in G_v, so G_v is 171-edge-connected. Since $v \notin U$, our assumption on H implies that the number of edges of $H(v)$ is divisible by 3. Note that G_v is exactly the link-graph over $H \smallsetminus F_1'$ when restricted to U. Therefore, and since F_1' is C_ℓ-decomposable, the number of edges in G_v is divisible by 3 as well.

By Theorem 7.7, G_v has a decomposition into paths $\mathcal{P}_v' = \{P_1, \dots, P_t\}$, each of length 3. Observe that these paths yields to a collection of (3-uniform) paths in J_2' by substituting each path $P_i = w_1w_2w_3w_4$ in $\mathcal{P}_v'$ by the tight path $w_1w_2vw_3w_4$. Let $\mathcal{P}_v$ be the collection of paths obtained in this way. Observe that for $u \neq v$ in $V(H) \smallsetminus U$, $\mathcal{P}_v$ and $\mathcal{P}_u$ are edge-disjoint. Let $\mathcal{P}_2 = \bigcup_{v \in V(H) \smallsetminus U} \mathcal{P}_v$. Note that $\mathcal{P}_2$ decomposes J_2' into paths on five vertices.

Let $\gamma_3, \varepsilon_3 > 0$ be such that $p_2 \ll \gamma_3 \ll \varepsilon_3 \ll \mu_3 \ll \mu, \varepsilon$. Recall that $|U| = \lfloor \varepsilon n \rfloor$. Since $J_2' \subseteq J_2 \cup R_2 \subseteq F \cup R_2$, we have $\Delta_2(J_2') \leqslant \Delta_2(R_2) + \Delta_2(J) \leqslant 2p_2 n + \gamma_1 n \leqslant \gamma_3 n$, so $\mathcal{P}_2$ is γ_3-sparse. Let $H_2' = H[U] \smallsetminus F_1'$. We have $F_1'[U] = F_1[U] \cup F_0[U]$. By (7.8)–(7.9), we have $\delta_2(H_2') \geqslant \delta_2(H[U]) - 2\mu_2 n \geqslant (2/3 + \varepsilon/2)|U|$. By Corollary 5.3, we deduce each $P \in \mathcal{P}_2$ can be extended to at least $2\varepsilon_2 n^{\ell-5}$ cycles C such that $C \smallsetminus P \subseteq H_2'$. Thus we can apply Lemma 7.1 with $\varepsilon_3, \mu_3, \gamma_3, 5, J_2', H_2', \mathcal{P}_2$ playing the rôles of $\varepsilon, \mu, \gamma_3, \ell', H_1, H_2, \mathcal{P}$ respectively, to obtain a C_ℓ-decomposable subgraph F_2 such that $F_2 \supseteq J_2'$, and

$$\Delta_2(F_2 \cap H_2') \leqslant \mu_3 n. \tag{7.10}$$

By construction, F_2 and F_1' are edge-disjoint, and then $F = F_1' \cup F_2$ is C_ℓ-decomposable. Moreover, all edges not contained in U are covered by F. In fact, we have that

$$H - H[U] = E(\mathcal{C}) \cup J_0 \cup (J_1 \cup R_1) \cup (J_2 \cup R_2) \subseteq E(\mathcal{C}) \cup F_0 \cup F_1 \cup F_2 = F.$$

Finally, inequalities (7.8)–(7.10) yield that $\Delta_2(F[U]) \leqslant \mu n$, as required. ■

§8. Absorbing Lemma

In this section we prove Lemma 4.2. We need to show that, given a sufficiently large H with $\delta_2(H) \geqslant (2/3+\varepsilon)n$ and a subgraph $R \subseteq H$ on at most m vertices, there is an C_ℓ-absorber A for R on at most $O(m^9\ell^9)$ vertices. We divide the proof in two main parts.

First, in Section 8.1 we shall find a bounded-size hypergraph $A_1 \subseteq H$, edge-disjoint from R, which admits a C_ℓ-decomposition. This subgraph will be chosen such that $R \cup A_1$ contains a *tour decomposition*, that is, a decomposition in which all subgraphs are tours (see Lemma 8.1). The second step is to transform the found tour decomposition in the remainder to a C_ℓ-decomposition (see details in Section 8.2). Finally, in Section 8.3 we combine both steps to prove Lemma 4.2.

8.1. **Tour decomposition.**

The main goal of this subsection is to prove the following lemma.

Lemma 8.1. *Let $\ell \geqslant 7$, $\varepsilon > 0$, and $n, m \in \mathbb{N}$ be such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be a 3-graph on n vertices with $\delta_2(H) \geqslant (2/3+\varepsilon)n$. Let $R \subseteq H$ be C_ℓ-divisible on at most m vertices. There exists a subgraph $A_1 \subseteq H$, edge-disjoint with R, such that*

(i) *A_1 has at most $30\binom{m}{3}\ell(6\ell+1)$ edges,*
(ii) *$A_1 \cup R$ spans at most $30\binom{m}{3}\ell(6\ell+1)$ vertices.*
(iii) *A_1 has a C_ℓ-decomposition, and*
(iv) *$A_1 \cup R$ has a tour decomposition,*

8.1.1. *Tour-trail decompositions.* We consider decompositions $\mathcal{T} = \{C_1, \dots, C_t, P_1, \dots, P_k\}$ in which C_i is a tour for every $i \in [t]$ and P_j is a trail for every $j \in [k]$. In this case we say $\mathcal{T}$ is a *tour-trail decomposition*. Note that every 3-graph has a tour-trail decomposition, since we can consider every single edge in a 3-graph as a trail on three vertices (by giving it an arbitrary ordering).

For a trail $P = u_1u_2\cdots u_{k-1}u_k$ we say that the ordered pairs (u_2, u_1) and (u_{k-1}, u_k) are the *ends* of P. We denote the those pairs as $\mathrm{ends}(P)$. Observe that the set of ends of a P depends on the edge-set of P only, i.e. is independent of order in which we transverse the trail. We remark that the ends differ from the start and terminus of P (as defined in Section 1.5) since they have different orderings.

Given H and a tour-trail decomposition $\mathcal{T} = \{C_1, C_2, \dots, C_t, P_1, P_2, \dots, P_k\}$ of some $R \subseteq H$, we define the *residual digraph of* $\mathcal{T}$, denoted as $D(\mathcal{T})$, as the multidigraph on the same vertex set as H, where the arcs correspond to the union of the ordered ends of each trail of $\mathcal{T}$, considered with repetitions. Thus $D(\mathcal{T})$ has exactly $2t$ arcs, counted with multiplicities, if and only if $\mathcal{T}$ has t trails. For a given pair of vertices $u, v \in V$ we denote the multiplicity of the pair (u, v) in $D(\mathcal{T})$ as $\mu_{\mathcal{T}}(u, v)$. Outdegrees and indegrees of a vertex x in $D(\mathcal{T})$ are denoted by $d^+{}_{D(\mathcal{T})}(x), d^-{}_{D(\mathcal{T})}(x)$ respectively, omitting subscripts from the notation if the underlying digraph is clear from context.

Remark 8.2. Observe that if $(x, y), (y, x) \in E(D_{\mathcal{T}})$ then, there are two trails P_i and P_j in $\mathcal{T}$ that can be merged into a trail (if $i \neq j$) or tour (if $i = j$) which contains all the edges contained in P_i and P_j. Thus there is another tour-trail decomposition $\mathcal{T}'$ of R with less trails than $\mathcal{T}$, obtained from $\mathcal{T}$ by removing P_i, P_j and adding the tour or trail born from joining P_1 and P_2.

We construct A_1 in Lemma 8.1 as follows. We begin with an arbitrary tour-trail decomposition $\mathcal{T}_0$ of R and we will find an increasing sequence of subgraphs $\varnothing = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_k \subseteq H$. Each $T_i \setminus T_{i-1}$ will be sufficiently small, C_ℓ-decomposable and edge-disjoint from T_{i-1}. Moreover, each $T_i \setminus T_{i-1}$ will be a 'gadget' 3-graph of a prescribed family, which is designed to modify T_{i-1} locally. More precisely, for each $i > 0$, each $T_i \cup R$ will contain a tour-trail decomposition $\mathcal{T}_i$, obtained from the tour-trail decomposition $\mathcal{T}_{i-1}$ of $T_{i-1} \cup R$, and the gadget $T_i \setminus T_{i-1}$ will be chosen carefully so the residual digraph is slightly modified and becomes 'simpler'. At the end, we will have found T_k and a tour-trail decomposition $\mathcal{T}_k$ of $R \cup T_k$ which has an empty residual digraph. Thus $\mathcal{T}_k$ is actually a tour decomposition, and we finish by setting $A_1 = T_k$.

The following lemma establishes a crucial property of residual digraphs.

Lemma 8.3. *Let $H = (V, E)$ be a 3-vertex-divisible hypergraph and let $\mathcal{T}$ be a tour-trail decomposition of H with residual digraph $D(\mathcal{T})$. For every $x \in V$ we have that*

$$d^+(x) \equiv d^-(x) \pmod 3.$$

Proof. For every vertex $x \in V(H)$, we need to show that $d^+(x) - d^-(x) \equiv 0 \bmod 3$ in the digraph $D(\mathcal{T})$. Consider the auxiliary digraph $F(\mathcal{T})$ obtained as follows: for every trail or tour $P = w_1 w_2 \cdots w_\ell$ in $\mathcal{T}$, to $F(\mathcal{T})$ add the arcs (w_i, w_{i+1}) and (w_{i+2}, w_{i+1}) for every $1 \leqslant i \leqslant \ell - 2$ (and for tours, add $(w_{\ell-1}, w_\ell), (w_1, w_\ell), (w_\ell, w_1), (w_2, w_1)$ as well), including all repetitions. In such a way (and since $\mathcal{T}$ is a decomposition) every edge of H contributes with exactly two arcs to $F(\mathcal{T})$. It is straightforward to check $D(\mathcal{T}) \subseteq F(\mathcal{T})$ and, crucially, that

$$d^+{}_{D(\mathcal{T})}(x) - d^-{}_{D(\mathcal{T})}(x) = d^+{}_{F(\mathcal{T})}(x) - d^-{}_{F(\mathcal{T})}(x),$$

so from now on we work with $F(\mathcal{T})$ only.

Let $x \in V(H)$. Each edge xyz in H contributes with two arcs to $F(\mathcal{T})$, which can be of type $\{(x, y), (x, z)\}, \{(y, x), (y, z)\}$, or $\{(z, x), (z, y)\}$. The edges of the first type contribute with 2 to $d^+(x) - d^-(x)$ in $F(\mathcal{T})$. The edges of second and third type contribute with -1 to $d^+(x) - d^-(x)$ in $F(\mathcal{T})$, which is congruent to $2 \bmod 3$. Thus we deduce $d^+(x) - d^-(x) \equiv 2|\deg_H(x)| \bmod 3$. Since H is 3-vertex-divisible, this is congruent to $0 \bmod 3$, and we are done. ∎

8.1.2. *Gadgets.* In the following three lemmata we describe the aforementioned gadgets, and their main properties.

First, for a given tour-trail decomposition $\mathcal{T}$ of $R \subseteq H$ and three distinct vertices v_1, v_2, v_3, the following lemma states that there is a subgraph $S_3 = S_3(v_1, v_2, v_3) \subseteq H$ edge-disjoint with R and which contains a C_ℓ-decomposition. Moreover, there is a tour-trail decomposition of $R \cup S_3$ such that its residual digraph is exactly $D(\mathcal{T})$ with the additional arcs (v_1, v_2), (v_2, v_3), and twice the arc (v_1, v_3). We define the multidigraph $\vec{S}_3(v_1, v_2, v_3) = \{(v_1, v_3), (v_1, v_3), (v_1, v_2), (v_2, v_3)\}$.

For two multidigraphs D_1, D_2, we set the notation $D_1 \sqcup D_2$ to mean the multigraph on $V(D_1) \cup V(D_2)$ obtained by adding all the arcs of D_2 to D_1, considering the multiplicities.

Lemma 8.4. *Let $\ell \geqslant 7$, $\varepsilon > 0$ and $n, m \in \mathbb{N}$ be such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be a 3-graph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Given three distinct vertices $v_1, v_2, v_3 \in V(H)$, $R \subseteq H$ on at most m vertices, and a tour-trail decomposition $\mathcal{T}$ of R the following holds. There is a subgraph $S_3 = S_3(v_1, v_2, v_3) \subseteq H$, edge-disjoint from R, and a tour-trail decomposition $\mathcal{T}_{S_3} = \mathcal{T}_{S_3}(\mathcal{T}, v_1, v_2, v_3)$ of $R \cup S_3$ such that*

- (i $_{S_3}$) *S_3 contains at most 2ℓ edges and $S_3 \cup R$ spans at most $m + 2\ell - 3$ vertices,*
- (ii $_{S_3}$) *S_3 has a C_ℓ-decomposition, and*
- (iii $_{S_3}$) *$D(\mathcal{T}_{S_3}) = D(\mathcal{T}) \sqcup \vec{S}_3(v_1, v_2, v_3)$.*

Proof. The minimum codegree condition on H implies that there is a vertex $x \in V(H)$ that lies in $N(v_1v_2) \cap N(v_1v_3) \cap N(v_2v_3)$. Considering the paths v_1v_3x and $v_3xv_2v_1$, two applications of Lemma 5.1 yield the existence of two edge-disjoint cycles C_1 and C_2 of length ℓ, edge-disjoint with R, and such that $v_1v_3x \in E(C_1)$ and $v_3xv_2, xv_2v_1 \subseteq E(C_2)$ (transversing the vertices in that order). Then $S_3 = C_1 \cup C_2$, clearly satisfies (i $_{S_3}$) and (ii $_{S_3}$). Hence, we only need to prove the existence of a tour-trail decomposition $\mathcal{T}_{S_3}$ of $R \cup S_3$ for which (iii $_{S_3}$) holds.

For this, consider the trail $P_1 = v_3v_2xv_1v_3$. Observe that $E(S_3) \smallsetminus E(P_1)$ consists exactly in the edges of a trail P_2 whose ends are (v_1, v_2) and (v_1, v_3). Indeed, the edges contained in the set $E(C_2) \smallsetminus \{v_3v_2x, v_2xv_1\}$ form a trail between (v_2, v_1) and (v_3, x), that we may merge with the trail with edges in $E(C_1) \smallsetminus \{xv_1v_3\}$ from (v_3, x) to (v_1, v_3). Therefore, $\mathcal{T}_{S_3} = \mathcal{T} \cup \{P_1, P_2\}$ is a tour-trail decomposition of $R \cup S_3$. We deduce (iii $_{S_3}$) by noticing that the ends of P_1 and P_2 are (v_2, v_3) and (v_1, v_3), and (v_1, v_2) and (v_1, v_3) respectively. ∎

The following is our second gadget. It is designed so we can add a small subgraph $C_4 \subseteq H$ to some R, such that $R \cup C_4$ has a tour-trail decomposition in which the residual digraph has an extra directed four-cycle. We use the notation $\vec{C}_4(v_1, v_2, v_3, v_4) = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$.

Lemma 8.5. *Let $\ell \geqslant 7$, $\varepsilon > 0$ and $n, m \in \mathbb{N}$ such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be a 3-graph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Given four distinct vertices $v_1, v_2, v_3, v_4 \in V(H)$, a subgraph $R \subseteq H$ on at most m vertices, and a tour-trail decomposition $\mathcal{T}$ of R the following holds. There is a subgraph $C_4 = C_4(v_1, v_2, v_3, v_4) \subseteq H$, edge-disjoint from R and a tour-trail decomposition $\mathcal{T}_{C_4} = \mathcal{T}_{C_4}(\mathcal{T}, v_1, v_2, v_3, v_4)$ of $R \cup C_4$ such that*

(i $_{C_4}$) *C_4 has at most 8ℓ edges and $C_4 \cup R$ spans at most $m + 4\ell - 6$ vertices,*
(ii $_{C_4}$) *C_4 has a C_ℓ-decomposition, and*
(iii $_{C_4}$) *$D(\mathcal{T}_{C_4}) = D(\mathcal{T}) \sqcup \vec{C}_4(v_1, v_2, v_3, v_4)$.*

Proof. Two consecutive applications of Lemma 8.4 yield the existence of edge-disjoint subgraphs $S_3(v_1, v_2, v_3)$ and $S_3(v_3, v_1, v_4)$. More precisely, first we apply Lemma 8.4 to obtain $S_3(v_1, v_2, v_3)$ edge-disjoint from R. Then, we apply it again with $R \cup S_3(v_1, v_2, v_3)$ in place of R to obtain $S_3(v_3, v_2, v_4)$ edge disjoint from $R \cup S_3(v_1, v_2, v_3)$ (here we use $1/n \ll 1/m$, to apply Lemma 8.4 to a larger subgraph with at most $m + 2\ell - 6$ vertices). It is not difficult to check that the subgraph $C_4 = S_3(v_1, v_2, v_3) \cup S_3(v_3, v_1, v_4)$ satisfies (i $_{C_4}$) and (ii $_{C_4}$)

Moreover, in the second application of Lemma 8.4 we obtain a tour-trail decomposition $\mathcal{T}'$ of $R \cup C_4$ equal to $\mathcal{T}' = \mathcal{T}_{S_3}\big(\mathcal{T}_{S_3}(\mathcal{T}, v_1, v_2, v_3), v_3, v_1, v_4\big)$, whose residual digraph is given by

$$D(\mathcal{T}') = D(\mathcal{T}) \sqcup \vec{S}_3(v_1, v_2, v_3) \sqcup \vec{S}_3(v_3, v_4, v_1).$$

Observe that $D(\mathcal{T}')$ contains both the arcs (v_1, v_3) and (v_3, v_1) twice. By Remark 8.2, we can obtain a tour-trail decomposition $\mathcal{T}_{C_4}$ which satisfies (iii $_{C_4}$). ∎

Our third and final gadget will add the arcs of two vertex-disjoint oriented triangles to the residual digraph. Set the notation $\vec{T}_3(v_1, v_2, v_3) = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$ for the oriented triangle on vertices v_1, v_2, v_3. Given six distinct vertices $v_1, v_2, v_3, v_4, v_5, v_6$, as a final result we wish for a residual digraph consisting of the two oriented triangles $\vec{T}_3(v_1, v_2, v_3)$ and $\vec{T}_3(v_4, v_5, v_6)$.

This can be done using the oriented 4-cycles of Lemma 8.5 three times, by considering the oriented 4-cycles $\vec{C}_4(v_1, v_2, v_5, v_6)$, $\vec{C}_4(v_2, v_3, v_4, v_5)$, and $\vec{C}_4(v_1, v_6, v_4, v_3)$. This can be thought geometrically, as the oriented 4-cycles forming the faces of a triangular prism, whose bases lie in the desired triangles. The arcs between the vertices of the two triangles will go in opposite directions, and therefore we will be able to "cancel" them.

To have an analogous notation as for the other two gadgets, set

$$\vec{P}_6(v_1, v_2, v_3, v_4, v_5, v_6) = \vec{T}_3(v_1, v_2, v_3) \sqcup \vec{T}_3(v_4, v_5, v_6).$$

Lemma 8.6. *Let $\ell \geqslant 7$, $\varepsilon > 0$ and $n, m \in \mathbb{N}$ be such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be a 3-graph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Given six distinct vertices $v_1, v_2, v_3, v_4, v_5, v_6 \in V(H)$, $R \subseteq H$ on at most m vertices, and a tour-trail decomposition $\mathcal{T}$ of R the following holds. There exists a subgraph $P_6 = P_6(v_1, v_2, v_3, v_4, v_5, v_6) \subseteq H$, edge-disjoint from R and a tour-trail decomposition $\mathcal{T}_{P_6} = \mathcal{T}_{P_6}(\mathcal{T}, v_1, v_2, v_3, v_4, v_5, v_6)$ of $R \cup P_6$ such that*

(i $_{P_6}$) *P_6 has at most 12ℓ edges and $P_6 \cup R$ spans at most $m + 12\ell - 18$ vertices,*
(ii $_{P_6}$) *P_6 has a C_ℓ-decomposition, and*
(iii $_{P_6}$) *$D(\mathcal{T}_{P_6}) = D(\mathcal{T}) \sqcup \vec{P}_6(v_1, v_2, v_3, v_4, v_5, v_6)$*

Proof. Using $1/n \ll 1/m$ we apply Lemma 8.5 iteratively three times, to obtain three edge-disjoint subgraphs $C_4(v_1, v_2, v_5, v_6)$, $C_4(v_2, v_3, v_4, v_5)$, and $C_4(v_1, v_6, v_4, v_3)$, which are also edge-disjoint from R. It is straightforward to check that $P_6 = C_4(v_1, v_2, v_5, v_6) \cup C_4(v_2, v_3, v_4, v_5) \cup C_4(v_1, v_6, v_4, v_3)$ satisfies (i $_{P_6}$) and (ii $_{P_6}$).

The last application of Lemma 8.5 yields a tour-trail decomposition $\mathcal{T}'$ of $R \cup P_6$ with residual digraph given by

$$D(\mathcal{T}') = D(\mathcal{T}) \sqcup \vec{C}_4(v_1, v_2, v_5, v_6) \sqcup \vec{C}_4(v_2, v_3, v_4, v_5) \sqcup \vec{C}_4(v_1, v_6, v_4, v_3).$$

$D(\mathcal{T}')$ contains the arcs (v_1, v_6), (v_6, v_1), (v_2, v_5), (v_5, v_2), (v_3, v_4), and (v_4, v_3), and by Remark 8.2 we can remove them to obtain a tour-trail $\mathcal{T}_{P_6}$ which satisfies (iii $_{P_6}$). ■

8.1.3. *The sea of triangles.* In what follows, we will use the previous gadgets to find, for any given $R \subseteq H$, an edge-disjoint small C_ℓ-decomposable $T \subseteq H$, the main property being that $R \cup T$ contains a tour-trail decomposition with residual digraph consisting only of vertex-disjoint oriented triangles.

The following definitions will be useful for this propose. Given a multidigraph $D = (V, E)$, a *triangle lake* $T \subseteq D$ is an induced subdigraph with vertices in $V' \subseteq V$ that consists only of vertex-disjoint (simple) oriented triangles and such that there is no arc between V' and $V \setminus V'$ or vice versa. Any D contains a unique vertex-maximal triangle lake (possibly empty), we call such subdigraph the *sea of triangles of D* and we denote it by $\vec{\triangle}(D)$. If $D = \vec{\triangle}(D)$ we say D is itself a sea of triangles. Given two directed digraphs D_1 and D_2 on the same vertex set, we establish the notation $D_1 - D_2$ to mean the multigraph resulting from subtracting the edges of D_2 from D_1 counting the multiplicities.

As for hypergraphs, we do not distinguish between the directed multigraph $D = (V, E)$ and the set of arcs E.

Lemma 8.7. *Let $\ell \geqslant 7$, and $\varepsilon > 0$ and $n, m \in \mathbb{N}$ be such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be a 3-graph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Let $R \subseteq H$ be C_ℓ-divisible on at most m vertices. There exists a subgraph $T \subseteq H$, edge-disjoint from R, such that*

- (i $_{8.7}$) *T has at most $30\binom{m}{3}\ell$ edges,*
- (ii $_{8.7}$) *$T \cup R$ spans at most $30\binom{m}{3}\ell$ vertices,*
- (iii $_{8.7}$) *T has a C_ℓ-decomposition, and*
- (iv $_{8.7}$) *there is a tour-trail decomposition $\mathcal{T}_\triangle$ of $T \cup R$ such that $D(\mathcal{T}_\triangle)$ is a sea of triangles.*

Proof. Set $k = 2\binom{m}{3} + 1$. To find T, we will iteratively find subgraphs $T_i \subseteq H$ for every $0 \leqslant i \leqslant k$ such that each T_i has a C_ℓ-decomposition, is edge-disjoint with respect to R, contains at most $14\ell i$ edges, and such that $T_i \cup R$ spans at most $m + i(14\ell - 38)$ vertices. We will see that the last subgraph T_k satisfies the desired properties. Note that in this case properties (i $_{8.7}$), (ii $_{8.7}$), and (iii $_{8.7}$) would follow directly since $14\ell i$ and $m + i(14\ell - 38)$ are smaller than $30\binom{m}{3}\ell$ for every $i \leqslant k$. Hence, most or our effort is dedicated to ensure (iv $_{8.7}$). To do so, at each step we will define a tour-trail decomposition $\mathcal{T}_i$ of $T_i \cup R$ such that its residual digraph will be almost identical to the one of $\mathcal{T}_{i-1}$ except for a few subtly chosen arcs. Additionally, we will define auxiliary vertex sets X_i of size at least $n/2 - 4i$ such that $(T_i \cup R)[X_i]$ is empty.

Since $1/n \ll 1/m, 1/\ell$ and $T_i \cup R$ spans at most $30\binom{m}{3}\ell$ vertices for every $i \in [k]$, n will be sufficiently large to apply lemmata 8.4–8.6 with $T_i \cup R$ in place of R, and we will do this without further comment.

For $i = 0$, take $T_0 = \varnothing$ and $\mathcal{T}_0$ to be an arbitrary tour-trail decomposition of R (this always exists). Also, let $X_0 \subseteq V(H)$ have size $\lceil n/2 \rceil$ such that $R[X_1]$ is empty, which can be done since $1/n \ll 1/m$. Now, for $0 \leqslant i < k$, given T_i, $\mathcal{T}_i$ and X_i define T_{i+1}, $\mathcal{T}_{i+1}$ and X_{i+1} using the following set of rules:

(I) Suppose there are vertices $a, b \in V$ such that $(a, b), (b, a) \in D(\mathcal{T}_i)$. In this case just set $T_{i+1} = T_i$ and $X_{i+1} = X_i$, and let $\mathcal{T}_{i+1}$ be a tour-trail decomposition such that

$$D(\mathcal{T}_{i+1}) = D(\mathcal{T}_i) - \{(a, b), (b, a)\}, \tag{8.1}$$

which exists by Remark 8.2.

(II) Suppose that (I) does not hold and $D(\mathcal{T}_i)$ contains an arc with multiplicity more than one, i.e. there are vertices $a, b \in V(H)$ with $\mu_{\mathcal{T}_i}(a,b) > 1$. Take $x \in X_i$ and apply Lemma 8.4 to $R \cup T_i$ on the vertices b, x, a to obtain the subgraph $S_3(b,x,a) \subseteq H$ and the tour-trail decomposition $\mathcal{T}_i' = \mathcal{T}_{S_2}(\mathcal{T}_i, b, x, a)$. Further, take new vertices $y, z, w \in X_i \setminus \{x\}$ and apply Lemma 8.6 on $T_i \cup R \cup S_3(b,x,a)$ to obtain $P_6(a,x,b,y,z,w)$ and a tour-trail decomposition $\mathcal{T}_i'' = \mathcal{T}_{P_6}(\mathcal{T}_i', a, x, b, y, z, w)$. Set

$$\begin{aligned} T_{i+1} &= T_i \cup S_3(b,x,a) \cup P_6(a,x,b,y,z,w), \text{ and} \\ X_{i+1} &= X_i \setminus \{x,y,z,w\}, \end{aligned}$$

and observe that (i $_{S_3}$) and (i $_{P_6}$) in Lemmata 8.4 and 8.6, we have that T_{i+1} has at most $14\ell i + 2\ell + 12\ell = 14\ell(i+1)$ edges and the subgraph $T_{i+1} \cup R$ spans at most

$$m + i(14\ell - 21) + 14\ell - 21 = m + (i+1)(14\ell - 21)$$

vertices. Moreover, since T_{i+1} is edge-disjoint union of subgraphs that contain C_ℓ-decomposition it also contains one. Additionally $|X_{i+1}| = |X_i| - 4$.

Observe that the resulting tour-trail decomposition $\mathcal{T}_i''$ has a residual digraph given by $D(\mathcal{T}_i'') = D(\mathcal{T}_i) \sqcup \vec{S}_3(b,x,a) \sqcup \vec{P}_6(a,x,b,y,z,w)$. Recall that the multiplicity of (a,b) is at least two in $D(\mathcal{T}_i)$ and, using Remark 8.2 to annihilate edges which have opposite directions, we obtain a tour-trail decomposition $\mathcal{T}_{i+1}$ such that

$$D(\mathcal{T}_{i+1}) = D(\mathcal{T}_i) \setminus \{(a,b),(a,b)\} \sqcup \{(b,a)\} \sqcup \vec{T}_3(y,z,w). \tag{8.2}$$

(III) Suppose cases (I) and (II) do not hold, and that there are three distinct vertices $a, b, c \in V(H)$ such that $(a,b), (b,c) \in D(\mathcal{T}_i) \setminus \vec{\triangle}(D(\mathcal{T}_i))$.

Consider vertices $x, y, z \in X_i$ and apply Lemma 8.6 on the vertices c, b, a, x, y, z to obtain $P_6(c,b,a,x,y,z)$ and the tour-trail decomposition $\mathcal{T}_i' = \mathcal{T}_{P_6}(\mathcal{T}_i, c, b, a, x, y, z)$. Setting

$$T_{i+1} = T_i \cup P_6(c,b,a,x,y,z) \quad \text{ and } \quad X_{i+1} = X_i \setminus \{x,y,z\},$$

we deduce from Lemma 8.6 that T_{i+1} has at most $14\ell i + 12\ell \leqslant 14\ell(i+1)$ edges and that $T_{i+1} \cup R$ spans at most $m + i(14\ell - 38) + 12\ell - 18 \leqslant m + (i+1)(14\ell - 21)$ vertices. Moreover, it is clear that T_{i+1} contains a C_ℓ-decomposition and that $|X_{i+1}| \geqslant |X_i| - 4$.

The residual digraph of $\mathcal{T}_i'$ is $D(\mathcal{T}_i') = D(\mathcal{T}_i) \sqcup \vec{P}_6(c,b,a,x,y,z)$ and therefore, using Remark 8.2, we obtain a tour-trail decomposition $\mathcal{T}_{i+1}$ such that

$$D(\mathcal{T}_{i+1}) = (D(\mathcal{T}_i) - \{(a,b),(b,c)\}) \sqcup \{(a,c)\} \sqcup \vec{T}_3(x,y,z). \tag{8.3}$$

(IV) Suppose that cases (I), (II) and (III) do not hold, and that there are vertices a, b, c, d such that $(a,b), (a,c), (a,d) \in D(\mathcal{T}_i)$. Apply Lemma 8.4 on the vertices c, d, a to obtain $S_3(c,d,a)$ and a tour-trail decomposition $\mathcal{T}_i' = \mathcal{T}_{S_3}(\mathcal{T}_i, c, d, a)$. Further, take $x, y, z \in X_i$ apply Lemma 8.6 to $T_i \cup R \cup S_3(c,d,a)$ on the vertices a, c, b, x, y, z to obtain $P_6(a,c,b,x,y,z)$ and the tour-trail decomposition $\mathcal{T}_i'' = \mathcal{T}_{P_6}(\mathcal{T}_i', a, c, b, x, y, z)$. Set

$$\begin{aligned} T_{i+1} &= T_i \cup S_3(c,d,a) \cup P_6(a,c,b,x,y,z) \text{ and} \\ X_{i+1} &= X_i \setminus \{x,y,z\}, \end{aligned} \tag{8.4}$$

and observe that T_{i+1} has at most $14\ell(i+1)$ edges and that $T_{i+1} \cup R$ spans at most $m + i(14\ell - 21) + 14\ell - 21 = m + (i+1)(14\ell - 21)$. Again, it is easy to check that T_{i+1} contains a C_ℓ-decomposition and that $|X_{i+1}| \geqslant |X_i| - 4$.

Observe that the residual digraph of the tour-trail decomposition $\mathcal{T}_i''$ is given by

$$D(\mathcal{T}_i'') = D(\mathcal{T}_i) \sqcup \vec{S}_3(c,d,a) \sqcup \vec{P}_6(a,c,b,x,y,z),$$

and again, by Remark 8.2 we can find a tour-trail decomposition $\mathcal{T}_{i+1}$ such that

$$D(\mathcal{T}_{i+1}) = (D(\mathcal{T}_i) - \{(a,b),(a,c),(a,d)\}) \sqcup \{(c,b),(c,d)\} \sqcup \vec{T}_3(x,y,z). \tag{8.5}$$

(V) If none of the previous cases takes place, then set $T_{i+1} = T_i$ and $\mathcal{T}_{i+1} = \mathcal{T}_i$.

Let $T = T_k$ and $\mathcal{T}_\triangle = \mathcal{T}_k$. As discussed before, we have ensured (i 8.7)–(iii 8.7) hold by construction. To prove (iv 8.7) we have to show all arcs of $D(\mathcal{T})$ are in its sea of triangles $\vec{\triangle}(D(\mathcal{T}_k))$. We shall require the following definition. For any tour-trail decomposition $\mathcal{T}$ define the parameter $\Phi(\mathcal{T}) = |E(D(\mathcal{T}))| - |E(\vec{\triangle}(D(\mathcal{T})))|$. In words, $\Phi(\mathcal{T})$ is the number of arcs in $\mathcal{T}$ which are not in its sea of triangles. Note $\Phi(\mathcal{T}) \geqslant 0$ always.

First, we claim that there exists some $0 \leqslant i < k$ such that case (V) happens when processing $\mathcal{T}_i$. Suppose this is not the case. Observe that if any of the cases (I)–(IV) happens when processing $\mathcal{T}_i$, due to the structure of $D(\mathcal{T}_{i+1})$ given in (I), (8.2), (8.3), and (8.5), we have

$$\Phi(\mathcal{T}_{i+1}) \leqslant \Phi(\mathcal{T}_i) - 1.$$

Hence, we have $\Phi(\mathcal{T}_k) \leqslant \Phi(\mathcal{T}_0) - k$. Note that the number of arcs in $D(\mathcal{T}_0)$ is twice the number of trails of $\mathcal{T}_0$, each trail uses at least one edge of R, and R has at most $\binom{m}{3}$ edges since it spans at most m vertices. Thus $\Phi(\mathcal{T}_0) \leqslant 2|E(R)| \leqslant 2\binom{m}{3}$. Since $\Phi(\mathcal{T}_k) \geqslant 0$, we deduce $2\binom{m}{3} < k \leqslant \Phi(\mathcal{T}_0) \leqslant 2\binom{m}{3}$, a contradiction. This proves the claim.

Now, let $\vec{G} = D(\mathcal{T}_\triangle) \smallsetminus \vec{\triangle}(D(\mathcal{T}_\triangle))$. To prove (iv 8.7) we need to show $\vec{G}$ is empty. Note that once the procedure falls in (V) in a step $i < k-1$, it will happen again in step $i+1$. Therefore, by the previous discussion, we know that case (V) happened when processing $\mathcal{T}_{k-1}$ to build $\mathcal{T}_k$. In particular, $\mathcal{T}_{k-1} = \mathcal{T}_k = \mathcal{T}_\triangle$ and we know cases (I)–(IV) did not hold when processing $\mathcal{T}_{k-1}$.

Denote the vertices spanned by the arcs of $\vec{G}$ as V. Observe first that there are no vertices $a, b \in V$ such that $(a,b), (b,a) \in D(\mathcal{T}_{k-1})$ otherwise case (I) would have hold. Then, notice that for every pair $a, b \in V$ we have that $\mu_{\mathcal{T}_{k-1}}(a,b) \leqslant 1$, otherwise $\mathcal{T}_{k-1}$ would have qualified for case (II). This implies that $\vec{G}$ is an oriented graph, with no multiple edges or directed 2-cycles. Moreover, for every vertex $b \in V$ we have either $d^+(b) = 0$ or $d^-(b) = 0$ in $\vec{G}$, otherwise the case (III) would have taken place. If $\vec{G}$ is non-empty, then there is a vertex $b \in V$ with $d^+(b) > 0$, which then implies $d^-(b) = 0$. Then Lemma 8.3 implies that $d^+(b) \geqslant 3$. Therefore, $\mathcal{T}_{k-1}$ would have fallen in case (IV), a contradiction. Thus $\vec{G}$ is empty, which finally shows (iv 8.7). ■

Now we are ready to prove the main lemma of this subsection.

Proof of Lemma 8.1. Let $T \subseteq H$ as found in Lemma 8.7 and let $\mathcal{T}_\triangle$ be a tour-trail decomposition of $R \cup T$ given in (iv 8.7) such that it residual digraph is a sea of triangles.

Since each trail of $\mathcal{T}_\triangle$ contributes two arcs to $\mathcal{D}(\mathcal{T}_\triangle)$ the number of arcs is even, and so is the number of oriented triangles in $\mathcal{D}(\mathcal{T}_\triangle)$. Suppose the number of triangles is $2k$ and let $D(\mathcal{T}_\triangle)) = \bigcup_{i \in [2k]} \vec{T}_3(a_i, b_i, c_i)$. Since the triangles are vertex-disjoint and by (ii 8.7) we have that $2k \leqslant 30\binom{m}{3}\ell$.

Apply Lemma 8.6 to obtain the prism $P_1 = P_6(c_1, b_1, a_1, c_2, b_2, a_2)$ and the tour-trail decomposition $\mathcal{T}'$ of $R \cup T \cup P_1$ whose residual digraph is given by

$$\begin{aligned} D(\mathcal{T}') &= \vec{T}_3(c_1, b_1, a_1) \sqcup \vec{T}_3(c_2, b_2, a_2) \sqcup \bigcup_{i=1}^{2k} \vec{T}_3(a_i, b_i, c_i) \\ &= \vec{T}_3(c_1, b_1, a_1) \sqcup \vec{T}_3(a_1, b_1, c_1) \sqcup \vec{T}_3(a_2, b_2, c_2) \sqcup \vec{T}_3(c_2, b_2, a_2) \sqcup \bigcup_{i=3}^{2k} \vec{T}_3(a_i, b_i, c_i). \end{aligned}$$

Using Remark 8.2 we can "cancel out" the arcs of triangles $\vec{T}_3(c_1, b_1, a_1)$, $\vec{T}_3(a_1, b_1, c_1)$, $\vec{T}_3(a_2, b_2, c_2)$, and $\vec{T}_3(c_2, b_2, a_2)$, and obtain a tour-trail decomposition $\mathcal{T}_i$ whose residual digraph is a sea of triangles with $2k-2$ triangles.

Since $1/n \ll 1/m$, and every prism spans at most $12\ell - 18$ new vertices, we may assume that n is large enough for $k - 1 \leqslant 15\binom{m}{3}\ell - 1$ extra applications of Lemma 8.6, adding the prism $P_i = P(c_{2i-1}, b_{2i-1}, a_{2i-1}, c_{2i}, b_{2i}, a_{2i})$ in each step $2 \leqslant i \leqslant k$. Therefore, we can repeat the previous argument until there are no more triangles in the residual digraph (and hence, no more arcs). Taking $A_1 = T \cup \bigcup_{i\in[k]} P_i$, it is easy to check that it satisfies all the desired properties. ■

8.2. **From a tour decomposition to a cycle decomposition.** In this section we prove the following lemma, which constructs an absorber given a C_ℓ-divisible remainder which has a tour decomposition.

Lemma 8.8. *Let $\ell \geqslant 7$, $\varepsilon > 0$, and $n, m \in \mathbb{N}$ be such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be a 3-graph on n vertices with $\delta_2(H) \geqslant (2/3+\varepsilon)n$. Let $R \subseteq H$ be a C_ℓ-divisible edge-disjoint collection of tours spanning at most m vertices in total. Then, there is a C_ℓ-absorber A_2 for R, such that $A_2 \cup R$ spans at most $10\binom{m}{3}\ell^2$ edges.*

Given two subgraphs R_1 and R_2, we say that a subgraph $T \subseteq H$ edge-disjoint from R_1 and R_2 is a (R_1, R_2)-*transformer* if $T[V(R_1)], T[V(R_2)]$ are empty and both $T \cup R_1$ and $T \cup R_2$ contain a C_ℓ-decomposition. Observe that if R_2 has a C_ℓ-decomposition, then $T \cup R_2$ is an absorber for R_1.

Lemma 8.9. *Let $\ell \geqslant 7$, $\varepsilon > 0$, and $n, m \in \mathbb{N}$ be such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be a 3-graph on n vertices with $\delta_2(H) \geqslant (2/3+\varepsilon)n$. Let $R \subseteq H$ be a tour and $C \subseteq H$ be a cycle. Suppose that R and C are edge-disjoint and contain the same number of edges, which is at most m. Then H contains an (R, C)-transformer L with at most $m\ell$ edges and spanning at most $m(\ell-4)$ vertices.*

Proof. Let $r_1, r_2, \dots, r_m$ and $c_1, c_2, \dots, c_m$ the sequence of vertices of R and C respectively (recall that while C does not contain repetitions, R may contain).

In the following, all operations on the indices are modulo m. We define iteratively the following paths P_i, Q_i for every $i \in [m]$. Apply Lemma 5.1 to obtain a path P_i on 5 vertices, edge-disjoint from $R \cup C$, from the pair (r_i, r_{i+1}) to the pair (c_{i-1}, c_i). Similarly, we can obtain a path Q_i on $\ell - 5$ vertices, from the pair (r_i, r_{i-1}) to the pair (c_i, c_{i-1}), edge disjoint from $R \cup C$, and with no interior vertex in common with the paths P_i, P_{i-1}.

We claim that $L = \bigcup_{i\in[m]} (P_i \cup Q_i)$ is the desired transformer. Indeed, observe that the edges of P_i and Q_i together with the edge $r_{i-1}r_ir_{i+1} \in E(R)$ form a cycle of length ℓ, thus $R \cup L$ can be decomposed into those ℓ-cycles. In the same way, the edges of P_{i-1} and Q_i together with the edge $c_{i-2}c_{i-1}c_i \in E(C)$ form a cycle of length ℓ, and therefore all those cycles form a C_ℓ-decomposition of $C \cup L$. ■

For any $k, \ell \in \mathbb{N}$ we define $B(k, \ell)$ to be the 3-graph resulting from a cycle of length $k\ell$ with vertices in $\{v_1, v_2, \dots, v_{k\ell}\}$ and identifying all vertices v_i with $i \equiv 1 \bmod \ell$ and all vertices v_j with $j \equiv 2 \bmod \ell$. This is to say that $B(k, \ell)$ consists of k copies of cycles of length ℓ glued through exactly two vertices, and those two vertices are consecutive in every cycle. Observe that $B(k, \ell)$ is a tour and admits a C_ℓ-decomposition.

Now we are ready to prove Lemma 8.8.

Proof of Lemma 8.8. Consider the tours $T_1, T_2, \dots, T_k$ in R and observe that $k \leqslant \binom{m}{3}/4$ (each tour has at least 4 edges). First, we want to reduce the proof to the case in which there is a single long tour. Suppose $k \geqslant 2$ and take a_i, b_i two consecutive vertices in T_i for $i = \{1, 2\}$. We can apply Lemma 5.1 to find a path P_1 on 5 vertices with ends (b_1, a_1) and (a_2, b_2) which is edge-disjoint to R. Similarly, we can find P_2 on $\ell - 5$ vertices with ends (a_1, b_1) and (b_2, a_2), edge-disjoint with R, and sharing no interior vertex with P_1. Starting in (a_1, b_2) and then traversing sequentially T_1, P_1, T_2, and P_2, one can check that $T_1 \cup T_2 \cup P_1 \cup P_2$ forms a tour spanning at most $|V(T_1 \cup T_2)| + \ell - 4$ vertices. Moreover, it is easy to see that $P_1 \cup P_2$ is a cycle of length ℓ. By repeating this argument we can obtain $A' \subseteq H$ edge-disjoint from R, C_ℓ-decomposable, and such that $R' = R \cup A'$ consists of a single tour spanning at most $m + k(\ell - 4)$ vertices. Observe that since R is C_ℓ-divisible, then so is R'. Let m' be the number of edges in R' and notice that

$$m' \leqslant \binom{m}{3} + k\ell \leqslant 2\binom{m}{3}\ell$$

Second, observe that by several applications of Lemma 5.1 we can find two edge-disjoint subgraphs $B, C \subseteq H$, vertex-disjoint to each other, both of them edge-disjoint with R', and such that B is a copy of $B(m'/\ell, \ell)$ and C is a cycle of length m' (observe that ℓ divides m' since R' is C_ℓ-divisible).

Now two suitable applications of Lemma 8.9 yield the result. More precisely, first apply Lemma 8.9 with R' in the rôle of R to obtain a (R', C)-transformer $L_1 \subseteq H$ with at most $m'\ell$ edges. For the second application of Lemma 8.9 observe that, since $R' \cup L_1$ contain at most $m'(\ell+1)$ we may assume n is large enough so that $\delta_2(H \smallsetminus (R' \cup L_1)) \geqslant (2/3 + \varepsilon/2)n$. Hence, another application of Lemma 8.9 now with B in the rôle of R and $H \smallsetminus (R' \cup L_1)$ in the rôle of H yields the existence of a (B, C)-transformer $L_2 \subseteq H$ edge disjoint with $R' \cup L_1$.

Putting all this together, and recalling that both A' and B contain a C_ℓ-decomposition, we have that the hypergraphs

$$R \cup A' \cup L_1 \cup C \cup L_2 \cup B \quad \text{and} \quad A' \cup L_1 \cup C \cup L_2 \cup B$$

contain C_ℓ-decompositions. To finish the proof take $A_2 = A' \cup L_1 \cup C \cup L_2 \cup B$ and observe that each of the hypergraphs A', L_1, L_2, C, and B contain at most $m'\ell \leqslant 2\binom{m}{3}\ell^2$ edges. ■

8.3. **Proof of Lemma 4.2.** We can finally give the short proof of Lemma 4.2.

Proof of Lemma 4.2. Given $R \subseteq H$, an application of Lemma 8.1 yields the existence of $A_1 \subseteq H$ edge disjoint from R such that

(i) A_1 has a C_ℓ-decomposition,
(ii) $A_1 \cup R$ contain a tour decomposition, and
(iii) $A_1 \cup R$ spans at most $30\binom{m}{3}\ell(6\ell + 1)$ vertices.

Then, we apply Lemma 8.8 to obtain $A_2 \subseteq H$, which is an absorber of $R \cup A_1$. It is straightforward to check that $A = A_1 \cup A_2$ has the desired properties. ■

§9. Final remarks

A natural question is what happens for the values of ℓ not covered by our Theorem 1.1. Our results do not cover C^3_ℓ-decompositions for small values of ℓ, i.e. $\ell \leqslant 8$. As in the graph case, for short cycles it is likely that the behaviour of the decomposition threshold is different.

For $\ell = 4$ the 3-uniform tight cycle C^3_4 is isomorphic to a tetrahedron K^3_4, i.e. a complete 3-graph on four vertices. Since every pair of vertices in K^3_4 has degree 2, the obvious necessary divisibility conditions in a host 3-graph which admits a C^3_4-decomposition are (i) total number of edges divisible by 4, (ii) every vertex degree divisible by 3, and (iii) every codegree divisible by 2. Say that a 3-graph satisfying all three conditions is *K^3_4-divisible*. We define $\delta_{K^3_4}$ as the asymptotic minimum codegree threshold ensuring a K^3_4-decomposition over K^3_4-divisible graphs (in analogy to δ_{C_ℓ} taken over C_ℓ-divisible graphs). The following construction shows that $\delta_{K^3_4} \geqslant 3/4$.

Example 9.1. Let $k \geqslant 1$ be arbitrary, $d = 6k + 2$ and $n = 12k + 9$. Let G_1 be an arbitrary d-regular graph on n vertices. Let G be the graph on $2n$ vertices obtained by taking two vertex-disjoint copies of G_1 and adding every edge between vertices belonging to different copies, say those edges are *crossing*. Now, form a 3-graph H as follows. Take a set Z on $2n$ vertices and edges forming a complete 3-uniform graph on Z. Then add two new vertices x_1, x_2. For each $z \in Z$, add the edge x_1x_2z. Identify a copy of the graph G in Z and, for each edge z_1z_2 of G add the edges $z_1z_2x_1$ and $z_1z_2x_2$.

H has $2n+2 = 24k+20$ vertices and $\delta_2(H) = d+n+1 = 18k+12$ (attained by any pair x_1z with $z \in Z$). It is tedious but straightforward to check H is K_4^3-divisible. To see H is not K_4^3-decomposable, we prove that the link graph $H(x_1)$ is not C_3^2-decomposable. Note $H(x_1)$ is isomorphic to the graph G' obtained from G by adding an extra universal vertex x. Suppose G' has a triangle decomposition. There are n^2 crossing edges in G, at most n of those can be covered with triangles using x. Thus at least $n(n-1)$ crossing edges are covered with triangles which use one edge in a copy of G_1 and two crossing edges. Thus we need at least $n(n-1)/2$ edges in the two copies of G_1, but those copies have $dn < n(n-1)/2$ edges, contradiction.

What is the smallest ℓ_0 such that $\delta_{C_\ell^3} = 2/3$ holds for all $\ell \geqslant \ell_0$? The previous example and Theorem 1.1 show that $5 \leqslant \ell_0 \leqslant 10^7$. Observe that our Absorbing Lemma works for all $\ell \geqslant 7$. The bottleneck is our use of Theorem 7.5 in the Cover-Down Lemma. New ideas are needed to close the gap.

Another question is what happens for k-graphs with $k \geqslant 4$. It is not clear for us if Theorem 1.4 indicates the emergence of a pattern where the necessary codegree to ensure cycle decompositions and Euler tours on n-vertex k-graphs is substantially larger than $(1/2 + o(1))n$.

Question 9.2. *For $k \geqslant 4$, let H be a k-graph on n vertices. Is $\delta_{k-1}(H) \geqslant ((k-1)/k + o(1))n$ a necessary and sufficient condition for the existence of cycle decompositions or Euler tours?*

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References

[1] N. Alon and R. Yuster. "On a hypergraph matching problem". In: *Graphs Combin.* 21.4 (2005), pp. 377–384. DOI: 10.1007/s00373-005-0628-x (cit. on pp. 14, 15).

[2] B. Barber, S. Glock, D. Kühn, A. Lo, R. Montgomery, and D. Osthus. "Minimalist designs". In: *Random Structures Algorithms* 57.1 (2020), pp. 47–63. DOI: 10.1002/rsa.20915 (cit. on pp. 2, 7, 8, 11).

[3] B. Barber, D. Kühn, A. Lo, and D. Osthus. "Edge-decompositions of graphs with high minimum degree". In: *Adv. Math.* 288 (2016), pp. 337–385. DOI: 10.1016/j.aim.2015.09.032 (cit. on pp. 2, 7, 13).

[4] F. Chung, P. Diaconis, and R. Graham. "Universal cycles for combinatorial structures". In: *Discrete Math.* 110.1-3 (1992), pp. 43–59. DOI: 10.1016/0012-365X(92)90699-G (cit. on p. 3).

[5] M. Delcourt and L. Postle. "Progress towards Nash-Williams' conjecture on triangle decompositions". In: *J. Combin. Theory Ser. B* 146 (2021), pp. 382–416. DOI: 10.1016/j.jctb.2020.09.008 (cit. on p. 2).

[6] S. Ehard, S. Glock, and F. Joos. "Pseudorandom hypergraph matchings". In: *Combin. Probab. Comput.* 29.6 (2020), pp. 868–885. DOI: 10.1017/s0963548320000280 (cit. on p. 14).

[7] P. Erdős. "On extremal problems of graphs and generalized graphs". In: *Israel J. Math.* 2 (1964), pp. 183–190. DOI: 10.1007/BF02759942 (cit. on pp. 13, 26).

[8] L. Euler. "Solutio problematis ad geometriam situs pertinentis". In: *Comment. Academiae Sci. I. Petropolitanae* 8 (1741), pp. 128–140. URL: https://scholarlycommons.pacific.edu/euler-works/53/ (cit. on p. 2).

[9] S. Glock, F. Joos, D. Kühn, and D. Osthus. "Euler tours in hypergraphs". In: *Combinatorica* 40.5 (2020), pp. 679–690. DOI: 10.1007/s00493-020-4046-8 (cit. on pp. 3, 7).

[10] S. Glock, D. Kühn, A. Lo, R. Montgomery, and D. Osthus. "On the decomposition threshold of a given graph". In: *J. Combin. Theory Ser. B* 139 (2019), pp. 47–127. DOI: 10.1016/j.jctb.2019.02.010 (cit. on pp. 2, 7).

[11] S. Glock, D. Kühn, A. Lo, and D. Osthus. "The existence of designs via iterative absorption: hypergraph F-designs for arbitrary F". In: *Mem. Am. Math. Soc.* (2020), to appear. URL: https://arxiv.org/abs/1611.06827v3 (cit. on pp. 1, 2).

[12] S. Glock, D. Kühn, and D. Osthus. "Extremal aspects of graph and hypergraph decomposition problems". In: *Surveys in Combinatorics, London Math. Soc. Lecture Note Ser.* (2020), to appear. URL: https://arxiv.org/abs/2008.00926v1 (cit. on pp. 1, 3).

[13] C. Hierholzer and C. Wiener. “Ueber die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechung zu umfahren”. In: *Math. Ann.* 6.1 (1873), pp. 30–32. DOI: 10.1007/BF01442866 (cit. on p. 2).

[14] S. Janson, T. Łuczak, and A. Rucinski. *Random graphs.* Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000, pp. xii+333. DOI: 10.1002/9781118032718 (cit. on p. 10).

[15] F. Joos and M. Kühn. *Fractional cycle decompositions in hypergraphs.* Preprint (2021). arXiv: 2101.05526 [math.CO] (cit. on pp. 2, 14).

[16] J. Kahn. “A linear programming perspective on the Frankl-Rödl-Pippenger theorem”. In: *Random Structures Algorithms* 8.2 (1996), pp. 149–157. DOI: 10.1002/(SICI)1098-2418(199603)8:2<149::AID-RSA5>3.3.CO;2-S (cit. on p. 14).

[17] P. Keevash. *The existence of designs.* Preprint (2014). arXiv: 1401.3665 [math.CO] (cit. on p. 1).

[18] C. Nash-Williams. “An unsolved problem concerning decomposition of graphs into triangles”. In: *Combinatorial Theory and its Applications, Vol. III, Proceedings of the Colloquium on Combinatorial Theory and its Applications held at Balatonfüred, August 24–29, 1969, North-Holland* (1970), pp. 1179–1183 (cit. on p. 2).

[19] R. Raman. “The power of collision: randomized parallel algorithms for chaining and integer sorting”. In: *Foundations of software technology and theoretical computer science (Bangalore, 1990).* Vol. 472. Lecture Notes in Comput. Sci. Springer, Berlin, 1990, pp. 161–175. DOI: 10.1007/3-540-53487-3_42 (cit. on p. 10).

[20] A. Taylor. “On the exact decomposition threshold for even cycles”. In: *J. Graph Theory* 90.3 (2019), pp. 231–266. DOI: 10.1002/jgt.22399 (cit. on p. 2).

[21] C. Thomassen. “Decompositions of highly connected graphs into paths of length 3”. In: *J. Graph Theory* 58.4 (2008), pp. 286–292. DOI: 10.1002/jgt.20311 (cit. on p. 15).

§Appendix A. Proof of Lemma 7.3

Proof. The proof proceeds in three steps. First, we find $H_p \subseteq H$ by including each edge with probability p, and in the remainder $H_0 = H \smallsetminus H_p$ we find an almost perfect C_ℓ-packing $\mathcal{C}_0$, let $L_0 = H_0 \smallsetminus E(\mathcal{C}_0)$ be the leftover edges. Secondly, we correct the leftover L_0 in the vertices incident with $\Omega(n^2)$ many edges of L_0 by constructing cycles with the help of the edges in H_p. This provides us with a new cycle packing $\mathcal{C}_1 \subseteq L_0 \cup H_p$ whose new leftover $L_1 = H_0 \smallsetminus E(\mathcal{C}_0 \cup \mathcal{C}_1)$ satisfies $\Delta_1(L_1) = o(n^2)$. Finally, we correct the new leftover L_1 in a similar way, fixing the pairs incident to $\Omega(n)$ edges in L_1. We get a cycle packing $\mathcal{C}_2 \subseteq L_1 \cup H_p$, and $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ will be the desired cycle packing.

Step 1: Random slice and aproximate decomposition. Note that $\delta_2^{(3)}(H) \geqslant 3\varepsilon n$. Now let $p = \gamma/4$, and let $H_p \subseteq H$ be obtained from H by including each edge independently with probability p. Using concentration inequalities (e.g. Theorem 5.4) we see that with non-zero probability

$$\Delta_2(H_p) \leqslant 2pn, \text{ and } \delta_2^{(3)}(H_p) \geqslant 2\varepsilon pn. \tag{A.1}$$

hold simultaneously for H_p. From now on we suppose H_p is fixed and satisfies (A.1).

Let $H_0 = H \smallsetminus H_p$. In H_0, construct a C_ℓ-packing by removing edge-disjoint cycles, one by one, until no longer possible. We get a C_ℓ-packing $\mathcal{C}_0$ in H_0, let $F_0 = E(\mathcal{C}_0)$. By Erdős' Theorem [7, Theorem 1] there exists $c > 0$ such that $L_0 = H_0 \smallsetminus F_0$ has at most n^{3-3c} edges.

Step 2: Eliminating bad vertices. Let $B_0 = \{v \in V : \deg_{L_0}(v) \geqslant n^{2-2c}\}$. Since $|L_0| \leqslant n^{3-3c}$, by double-counting we have $|B_0| \leqslant 3n^{1-c}$.

For each $b \in B_0$, let G_b be the subgraph of $L_0(b)$ obtained after removing the vertices of B_0. Note that $L_0(b) - G_0$ has at most $|B_0|n \leqslant 3n^{2-c}$ edges. Now, let $\mathcal{P}_b$ be a maximal edge-disjoint collection of paths of length 3 in G_b. Since every graph on n vertices with at least $n+1$ edges contains a path of length 3, then $G_b - E(\mathcal{P}_b)$ has at most n edges. All together, we deduce that the number of edges in $L_0(b) - E(\mathcal{P}_b)$ satisfies

$$|L_0(b)| - |E(\mathcal{P}_b)| \leqslant 3n^{2-c} + n \leqslant 4n^{2-c}. \tag{A.2}$$

Since G_b contains at most n^2 edges, we certainly have $|\mathcal{P}_b| \leqslant n^2$. Let $\mathcal{P}_b$ be a collection of tight paths on five vertices obtained by replacing each $v_0v_1v_2v_3$ in $\mathcal{P}_b$ with the tight path $v_0v_1bv_2v_3$ in L_0. Note that any two distinct $P_1, P_2 \in \mathcal{P}_b$ are edge-disjoint, and for two distinct $b, b' \in B_0$, and $P \in \mathcal{P}_b$, $P' \in \mathcal{P}_{b'}$, since $b' \notin V(G_b)$ we have P, P' are edge-disjoint. Thus the union $\mathcal{P} = \bigcup_{b \in B_0} \mathcal{P}_b$ is an edge-disjoint collection of tight paths on 5 vertices.

Select $\gamma', \mu', \varepsilon'$ such that $1/n \ll \gamma' \ll \mu' \ll \varepsilon' \ll \gamma, \varepsilon, 1/\ell$. We wish to apply Lemma 7.1 to extend $\mathcal{P}$ into cycles. We claim $\mathcal{P}$ is γ'-sparse. Let $S \in \binom{V(H)}{2}$. Since $|\mathcal{P}| \leqslant |B_0|n^2 \leqslant 3n^{3-c} \leqslant \gamma' n^3$, certainly $\mathcal{P}$ contains at most $|\mathcal{P}| \leqslant \gamma' n^3$ paths of type 0 for S. Now, note that for each $b \in B_0$, $P \in \mathcal{P}_b$ can have at most $2n$ paths of type 1 for S, thus $\mathcal{P}$ has at most $|B_0|2n \leqslant 6n^{2-c} \leqslant \gamma' n^2$ paths of type 1 for S. Analogously, for each $b \in B_0$, $P \in \mathcal{P}_b$ can have at most 1 path of type 2 for S, thus $\mathcal{P}$ has at most $|B_0| \leqslant 3n^{1-c} \leqslant \gamma' n$ paths of type 2 for S. Thus $\mathcal{P}$ is γ'-sparse.

Recall that L_0 is edge-disjoint with H_p. Inequalities (A.1) together with $p = \gamma/4$ and $\varepsilon' \ll \gamma, 1/\ell$, show that we can use Corollary 5.3 (with $U = V(H)$) and deduce that for each $P \in \mathcal{P}$, there exists at least $\varepsilon' n^{\ell-5}$ copies of C_ℓ in $L_0 \cup H_p$ which extend P_i using extra edges of H_p only.

We apply Lemma 7.1 with $\varepsilon', \mu', \gamma', \ell, 5, L_0, H_p, \mathcal{P}$ playing the rôle of $\varepsilon, \mu, \gamma, \ell, \ell', H_1, H_2, \mathcal{P}$ respectively, to obtain a C_ℓ-decomposable graph $F_1 \subseteq L_0 \cup H_p$ such that $E(\mathcal{P}) \subseteq F_1$ and

$$\Delta_2(F_1 \smallsetminus E(\mathcal{P})) \leqslant \mu' n. \tag{A.3}$$

Since F_0, F_1 are edge-disjoint, $F_0 \cup F_1$ is C_ℓ-decomposable. Let $L_1 = H_0 \smallsetminus (F_0 \cup F_1)$. Observe that, if $v \notin B_0$, then $\deg_{L_1}(v) \leqslant \deg_{L_0}(v) < n^{2-2c}$ by definition. Moreover, if $v \in B_0$, then each edge in $E(\mathcal{P}_v)$ is in F_1, and hence (A.2) implies $\deg_{L_1}(v) \leqslant |L_0(v)| - |E(\mathcal{P}_v)| \leqslant 4n^{2-c}$. Therefore,

$$\Delta_1(L_1) \leqslant 4n^{2-c}. \tag{A.4}$$

Step 3: Eliminating bad pairs. Let $f = c/2$ and $B_1 = \{xy \in \binom{V}{2} : \deg_{L_1}(xy) \geqslant n^{1-f}\}$. From $|L_1| \leqslant |L_0| \leqslant n^{3-3c} \leqslant n^{3-6f}$ we deduce $|B_1| \leqslant n^{2-4f}$. Now consider B_1 as the set of edges of a 2-graph in V. Each edge of B_1 incident to a vertex x implies that x belongs to at least n^{1-f} edges in L_1, and each of those edges participates in at most two of the edges in B_1 incident to x. So we have $\deg_{L_1}(x) \geqslant \frac{1}{2} n^{1-f} \deg_{B_1}(x)$. Together with inequality (A.4) we deduce $\Delta(B_1) \leqslant 8n^{1-f}$.

A path P on L_1 is B_1-*based* if $P = zxyw$ and $xy \in B_1$. Let $\mathcal{P}_2$ be a maximal packing of B_1-based paths. For all $xy \in B_1$, it holds that $\deg_{L_1}(xy) - \deg_{E(\mathcal{P}_2)}(xy) \leqslant 1$. Otherwise it would exist distinct $z, w \in N_{L_1 \smallsetminus E(\mathcal{P}_2)}(xy)$, and then $zxyw$ would a B_1-based path not in $\mathcal{P}_2$ which contradicts its maximality.

We claim $\mathcal{P}_2$ is γ'-sparse. For each $xy \in B_1$, let $\mathcal{P}_{xy} \subseteq \mathcal{P}_2$ be the paths whose two interior vertices are precisely xy. Clearly $|\mathcal{P}_{xy}| \leqslant n$ and $\mathcal{P}_2 = \bigcup_{xy \in B_1} \mathcal{P}_{xy}$. Let $e \in \binom{V}{2}$. Since $|\mathcal{P}_2| \leqslant \sum_{xy \in B_1} |\mathcal{P}_{xy}| \leqslant n|B_1| \leqslant n^{3-4f} \leqslant \gamma' n^3$, there are at most $\gamma' n^3$ paths of type 0 for e in $\mathcal{P}_2$. Recall that if $P = zxyw$ is a path of type 1 for e, then we have $|e \cap \{z, x, y, w\}| = 1$. If $xy \in B_1$ satisfies $e \cap \{x, y\} = \varnothing$, then at most two paths in $\mathcal{P}_{xy}$ can be of type 1 for e and therefore there are at most $2|B_1| \leqslant 2n^{2-4f}$ paths of type 1 for e in $\mathcal{P}_2$. We estimate the contribution of the pairs $xy \in B_1$ such that $|e \cap \{x, y\}| = 1$. Each such xy contributes with at most n paths of type 1 for e in $\mathcal{P}_{xy}$. By (A.4), the number of such xy is at most $2\Delta(B_1) \leqslant 16n^{1-f}$, thus the total contribution of those pairs is at most $16n^{2-f}$. All together, the total number of paths of type 1 for e in $\mathcal{P}_2$ is at most $2n^{2-4f} + 16n^{2-f} \leqslant \gamma' n^2$. If $e = \{a, b\}$ then $\mathcal{P}_{a,b}$ does not contain any path of type 2 for e, by definition of the path types. Thus the only possible contributions come from the pairs in $\mathcal{P}_{a,x}$ and $\mathcal{P}_{b,y}$ for some $x, y \in V(H)$; and each one of those sets contains at most 1 path of type 2 for e. Thus the total number of pairs of type 2 for e in $\mathcal{P}_2$ is at most $2\Delta(B_1) \leqslant 16n^{1-f} \leqslant \gamma' n$. Thus $\mathcal{P}_2$ is γ'-sparse.

Let $H'_p = H_p \setminus (F_0 \cup F_1)$. (A.1) and (A.3), together with $\mu' \ll \varepsilon' \ll \gamma, 1/\ell$, allow us to use Corollary 5.3 with $U = V(H)$, thus for each $P \in \mathcal{P}_2$, there exists at least $\varepsilon' n^{\ell-4}$ copies of C_ℓ in $L_1 \cup H'_p$ which extend P using extra edges of H'_p only. Apply Lemma 7.1 with the parameters $\varepsilon', \mu', \gamma', \ell, 4, L_1, H'_p, \mathcal{P}_2$ playing the rôles of $\varepsilon, \mu, \gamma, \ell, \ell', H_1, H_2, \mathcal{P}$ respectively, to obtain a C_ℓ-decomposable $F_2 \subseteq L_1 \cup H'_p$ such that $E(\mathcal{P}_2) \subseteq F_2$ and $\Delta_2(F_2 \setminus E(\mathcal{P}_2)) \leqslant \mu' n$.

We claim that $\Delta_2(L_1 \setminus F_2) \leqslant n^{1-f}$. Indeed, if $xy \in B_1$, $\deg_{L_1 \setminus F_2}(xy) \leqslant \deg_{L_1}(xy) \leqslant n^{1-f}$ follows by definition, otherwise, $E(\mathcal{P}_2) \subseteq F_2$ implies $\deg_{L_1 \setminus F_2}(xy) \leqslant \deg_{L_1}(xy) - \deg_{F_2}(xy) \leqslant 1$. Since F_2 and $F_0 \cup F_1$ are edge-disjoint, $F = F_0 \cup F_1 \cup F_2$ is a C_ℓ-decomposable subgraph of H. We claim $L = H \setminus F$ satisfies $\Delta_2(L) \leqslant \gamma n$. Indeed, an edge not covered by F is either in H_p or in $L_1 \setminus F_2$. Thus we have $\Delta_2(L) \leqslant \Delta_2(H_p) + \Delta_2(L_1 \setminus F_2) \leqslant 2pn + n^{1-f} \leqslant \gamma n$, as required. ■

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