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DOI:
10.1016/j.procs.2021.11.055

Available online at www.sciencedirect.com

ScienceDirect

Procedia Computer Science 195 (2021) 453–458

Procedia Computer Science

www.elsevier.com/locate/procedia

XI Latin and American Algorithms, Graphs and Optimization Symposium

Maximum size of r-cross t-intersecting families

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Abstract

Given r families of subsets of a fixed n-set, we say that they are r-cross t-intersecting if for every choice of representatives, exactly one from each family, the common intersection of these representatives is of size at least t. We obtain a generalisation of a result by Hilton and Milner on cross intersecting families. In particular, we determine the maximum possible sum of the sizes of non-empty r-cross t-intersecting families in the case when all families are k-uniform and in the case when they are arbitrary subfamilies of the power set. Only some special cases of these results had been proved before. The method we use also yields more general results concerning measures of families instead of their sizes.

Keywords: Extremal set theory; Erdős-Ko-Rado; Hilton-Milner; intersecting families

1. Introduction

For a set A, we denote the power set of A by $\mathscr{P}(A) = \{B : B \subseteq A\}$. Let $j \in \mathbb{N}$, then define $[j] = \{1, \dots, j\}$, $[j]_0 = [j] \cup \{0\}$, and for $i \in [j]_0$ define $[i, j] = \{i, i+1, \dots, j\}$. For a set with a single element, say $\{i\}$, we sometimes just write i. Given a set A, we write $A^{(k)}$ for the set of k-element subsets of A. For $n \in \mathbb{N}$, we say that a family of subsets $\mathcal{F} \subseteq \mathscr{P}([n])$ is *intersecting* if for all $F, F' \in \mathcal{F}$ we have $F \cap F' \neq \emptyset$. The celebrated result of Erdős, Ko, and Rado [5] establishes the maximum possible size of k-uniform intersecting families.

Theorem 1.1. *Let $k, n \in \mathbb{N}$ with $2k \leqslant n$ and let $\mathcal{F} \subseteq [n]^{(k)}$ be an intersecting family. Then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$ and this bound is sharp.*

Observe that this maximum is attained by a family which contains all the sets of size k that contain one fixed element, for instance $\mathcal{F} = \{F \in [n]^{(k)} : 1 \in F\}$.

As a variation of this classical result *cross intersecting* families can be considered. For $r, t, n \in \mathbb{N}$ we say that the families $\mathcal{F}_1, \dots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ are *r-cross t-intersecting* if for all $F_1 \in \mathcal{F}_1, \dots, F_r \in \mathcal{F}_r$ we have $|\bigcap_{i\in[r]} F_i| \geqslant t$. Now it

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[2] Supported by ANID/CONICYT Acuerdo Bilateral DAAD/62170017 through a Ph.D. Scholarship.

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10.1016/j.procs.2021.11.055

is natural to ask for the maximum of $\sum_{i\in[r]}|\mathcal{F}_i|$ taken over all non-empty r-cross t-intersecting families $\mathcal{F}_1,\dots,\mathcal{F}_r$. In this regime there are several partial results concerning the maximal sizes of r-cross t-intersecting families for specific instances of r and t, starting with theorems by Hilton [14] and by Hilton and Milner [15] and continued, for instance, in [10, 9, 17, 18] (also see the references therein). Here we determine $\sum_{i\in[r]}|\mathcal{F}_i|$ for every $r \geqslant 2$ and $t \geqslant 1$ for uniform families and also for arbitrary subfamilies of the power set.

Our first result determines the maximum sum of the sizes for k-uniform r-cross t-intersecting families when $n \geqslant 3k-t$.

Theorem 1.2. *Let $n,t \geqslant 1$, and $r \geqslant 2$ be integers, $k \in [n]$, and let $\mathcal{F}_1,\dots,\mathcal{F}_r \subseteq [n]^{(k)}$ be non-empty r-cross t-intersecting families. If $n \geqslant 3k-t$, then*

$$\sum_{j\in[r]}|\mathcal{F}_j| \leqslant \max_{m\in[t,k]}\Big\{\sum_{i\in[t,k]}\binom{m}{i}\cdot\binom{n-m}{k-i}+(r-1)\binom{n-m}{k-m}\Big\}$$

and this bound is attained.

To see that this maximum is attained, consider the following families:

$$\mathcal{A}(n,a,t) = \{F \in \mathscr{P}([n])\colon |F\cap[a]| \geqslant t\}$$
$$\mathcal{B}(n,a) = \{F \in \mathscr{P}([n])\colon [a]\subseteq F\}.$$

It is easy to see that $\mathcal{A}(n,m,t)\cap[n]^{(k)}$ and $r-1$ copies of $\mathcal{B}(n,m)\cap[n]^{(k)}$ are r-cross t-intersecting for $m\in[t,k]$. For the appropriate $m\in[t,k]$, these families attain the maximum in Theorem 1.2.

There are also some results for families of arbitrary subsets (instead of k-uniform). Frankl and Wong H.W. [12] determined the maximum of $|\mathcal{F}_1|+|\mathcal{F}_2|$ if $\mathcal{F}_1,\mathcal{F}_2\subseteq\mathscr{P}([n])$ are 2-cross t-intersecting for $t\geqslant 1$. In our second result we establish this maximum for all $r\geqslant 2$ and $t\geqslant 1$.

Theorem 1.3. *Let $n,t \geqslant 1$, and $r \geqslant 2$ be integers and let $\mathcal{F}_1,\dots,\mathcal{F}_r \subseteq \mathscr{P}([n])$ be non-empty r-cross t-intersecting families. Then,*

$$\sum_{j\in[r]}|\mathcal{F}_j| \leqslant \max_{m\in[t,n]}\Big\{2^{n-m}\sum_{i\in[t,m]}\binom{m}{i}+(r-1)2^{n-m}\Big\}$$

and this bound is attained.

Again, the maximum is attained by $\mathcal{A}(n,m,t)$ and $(r-1)$ copies of $\mathcal{B}(n,m)$ for the appropriate $m\in[t,n]$.

The proofs of both these results are based on the same method. In the next section, we sketch the main ideas of this technique and in Section 3 we present the full proof of Theorem 1.3. The proof of Theorem 1.2 uses the same technique but requires some more work. It is included in the full version of this article [13].

We remark that based on the same technique we can obtain a more general result which concerns *measures* (or weights) of families instead of sizes of families. More precisely, consider a function $\mu : [n]_0 \to \mathbb{R}_{\geqslant 0}$ and assign the weight $\mu(|F|)$ to each $F\in\mathscr{P}([n])$. This notion has been studied by several authors, in particular in relation with finding maximum possible measure of families with intersection properties (see for instance [1, 4] and Chapter 12 in [11] for a more thorough overview). Given the measures $\mu_1,\dots,\mu_r$, instead of asking for the maximal sum of the sizes of r-cross t-intersecting families, we ask for the maximum of $\sum_{i\in[r]}\mu_i(\mathcal{F}_i)$, where $\mu_i(\mathcal{F}_i) = \sum_{F\in\mathcal{F}_i}\mu_i(|F|)$. We show that for a broad range of measures, including the commonly studied measures, the maximum is attained by the families $\mathcal{A}(n,m,t)$ and $\mathcal{B}(n,m)$. This more general result also determines the maximum of $\sum_{i\in[r]}|\mathcal{F}_i|$ for families $\mathcal{F}_1,\dots,\mathcal{F}_r$, in which each family $\mathcal{F}_i$ is k_i-uniform, where $k_1,\dots,k_r$ are allowed to differ. It is included in the full version of this article [13].

2. Sketch of the Proof

In this subsection, we summarise the general strategy that we use for the proofs of all the aforementioned results.

First, we introduce one of the most common techniques in the area, namely the *shifting* operation (see [7] for a survey). For $F \subseteq [n]$ and $i, j \in [n]$, set

$$\sigma_{ij}(F) = \begin{cases} F \setminus \{j\} \cup \{i\}, & \text{if } j \in F \text{ and } i \notin F; \\ F, & \text{otherwise} \end{cases}$$

and note that $|\sigma_{ij}(F)| = |F|$. Moreover, given a family $\mathcal{F} \subseteq \mathscr{P}([n])$ we define $\sigma_{ij}(\mathcal{F})$ by

$$\sigma_{ij}(\mathcal{F}) = \{\sigma_{ij}(F) : F \in \mathcal{F}\} \cup \{F \in \mathcal{F} : \sigma_{ij}(F) \in \mathcal{F}\},$$

and note that $|\sigma_{ij}(\mathcal{F})| = |\mathcal{F}|$. We say that $\mathcal{F} \subseteq \mathscr{P}([n])$ is *shifted* if for all $i, j \in [n]$ with $i < j$ we have that $\sigma_{ij}(\mathcal{F}) = \mathcal{F}$, i.e., for all $F \in \mathcal{F}$ we have that $\sigma_{ij}(F) \in \mathcal{F}$.

It is not difficult to see that if the families $\mathcal{F}_1, \dots, \mathcal{F}_r$ are r-cross t-intersecting, then $\sigma_{ij}(\mathcal{F}_1), \dots, \sigma_{ij}(\mathcal{F}_r)$ are also r-cross t-intersecting. Thus, since the size of a family does not change by shifting, we can assume that the families considered are shifted.

Our proof is based on what we call *necessary intersection points* (see Definition 3.1). Roughly speaking, we say that a vertex is a necessary intersection point if there are sets in the families which "depend" on this vertex to fulfil the intersection property. For example, if we consider the cross intersecting families $\mathcal{A}(n, 2, 1)$ and $\mathcal{B}(n, 2)$, the vertex 2 is a necessary intersection point because there are pairs of sets that intersect only in 2. In this case, 1 and 2 are the only necessary intersection points of these families.

For r-cross t-intersecting families $\mathcal{F}_1, \dots, \mathcal{F}_r$, we will construct new families which are still r-cross t-intersecting but with smaller maximal necessary intersection point and for which the sum of the sizes is not smaller. More precisely, let $a_* \in [n]$ be the maximal necessary intersection point of $\mathcal{F}_1, \dots, \mathcal{F}_r$. To construct the new families we first remove all sets that "depend" on a in one family, say $\mathcal{F}_r$; we call the family of these sets $\mathcal{F}_r(a)$. Then a will no longer be a necessary intersection point. Potentially, there are some subsets of $[n]$ which could not be in any of the other families because they would not intersect "correctly" with some set in $\mathcal{F}_r(a)$. However, after removing $\mathcal{F}_r(a)$ from $\mathcal{F}_r$ and depending on how such a set relates with $\mathcal{F}_r \setminus \mathcal{F}_r(a)$, it may be added to one of the other families without breaking the intersection property.

There are some structural properties that follow from a being the maximal necessary intersection point and the fact that the families are shifted. These will help us to analyse which new sets can actually be added to the families $\mathcal{F}_1, \dots, \mathcal{F}_{r-1}$ and to prove that in fact the number of the newly added sets is at least as large as the number of the removed sets. Moreover, this analysis guarantees that the new maximal necessary intersection point is at most $a - 1$.

We can iterate this construction and decrease the maximal necessary intersection point in every step. This process has to stop at a certain point, and we show that then the resulting families are contained in families with the desired structure (namely $\mathcal{A}(n, m, t)$ and $\mathcal{B}(n, m)$).

3. Proof of Theorem 1.3

As mentioned above, our proof is based on necessary intersection points.

Definition 3.1. *Let $n, t \geqslant 1$, and $r \geqslant 2$ be integers and let $\mathcal{F}_1, \dots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ be r-cross t-intersecting families. We say $a \in [n]$ is a* necessary intersection point *of $\mathcal{F}_1, \dots, \mathcal{F}_r$ if for all $j \in [r]$ there is an $F_j \in \mathcal{F}_j$ such that*

$$|[a] \cap \bigcap_{j\in[r]} F_j| = t \text{ and } a \in \bigcap_{j\in[r]} F_j\,. \tag{1}$$

In addition, we introduce the following notation. Let $\mathcal{F}_1, \dots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ be r-cross t-intersecting families and let a be their maximal necessary intersection point. For every $j \in [r]$ define $\mathcal{F}_j(a)$ to be the set of all $F_j \in \mathcal{F}_j$ for which there exist $F_i \in \mathcal{F}_i$ for every $i \in [r] \setminus j$ such that (1) holds. We also refer to the sets in $\mathcal{F}_j(a)$ as the sets in $\mathcal{F}_j$ *depending on a*. Note that a is *not* a necessary intersection point for the families $\mathcal{F}_1, \dots, \mathcal{F}_i \setminus \mathcal{F}_i(a), \dots, F_r$.

The following lemma states that for certain r-cross t-intersecting families, we can find new r-cross t-intersecting families with at least the same total number of sets and with a smaller maximal necessary intersection point.

Lemma 3.2. *Let $n, t \geqslant 1$ and $r \geqslant 2$ be integers and let $\mathcal{F}_1, \dots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ be non-empty shifted r-cross t-intersecting families with maximal necessary intersection point $a \geqslant t+1$. Assume that for all $i \in [r]$, we have $\mathcal{F}_i \setminus \mathcal{F}_i(a) \neq \emptyset$. Then there are families $\mathcal{H}_1, \dots, \mathcal{H}_r \subseteq \mathscr{P}([n])$ such that*

(a) $\mathcal{H}_1, \dots, \mathcal{H}_r$ are r-cross t-intersecting families,

(b) $\sum_{j\in[r]} |\mathcal{H}_j| \geqslant \sum_{j\in[r]} |\mathcal{F}_j|$, and

(c) the maximal necessary intersection point of $\mathcal{H}_1, \dots, \mathcal{H}_r$ is smaller than a.

Proof. First observe that for every $F_i \in \mathcal{F}_i(a)$,

$$F_i \setminus a \notin \mathcal{F}_i. \tag{2}$$

Indeed, assume there is an $F_i \in \mathcal{F}_i(a)$ such that $F_i \setminus a \in \mathcal{F}_i$. Consider the sets $F_j \in \mathcal{F}_j$ for $j \neq i$ which exist by the definition of $\mathcal{F}_i(a)$ and notice that $|[a] \cap \bigcap_{j\in[r]\setminus i} F_j \cap (F_i \setminus a)| < t$. Then, we deduce that $|\bigcap_{j\in[r]\setminus i} F_j \cap (F_i \setminus a)| < t$ which contradicts the fact that $\mathcal{F}_1, \dots, \mathcal{F}_r$ are r-cross t-intersecting with maximal necessary intersection point a.

We define the families $\mathcal{H}_1, \dots, \mathcal{H}_r$ depending on two cases. First, if $|\mathcal{F}_r(a)| \leqslant \sum_{i\in[r-1]} |\mathcal{F}_i(a)|$, we set

$$\mathcal{H}_i = \begin{cases} \mathcal{F}_i \dot\cup \{F \setminus a \colon F \in \mathcal{F}_i(a)\} & \text{for } i \neq r \\ \mathcal{F}_r \setminus \mathcal{F}_r(a) & \text{for } i = r\,. \end{cases} \tag{3}$$

Assume there are sets $H_i \in \mathcal{H}_i$ for $i \in [r]$ with $|[a-1] \cap \bigcap_{i\in[r]} H_i| < t$. Take $I \subseteq [r]$ to be the set of indices for which $H_i \notin \mathcal{F}_i$. Observe that since $\mathcal{F}_1, \dots, \mathcal{F}_r$ are r-cross t-intersecting with a as their maximal necessary intersection point, and since $H_r \in \mathcal{H}_r = \mathcal{F}_r \setminus \mathcal{F}_r(a)$, we have that $I \neq \emptyset$ and $r \notin I$. For $i \in I$, it follows from (3) that there is an $F_i \in \mathcal{F}_i$ such that $H_i \dot\cup a = F_i$. For $i \in [r] \setminus I$, set $F_i = H_i$. Then $|[a-1] \cap \bigcap_{i\in[r]} H_i| < t$ and $\mathcal{F}_1, \dots, \mathcal{F}_r$ being r-cross t-intersecting with maximal necessary intersection point a imply $|[a] \cap \bigcap_{i\in[r]} F_i| = t$ and $a \in \bigcap_{i\in[r]} F_i$. This yields $H_r = F_r \in \mathcal{F}_r(a)$, which contradicts (3). Thus, for all $H_1 \in \mathcal{H}_1, \dots, H_r \in \mathcal{H}_r$, we have that $|[a-1] \cap \bigcap_{i\in[r]} H_i| \geqslant t$, that is, the families $\mathcal{H}_1, \dots, \mathcal{H}_r$ satisfy (a) and (c).

The fact that they satisfy (b) comes from (2) and the assumption of this case, namely that $|\mathcal{F}_r(a)| \leqslant \sum_{i\in[r-1]} |\mathcal{F}_i(a)|$.

For the other case, in which $|\mathcal{F}_r(a)| > \sum_{i\in[r-1]} |\mathcal{F}_i(a)|$, we define the families $\mathcal{H}_1, \dots \mathcal{H}_r$ in a similar way, namely

$$\mathcal{H}_i = \begin{cases} \mathcal{F}_i \setminus \mathcal{F}_i(a) & \text{for } i \neq r \\ \mathcal{F}_r \dot\cup \{F \setminus a \colon F \in \mathcal{F}_r(a)\} & \text{for } i = r\,. \end{cases} \tag{4}$$

The proof that they satisfy (a), (b), and (c) is analogous to the proof of the previous case. □

Remark 3.3. *To prove Theorem 1.2 we show a result similar to Lemma 3.2. However, observe that for Theorem 1.2 the sets considered in (2) are not allowed in the families. To construct families similar to $\mathcal{H}_1, \dots, \mathcal{H}_r$, we need to proceed slightly differently using the bound on n, see [13] for more details.*

We now present the proof of Theorem 1.3.

Proof. We will in fact prove the following slightly stronger statement.

($\star$) Let $n, t \geqslant 1$, and $r \geqslant 2$ be integers and let $\mathcal{F}_1, \dots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ be non-empty r-cross t-intersecting families with maximal necessary intersection point at most a. Then, it holds that

$$\sum_{j\in[r]} |\mathcal{F}_j| \leqslant \max_{a_*\in[t,a]} \Big\{ |\mathcal{A}(n, a_*, t)| + \sum_{j\in[r-1]} |\mathcal{B}(n, a_*)| \Big\}. \tag{5}$$

We perform an induction on r. The beginning is the same for the induction start and the induction step. Let $a \in [n]$ and $r \geqslant 2$ and let $\mathcal{F}_1, \dots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ be non-empty families such that

1. they are r-cross t-intersecting with maximal necessary intersection point at most a,
2. they maximise $\sum_{j\in[r]} |\mathcal{F}_j|$ among all families satisfying (1), and
3. their maximal necessary intersection point is minimal among those families that fulfil (1) and (2).

It is easy to see that the properties (1), (2), and (3) are preserved when shifting, so we may assume that $\mathcal{F}_1, \dots, \mathcal{F}_r$ are shifted. Denote the maximal necessary intersection point of $\mathcal{F}_1, \dots, \mathcal{F}_r$ by a_* and observe that if $a_* = t$, we are done. So we assume that $a_* \geqslant t + 1$.

If for all $i \in [r]$ we have that $\mathcal{F}_i \setminus \mathcal{F}_i(a_*) \neq \emptyset$, then Lemma 3.2 yields the existence of $\mathcal{H}_1, \dots, \mathcal{H}_r \subseteq \mathscr{P}([n])$ which still satisfy (1) and (2), but have a smaller necessary intersection point. This contradicts our choice of $\mathcal{F}_1, \dots, \mathcal{F}_r$. Therefore, there is an $i_0 \in [r]$ such that $\mathcal{F}_{i_0} \setminus \mathcal{F}_{i_0}(a_*) = \emptyset$. Without loss of generality, assume that $i_0 = r$.

So all sets in $\mathcal{F}_r$ depend on a_*. Assume that there is a $b \in [a_* - 1]$ and $F \in \mathcal{F}_r$ such that $b \notin F$. As F_r is shifted, we have that $\sigma_{ba_*}(F) \in \mathcal{F}_r$, but this set does not depend on a_*. Hence, for every $F \in \mathcal{F}_r$, we have $[a_*] \subseteq F$, in other words $\mathcal{F}_r \subseteq \mathcal{B}(n, a_*)$.

For the base case of the induction, $r = 2$, notice that since a_* is the maximal necessary intersection point, every $F_1 \in \mathcal{F}_1$ has at least t elements in $[a_*]$. This yields $\mathcal{F}_1 \subseteq \mathcal{A}(n, a_*, t)$ and hence

$$|\mathcal{F}_1| + |\mathcal{F}_2| \leqslant |\mathcal{A}(n, a_*, t)| + |\mathcal{B}(n, a_*)|\,,$$

which finishes the proof of the induction start.

For $r \geqslant 3$, observe that the families $\mathcal{F}_1, \dots, \mathcal{F}_{r-1}$ are $(r-1)$-cross t-intersecting families with maximal necessary intersection point at most a_* (which maximise $\sum_{j\in[r-1]} |\mathcal{F}_j|$ among all $(r-1)$-cross t-intersecting families with maximal necessary intersection point at most a_*). Thus, the induction hypothesis implies that there is an $a_{**} \in [a_*]$ such that

$$\sum_{j\in[r-1]} |\mathcal{F}_j| \leqslant |\mathcal{A}(n, a_{**}, t)| + (r-2)|\mathcal{B}(n, a_{**})|\,.$$

Since $\mathcal{F}_r \subseteq \mathcal{B}(n, a_*) \subseteq \mathcal{B}(n, a_{**})$, this entails

$$\sum_{j\in[r]} |\mathcal{F}_j| \leqslant |\mathcal{A}(n, a_{**}, t)| + (r-1)|\mathcal{B}(n, a_{**})|$$

which concludes the induction step and the proof. □

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