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ON THE EXPECTED NUMBER OF REAL ROOTS OF RANDOM POLYNOMIALS ARISING FROM EVOLUTIONARY GAME THEORY[*]

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Abstract. In this paper, we obtain finite estimates and asymptotic formulas for the expected number of real roots of two classes of random polynomials arising from evolutionary game theory. As a consequence of our analysis, we achieve an asymptotic formula for the expected number of internal equilibria in multi-player two-strategy random evolutionary games. Our results contribute both to evolutionary game theory and random polynomial theory.

Keywords. evolutionary game theory, random polynomials, multi-player two-strategy games, equilibrium points

AMS subject classifications. 91A22; 60F99; 92D25

1. Introduction

1.1. Motivation from evolutionary game theory and random polynomial theory

Large random systems, in particular random polynomials and systems of random polynomials, arise naturally in a variety of applications in physics (such as in quantum chaotic dynamics [3]), biology (such as in theoretical ecology [23], evolutionary game theory and population dynamics [19]), computer science (such as in the theory of computational complexity [33]) and in social sciences (such as in social/complex networks [25]). They are indispensable in the modelling and analysis of complex systems in which very limited information is available or where the environment changes so rapidly and frequently that one cannot describe the payoffs of their inhabitants' interactions [16,18–20,24]. The study of statistics of equilibria in large random systems provides important insight into the understanding of the underlying physical, biological and social system such as the complexity-stability relationship in ecosystems [17, 20, 23, 32], biodiversity and maintenance of polymorphism in multi-player multi-strategy games [19], and the learning dynamics [18]. A key challenge in such study is due to the large (but finite) size of the underlying system (such as the population in an ecological system, the number of players and strategies in an evolutionary game and the number of nodes and connections in a social network). Understanding the behaviour of the system at finite size or characterizing its asymptotic behaviour when the size tends to infinity are of both theoretical and practical interest, see for instance [29, 31].

In this paper we are interested in the number of internal equilibria in $(n+1)$-player two-strategy random evolutionary games as in [8, 9, 11, 12]. We consider an infinitely large population that consists of individuals using two strategies, A and B. We denote by y, $0 \le y \le 1$, the frequency of strategy A in the population. The frequency of strategy B is thus $(1-y)$. The interaction of the individuals in the population is in randomly selected groups of $(n+1)$ participants, that is, they interact and obtain their fitness from $(n+1)$-player games. In this paper, we consider symmetric games where the payoffs do

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not depend on the ordering of the players. Suppose that a_i (respectively, b_i) is the payoff that an A-strategist (respectively, B) achieves when interacting with a group of n other players consisting i ($0 \le i \le n$) A strategists and $(n-i)$ B strategists. In other words, the payoff matrix is given by

Opossing A players	0	1	...	i	...	n
A	a_0	a_1	...	a_i	...	a_n
B	b_0	b_1	...	b_i	...	b_n

The average payoffs (fitnesses) of strategies A and B are respectively given by

$$\pi_A = \sum_{i=0}^{n} a_i \binom{n}{i} y^i (1-y)^{n-i} \quad \text{and} \quad \pi_B = \sum_{i=0}^{n} b_i \binom{n}{i} y^i (1-y)^{n-i}.$$

Internal equilibria in $(n+1)$-player two-strategy games can be derived using the replicator dynamic approach [19] or the definition of an evolutionary stable strategy, see e.g., [4]. They are those points $0 < y < 1$ (note that $y = 0$ and $y = 1$ are trivial equilibria in the replicator dynamics) such that the fitnesses of the two strategies are the same $\pi_A = \pi_B$, that is

$$\sum_{i=0}^{n} \xi_i \binom{n}{i} y^i (1-y)^{n-i} = 0 \quad \text{where} \quad \xi_i = a_i - b_i.$$

In the literature, the sequence of the difference of payoffs $\{\xi_i\}_i$ is called the gain sequence [1, 30]. Dividing the above equation by $(1-y)^n$ and using the transformation $x = \frac{y}{1-y}$, we obtain the following polynomial equation for x ($x > 0$)

$$P_n(x) := \sum_{i=0}^{n} \xi_i \binom{n}{i} x^i = 0, \tag{1.1}$$

In random games, the payoff entries $\{a_i\}_i$ and $\{b_i\}$ are random variables, thus so are the gain sequence $\{\xi_i\}_i$. Therefore, the expected number of internal equilibria in a $(n+1)$-player two-strategy random game is the same as the expected number of positive roots of the random polynomial P_n, which is half of the expected number of the real roots of P_n due to the symmetry of the distributions. This connection between evolutionary game theory and random polynomial theory has been revealed and exploited in recent serie of papers [8, 9, 11, 12]. It has been shown that, if $\{\xi_i\}_i$ are i.i.d normal (Gaussian) distributions then [8, 9]

$$\frac{2n}{\pi\sqrt{2n-3}} \le \mathbb{E} N_n \le \frac{2\sqrt{n}}{\pi} \left(1 + \log 2 + \frac{1}{2} \log(n)\right) \quad \forall n, \tag{1.2}$$

where N_n is the number of real roots of P_n. We emphasize that (1.2) is true for all finite group size n, which is useful for practical purposes, for instance when doing simulations. A direct consequence of this estimate is the following asymptotic limit

$$\lim_{n \to \infty} \frac{\log \mathbb{E} N_n}{\log n} = \frac{1}{2}. \tag{1.3}$$

On the other hand, the expected number of real roots of random polynomials has been a topic of intensive research over the last hundred years. Three most well-known classes studied in the literature are

(i) Kac polynomials: $\sum_{i=0}^{n} \xi_i x^i$,
(ii) Weyl (or flat) polynomials: $\sum_{i=0}^{n} \frac{1}{i!}\xi_i x^i$,

(iii) Elliptic (or binomial) polynomials: $\sum_{i=0}^{n} \sqrt{\binom{n}{i}}\xi_i x^i$.

When $\{\xi_i\}$ are Gaussian distributions, it has been proved, see for instance [13], that

$$\mathbb{E}N_n = \begin{cases} \frac{2}{\pi}\log(n) + C_1 + \frac{2}{n\pi} + O(1/n^2) & \text{for Kac polynomials,} \\ \sqrt{n} & \text{for elliptic polynomials,} \\ \sqrt{n}\left(\frac{2}{\pi} + o(1)\right) & \text{for Weyl polynomials.} \end{cases} \tag{1.4}$$

We refer the reader to standard monographs [2, 15] for a detailed account and [7, 26] for recent developments of the topic. The asymptotic formulas (1.4) are much stronger than the limit (1.3) because they provide precisely the leading order of the quantity $\mathbb{E}N_n$. A natural question arises: *can one obtain an asymptotic formula akin to* (1.4) *for the random polynomial from random multi-player evolutionary games?* It has been conjectured, in a study on computational complexity, by Emiris and Galligo [14] and formally shown in [12] that

$$\mathbb{E}N_n \sim \sqrt{2n} + O(1). \tag{1.5}$$

In this paper, we rigorously prove generalizations of the asymptotic formula (1.5) and of the finite group size estimates (1.2) for two more general classes of random polynomials

$$P_n^{(\gamma)}(x) = \sum_{i=0}^{n} \xi_i \binom{n}{i}^{\gamma} x^i \quad \text{and} \quad P_n^{(\alpha,\beta)}(x) = \sum_{i=0}^{n} \binom{n+\alpha}{n-i}^{\frac{1}{2}} \binom{n+\beta}{i}^{\frac{1}{2}} \xi_i x^i. \tag{1.6}$$

Here $\gamma > 0, \alpha, \beta > -1$ are given real numbers, $\{\xi_i\}_{i=0,\ldots,n}$ are standard normal i.i.d. random variables. The class of random polynomials P_n arising from evolutionary game theory is a special case of both $P_n^{(\gamma)}$ (when $\gamma = 1$) and $P_n^{(\alpha,\beta)}$ (when $\alpha = \beta = 0$). For general values of α, β and γ, $P_n^{(\gamma)}$ and $P_n^{(\alpha,\beta)}$ are related to more complex models in evolutionary game theory where the gain sequence $\{\xi_i\}_i$ depends not only on i but also on group size n. An example for such scenarios is in a public goods game in which the benefit from cooperation are shared among all group members rather than accruing to each individual [21, 28, 29]. From a mathematical point of view, the class $P_n^{(\gamma)}$ is a natural extension of P_n and covers both Kac polynomials and elliptic polynomials as special cases (corresponding to $\gamma = 0$ and $\gamma = \frac{1}{2}$ respectively). In addition, as previously shown in [9], P_n is connected to Legendre polynomials. As will be shown in Section 3.1, the class $P_n^{(\alpha,\beta)}$ is intrinsically related to Jacobi polynomials, which contain Legendre polynomials as special cases. The link between P_n and Legendre polynomials in [9] is extended to that of between $P_n^{(\alpha,\beta)}$ and Jacobi polynomials in the present paper.

1.2. Main results

Throughout this paper, we suppose that $\{\xi_i\}$ are i.i.d standard normal distributions. We denote by $\mathbb{E}N_n^{(\gamma)}$ and $\mathbb{E}N_n^{(\alpha,\beta)}$ the expected number of real roots of $P_n^{(\gamma)}$ and $P_n^{(\alpha,\beta)}$ respectively. The main results of the present paper are the following theorems.

THEOREM 1.1 (Estimates of $EN_n^{(\alpha,\beta)}$ for any n). *Suppose that* $\alpha, \beta > -1$.

(1) *(estimates in terms of roots of Jacobi polynomial)* *Let $0 < s_{n,max} < 1$ be the maximum root of the Jacobi's polynomial of degree n as defined in (3.1) . Then*

$$\sqrt{n} \leq \mathbb{E} N_n^{(\alpha,\beta)} \leq \sqrt{n} \frac{1 + s_{n,max}}{1 - s_{n,max}}. \tag{1.7}$$

(2) *(explicit estimates for finite n) For all $\alpha = \beta > -1$, it holds that*

$$\frac{2}{\pi} \sqrt{\frac{n(n+2\alpha)}{2n+2\alpha-1}} \leq \mathbb{E} N_n^{(\alpha,\alpha)} \leq \frac{2\sqrt{n}}{\pi} \left(1 + \log(2) + \frac{1}{2} \log \frac{n+\alpha}{1+\alpha} \right). \tag{1.8}$$

Theorem 1.1, which combines Theorems 3.1 and 3.2, provides lower and upper bounds for $EN_n^{(\alpha,\beta)}$ in terms of the group size n. It is only applicable to the class $P_n^{\alpha,\beta}$ since our proof makes use of a connection between $P_n^{\alpha,\beta}$ and Jacobi polynomials. In addition, in the second part, we use a symmetry condition on the coefficients of the polynomial $P_n^{(\alpha,\beta)}$ which requires $\alpha = \beta$. The next result characterizes the asymptotic limits, as the group size n tends to infinity, of both $\mathbb{E} N_n^{(\gamma)}$ and $\mathbb{E} N_n^{(\alpha,\beta)}$.

THEOREM 1.2 (Asymptotic behaviour as $n \to +\infty$). *We have*

$$\mathbb{E} N_n^{(\gamma)} \sim \sqrt{2\gamma n}(1 + o(1)) \quad and \quad \mathbb{E} N_n^{(\alpha,\beta)} \sim \sqrt{2n}(1 + o(1)) \quad as \quad n \to \infty. \tag{1.9}$$

As a consequence, there is a phase transition (discontinuity) in the expected number of roots of $\mathbb{E} N_n^{(\gamma)}$ as a function of γ as $n \to \infty$

$$\mathbb{E} N_n^{(\gamma)} \sim \begin{cases} \frac{2}{\pi} \log(n) & for \ \ \gamma = 0, \\ \sqrt{2\gamma n} & for \ \ \gamma > 0. \end{cases} \tag{1.10}$$

Our study on the expected number of real roots of $P_n^{(\gamma)}$ and $P_n^{(\alpha,\beta)}$ contributes to both evolutionary game theory and random polynomial theory. From an evolutionary game theory point of view, our results show surprisingly that in random multiplayer evolutionary games, one expects much less number of equilibria, which is proportional to the square root of the group size, than in deterministic games (recalling that the expected number of internal equilibria is the same as the expected number of positive roots, which is half of the expected number of real roots). In addition, since for a polynomial equation, the number of stable equilbria is half of that of equilibria, our results also apply to stable equilibria. From a random polynomial theory point of view, the present paper introduces two meaningful classes of random polynomials that have not been studied in the literature. In particular, the fact that the asymptotic behavour of $\mathbb{E} N_n^{(\alpha,\beta)}$ is independent from α and β is rather unexpected and is interesting on its own right. In addition the phase transition phenomenon (1.10), to the best of our knowledge, is shown for the first time.

1.3. Organization of the paper

The rest of the paper is organized as follows. In Section 2 we recall the Kac-Rice formula for computing the expected number of real roots of a random polynomial. In Section 3, we establish connections between $P_n^{(\alpha,\beta)}$ and Jacobi polynomials and prove Theorem 1.1. Proof of Theorem 1.2 is presented in Section 4 and Section 5. In Section 6 we provide further discussions and outlook. Finally, detailed proofs of technical lemmas are given in Appendix 7.

2. Kac-Rice formula In this section, we recall the celebrated Kac-Rice formula for computing the expected number of real roots of a random polynomials, which is the starting point of our analysis. Consider a general random polynomial

$$p_n(x) = \sum_{i=0}^{n} a_i \xi_i x^i.$$

Let $\{\xi\}$ are standard i.i.d. random variables. Let $\mathbb{E}N_n(a,b)$ be the expected number of real roots of p_n in the interval (a,b). Then the Kac-Rice formula is given by, see for instance [13]

$$\mathbb{E}N_n(a,b) = \frac{1}{\pi} \int_a^b \frac{\sqrt{A_n(x)M_n(x) - B_n^2(x)}}{M_n(x)} \, dx \qquad (2.1)$$

where

$$M_n(x) = \mathrm{var}(p_n(x)), \quad A_n(x) = \mathrm{var}(p_n'(x)), \quad B = \mathrm{cov}(p_n(x)p_n'(x)).$$

We can find M_n, A_n and B_n explicitly in terms of the coefficients $\{a_i\}$ of p_n as follows. Since $\{\xi_i\}$ are standard i.i.d. random variables, we have

$$p_n'(x) = \sum_{i=0}^{n} a_i i \xi_i x^{i-1}, \quad p_n(x)^2 = \sum_{i,j=0}^{n} a_i a_j \xi_i \xi_j x^{i+j}, \quad p_n(x)p_n'(x) = \sum_{i,j=0}^{n} a_i a_j i \xi_i \xi_j x^{i+j-1},$$

$$\mathbb{E}(p_n(x)) = \sum_{i=0}^{n} a_i x^i \mathbb{E}(\xi_i) = 0, \quad \mathbb{E}(p_n'(x)) = 0$$

$$M_n(x) = \mathrm{var}(p_n(x)) = \mathbb{E}(p_n^2(x)) - (\mathbb{E}(p_n(x)))^2 = \sum_{i,j=0}^{n} a_i a_j x^{i+j} \mathbb{E}(\xi_i \xi_j) = \sum_{i=0}^{n} a_i^2 x^{2i},$$

$$A_n(x) = \mathrm{var}(p_n'(x)) = \mathbb{E}((p_n'(x))^2) - (\mathbb{E}(p_n'(x)))^2$$

$$= \sum_{i,j=0}^{n} a_i a_j ij x^{i+j-2} \mathbb{E}(\xi_i \xi_j) = \sum_{i=0}^{n} a_i^2 i^2 x^{2(i-1)},$$

$$B_n(x) = \mathrm{cov}(p_n(x)p_n'(x)) = \mathbb{E}(p_n(x)p_n'(x)) = \sum_{i,j=0}^{n} i a_i a_j x^{i+j} \mathbb{E}(\xi_i \xi_j) = \sum_{i=0}^{n} i a_i^2 x^{2i-1}.$$

In conclusion, we have

$$M_n(x) = \sum_{i=0}^{n} a_i^2 x^{2i}, \quad A_n(x) = \sum_{i=0}^{n} a_i^2 i^2 x^{2(i-1)}, \quad B_n(x) = \sum_{i=0}^{n} i a_i^2 x^{2i-1}. \qquad (2.2)$$

Furthermore, the following relations between M_n, A_n and B_n, which follow directly from the above formulas, will also be used in the subsequent sections:

$$B_n(x) = \frac{1}{2} M_n'(x), \quad A_n(x) = \frac{1}{4x}\left(xM_n'(x)\right)',$$

$$\frac{A_n(x)M_n(x) - B_n^2(x)}{M_n^2(x)} = \frac{1}{4}\left(\frac{M_n''(x)}{M_n(x)} + \frac{1}{x}\frac{M_n'(x)}{M_n(x)} - \left(\frac{M_n'(x)}{M_n(x)}\right)^2\right)$$

$$= \frac{1}{4}\left(\frac{1}{x}\frac{M_n'(x)}{M_n(x)} + \left(\frac{M_n'(x)}{M_n(x)}\right)'\right) = \frac{1}{4x}\left(x\frac{M_n'(x)}{M_n(x)}\right)',$$

where the prime $'$ notation denotes a derivative with respect to the variable x.

Let $\mathbb{E}N_n^{(\gamma)}(a,b)$ and $\mathbb{E}N_n^{(\alpha,\beta)}(a,b)$ be respectively the expected number of real roots of $P_n^{(\gamma)}$ and of $P_n^{(\alpha,\beta)}$ in a given interval $[a,b]$. Applying (2.1)-(2.2) to $P_n^{(\gamma)}$ and to $P_n^{(\alpha,\beta)}$, we obtain the following common formula for $\mathbb{E}N_n^{(\gamma)}(a,b)$ and $\mathbb{E}N_n^{(\gamma)}(a,b)$ but with different triples $\{A_n, B_n, M_n\}$

$$\mathbb{E}N_n^{(*)}(a,b) = \frac{1}{\pi} \int_a^b \frac{\sqrt{A_n(x)M_n(x) - B_n^2(x)}}{M_n(x)} \, dx, \qquad (2.3)$$

where $(*) \in \{(\gamma), (\alpha,\beta)\}$. For $EN_n^{(\gamma)}(a,b)$:

$$M_n(x) = \sum_{k=0}^n \binom{n}{k}^{2\gamma} x^{2k}, \ A_n(x) = \sum_{k=0}^n k^2 \binom{n}{k}^{2\gamma} x^{2(k-1)}, \ B_n(x) = \sum_{k=0}^n k \binom{n}{k}^{2\gamma} x^{2k-1}. \qquad (2.4)$$

For $EN_n^{(\alpha,\beta)}$:

$$M_n(x) = \sum_{k=0}^n \binom{n+\alpha}{n-k}\binom{n+\beta}{k} x^{2k}, \ A_n(x) = \sum_{k=0}^n k^2 \binom{n+\alpha}{n-k}\binom{n+\beta}{k} x^{2(k-1)},$$

$$B_n(x) = \sum_{k=0}^n k \binom{n+\alpha}{n-k}\binom{n+\beta}{k} x^{2k-1}. \qquad (2.5)$$

By writing $\mathbb{E}N_n^{(\gamma)}$ or $\mathbb{E}N_n^{(\alpha,\beta)}$ it becomes clear which class of random polynomials is under consideration; therefore, for notational simplicity, we simply write $\{A_n, B_n, M_n\}$ without superscripts (γ) or (α,β). The above Kac-Rice formulas are starting points for our analysis. The difficulty now is to analyze the integrand in (2.3) for each class of random polynomials.

3. Finite group-size estimates   In this section, we show a connection between the class $P_n^{(\alpha,\beta)}$ and Jacobi polynomials which extends that of between P_n and Legendre polynomials in [9]. Using this connection, we will prove Theorem 1.1 on the estimates of $\mathbb{E}N_n^{(\alpha,\beta)}$ for finite n.

3.1. Connections to Jacobi polynomials and finite estimates of $\mathbb{E}N_n^{(\alpha,\beta)}$
We recall that the Jacobi polynomial is given by

$$J_n^{(\alpha,\beta)}(x) = \sum_{i=0}^n \binom{n+\alpha}{n-i}\binom{n+\beta}{i}\left(\frac{x-1}{2}\right)^i \left(\frac{x+1}{2}\right)^{n-i}. \qquad (3.1)$$

If $\alpha = \beta$, Jacobi's polynomial $J_n^{(\alpha,\beta)}(x)$ is called an ultraspherical polynomial. Legendre's polynomial is a special case of Jacobi's polynomial when $\alpha = \beta = 0$. It is well-known that the zeros of $J_n^{(\alpha,\beta)}$ are real, distinct and are located in the interior of the interval $[-1,1]$ [34]. The following lemma links $M_n^{(\alpha,\beta)}$ to Jacobi polynomials. Its proof is given in Appendix 7.

LEMMA 3.1. *It holds that*

$$M_n^{(\alpha,\beta)}(x) = (1-x^2)^n J_n^{(\alpha,\beta)}\left(\frac{1+x^2}{1-x^2}\right). \qquad (3.2)$$

 The following theorem provides estimates of $\mathbb{E}N_n^{(\alpha,\beta)}$ in terms of the maximum root of the Jacobi polynomial.

THEOREM 3.1. *Let $0 < s_{n,max} < 1$ be the maximum root of the Jacobi's polynomial of degree n. Then the expected number of real roots, $\mathbb{E}N_n^{(\alpha,\beta)}$, of $P_n^{(\alpha,\beta)}$ satisfies*

$$\sqrt{n} \leq \mathbb{E}N_n^{(\alpha,\beta)} \leq \sqrt{n}\frac{1+s_{n,max}}{1-s_{n,max}}. \tag{3.3}$$

Proof. Let $\{-1 < s_1 < s_2 < \ldots < s_n < 1\}$ be the zeros of the Jacobi polynomial of degree n. Note that $s_k = -s_{n+1-k} < 0$ for $k = 1,\ldots,\lfloor\frac{n}{2}\rfloor$. We deduce from Lemma 3.1 that M_n has $2n$ distinct zeros given by $\{\pm i\sqrt{\frac{1-s_k}{1+s_k}}, \ 1 \leq k \leq n\}$ which are purely imaginary. Thus M_n can be written as

$$M_n(x) = m_n \prod_{k=1}^{n}(x^2 + r_k), \tag{3.4}$$

where m_n is the leading coefficient and for $1 \leq k \leq n$

$$r_k = \frac{1-s_k}{1+s_k} > 0. \tag{3.5}$$

It follows from the properties of $\{s_k\}$ that $r_1 > r_2 > \ldots > r_n > 0$ and $r_k r_{n+1-k} = 1$ for $k = 1,\ldots,\lfloor\frac{n}{2}\rfloor$. Using the representation (3.4) of M_n we have

$$M_n'(x) = 2xm_n\sum_{k=1}^{n}\prod_{j\neq k}(x^2+r_j), \quad \frac{M_n'(x)}{M_n(x)} = \sum_{k=1}^{n}\frac{2x}{x^2+r_k}, \quad \left(x\frac{M_n'(x)}{M_n(x)}\right)' = \sum_{k=1}^{n}\frac{4xr_k}{(x^2+r_k)^2}.$$

Hence the density function can be represented as

$$f_n(x)^2 = \frac{1}{4x}\left(x\frac{M_n'(x)}{M_n(x)}\right)' = \sum_{k=1}^{n}\frac{r_k}{(x^2+r_k)^2}. \tag{3.6}$$

Since $0 < r_n < \ldots < r_1$, we deduce that

$$f_n(x)^2 = \sum_{k=1}^{n}\frac{r_k}{(x^2+r_k)^2} \leq n\frac{r_1}{(x^2+r_n)^2}, \tag{3.7}$$

that is

$$f_n(x) \leq \sqrt{n}\frac{\sqrt{r_1}}{x^2+r_n}.$$

Since

$$\mathbb{E}N_n^{(\alpha,\beta)} = \frac{1}{\pi}\int_{-\infty}^{\infty}f_n(x)\,dx$$

we have

$$\mathbb{E}N_n^{(\alpha,\beta)} \leq \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sqrt{nr_1}}{x^2+r_n}\,dx,$$

that is, since $\int_{-\infty}^{\infty}\frac{1}{x^2+a}\,dx = \frac{\pi}{\sqrt{a}}$ for $a > 0$,

$$\mathbb{E}N_n^{(\alpha,\beta)} \leq \sqrt{n}\sqrt{\frac{r_1}{r_n}}.$$

Since $r_1 r_n = 1$, the above expression can be written as

$$\mathbb{E}N_n^{(\alpha,\beta)} \leq \sqrt{n}\, r_1.$$

From (3.5), we obtain the following upper estimate for $\mathbb{E}N_n^{(\alpha,\beta)}$ in terms of roots of Jacobi's polynomials

$$\mathbb{E}N_n^{(\alpha,\beta)} \leq \sqrt{n}\frac{1-s_1}{1+s_1} = \sqrt{n}\frac{1+s_n}{1-s_n}.$$

It remains to derive a lower bound for $\mathbb{E}N_n^{(\alpha,\beta)}$. From (3.6) and using Cauchy-Schwartz inequality we have

$$\left(\sum_{k=1}^{n}\frac{\sqrt{r_k}}{x^2+r_k}\right)^2 \leq n\sum_{k=1}^{n}\frac{r_k}{(x^2+r_k)^2} = nf_n(x)^2,$$

from which we deduce

$$f_n(x) \geq \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{\sqrt{r_k}}{x^2+r_k}.$$

Therefore,

$$\mathbb{E}N_n^{(\alpha,\beta)} = \frac{1}{\pi}\int_{-\infty}^{\infty}f_n(x)\,dx \geq \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\int_{-\infty}^{\infty}\frac{\sqrt{r_k}}{\pi(x^2+r_k)}\,dx = \sqrt{n}.$$

It is worth noticing that this lower bound is independent of α,β. This completes the proof of the theorem. $\square$

The following theorem provides an explicit finite estimate for $\mathbb{E}N_n^{(\alpha,\beta)}$ in the ultraspherical case. It generalizes a previous result for $\alpha = 0$ (see (1.2)) obtained in [9].

THEOREM 3.2. *Consider the ultraspherical case (i.e., $\alpha = \beta$). We have*

$$\frac{2}{\pi}\sqrt{\frac{n(n+2\alpha)}{2n+2\alpha-1}} \leq \mathbb{E}N_n^{(\alpha,\alpha)} \leq \frac{2\sqrt{n}}{\pi}\left(1+\log(2)+\frac{1}{2}\log\frac{n+\alpha}{1+\alpha}\right). \tag{3.8}$$

As a consequence,

$$\lim_{n\to+\infty}\frac{\log(\mathbb{E}N_n^{(\alpha,\alpha)})}{\log(n)} = \frac{1}{2}. \tag{3.9}$$

Proof. Since $\alpha = \beta$ changing x to $1/x$ and x to $-x$ leaves the distribution of the coefficients of $P_n^{(\alpha,\alpha)}(x)$ invariant. Thus we obtain that

$$\mathbb{E}N_n^{(\alpha,\beta)} = 4\mathbb{E}N_n^{(\alpha,\beta)}(-\infty,-1) = 4\mathbb{E}N_n^{(\alpha,\beta)}(-1,0) = 4\mathbb{E}N_n^{(\alpha,\beta)}(0,1) = 4\mathbb{E}N_n^{(\alpha,\beta)}(1,\infty).$$

It follows from (3.6) that $f_n(x)$ is decreasing on $(0,+\infty)$. Thus for any $x \in [0,1]$, we have

$$f_n(0) = \sqrt{\frac{n(n+\alpha)}{1+\alpha}} \geq f_n(x) \geq f_n(1) = \frac{1}{2}\sqrt{\frac{n(n+2\alpha)}{2n+2\alpha-1}}. \tag{3.10}$$

In addition, since $(x^2 + r_k)^2 \geq 4 r_k x^2$ for all $x > 0$, we also deduce from (3.6) that

$$f_n(x)^2 \leq \frac{n}{4x^2} \quad \text{for} \quad x > 0,$$

that is

$$f_n(x) \leq \frac{\sqrt{n}}{2x} \quad \text{for} \quad x > 0. \tag{3.11}$$

Using the second inequality in (3.10) we obtain the lower bound for $\mathbb{E}N_n^{(\alpha,\beta)}$ as follows

$$\mathbb{E}N_n^{(\alpha,\beta)} = \frac{4}{\pi} \int_0^1 f_n(x)\,dx \geq \frac{4}{\pi} \int_0^1 f_n(1)\,dx = \frac{4f_n(1)}{\pi} = \frac{2}{\pi}\sqrt{\frac{n(n+2\alpha)}{2n+2\alpha-1}}.$$

Using the first inequality in (3.10) and (3.11) we obtain the following upper bound for $\mathbb{E}N_n^{(\alpha,\beta)}$ for any $0 < \gamma < 1$

$$\mathbb{E}N_n^{(\alpha,\beta)} = \frac{4}{\pi} \int_0^1 f_n(x)\,dx = \frac{4}{\pi}\left(\int_0^\gamma f_n(x)\,dx + \int_\gamma^1 f_n(x)\,dx\right)$$

$$\leq \frac{4}{\pi}\left(\int_0^\gamma f_n(0)\,dx + \int_\gamma^1 \frac{\sqrt{n}}{2x}\,dx\right)$$

$$= \frac{4}{\pi}\left(\gamma\sqrt{\frac{n(n+\alpha)}{1+\alpha}} - \frac{\sqrt{n}}{2}\log(\gamma)\right).$$

We choose $\gamma \in (0,1)$ that minimizes the right-hand side of the above expression. That is

$$\gamma = \frac{1}{2}\sqrt{\frac{1+\alpha}{n+\alpha}},$$

which gives

$$\mathbb{E}N_n^{(\alpha,\beta)} \leq \frac{2\sqrt{n}}{\pi}\left(1 + \log(2) + \frac{1}{2}\log\frac{n+\alpha}{1+\alpha}\right).$$

This completes the proof of the theorem. $\square$

REMARK 3.1. *The connections to Jacobi polynomials is crucial in our approach to obtain finite n estimates for the expected number of the real roots of the class $P_n^{\alpha,\beta}$. It is an interesting question to derive such estimates for the class P_n^γ. In the special case $\gamma = 0$, this problem has been studied intensively in the literature [6, 36–38] using the explicit formula*

$$\mathbb{E}(N_n^{(0)}) = \frac{4}{\pi}\int_0^1 \frac{\sqrt{1-h_n^2(x)}}{1-x^2}\,dx$$

where $h_n(x) = (n+1)x^n(1-x^2)/(1-x^{2n+2})$ computed earlier by Kac [22]. The explicit and benign formulation of the density function enables the authors in the mentioned papers to get lower and upper bounds for $\mathbb{E}(N_n^{(0)})$ in terms of the degree n. In the general case, the density function (2.4) is much more complicated, and to the best of our knowledge, there is no existing connection between $P_n^{(\gamma)}$ with a well-known special polynomial as in the class $P_n^{\alpha,\beta}$. As such, currently it is not clear to us how to obtain similar results in Theorem 1.1 for the class $P_n^{(\gamma)}$ for an arbitrary $\gamma > 0$. We leave this interesting problem for future research.

4. Asymptotic behaviour of $EN_n^{(\gamma)}$ In this section, we prove Theorem 1.2 obtaining asymptotic formulas for $\mathbb{E}N_n^{(\gamma)}$.

Strategy of the the proof. Let us first explain the main idea of the proof since it requires a rather delicate analysis. The first observation is that, similarly as the proof of Theorem 3.2, since changing x to $1/x$ and x to $-x$ leaves the distribution of the coefficients of $P_n^{(\gamma)}(x)$ invariant, we have

$$\mathbb{E}N_n^{(\gamma)}(-\infty,-1)=\mathbb{E}N_n^{(\gamma)}(-1,0)=\mathbb{E}N_n^{(\gamma)}(0,1)=\mathbb{E}N_n^{(\gamma)}(1,\infty).$$

Thus $\mathbb{E}N_n^{(\gamma)}=4\mathbb{E}N_n^{(\gamma)}(0,1)$ and it suffices to analyze $\mathbb{E}N_n^{(\gamma)}(0,1)$. We then split the interval $(0,1)$ further into two smaller intervals $(0,\eta)$ and $(\eta,1)$, $\mathbb{E}N_n^{(\gamma)}(0,1)=\mathbb{E}N_n^{(\gamma)}(0,\eta)+\mathbb{E}N_n^{(\gamma)}(\eta,1)$ for a carefully chosen $0<\eta<1$ (which may depend on n) such that $\mathbb{E}N_n^{(\gamma)}(0,\eta)$ is negligible. To select a suitable η, we will use Jensen's inequality (see Lemma 4.1) that provides an upper bound on the number of roots of an analytic function (including polynomials) in an open ball. As such, we obtain $\eta=n^{-3\gamma/4}$ and write

$$\mathbb{E}N_n^{(\gamma)}(0,1)=\mathbb{E}N_n^{(\gamma)}(0,n^{-3\gamma/4})+\mathbb{E}N_n^{(\gamma)}(n^{-3\gamma/4},1).$$

In fact as will be shown, $\mathbb{E}N_n^{(\gamma)}(0,n^{-3\gamma/4})$ is of order $o(\sqrt{n})$, which is negligible (see Proposition 4.1). The next step is to obtain precisely the leading order in $\mathbb{E}N_n^{(\gamma)}(n^{-3\gamma/4},1)$. We recall that by Kac-Rice formula (see Section 2) we have

$$\mathbb{E}N_n^{(\gamma)}(n^{-3\gamma/4},1)=\int_{n^{-3\gamma/4}}^{1}\frac{\sqrt{A_n(x)M_n(x)-B_n^2(x)}}{M_n(x)}\,dx \tag{4.1}$$

where A_n, B_n and M_n are given in (2.4). Therefore, we need to understand thoroughly the asymptotic behaviour of $A_n(x)M_n(x)-B_n^2(x)$ and of $M_n(x)$ in the interval $[n^{-3\gamma/4},1]$. This will be the content of Proposition 4.2. Its proof requires a series of technical lemmas and will be presented in Appendix 7.

We now follow the strategy, starting with Jensen's inequality.

LEMMA 4.1 (Jensen's inequality). *Let $R>r>0$ and f be an analytic function on an open domain that contains the closed disk $\mathbb{B}(R)=\{z\in\mathbb{C}:|z|\leq R\}$. Then the number of roots (including multiplicities) of f in $\mathbb{B}(r)=\{z\in\mathbb{C}:|z|\leq r\}$, denoted by $N_f(r)$, satisfies*

$$N_f(r)\leq\frac{\log\frac{M_R}{M_r}}{\log\frac{R^2+r^2}{2Rr}}, \tag{4.2}$$

where $M_t=\max_{|z|\leq t}|f(z)|$ for $t>0$.

An elementary proof of Jensen's inequality can be found in [27, Section 15.5]. Now we show that $\mathbb{E}N_n^{(\gamma)}(0,n^{-3\gamma/4})$ is negligible as an interesting application of Jensen's inequality.

PROPOSITION 4.1. *We have*

$$\mathbb{E}N_n^{(\gamma)}(0,n^{-3\gamma/4})=o(\sqrt{n}).$$

Proof. We aim to apply (4.2) to $P_n^{(\gamma)}(z)$, which is indeed an entire function. Let $r=n^{-3\gamma/4}$ and $R=n^{-2\gamma/3}$. Then

$$\log\frac{R^2+r^2}{2Rr}\asymp\log n. \tag{4.3}$$

Moreover,

$$M_r = \max_{|z| \leq r} |P_n^{(\gamma)}(z)| \geq |P(0)| = |\xi_0|, \tag{4.4}$$

and

$$M_R = \max_{|z| \leq R} |P_n^{(\gamma)}(z)| \leq \sum_{i=0}^{n} |\xi_i| R^i \binom{n}{i}^\gamma \leq \max_{0 \leq i \leq n} |\xi_i| \times \sum_{i=0}^{n} \left(\sum_{i=0}^{n} R^{i/\gamma} \binom{n}{i} \right)^\gamma$$

$$\leq n \max_{0 \leq i \leq n} |\xi_i| \times (1 + R^{1/\gamma})^{n\gamma}$$

$$\leq n \max_{0 \leq i \leq n} |\xi_i| \times \exp(\gamma O(n^{1/3})). \tag{4.5}$$

We define the event

$$\mathcal{E} = \big\{ \max_{1 \leq i \leq n} |\xi_i| \leq n \big\} \cap \{ n^{-1} \leq |\xi_0| \leq n \}.$$

Since $\{\xi_i\}_{i=0,\dots,n}$ are standard normal i.i.d. random variables,

$$\mathbb{P}(\mathcal{E}) \geq 1 - O(1/n). \tag{4.6}$$

By combining (4.2)–(4.5), we obtain

$$N_n^{(\gamma)}(r)\mathbb{I}(\mathcal{E}) \leq \frac{Cn^{1/3}}{\log n}, \tag{4.7}$$

for some positive constant C, where $N_n^{(\gamma)}(r)$ is the number of roots of $P_n^{(\gamma)}$ in the ball $\mathbb{B}(r)$ defined above. We notice also that $N_n^{(\gamma)}(r) \leq n$. Therefore, by (4.6) and (4.7)

$$\mathbb{E}N_n^{(\gamma)}(0, n^{-3\gamma/4}) \leq \mathbb{E}N_n^{(\gamma)}(r) = \mathbb{E}[N_n^{(\gamma)}(r)\mathbb{I}(\mathcal{E})] + \mathbb{E}[N_n^{(\gamma)}(r)\mathbb{I}(\mathcal{E}^c)]$$

$$\leq \frac{Cn^{1/3}}{\log n} + n\mathbb{P}(\mathcal{E}^c) \leq Cn^{1/3}.$$

As a consequence, we obtain

$$\mathbb{E}N_n^{(\gamma)}(0, n^{-3\gamma/4}) = o(\sqrt{n}). \tag{4.8}$$

□

As already mentioned, the following proposition characterizes precisely the asymptotic behaviour of $A_n M_n - B_n^2$ and of M_n, the two quantities appearing in (4.1). The proof of this proposition is presented in Appendix 7.

PROPOSITION 4.2. *If* $1 \geq x \geq \frac{(\log n)^{4\gamma}}{n^\gamma}$ *then*

$$M_n(x) = \sum_{i=0}^{n} \binom{n}{i}^{2\gamma} x^{2i} = \binom{n}{i_{\gamma,x}}^{2\gamma} x^{2i_{\gamma,x}} \times (\sqrt{\pi} + o(1)) \sqrt{\frac{nx^{1/\gamma}}{\gamma(1 + x^{1/\gamma})^2}}, \tag{4.9}$$

and

$$A_n(x)M_n(x) - B_n^2(x) = \binom{n}{i_{\gamma,x}}^{2\gamma} x^{4i_{\gamma,x}-2} \times \left(\frac{\pi}{2} + o(1) \right) \left(\frac{nx^{1/\gamma}}{\gamma(1 + x^{1/\gamma})^2} \right)^2, \tag{4.10}$$

where $i_{\gamma,x} = [nt_{\gamma,x}]$ with $t_{\gamma,x} = \frac{x^{1/\gamma}}{1+x^{1/\gamma}}$.

We are now ready to prove the asymptotic behaviour of $\mathbb{E}N_n^{(\gamma)}$ (the first part of (1.9) in Theorem 1.2).

Proof. [Proof of asymptotic formula of $\mathbb{E}N_n^{(\gamma)}$] From Proposition 4.1, Proposition 4.2 and Kac-Rice formula, we get

$$\mathbb{E}N_n^{(\gamma)}(0,1) = \mathbb{E}N_n^{(\gamma)}(0,n^{-3\gamma/4}) + \mathbb{E}N_n^{(\gamma)}(n^{-3\gamma/4},1)$$

$$= \frac{1}{\pi}\int_{n^{-3\gamma/4}}^{1} \frac{\sqrt{A_n(x)M_n(x) - B_n^2(x)}}{M_n(x)}\,dx + o(\sqrt{n})$$

$$= \frac{\sqrt{n}}{\sqrt{2\pi}}\int_0^1 \frac{x^{\frac{1}{2\gamma}-1}}{\sqrt{\gamma}\left(1+x^{1/\gamma}\right)}\,dx + o(\sqrt{n})$$

$$= \frac{\sqrt{n}}{\sqrt{2\pi}} \times \frac{2\sqrt{\gamma}\pi}{4} + o(\sqrt{n}) = \frac{\sqrt{2\gamma n}}{4} + o(\sqrt{n}),$$

where the last line follows from the change of variable $u = x^{1/(2\gamma)}$ and the equality $\int_0^1 \frac{du}{1+u^2} = \frac{\pi}{4}$. Hence

$$\mathbb{E}N_n^{(\gamma)} = 4\mathbb{E}N_n^{(\gamma)}(0,1) = \sqrt{2\gamma n} + o(\sqrt{n}).$$

The proof is complete. □

5. Asymptotic behaviour of $EN_n^{(\alpha,\beta)}$

This section deals with the asymptotic formula of $\mathbb{E}N_n^{(\alpha,\beta)}$ (the second part of (1.9) in Theorem 1.2). The strategy of the proof is as follows. We will first relate the asymptotic behaviour of $\mathbb{E}N_n^{(\alpha,\beta)}$ for general (α,β) to that of $\mathbb{E}N_n^{(0,0)}$ for $\alpha = \beta = 0$. We then exploit the relation that $\mathbb{E}N_n^{(0,0)} = \mathbb{E}N_n^{(1)}$ and use the result from the previous section.

The negligible interval $[0,n^{-3/4}]$. We use the same argument as in Proposition 4.1. The estimate for M_R can be replaced as

$$M_R = \max_{|z|\le R}|P_n^{(\alpha,\beta)}(z)| \le \sum_{i=0}^{n}|\xi_i|R^i\binom{n+\alpha}{n-i}^{\frac{1}{2}}\binom{n+\beta}{i}^{\frac{1}{2}}$$

$$\le \max_{0\le i\le n}|\xi_i|(n+|\alpha|)^{|\alpha|}(n+|\beta|)^{|\beta|} \times \sum_{i=0}^{n}R^i\binom{n}{i}$$

$$\le \max_{0\le i\le n}|\xi_i| \times \exp(O(n^{1/3})),$$

where for the second line we used the inequality that $\binom{n+\alpha}{k} \le \binom{n}{k}(n+|\alpha|)^{2|\alpha|}$. Then by repeating the same argument in Proposition 4.1, we can show that

$$\mathbb{E}[N_n^{(\alpha,\beta)}(0,n^{-3/4})] = o(\sqrt{n}). \tag{5.1}$$

The main interval $[n^{-3/4},1]$. We first study the coefficients. It follows from Stirling formula that as $i \wedge (n-i) \to \infty$,

$$a_i^{(\alpha,\beta)} = \binom{n+\alpha}{n-i}\binom{n+\beta}{i}$$

$$= (1+o(1))\sqrt{\frac{(n+\alpha)(n+\beta)}{4\pi^2 i(i+\alpha)(n-i)(n+\beta-i)}} \exp\left((n+\alpha)I(\tfrac{\alpha+i}{n+\alpha}) + (n+\beta)I(\tfrac{i}{n+\beta})\right)$$

$$= (1+o(1))\frac{n}{2\pi i(n-i)} \exp\left((n+\alpha)I(\tfrac{\alpha+i}{n+\alpha}) + (n+\beta)I(\tfrac{i}{n+\beta})\right),$$

where $I(t) = -t\log t + (t-1)\log(1-t)$. By Taylor expansion,

$$I(\tfrac{i+\alpha}{n+\alpha}) = I(\tfrac{i}{n}) + I'(\tfrac{i}{n})\left(\frac{i+\alpha}{n+\alpha} - \frac{i}{n}\right) + O(I''(i/n))n^{-2}$$

$$= I(\tfrac{i}{n}) + I'(\tfrac{i}{n})\frac{\alpha(n-i)}{n^2} + O(\tfrac{1}{i(n-i)}).$$

Note that $I''(t) = -t^{-1}(1-t)^{-1}$. Therefore, as $i \wedge (n-i) \to \infty$,

$$(n+\alpha)I(\tfrac{i+\alpha}{n+\alpha}) - nI(\tfrac{i}{n}) = (1+o(1))\left(\alpha I(\tfrac{i}{n}) + \alpha I'(\tfrac{i}{n})\frac{(n-i)}{n}\right). \tag{5.2}$$

Similarly,

$$(n+\beta)I(\tfrac{i}{n+\beta}) - nI(\tfrac{i}{n}) = (1+o(1))\left(\beta I(\tfrac{i}{n}) - \beta I'(\tfrac{i}{n})\frac{i}{n}\right).$$

Hence,

$$a_i^{(\alpha,\beta)} = (1+o(1))\frac{n}{2\pi i(n-i)}\exp\left((\alpha+\beta)I(\tfrac{i}{n}) + I'(\tfrac{i}{n})\frac{\alpha(n-i)-\beta i}{n}\right)\exp(2nI(\tfrac{i}{n})),$$

so that

$$a_i^{(\alpha,\beta)} = (1+o(1))\exp\left((\alpha+\beta)I(\tfrac{i}{n}) + I'(\tfrac{i}{n})\frac{\alpha(n-i)-\beta i}{n}\right)a_i^{(0,0)}$$

$$= (1+o(1))h_{(\alpha,\beta)}(\tfrac{i}{n})a_i^{(0,0)},$$

where for $t \in (0,1)$

$$h_{(\alpha,\beta)}(t) = (\alpha+\beta)I(t) + I'(t)(\alpha(1-t) - \beta t).$$

Suppose that $x \in [n^{-3/4}, 1]$. In the case $(\alpha,\beta) = (0,0)$, or equivalently the case $\gamma = 1$, in Lemma 7.2 below, we show that the terms $a_i^{(0,0)}x^i$ attain the maximum around $i = i_x \pm i_x^{3/4}$ with $i_x = [nx/(x+1)]$, and the other terms are negligible. Here, the asymptotic behavior of $a_i^{(\alpha,\beta)}$ differs from that of $a_i^{(0,0)}$ only on the term $h_{(\alpha,\beta)}(\tfrac{i}{n})$ which is minor compared with $a_i^{(0,0)}$. Hence, using exactly the same analysis in Lemma 7.2, we can also show that these terms $a_i^{(\alpha,\beta)}x^i$ when $|i - i_x| \geq i_x^{3/4}$ are negligible. Therefore,

$$M_n^{(\alpha,\beta)}(x) = (1+o(1))\sum_{i:|i-i_x|\leq i_x^{3/4}} a_i^{(\alpha,\beta)}x^i = (1+o(1))\sum_{i:|i-i_x|\leq i_x^{3/4}} h_{(\alpha,\beta)}(\tfrac{i}{n})a_i^{(0,0)}x^i$$

$$= (1+o(1))h_{(\alpha,\beta)}(\tfrac{x}{x+1})\sum_{i:|i-i_x|\leq i_x^{3/4}} a_i^{(0,0)}x^i = (1+o(1))h_{(\alpha,\beta)}(\tfrac{x}{x+1})M_n^{(0,0)}(x),$$

since when $|i - i_x| \leq i_x^{3/4}$, we have $h_{(\alpha,\beta)}(\tfrac{i}{n}) = (1+o(1))h_{\alpha,\beta}(\tfrac{x}{x+1})$. Similarly,

$$A_n^{(\alpha,\beta)}(x) = (1+o(1))h_{(\alpha,\beta)}(\tfrac{x}{x+1})A_n^{(0,0)}(x), \quad B_n^{(\alpha,\beta)}(x) = (1+o(1))h_{(\alpha,\beta)}(\tfrac{x}{x+1})B_n^{(0,0)}(x).$$

Thus for $x \in [n^{-3/4}, 1]$,

$$f_n^{(\alpha,\beta)}(x) = (1+o(1))f_n^{(0,0)}(x),$$

and hence

$$\mathbb{E}N_n^{(\alpha,\beta)}(n^{-3/4},1) = (1+o(1))\mathbb{E}N_n^{(0,0)}(n^{-3/4},1) = (1+o(1))\frac{\sqrt{2n}}{4}. \qquad (5.3)$$

Combining (5.1) and (5.3), we obtain that $\mathbb{E}N_n^{(\alpha,\beta)}(0,1) = (1+o(1))\frac{\sqrt{2n}}{4}$, and hence

$$\mathbb{E}N_n^{(a,b)} = 4\mathbb{E}N_n^{(a,b)}(0,1) = (1+o(1))\sqrt{2n}.$$

6. Summary and outlook

In this paper, we have proved asymptotic formulas for the expected number of real roots of two general class of random polynomials. As a consequence, we have obtained an asymptotic formula for the expected number of internal equilibria in multi-player two-strategy random evolutionary games. Our results deepen the connection between evolutionary game theory and random polynomial theory which was discovered previously in [9, 11]. Below we discuss some important directions for future research.

Extensions to other models in EGT. The class of random polynomials that we studied in this paper arises from the replicator dynamics. It would be interesting to generalize our results to more complex models in evolutionary game theory and population dynamics. The most natural model to study next is the replicator-mutator dynamics where mutation is present. Equilibria for the replicator-mutator dynamics are positive roots of a much more complicated class of random polynomials, which depend on the mutation. Studying the effect of mutation on the equilibrium properties, in particular on the expected number of internal equilibria, is a challenging problem, see [10] for an initial attempt. One can also ask whether our results can be extended to multi-player multi-strategy evolutionary games whose equilibria are positive roots of a system of random polynomials. In this case, the assumption that the gain sequence is independent is not realistic from evolutionary game theory's point of view (see [8, Remark 4] for a detailed explanation). Therefore, one needs to deal with a dependent system of random polynomials, which is very challenging.

Universality and other statistical properties. The assumption that the random variables $\{\xi_i\}_{i=0}^n$ are Gaussian distributions is crucial in the present paper. It allowed us to employ the fundamental tool of Kac-Rice formula in Section 2. What happens if $\{\xi_i\}$ are not Gaussian? Very recently, it has been shown that *universality phenomena* hold for three classes of random polynomials: Kac polynomials, elliptic polynomials, and Weyl polynomials (recall the Introduction for their explicit expressions) [7, 26, 35]. The universality results state that the expectation of the number of real roots of these classes of random polynomials depend only on the mean and variance of the coefficients $\{\xi_i\}$ but not on their type of the distributions. It would be very interesting to obtain such a universality theorem for the class of random polynomials arising from evolutionary game theory studied in this paper. The distributions of the roots in different classes are different, and the methods to study them need to be tailored to each of the class. It remains elusive to us whether the techniques in [7, 26, 35] can be applied to the class of random polynomials in this paper. Furthermore, studying other statistical properties such as central limit theorem and the distribution of the number of equilibria also demands future investigations, see for instance [5] for a characterization of the probability that a multiplayer random evolutionary random game has no internal equilibria.

7. Appendix   In this appendix, we present detailed computations and proofs of technical results used in previous sections.

7.1. Proof of Lemma 3.1 and detailed computations of $f_n(0)$ and $f_n(1)$, In this section, we prove Lemma 3.1 and compute $f_n(0)$ and $f_n(1)$.

Proof. [Proof of Lemma 3.1] It follows from the definition of Jacobi polynomial (3.1) that for any $q \in \mathbb{R}$

$$J_n^{\alpha,\beta}\left(\frac{1+q}{1-q}\right) = \frac{1}{2^n}\sum_{i=0}^n \binom{n+\alpha}{n-i}\binom{n+\beta}{i}\left(\frac{1+q}{1-q}-1\right)^i\left(\frac{1+q}{1-q}+1\right)^{n-i}$$

$$= \frac{1}{2^n}\sum_{i=0}^n \binom{n+\alpha}{n-i}\binom{n+\beta}{i}\left(\frac{2q}{1-q}\right)^i\left(\frac{2}{1-q}\right)^{n-i}$$

$$= \frac{1}{(1-q)^n}\sum_{i=0}^n \binom{n+\alpha}{n-i}\binom{n+\beta}{i}q^{n-i}.$$

Taking $q = x^2$ yields the statement of Lemma 3.1. $\square$ Next we compute $f_n(0)$ and $f_n(1)$. We have

$$M_n(0) = \binom{n+\alpha}{n}, \quad A_n(0) = \binom{n+\alpha}{n-1}\binom{n+\beta}{1} = (n+\beta)\binom{n+\alpha}{n-1}, \quad B_n(0) = 0. \quad (7.1)$$

Thus

$$f_n(0)^2 = \frac{A_n(0)M_n(0) - B_n(0)^2}{M_n(0)^2} = \frac{A_n(0)}{M_n(0)} = \frac{n(n+\beta)}{\alpha+1},$$

that is

$$f_n(0) = \sqrt{\frac{n(n+\beta)}{1+\alpha}}.$$

Next we compute $f_n(1)$. We have

$$M_n(1) = \sum_{i=0}^n \binom{n+\alpha}{n-i}\binom{n+\beta}{i} = \binom{2n+\alpha+\beta}{n}$$

Using the following formula for the derivative of Jacobi polynomials [34, Section 4.5]

$$(2n+\alpha+\beta)(1-x^2)\frac{d}{dx}J_n^{(\alpha,\beta)}(x)$$

$$= -n\Big((2n+\alpha+\beta)x+\beta-\alpha\Big)J_n^{(\alpha,\beta)}(x) + 2(n+\alpha)(n+\beta)J_{n-1}^{(\alpha,\beta)}(x)$$

and Lemma 3.1 we obtain the following formula for the derivative of $M_n^{\alpha,\beta}(x)$.

$$x(2n+\alpha+\beta)M_n'(x) = \Big(n(2n+\alpha+\beta)+\beta-\alpha\Big)M_n^{(\alpha,\beta)}(x)$$

$$- 2(1-x^2)(n+\alpha)(n+\beta)M_{n-1}^{(\alpha,\beta)}(x). \quad (7.2)$$

Applying (7.2) for $x = 1$, we obtain

$$B_n(1) = \frac{1}{2}M_n'(1) = \frac{n(2n+\alpha+\beta)+\beta-\alpha)M_n(1)}{2(2n+\alpha+\beta)} = \frac{1}{2}\Big(n+\frac{\beta-\alpha}{2n+\alpha+\beta}\Big)\binom{2n+\alpha+\beta}{n}.$$

Taking derivative of (7.2) we have

$$(2n+\alpha+\beta)(M_n'(x)+xM_n''(x))=(n(2n+\alpha+\beta)+\beta-\alpha)M_n'(x)$$
$$-2(n+\alpha)(n+\beta)\left[-2xM_{n-1}(x)+(1-x^2)M_{n-1}'(x)\right].$$

It follows that

$$A_n(1)=\frac{1}{4}(M_n'(1)+M_n''(1))$$
$$=\frac{1}{4(2n+\alpha+\beta)}\left[(n(2n+\alpha+\beta)+\beta-\alpha)M_n'(1)+4(n+\alpha)(n+\beta)M_{n-1}(1)\right]$$
$$=\frac{1}{4}\left(n+\frac{\beta-\alpha}{2n+\alpha+\beta}\right)^2\binom{2n+\alpha+\beta}{n}+\frac{(n+\alpha)(n+\beta)}{2n+\alpha+\beta}\binom{2n+\alpha+\beta-2}{n-1}.$$

Hence

$$f_n(1)=\frac{\sqrt{A_n(1)M_n(1)-B_n(1)^2}}{M_n(1)}$$

$$=\sqrt{\frac{(n+\alpha)(n+\beta)}{2n+\alpha+\beta}}\sqrt{\frac{\binom{2n+\alpha+\beta-2}{n-1}}{\binom{2n+\alpha+\beta}{n}}}$$

$$=\frac{1}{2n+\alpha+\beta}\sqrt{\frac{n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{2n+\alpha+\beta-1}}.$$

In particular, when $\alpha=\beta$,

$$f_n(1)=\frac{1}{2}\sqrt{\frac{n(n+2\alpha)}{2n+2\alpha-1}}.$$

7.2. Proof of Proposition 4.2   In this section we prove Proposition 4.2. The proof will be established after a series of technical lemmas. We start with the following lemma that provides an estimate for a power of the binomial coefficient, which is a key factor appearing in the expressions of A_n, B_n and M_n.

LEMMA 7.1. *For* $0<t<1$ *and* $x>0$ *we define* $I(t):=t\log\frac{1}{t}+(1-t)\log\frac{1}{1-t}$ *and* $J_{\gamma,x}(t):=$
$\gamma I(t)+t\log x$. *Then*

$$\binom{n}{i}^\gamma x^i=\left(\frac{n}{2\pi i(n-i)}\right)^{\gamma/2}\left(1+O\left(\frac{1}{i}+\frac{1}{n-i}\right)\right)^\gamma e^{nJ_{\gamma,x}\left(\frac{i}{n}\right)}. \qquad (7.3)$$

Proof. [Proof of Lemma 7.1] It follows from Stirling formula

$$i!=\sqrt{2\pi i}(1+O(i^{-1}))\left(\frac{i}{e}\right)^i$$

that

$$\binom{n}{i}=\sqrt{\frac{n}{2\pi i(n-i)}}\left(1+O\left(\frac{1}{i}+\frac{1}{n-i}\right)\right)e^{nI\left(\frac{i}{n}\right)},$$

where

$$I(t) = t\log\frac{1}{t} + (1-t)\log\frac{1}{1-t}$$

Therefore,

$$\binom{n}{i}^{\gamma} x^i = \left(\frac{n}{2\pi i(n-i)}\right)^{\gamma/2}\left(1 + O\left(\frac{1}{i} + \frac{1}{n-i}\right)\right)^{\gamma} e^{nJ_{\gamma,x}\left(\frac{i}{n}\right)},$$

which is the statement of the lemma. □

LEMMA 7.2. *We define*

$$t_{\gamma,x} := \frac{x^{1/\gamma}}{1 + x^{1/\gamma}} \quad \text{and} \quad i_{\gamma,x} := \lfloor nt_{\gamma,x} \rfloor, \tag{7.4}$$

We note that $t_{\gamma,x}$ is the unique solution of the equation $J'_{\gamma,x}(t) = 0$, where

$$J'_{\gamma,x}(t) = \gamma\log\frac{1-t}{t} + \log x \text{ and } J''_{\gamma,x}(t) = \frac{-\gamma}{t(1-t)}. \tag{7.5}$$

Assume that $1 \geq x \geq (\log n)^{4\gamma}/n^{\gamma}$. Then the following holds for all n large enough:
  a. If $|i - i_{\gamma,x}| \geq i_{\gamma,x}^{3/4}$ then

$$\frac{\binom{n}{i}^{\gamma} x^i}{\binom{n}{i_{\gamma,x}}^{\gamma} x^{i_{\gamma,x}}} \leq \frac{1}{n^{10}}.$$

  b. If $|i - i_{\gamma,x}| < i_{\gamma,x}^{3/4}$ then

$$\frac{\binom{n}{i}^{\gamma} x^i}{\binom{n}{i_{\gamma,x}}^{\gamma} x^{i_{\gamma,x}}} = \left(1 + O\left(\frac{1}{\log n}\right)\right)\exp\left\{\left[J''_{\gamma,x}(t_{\gamma,x}) + O\left(\frac{(nt_{\gamma,x})^{3/4}}{nt_{\gamma,x}^2}\right)\right]\frac{(i - i_{\gamma,x})^2}{2n}\right\}.$$

  Proof.
  a. Since $\binom{n}{i} \leq \exp(nI(i/n))$, using (7.3) we have

$$\frac{\binom{n}{i}^{\gamma} x^i}{\binom{n}{i_{\gamma,x}}^{\gamma} x^{i_{\gamma,x}}} \leq (2\pi i_{\gamma,x})^{\gamma/2}\exp\left(n[J_{\gamma,x}(i/n) - J_{\gamma,x}(i_{\gamma,x}/n)]\right)$$

$$\leq (2\pi i_{\gamma,x})^{\gamma/2}\exp\left(n\left[J_{\gamma,x}\left(\frac{i_{\gamma,x} \pm i_{\gamma,x}^{3/4}}{n}\right) - J_{\gamma,x}(i_{\gamma,x}/n)\right]\right),$$

where the second lines follows from the fact that the function $J_{\gamma,x}(t)$ is concave and attains maximum at $t_{\gamma,x}$. By Taylor expansion, there exists $\theta \in \left(\frac{i_{\gamma,x} - i_{\gamma,x}^{3/4}}{n}, \frac{i_{\gamma,x} + i_{\gamma,x}^{3/4}}{n}\right)$ such that

$$J_{\gamma,x}\left(\frac{i_{\gamma,x} \pm i_{\gamma,x}^{3/4}}{n}\right) - J_{\gamma,x}\left(\frac{i_{\gamma,x}}{n}\right) = \frac{\pm i_{\gamma,x}^{3/4}}{n}J'_{\gamma,x}\left(\frac{i_{\gamma,x}}{n}\right) + J''_{\gamma,x}(\theta)\frac{i_{\gamma,x}^{3/4}}{2n^2}.$$

Notice that

$$\left|J'_{\gamma,x}\left(\frac{i_{\gamma,x}}{n}\right)\right| = \left|J'_{\gamma,x}\left(\frac{i_{\gamma,x}}{n}\right) - J'_{\gamma,x}(t_{\gamma,x})\right| \leq \sup_{y \in \left(\frac{i_{\gamma,x}}{n}, t_{\gamma,x}\right)}|J''_{\gamma,x}(y)|\left|\frac{i_{\gamma,x}}{n} - t_{\gamma,x}\right|$$

$$\leq \frac{1}{n} \sup_{y \in (\frac{i_{\gamma,x}}{n}, t_{\gamma,x})} \left| \frac{-\gamma}{y(1-y)} \right| \leq \frac{C}{nx^{1/\gamma}}. \quad (7.6)$$

Combining this with the fact that

$$J''_{\gamma,x}(\theta) = \frac{-\gamma}{\theta(1-\theta)} \leq \frac{-\gamma}{\theta} \leq \frac{-n\gamma}{i_{\gamma,x} + i_{\gamma,x}^{3/4}} \leq \frac{-cn}{i_{\gamma,x}},$$

we get that for n large enough,

$$\frac{\binom{n}{i}^{\gamma} x^i}{\binom{n}{i_{\gamma,x}}^{\gamma} x^{i_{\gamma,x}}} \leq (2\pi i_{\gamma,x})^{\gamma/2} \exp\left(\frac{C i_{\gamma,x}^{3/4}}{nx^{1/\gamma}} - c i_{\gamma,x}^{1/2} \right)$$

$$\leq (2\pi n)^{\gamma/2} \exp\left(-c'(\log n)^2 \right) \leq \frac{1}{n^{10}},$$

where in the last line we use the estimate $i_{\gamma,x} \simeq nx^{1/\gamma} = (\log n)^4$. This ends the proof of Part a.

b. Suppose that $|i - i_{\gamma,x}| < i_{\gamma,x}^{3/4}$. By using (7.3) and Taylor expansion,

$$\frac{\binom{n}{i}^{\gamma} x^i}{\binom{n}{i_{\gamma,x}}^{\gamma} x^{2\gamma,x}} = \left[(1 + O(i_{\gamma,x}^{-1})) \frac{i_{\gamma,x}(n - i_{\gamma,x})}{i(n-i)} \right]^{\gamma/2} \exp\left(n \left[J_x\left(\frac{i}{n}\right) - J_x\left(\frac{i_x}{n}\right) \right] \right)$$

$$= \left(1 + O\left(\frac{1}{\log n}\right) \right) \exp\left(n \left[J'_{\gamma,x}\left(\frac{i_{\gamma,x}}{n}\right) \frac{(i - i_{\gamma,x})}{n} + J''_{\gamma,x}\left(\frac{i_{\gamma,x}}{n}\right) \frac{(i - i_{\gamma,x})^2}{2n^2} + J'''_{\gamma,x}(\theta) \frac{(i - i_{\gamma,x})^3}{6n^3} \right] \right),$$

for some $\theta \in (\frac{i_{\gamma,x}}{n}, \frac{i}{n})$.

Observe that

- $\left| J'_{\gamma,x}\left(\frac{i_{\gamma,x}}{n}\right) \right| \leq \frac{C'}{nx^{1/\gamma}} \leq \frac{C}{i_{\gamma,x}}$ as in the proof of Part a.
- $J'''_{\gamma,x}(y) = O(y^{-2})$ for all $y \in \mathbb{R}$. Therefore $J'''_{\gamma,x}(\theta) = O(t_{\gamma,x}^{-2})$.
- $J''_{\gamma,x}\left(\frac{i_{\gamma,x}}{n}\right) = J''_x(t_{\gamma,x}) + O\left(\frac{1}{t_{\gamma,x}^2}\right)\left(\frac{i_{\gamma,x}}{n} - t_{\gamma,x}\right) = J''_x(t_{\gamma,x}) + O\left(\frac{1}{nt_{\gamma,x}^2}\right)$.

Combining these estimates,

$$n \left[J_x\left(\frac{i}{n}\right) - J_x\left(\frac{i_x}{n}\right) \right] = O\left(\frac{i_{\gamma,x}^{3/4}}{i_{\gamma,x}} \right) + \left[J''_{\gamma,x}(t_{\gamma,x}) + O\left(\frac{1}{nt_{\gamma,x}^2}\right) + O\left(\frac{1}{t_{\gamma,x}^2}\right) \frac{i - i_{\gamma,x}}{n} \right] \frac{(i - i_{\gamma,x})^2}{2n}$$

$$= O\left(i_{\gamma,x}^{(\delta-1)/2} \right) + \left[J''_{\gamma,x}(t_{\gamma,x}) + O\left(\frac{(nt_{\gamma,x})^{3/4}}{nt_{\gamma,x}^2} \right) \right] \frac{(i - i_{\gamma,x})^2}{2n}.$$

Then Part b. follows. □

LEMMA 7.3. *We have*

$$A_n(x)M_n(x) - B_n(x)^2 = \frac{1}{2} \sum_{i,j=0}^{n} (i - j)^2 \binom{n}{i}^{2\gamma} \binom{n}{j}^{2\gamma} x^{2(i+j-1)}. \quad (7.7)$$

Proof. [Proof of Lemma 7.3] Using formulas of M_n, A_n and B_n given in (2.4), we obtain

$$A_n(x)M_n(x) - B_n(x)^2$$

$$= \sum_{i,j=0}^{n} i^2 \binom{n}{i}^{2\gamma} x^{2(i-1)} \binom{n}{j}^{2\gamma} x^{2j} - \sum_{i,j=0}^{n} i \binom{n}{i}^{2\gamma} x^{2i-1} j \binom{n}{j}^{2\gamma} x^{2j-1}$$

$$= \frac{1}{2} \sum_{i,j=0}^{n} [i^2 + j^2 - 2ij] \binom{n}{i}^{2\gamma} \binom{n}{j}^{2\gamma} x^{2(i+j-1)}$$

$$= \frac{1}{2} \sum_{i,j=0}^{n} (i-j)^2 \binom{n}{i}^{2\gamma} \binom{n}{j}^{2\gamma} x^{2(i+j-1)}.$$

The last step is the desired equality. □

The following lemma will be used to obtain asymptotic behaviour of $A_n M_n - B_n^2$ later on.

LEMMA 7.4. *Let $f(x,y)$ be a bivariate function such that $f(x,y) = O(x^2 + y^2)$. Consider two sequences (a_n) and (b_n) such that $a_n \to 0$ and $\frac{b_n (\log n)^4}{a_n} \to 0$. Then for $k < i_{\gamma,x} - a_n^{-1}$ and $l > i_{\gamma,x} + a_n^{-1}$,*

$$\sum_{i,j=k}^{l} f(i,j) \exp\left(-(a_n + b_n)\left((i - i_{\gamma,x})^2 + (j - i_{\gamma,x})^2\right)\right)$$

$$= \left(1 + O\left(\frac{1}{\log n}\right)\right) \sum_{\substack{|i - i_{\gamma,x}| < (a_n b_n)^{-1/4} \\ |j - i_{\gamma,x}| < (a_n b_n)^{-1/4}}} f(i,j) \exp\left(-a_n\left((i - i_{\gamma,x})^2 + (j - i_{\gamma,x})^2\right)\right) + O(1).$$

Proof. [Proof of Lemma 7.4] Denote by $\theta = (a_n b_n)^{-1/4}$. If $|i - i_{\gamma,x}| \geq \theta$ or $|j - i_{\gamma,x}| \geq \theta$ then

$$-(a_n + b_n)\left((i - i_{\gamma,x})^2 + (j - i_{\gamma,x})^2\right) \leq -a_n \theta^2 \leq -c(\log n)^2.$$

Therefore

$$\sum_{\substack{|i - i_{\gamma,x}| < (a_n b_n)^{-1/4} \text{ or} \\ |j - i_{\gamma,x}| < (a_n b_n)^{-1/4}}} f(i,j) \exp\left(-(a_n + b_n)\left((i - i_{\gamma,x})^2 + (j - i_{\gamma,x})^2\right)\right)$$

$$\leq C n^4 e^{-c(\log n)^2} = O(1).$$

Consider $|i - i_{\gamma,x}| < \theta$ and $|j - i_{\gamma,x}| < \theta$. Then

$$b_n\left((i - i_{\gamma,x})^2 + (j - i_{\gamma,x})^2\right) = O(b_n \theta^2) = O\left(\frac{1}{(\log n)^2}\right).$$

It implies that

$$\exp\left(-(a_n + b_n)\left((i - i_{\gamma,x})^2 + (j - i_{\gamma,x})^2\right)\right)$$

$$= \left(1 + O\left(\frac{1}{\log n}\right)\right) \exp\left(-a_n\left((i - i_{\gamma,x})^2 + (j - i_{\gamma,x})^2\right)\right).$$

Then the result follows. □

LEMMA 7.5. *Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that*

$$\sup_{x \in \mathbb{R}} \left[(x^4 + 1)|g(x)| + |g'(x)|\right] \leq C < \infty.$$

Then for any K,l,m such that $K,\frac{l}{\sqrt{K}},\frac{m}{\sqrt{K}}\to\infty$, we have

$$\frac{1}{\sqrt{K}}\sum_{i=-l}^{m}g\left(\frac{i}{\sqrt{K}}\right)=(1+o(1))\int_{\mathbb{R}}g(x)dx, \tag{7.8}$$

and

$$\frac{1}{K^2}\sum_{i,j=-l}^{m}(i-j)^2g\left(\frac{i}{\sqrt{K}}\right)g\left(\frac{j}{\sqrt{K}}\right)=(1+o(1))\int_{\mathbb{R}^2}(x-y)^2g(x)g(y)dxdy. \tag{7.9}$$

Proof. We start by proving (7.8). Let

$$m_1=\min\{m,K^{2/3}\}, \qquad l_1=\min\{l,K^{2/3}\}.$$

Then using $|g(x)|\leq C/(x^4+1)$ we have

$$\left|\frac{1}{\sqrt{K}}\sum_{i=-l}^{m}g\left(\frac{i}{\sqrt{K}}\right)-\frac{1}{\sqrt{K}}\sum_{i=-l_1}^{m_1}g\left(\frac{i}{\sqrt{K}}\right)\right|\leq\frac{1}{\sqrt{K}}\sum_{|i|\geq K^{2/3}}\left|g\left(\frac{i}{\sqrt{K}}\right)\right|$$

$$\leq\frac{C}{\sqrt{K}}\sum_{|i|\geq K^{2/3}}\left(\frac{\sqrt{K}}{|i|}\right)^4=O(K^{-1/2}). \tag{7.10}$$

For $-l_1\leq i,j\leq m_1$, by the mean value theorem,

$$\left|\frac{1}{\sqrt{K}}g\left(\frac{i}{\sqrt{K}}\right)-\int_{\frac{i}{\sqrt{K}}}^{\frac{i+1}{\sqrt{K}}}g(x)dx\right|\leq\sup_{x\in\mathbb{R}}|g'(x)|\int_{\frac{i}{\sqrt{K}}}^{\frac{i+1}{\sqrt{K}}}\left|x-\frac{i}{\sqrt{K}}\right|dx\leq\frac{C}{2K}.$$

Therefore,

$$\left|\frac{1}{\sqrt{K}}\sum_{i=-l_1}^{m_1}g\left(\frac{i}{\sqrt{K}}\right)-\int_{-l_1}^{\frac{m_1+1}{\sqrt{K}}}g(x)dx\right|\leq\frac{C(l_1+m_1)}{2K}=O(K^{-1/3}). \tag{7.11}$$

Since $\int_{\mathbb{R}}|g(x)|dx<\infty$ (as $|g(x)|\leq C/(|x^4+1|)$), and $m_1/\sqrt{K},l_1/\sqrt{K}\to\infty$,

$$\left|\int_{\mathbb{R}}g(x)dx-\int_{\frac{-l_1}{\sqrt{K}}}^{\frac{m_1+1}{\sqrt{K}}}g(x)dx\right|=o(1).$$

Combining this with (7.10) and (7.11) we obtain (7.8).

Next, we prove (7.9). Define

$$m_2=\min\{m,K^{5/9}\}, \qquad l_2=\min\{l,K^{5/9}\},$$

and

$$h(x,y)=(x-y)^2g(x)g(y).$$

Then using the inequality

$$|h(x,y)|\leq 2(x^2+y^2)|g(x)g(y)|\leq 2C^2(x^2+y^2)/((x^4+1)(y^4+1)), \tag{7.12}$$

we have

$$\left| \frac{1}{K^2} \sum_{i,j=-l}^{m} h\left(\frac{i}{\sqrt{K}}, \frac{j}{\sqrt{K}}\right) - \frac{1}{K^2} \sum_{i,j=-l_2}^{m_2} h\left(\frac{i}{\sqrt{K}}, \frac{j}{\sqrt{K}}\right) \right| \leq \frac{1}{K^2} \sum_{|i|,|j| \geq K^{5/9}} \left| h\left(\frac{i}{\sqrt{K}}, \frac{j}{\sqrt{K}}\right) \right|$$

$$\leq \frac{2C^2}{K^2} \sum_{|i|,|j| \geq K^{5/9}} (i^2 + j^2) \left(\frac{K}{|i||j|}\right)^4 = O(K^{-2/9}). \tag{7.13}$$

For $l_2 \leq i, j \leq m_2$, we define $\square_{ij} = [\frac{i}{\sqrt{K}}, \frac{i+1}{\sqrt{K}}] \times [\frac{j}{\sqrt{K}}, \frac{j+1}{\sqrt{K}}]$. Using the trivial bound if $(u,v) \in \square_{ij}$ then $|u-v| \leq \frac{l_2 + m_2}{\sqrt{K}}$ and the assumption that $\sup_{x \in \mathbb{R}}(|g(x)| + |g'(x)|) \leq C$, we have

$$\sup_{(u,v) \in \square_{ij}} |\partial_x h(u,v)|, \quad \sup_{(u,v) \in \square_{ij}} |\partial_y h(u,v)| \leq 3C^2 \left(\frac{l_2 + m_2}{\sqrt{K}}\right)^2.$$

Thus using the mean value theorem again, we have

$$\left| \int_{\square_{ij}} \left[h\left(\frac{i}{\sqrt{K}}, \frac{j}{\sqrt{K}}\right) - h(x,y) \right] dx dy \right| \leq 3C^2 \left(\frac{l_2 + m_2}{\sqrt{K}}\right)^2 \int_{\square_{ij}} \left[\left| x - \frac{i}{\sqrt{K}} \right| + \left| y - \frac{j}{\sqrt{K}} \right| \right] dx dy$$

$$= \frac{3C^2}{2K\sqrt{K}} \left(\frac{l_2 + m_2}{\sqrt{K}}\right)^2.$$

Therefore, with $\square(l_2, m_2) = \left[\frac{-l_2}{\sqrt{K}}, \frac{m_2+1}{\sqrt{K}}\right]^2$ we have

$$\left| \sum_{i,j=-l_2}^{m_2} h\left(\frac{i}{\sqrt{K}}, \frac{j}{\sqrt{K}}\right) - \int_{\square(l_2,m_2)} h(x,y) dx dy \right| \leq \sum_{i,j=-l_2}^{m_2} \frac{3C^2(m_2+l_2)^2}{2K^{5/2}}$$

$$= O(1) \frac{(m_2+l_2)^4}{K^{5/2}} = o(1), \tag{7.14}$$

since $m_2, l_2 \leq K^{5/9}$. Notice that by (7.12), $\int_{\mathbb{R}^2} |h(x,y)| dx dy < \infty$, and $\square(l_2, m_2) \to \mathbb{R}^2$ (as $\frac{l_2}{\sqrt{K}}, \frac{m_2}{\sqrt{K}} \to \infty$). Hence,

$$\int_{\mathbb{R}^2 \setminus \square(l_2,m_2)} h(x,y) dx dy = o(1).$$

This estimates together with (7.13) and (7.14) implies (7.9). $\square$

Finally, we now bring all previous technical lemmas to prove Proposition 4.2.

Proof. [Proof of Proposition 4.2] We have

$$M_n(x) = \sum_{|i-i_{\gamma,x}| \geq i_{\gamma,x}^{3/4}} \binom{n}{i}^{2\gamma} x^{2i} + \sum_{|i-i_{\gamma,x}| < i_{\gamma,x}^{3/4}} \binom{n}{i}^{2\gamma} x^{2i} =: M_{1,n} + M_{2,n}.$$

By Lemma 7.2 Part a.,

$$\frac{M_{1,n}}{\binom{n}{i_{\gamma,x}}^{2\gamma} x^{2i_{\gamma,x}}} \leq \frac{n}{n^{20}} \leq \frac{1}{n^{10}};$$

and by Lemma 7.2 Part b., Lemma 7.4 and Lemma 7.5,

$$\frac{M_{2,n}}{\binom{n}{i_{\gamma,x}}^{2\gamma}x^{2i_{\gamma,x}}} = (1+o(1)) \sum_{|i-i_{\gamma,x}|<i_{\gamma,x}^{3/4}} \exp\left(\left[J_{\gamma,x}''(t_{\gamma,x}) + O\left(\frac{(nt_{\gamma,x})^{3/4}}{nt_{\gamma,x}^2}\right)\right]\frac{(i-i_{\gamma,x})^2}{n}\right)$$

$$= (1+o(1))\int_{\mathbb{R}} e^{-x^2}dx \times \sqrt{n/|J''(t_{\gamma,x})|}$$

$$= (\sqrt{\pi}+o(1))\sqrt{\frac{nx^{1/\gamma}}{\gamma(1+x^{1/\gamma})^2}}.$$

Now according to Lemma 7.3, we have

$$A_n(x)M_n(x) - B_n^2(x) = \frac{1}{2}\sum_{i,j=0}^{n}(i-j)^2\binom{n}{i}^{2\gamma}x^{2i}\binom{n}{j}^{2\gamma}x^{2i+2j-2}.$$

Then by the same argument as above, we get

$$A_n(x)M_n(x) - B_n^2(x) = \frac{1}{2}(1+o(1))\int_{\mathbb{R}^2}(x-y)^2 e^{-(x^2+y^2)}dxdy \times (n/|J''(t_{\gamma,x})|)^2$$

$$= \left(\frac{\pi}{2}+o(1)\right)\left(\frac{nx^{1/\gamma}}{\gamma(1+x^{1/\gamma})^2}\right)^2,$$

which completes the proof. □

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