Mean-field game for collective decision-making in honeybees via switched systems

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Mean-Field Game for Collective Decision-Making in Honeybees via Switched Systems

Leonardo Stella, Dario Bauso and Patrizio Colaneri, *Fellow, IEEE*

Abstract—In this paper, we study the optimal control problem arising from the mean-field game formulation of the collective decision-making in honeybee swarms. A population of homogeneous players (the honeybees) has to reach consensus on one of two options. We consider three states: the first two represent the available options (or strategies), and the third one represents the uncommitted state. We formulate the continuous-time discrete-state mean-field game model. The contributions of this paper are the following: i) we propose an optimal control model where players have to control their transition rates to minimize a running cost and a terminal cost, in the presence of an adversarial disturbance; ii) we develop a formulation of the micro-macro model in the form of an initial-terminal value problem (ITVP) with switched dynamics; iii) we study the existence of stationary solutions and the mean-field Nash equilibrium for the resulting switched system; iv) we show that under certain assumptions on the parameters, the game may admit periodic solutions; and v) we analyze the resulting microscopic dynamics in a structured environment where a finite number of players interact through a network topology.

Index Terms—Mean-Field Game Theory, Social Networks, Multi-Agent Systems, Switched Systems.

I. INTRODUCTION

We study a collective decision-making problem where a large population of individuals has to reach consensus on one of two options. In recent years, the study of the principles underpinning collective decision-making has seen a growing interest in cross-disciplinary research on behavioral ecology, psychology and neurosciences because of the similarities between the neural correlations of brains in vertebrates and the group cognition of social animals such as honeybees [1]. Motivated by the collective decision-making in honeybee swarms, we study the problem where the cognitive task of nest-site selection is investigated, see [2], [3], [4]. Two main behavioral traits were found to stir the decision-making process toward one of the options: the so-called waggle dance, performed by scout bees to share information about the nest sites and to recruit other bees; and the cross-inhibitory stop signal, which is used to prevent other bees to advertize the competing options and prevent deadlocks [5]. In [5], a critical value of the stop signal is found via bifurcation theory, and it is shown that, when the value of the stop signal is above this critical threshold, the swarm adaptively breaks a deadlock

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when faced with two near-equal value options. Through a different approach, namely Lyapunov theory for nonlinear systems, the same threshold was found in [6] and the symmetric and asymmetric cases where studied. The symmetric case describes the situation where the options have near-equal value, while the asymmetric case captures the case where one of the options is better than the other one. The symmetric case has been the testbed to study the impact of the cross-inhibitory signalling in preventing a deadlock during the decision-making process. The work by Pais *et al.*, i.e. [7], extended the previous research by investigating the value-sensitivity of the dynamics on the cross-inhibitory parameter. In [8], we studied the case where the interactions in the population are modeled using the Barabási-Albert complex network.

While a seminal study on the decision-making in house-hunting honeybees with more than two options was conducted in 1997 and 1998 and reported in [9], recent research, e.g. see [10], has extended the previous model to the best-of-N case, analysing the symmetric and asymmetric models. For an insight on consensus and swarm dynamics, the reader is referred to [11] and [12]. For a better understanding of the impact of noise on the decision-making process in networks we refer the reader to [13]. Recent applications of the decision-making in honeybee swarms include multi-agent decision-making and network design, e.g. see [14] and [15].

Mean-field game theory studies the strategic decision-making where the number of the individuals in the population is large. The origin of the theory of mean-field games can be found in [16], [17], [18] by M. Y. Huang, P. E. Caines and R. P. Malhamé and independently in [19], [20], [21] by J. M. Lasry and P. L. Lions. Huang, Caines and Malhamé developed the theory by extending stochastic dynamic games to a large population of players and by approximating their behavior via the average of the players' strategies. At the same time, Lasry and Lions introduced mean-field game theory as an extension to mean-field theory in physics and statistical mechanics: their intuition was to approximate the complexity of the behavior in high-dimensional models by averaging over all the components in the system. For a survey on mean-field games we refer the reader to [22]. Robust mean-field games are introduced in [23] and studied further in [24]. Robustness is also discussed in [25]. Mean-field games apply to a variety of domains, including economics, engineering, physics and biology; for details we refer the reader to [18], [26], [27], [28], [29], [30]. Mean-field games have predecessors in *anonymous games* and *aggregative games*, which are nonatomic games where each individual cannot influence the evolution of the game and the interaction occurs through a mass function. Discrete-time finite-state mean-field games were first introduced in [31]

while the continuous-time counterpart was introduced in [32].

In [33], the authors use mean-field game theory to study the mean-field trajectories of all players in the game. The resulting game takes inspiration from the collective decision-making in honeybee swarms [33]. When the population is sufficiently large, the best response strategies become epsilon-Nash equilibria and can be calculated solely via the joint probability of the initial conditions in the form of random variables. For a linear quadratic structure, the author in [34] provides explicit solutions in terms of mean-field equilibria.

A recent line of research, originated in the work by Weitz *et al.* [35], proposes a bidirectional game-environment feedback, where the environment plays a role in the game and in the strategy selection process. Within the framework of evolutionary game theory, replicator dynamics are the ones that have extensively been used to model these interactions between the state and the environment in a variety of contexts, from micro-economics to animal behavior [36]. The environmental feedback is motivated by the observed behavior of real systems and their complexity [36]. Other lines of research have used replicator-mutator dynamics to model social dynamics and decision-making [37]. Most of these studies report oscillations and limit cycles in the dynamics, either via bifurcation theory [37] or via the time-scale difference between game and environment dynamics [36].

Highlights of contributions. We propose an optimal control approach to model the collective decision-making inspired by honeybees and we formulate the corresponding mean-field game. A novel element of this approach is the modeling of the cross-inhibitory stop signal through an adversarial disturbance in the game dynamics. When we assume that the parameters take specific values, the mean-field game turns into the collective decision-making model used to describe the behavior in honeybee swarms. Additionally, by using the theory for discrete-state mean-field games, the addition of an adversarial disturbance is an element of novelty as well as the use of switched systems to prove the uniqueness of the mean-field Nash equilibrium for the corresponding initial-terminal value problem (ITVP). The ITVP brings together the macroscopic dynamics, namely how the population evolves as a whole, and the microscopic dynamics, namely how a single player responds to the population behavior.

Driven by the need to understand the conditions for periodic solutions to arise, we extend our model to include a form of game-environment feedback and a dependence of the parameters on the state itself through the value function. By doing so, we find that the system exhibits an oscillatory behavior that approximates the one found in the literature on environmental feedback. Finally, we propose a networked model where a finite number of players interact by means of a complete fully-connected network. In this model, a finite number of players interact through a network topology and a stability analysis is carried out on this system to prove that the whole population converges to the same equilibrium obtained without interaction topology. We also provide a link between the initial mean-field game and the networked model with finite population. This paper has extensively reworked the topic firstly presented in the conference version, see [38], and

the overlaps are now minimal, mostly related to the calculation of the optimal control and disturbance in Theorem 1.

This paper is organized as follows. In Section II, we provide a general mean-field model for the collective decision-making inspired by honeybees. In Section III, we study the stationary solutions and the stability of the mean-field Nash equilibrium corresponding to the initial-terminal value problem in the form of a switched system. In Section IV, we present the link between our model and the collective decision-making in honeybee swarms that motivated our study. In Section V, we study the periodic solutions under the assumption that the parameters act as an environmental feedback through the value function and prove conditions for oscillations to occur and we calculate the corresponding basin of attraction for this system. In Section VI, we investigate the networked model for a finite population where each player can be in any of the three states in probability. In Section VII, we draw conclusions and discuss future directions of research. Except for Theorem 6, all the proofs are given in the Appendix.

NOTATION

We use the following notations throughout this paper. The set $\mathcal{I}^3$ is the set of three possible states, state 1, state 2 and state 3. The set $\mathcal{S}^3$ is the probability simplex in $\mathbb{R}^3$ and $t > 0$ is the time index. The notation $\mathbb{R}_0^+$ indicates the set of all positive real numbers including zero, and $(\mathbb{R}_0^+)^3$ denotes the set of 3-dimensional vectors with non-negative entries. By i_τ, we mean the continuous-time Markov chain that describes the state of a player at time τ. We denote by $\text{diag}(a)$ the diagonal matrix with diagonal a, for any generic vector a. $\Delta_i : \mathbb{R}^3 \to \mathbb{R}^3$ is the difference operator on $i \in \mathcal{I}^3$ given by $\Delta_i v = (v_1 - v_i, v_2 - v_i, v_3 - v_i)^T$. For a generic function $f(x)$, the notation $[f(x)]^- = \max(-f(x), 0)$ and $[f(x)]^+ = \max(f(x), 0)$ denote the negative part and the positive part of $f(x)$, respectively. The notation $[0, 1]_N$ denotes the N-Cartesian product of $[0, 1]$. We denote by $\mathbf{1}_N$ the N-dimensional vector whose entries are all 1 and by $\mathbb{I}_N$ the N-dimensional identity matrix. For a square matrix X of dimension N the 1-Lozinski measure $\mu_1(X)$ is given by $\max_i(X_{ii} + \sum_{j=1, j \neq i}^N |X_{ji}|)$. The symbol '$\gg$' in $X \gg 0$ means that X is element-wise positive and '$\ll$' in $X \ll 0$ means that X is element-wise negative. Finally, the symbol $\otimes$ denotes the Kronecker product.

II. MEAN-FIELD MODEL

In this section, we study the mean-field model for a collective decision-making problem with three possible states. First, a general formulation of the problem is given for the macroscopic dynamics. Then, the perspective of a reference player is studied: the optimal control problem is analysed and finally the mean-field response for the reference player is presented.

We start by looking at the macroscopic dynamics, and consider a large population of players that can be in any of three possible states in a continuous-time dynamic game framework. These players control their state, according to some optimality criteria. Furthermore, we assume that, in the

same circumstances, all the players behave in the same way, i.e. they are *homogeneous*. The game is symmetric with respect to any permutation of players, i.e. the decision of each player does not take into account the other players individually but rather the population distribution. The players' controls depend on the knowledge of their own state and of the distribution of the population across the three states. The players in state 1 are committed to option 1 and those in state 2 are committed to option 2; finally, the players in state state 3 are simply uncommitted. For consistency with the terminology used for the microscopic dynamics, we use the term *commit* to indicate the action of a single player that chooses one of the options.

We model the distribution of the population with the probability vector $x(t) = [x_1, x_2, x_3]^T \in \mathcal{S}^3$. Players change state according to a continuous time Markov process with transition rate matrix $\beta(t) \in \mathbb{R}^{3 \times 3}$, which depends on the state $x(t)$. The elements of matrix β are indicated by β_{ij}, each of which represents the transition rate from node i to node j for any generic pair of states $i, j \in \mathcal{I}^3$. The transition rates can be split in two components: ρ_{ij} and w_{ij}. Analogously, each column of the matrix β has two components, i.e. $\beta_i = \rho_i + w_i$, where $\rho_i \in (\mathbb{R}_0^+)^3$ is controlled by the players and $w_i \in (\mathbb{R}_0^+)^3$ is controlled by an adversarial disturbance. In the mean-field limit when the number of players tends to infinity, the model is described by the following Kolmogorov equations:

$$\begin{aligned} \dot{x}_1 &= x_3 \beta_{31} - x_1 \beta_{13}, \\ \dot{x}_2 &= x_3 \beta_{32} - x_2 \beta_{23}, \\ \dot{x}_3 &= x_1 \beta_{13} + x_2 \beta_{23} - x_3 \beta_{31} - x_3 \beta_{32}. \end{aligned} \tag{1}$$

The above system has an initial condition for the population distribution $x(0) = x_0$.

We tackle the problem from the perspective of a single player, hereafter referred to as the *reference player* [32]. Here, the reference player is used to describe a general player that plays against the rest of the population, and his/her identity is anonymous, as the game is symmetric. The only information available to the reference player is the distribution of the other players across the three states, which follows from the mean-field hypothesis. This assumption on the available information is common to all the players. To study the mean-field response, let us consider the reference player and model the microscopic dynamics under the assumption that the population distribution over the time horizon is given. The state of the player takes value in a finite discrete set with cardinality 3, which describes the three possible states. The evolution of the state is described using a continuous-time Markov chain. The transition rates are chosen to minimize a total cost that consists of a running cost and a terminal penalty. The running cost depends on the state of the player, on the distribution of the population and on the transition rate, i.e. $g(i, x, \rho_i) : \mathcal{I}^3 \times \mathcal{S}^3 \times (\mathbb{R}_0^+)^3 \to \mathbb{R}$, and it is defined as:

$$g(i, x, \rho_i) = \frac{1}{2} \sum_{j \neq i} \rho_{ij}^2 R_{ij} + f_i(x), \tag{2}$$

where $\rho_i = [\rho_{i1}, \rho_{i2}, \rho_{i3}]^T \in (\mathbb{R}_0^+)^3$ is the control of the reference player, ρ_{ij} are the transition rates from state i to state j and depend on the population distribution, and $R_{ij} > 0$.

The functions $f_i(x) : \mathcal{S}^3 \to \mathbb{R}$ depend on the state and we assume it is continuous wrt $x \in \mathcal{S}^3$ (and therefore it is bounded). The first part of the cost function takes the usual form of a standard linear quadratic optimal control problem, but instead of a quadratic form for the state, we use a more general nonlinear function. Since this function is a cost term, our model accommodates crowd-seeking dynamics when this term is a monotonically decreasing function, but it can also accommodate crowd-averse dynamics when the same term is a monotonically increasing function. The model can accommodate other application domains, such as opinion dynamics [6].

Let us consider a finite horizon formulation of the game and use $[0, T]$ to indicate the horizon window. The reference player incurs also in a terminal cost which depends on the objective they seek to minimize. Let the terminal cost be $\psi(i, x) : \mathcal{I}^3 \times \mathcal{S}^3 \to \mathbb{R}$, and assume it is Lipschitz continuous in $x \in \mathcal{S}_3$.

Each player minimizes the cost functional

$$\begin{aligned} &J_x^i(\rho, w, t) \\ &= \mathbb{E}_{i_t = i}^{\rho, w} \left[\int_t^T \left[g(i_\tau, x(\tau), \rho_{i_\tau}) - \frac{1}{2} \sum_{j \neq i_\tau} w_{i_\tau j}^2 \Gamma_{i_\tau j} \right] d\tau \right. \\ &\quad \left. + \psi(i_T, x(T)) \right], \end{aligned}$$
$$\tag{3}$$

where $\mathbb{E}_{i_t = i}^{\rho, w}$ is the expectation for the event $i_t = i$ and $\Gamma_{ij} > 0$. The positive term on the control vector ρ_i and the negative term on the disturbance vector $w_i = [w_{i1}, w_{i2}, w_{i3}]^T \in (\mathbb{R}_0^+)^3$ give to the cost functional the structure of a robust mean-field game in spirit with H_∞-optimal control, see [23].

Problem 1: Consider the population dynamics in (1) where $x(t) : [0, T] \to \mathcal{S}^3$ and $\beta_{ij_\tau} = \rho_{ij_\tau} + w_{ij_\tau}$, $i \neq j_\tau$; $\rho_{ij}(\cdot) : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $w_{ij}(\cdot) : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are measurable functions that return a transition rate and a penalty of the disturbance at any given time t, respectively. Find the optimal control of the reference player which minimizes the cost functional:

$$v_i(x, t) = \inf_{\rho_i} \sup_{w_i} J_x^i(\rho_i, w_i, t), \tag{4}$$

where $v_i(x, t)$ is the value function, in the rest denoted also $v_i(t)$ or v_i for brevity and the minimization is performed over the Markovian controls for the reference player's control problem β_{ij_τ}.

For an interpretation of the controls as infinitesimal transition rates, we provide the definition of the Markov chain for the single player:

$$\mathbb{P}[i_{\tau+h} = j | i_\tau = i] = [\rho_{ij_\tau}(\tau) + w_{ij_\tau}(\tau)]h + o(h). \tag{5}$$

We define the Legendre transform of a convex function $G(p)$ as

$$G^*(q) = \sup -q^T p - G(p),$$

and when G is strictly convex and the previous supremum is achieved, then the following holds: $q = -\nabla G(p)$, or equivalently $p = -\nabla G^*(q)$ [32]. In line with [32], the generalized Legendre transform of the cost $g(i, x, \rho_i)$ for the difference operator on a general function z is

$$h(x, z, i) = \min_{\rho_i \in (\mathbb{R}_0^+)^3} g(i, x, \rho_i) + \nu_i^T \cdot \Delta_i z. \tag{6}$$

When z is the value function, we have that the Hamiltonian function $\mathcal{H}(\cdot)$ is given by:

$$\mathcal{H}(x, v, i, \rho_i, w_i) = g(\cdot) - \frac{1}{2} \sum_{j \neq i} w_{ij}^2 \Gamma_{ij} + (\rho_i + w_i)^T \Delta_i v,$$

and when it is supremized over the disturbance and infimized over the control, we obtain:

$$\bar{\mathcal{H}}(x, v, i) = \inf_{\rho_i \in (\mathbb{R}_0^+)^3} \sup_{w_i} g(\cdot) - \frac{1}{2} \sum_{j \neq i} w_{ij}^2 \Gamma_{ij} + (\rho_i + w_i)^T \Delta_i v. \tag{7}$$

Notice that in the cost the terms ρ_{ii} and w_{ii} are not present. In accordance with the structure of a transition probability matrix we let

$$\rho_{ii}^*(x, v, i) = -\sum_{j \neq i} \rho_{ij}^*(x, v, i), \tag{8}$$

$$w_{ii}^*(x, v, i) = -\sum_{j \neq i} w_{ij}^*(x, v, i). \tag{9}$$

We can now introduce the Hamilton-Jacobi equation as in the following:

$$\begin{cases} -\dot{v}_i = \bar{\mathcal{H}}(x, v, i), \\ v_i(T) = \psi(i, x(T)). \end{cases} \tag{10}$$

A system of coupled ODEs with a terminal condition like the one in (10) is referred to as terminal value problem. Because of the closed-loop structure of the game, our game is in feedback form. In preparation to the next result, we highlight that w_i^* is an adversarial but not an arbitrary signal. It is the worst-case deterministic time-varying signal which depends on the aggregate behavior of the players that choose symmetrically opposite options, namely state 1 and state 2. We are now ready to present the next result, which establishes that the solution of (10) is the value function. This is in accordance with [32], with the major difference that our dynamics include an adversarial disturbance, when assuming the given characterization. To prove the theorem, we recall that the running cost is concave in the disturbance w, which is in accordance with the coercivity condition, see [23].

Theorem 1: Let $v(t) : \mathcal{S}^3 \times [0, T] \to \mathbb{R}$ be a solution to the Hamilton-Jacobi-Bellman terminal value problem in (10). Then

$$\rho_i^* = -R_i^{-1} \left[\Delta_i v \right]^- = - \begin{bmatrix} R_{i1}^{-1}(v_1 - v_i)^- \\ R_{i2}^{-1}(v_2 - v_i)^- \\ R_{i3}^{-1}(v_3 - v_i)^- \end{bmatrix} \tag{11}$$

is the minimizer and

$$w_i^* = \Gamma_i^{-1} \left[\Delta_i v \right]^+ = \begin{bmatrix} \Gamma_{i1}^{-1}(v_1 - v_i)^+ \\ \Gamma_{i2}^{-1}(v_2 - v_i)^+ \\ \Gamma_{i3}^{-1}(v_3 - v_i)^+ \end{bmatrix} \tag{12}$$

is the maximizer over the control and disturbance, respectively. In the above, $R_i = \text{diag}(R_{ij})$ and $\Gamma_i = \text{diag}(\Gamma_{ij})$, for $j = 1, 2, 3$, and the values of ρ_{ii}^*, w_{ii}^* are obtained from (8)-(9), respectively, whereas only the values of ρ_{ij}^*, w_{ij}^* with $i \neq j$ are obtained from the above formulae. $\qquad \square$

We now tackle the problem of a reference player whose strategy profile takes into account other players' controls. Again, we assume that players are homogeneous and that a single player does not affect the evolution of the game and the interaction occurs through a mass function. In order to investigate this scenario, it is enough to replace the weighting coefficients of the diagonal matrices R_i and Γ_i, i.e. R_{ij}, Γ_{ij}, $i \neq j$, as follows :

$$R_{ij} \to R_{ij}(\Delta_i(z)), \quad \Gamma_{ij} \to \Gamma_{ij}(\Delta_i(z)), \quad i \neq j. \tag{13}$$

The difference of function $z_i(x, t)$ at two nodes can be interpreted as the resistance of transitions from one node to another. Now, we consider the cost functional (3), in the following denoted as $J_x^i(\rho_i, w_i, t, \Delta z_i)$, with weighting coefficients given by (13). The following problem can be stated.

Problem 2: Consider Problem 1 and let $\Delta_i z$ be the difference operator on the function that captures the resistance of transitions from a state to another $z_i(x_i, i) : \mathcal{S}^3 \times \mathbb{R}_0^+ \to \mathbb{R}$. Find the optimal control for the reference player which minimizes the cost functional (3) that depends on the distribution and on the control of the other players:

$$v_i(x, t) = \inf_{\rho_i} \sup_{w_i} J_{x_i}^i(\rho_i, w_i, t, \Delta_i z), \tag{14}$$

where $v_i(x_i, t)$ is the value function and the minimization is performed over the Markovian controls for the single player control problem β_{ij_τ}.

Writing the corresponding Hamiltonian function $\mathcal{H}(\cdot)$ and the corresponding Hamilton-Jacobi-Bellman ODEs, the solution of the problem is given by (11)-(12), where R_i and Γ_i are substituted by $R_i(\Delta_i z) = \text{diag}(R_{ij}(\Delta_i z))$ and $\Gamma_i(\Delta_i z) = \text{diag}(\Gamma_{ij}(\Delta_i z))$, for $j = 1, 2, 3$, respectively, and the dependence is on the difference operator on z, namely $\Delta_i z$. As before, for the values of ρ_{ii}^* and w_{ii}^* we use (8) and (9), respectively.

Remark. The main difference of this formulation of the problem is the role of the difference operator in the strategy profile of the reference player. The adversarial disturbance is treated again as a worst-case deterministic time-varying signal.

III. STATIONARY MEAN-FIELD EQUILIBRIUM

In this section, we use the general result found in Theorem 1 to obtain a fully worked-out characterization of the stationary solutions of our game. First, we find the equilibrium for the Kolmogorov equations in terms of the population distribution. Second, we find the solution of the optimal stochastic control problem corresponding to the reference player playing against a stationary population distribution. Finally, when the solution of the mass distribution is obtained by the optimal control and the solution of the optimal control is obtained against the same distribution, we obtain a fixed point solution that is, by the consistency condition, the mean-field game Nash equilibrium. This solution exists and is unique as proved in Theorem 5.

We start by showing that the equilibrium obtained by maximizing over the disturbance and minimizing over the control exists and it is unique. Because of the superlinearity

and uniform convexity of the cost function $g(\cdot)$, the following function

$$\eta_i^*(x,v,i) = \underset{\rho_i}{\operatorname{argmin}}\,\underset{w_i}{\max}\;g(\cdot) - \frac{1}{2}\sum_{j\neq i} w_{ij}^2 \Gamma_{ij} + (\rho_i + w_i)^T \Delta_i v \tag{15}$$

is well defined, as the saddle point exists and is unique. When background players use strategy ρ and the best response for the reference player is also ρ, we say that the current solution is a mean-field game Nash equilibrium. The corresponding mean-field game Nash equilibrium is given by the following system which combines the Kolmogorov equations and the Hamilton-Jacobi-Bellman equations:

$$\begin{cases} \dot{x}_i(t) = (1 - x_1(t) - x_2(t))\beta_{3i} - x_i\beta_{i3}, & \forall i \in \mathcal{I}^2 \\ \dot{x}_3(t) = -\dot{x}_1(t) - \dot{x}_2(t), \\ -\dot{v}_i(t) = \bar{\mathcal{H}}(x(t), \Delta_i v(t), i_t), & \forall i \in \mathcal{I}^3 \\ x(0) = x_0, \\ v_{i_T}(T) = \psi(i_T, x(T)), \end{cases} \tag{16}$$

where β is obtained from (11)-(12), and the above system is obtained from bringing together (1) and (15). Equation (15) models the way in which players respond to the evolution of the population defined by (1), and (1) describes the way in which the population evolves as a whole under the assumption that all the players behave according to (15). The macroscopic dynamics are modeled by (1), while the microscopic best response of each player is given by (15). This problem is called *initial-terminal value problem* (ITVP) for the mean-field game, see [32]. We expand the Hamiltonian according to (7) and use the optimal control and disturbance from (11)-(12). The calculation of the value function is the following:

$$\begin{aligned} -\dot{v}_i &= \tfrac{1}{2}\sum_{j\neq i}\rho_{ij}^2 R_{ij} - \tfrac{1}{2}\sum_{j\neq i} w_{ij}^2 \Gamma_{ij} + (\rho_i + w_i)^T \Delta_i v \\ &\quad + f_i(x) \\ &= -\tfrac{1}{2}(\Delta_i v)^{-T} R_i^{-1}(\Delta_i v)^- + \tfrac{1}{2}(\Delta_i v)^{+T}\Gamma_i^{-1}(\Delta_i v)^+ \\ &\quad + f_i(x). \end{aligned} \tag{17}$$

For consistency between the macroscopic and microscopic dynamics, we assume that the parameters $R_{12}, R_{21}, \Gamma_{12}, \Gamma_{21}$ are such that asymptotically the following holds: $R_{12}^{-1}, R_{21}^{-1}, \Gamma_{12}^{-1}, \Gamma_{21}^{-1} \to 0$. Therefore, the transition rates between state 1 and state 2 are negligible. In order to investigate the transitions from the two committed states to the uncommitted state and vice versa, we subtract the third equation to both the first and the second equations in the above system and replace the closed-loop expressions with the transition rate parameters in (11) and (12). Letting

$$y_1 = v_3 - v_1, \quad y_2 = v_3 - v_2, \quad \xi_1 = x_1, \quad \xi_2 = x_2, \tag{18}$$

we obtain the following switched system:

$$\dot{\xi} = \operatorname{diag}(y)\left(A_\sigma \xi + g_\sigma\right), \tag{19}$$

$$\dot{y} = -\frac{1}{2}A_\sigma^T \operatorname{diag}(y)y + b(\xi), \tag{20}$$

$$\xi(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x_0, \tag{21}$$

$$y(T) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \Psi(x(T)), \tag{22}$$

where $\sigma = \{1,2,3,4\}$ corresponds to the four quadrants $(y_1 > 0, y_2 > 0)$, $(y_1 < 0, y_2 > 0)$, $(y_1 < 0, y_2 < 0)$, $(y_1 > 0, y_2 < 0)$, respectively and

$$A_1 = \begin{bmatrix} -R_{31}^{-1} - \Gamma_{13}^{-1} & -R_{31}^{-1} \\ -R_{32}^{-1} & -R_{32}^{-1} - \Gamma_{23}^{-1} \end{bmatrix}, \quad g_1 = \begin{bmatrix} R_{31}^{-1} \\ R_{32}^{-1} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} R_{13}^{-1} + \Gamma_{31}^{-1} & \Gamma_{31}^{-1} \\ -R_{32}^{-1} & -R_{32}^{-1} - \Gamma_{23}^{-1} \end{bmatrix}, \quad g_2 = \begin{bmatrix} -\Gamma_{31}^{-1} \\ R_{32}^{-1} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} R_{13}^{-1} + \Gamma_{31}^{-1} & \Gamma_{31}^{-1} \\ \Gamma_{32}^{-1} & R_{23}^{-1} + \Gamma_{32}^{-1} \end{bmatrix}, \quad g_3 = \begin{bmatrix} -\Gamma_{31}^{-1} \\ -\Gamma_{32}^{-1} \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -R_{31}^{-1} - \Gamma_{13}^{-1} & -R_{31}^{-1} \\ \Gamma_{32}^{-1} & R_{23}^{-1} + \Gamma_{32}^{-1} \end{bmatrix}, \quad g_4 = \begin{bmatrix} R_{31}^{-1} \\ -\Gamma_{32}^{-1} \end{bmatrix},$$

$$b(\xi) = \begin{bmatrix} f_1(x) - f_3(x) \\ f_2(x) - f_3(x) \end{bmatrix},$$

where $\xi = [x_1, x_2]^T$ is the reduced state vector.

For the sake of conciseness we denote the quadrants $\mathcal{Q}_k$, $k = 1,2,3,4$ and say that $y \in \mathcal{Q}_k$ if $J_k y \ll 0$, where J_k are defined as the quaternion matrices

$$J_1 = -\mathbb{I}_2, \; J_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; J_3 = \mathbb{I}_2, \; J_4 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The switched system in (19)-(20) is used to establish the existence of a stationary solution. For the game under consideration, stationary solutions, also called stationary mean-field equilibrium points, are defined as:

$$\begin{cases} \sum_k x_k\beta_{ki}^* - \sum_j x_i\beta_{ij}^* = 0, & \forall i \in \mathcal{I}^3, \\ \mathcal{H}(x, \Delta_i v, i) = \kappa, \end{cases} \tag{23}$$

where κ is a constant and β_{ij}^* is the optimal Markovian control obtained from (11)-(12). It is worth noting that functions $f_i(x)$ are fixed and constant in a stationary mean-field equilibrium. This is due to the fact that the population distribution x is at an equilibrium of (1) and thus constant. For the existence and stability property of the first equation of (23), namely stationary distributions $x(t)$, we studied existence and stability in [6]. With regards to this, notice that for a fixed y, the equilibrium point of (19) depends only on quadrant $k = 1,2,3,4$ in which y is settled. Indeed, let $\bar{y}^{[k]}$ be the fixed value of y in the k-th quadrant, then the associated equilibrium point for ξ is

$$\bar{\xi}^{[k]} = -A_k^{-1}g_k, \tag{24}$$

that depends only on $\sigma = k$. Moreover, it is easy to recognize that

$$\dot{x} = \begin{bmatrix} \operatorname{diag}(y)(A_\sigma + g_\sigma \mathbf{1}_2^T) & \operatorname{diag}(y)g_\sigma \\ -y^T(A_\sigma + g_\sigma \mathbf{1}_2^T) & -y^T g_\sigma \end{bmatrix} x \tag{25}$$

is a switched forward Kolmogov equation so that $\operatorname{diag}(y)A_k$ is Hurwitz for any k and for constant y. We have:

$$\lim_{t\to\infty} x(t) = \bar{x}^{[k]} = \begin{bmatrix} -A_k^{-1}g_k \\ 1 + \mathbf{1}_2^T A_k^{-1}g_k \end{bmatrix}.$$

It is worth noting that even if $y(t)$ is (bounded) time-varying, but remains within a certain quadrant k, then $x(t)$ converges to the constant distribution $\bar{x}^{[k]}$.

Theorem 2: Consider the Kolmogorov system (25) and assume that $y(t)$ is bounded and $y(t) \in \mathcal{Q}_k$, $t \geq 0$. Then

$$\lim_{t\to\infty} x(t) = \bar{x}^{[k]} = \begin{bmatrix} -A_k^{-1}g_k \\ 1 + \mathbf{1}_2^T A_k^{-1}g_k \end{bmatrix}.$$

The aim of the rest of this section is to investigate the solutions of the second equation of (23). We now seek to study the asymptotic stability of our system under a deterministic adversarial disturbance. By fixing the equilibrium point for the population distribution we study the evolution of the value function under that specification. Therefore, the next result establishes the existence of stationary value functions, namely value functions that satisfy the second equation of (23).

Theorem 3: Let a stationary distribution $\bar{\xi}$ be given. Then there exists k such that

$$J_k b(\bar{\xi}) \gg 0, \tag{26}$$

and a stationary value function exists, which is given by

$$\bar{y}^{[k]} = -J_k \sqrt{2A_k^{-1^T} b(\bar{\xi})} \in \mathcal{Q}_k. \tag{27}$$

Moreover, $\bar{y}^{[k]}$ is backward stable. □

Remark. Equation (26) can be justified by (20) where the equations are positive from $\mathrm{diag}(y)y$. The equilibrium points can be found at the intersections of two ellipses centered at the origin obtained from setting (20) equal to zero. Different values of σ change the parameters $b(\xi)$ of the switched system and therefore change the shape of the ellipses. If we view the difference of value functions between two nodes as a difference of potentials, this difference is constant, i.e. the potential at the nodes is the same. It is easier for the disturbance to make the player go back to the node where their gain is lower. Additionally, we can say that $b(\xi)$ affect the distance of the ellipses from the center of axis, the equivalent of the radius in the circle.

We now prove that the backward solution of the equation of the value function in (20) is bounded for any solution $\xi(t)$ of (19).

Theorem 4: Let $T > 0$ and $y(T) = y_T$. Consider the backward solution $y(t)$, $t \leq T$ of (20). This solution is bounded for $T \to \infty$. □

We are now ready to prove the existence of a mean-field equilibrium for the coupled initial-terminal value problem, under the following assumption for the cost function $g(\xi)$.

Assumption 1: There exists one and only one $k \in \{1, 2, 3, 4\}$ such that

$$J_k b(\xi^{[k]}) \gg 0. \tag{28}$$

Theorem 5: Under Assumption 1, the unique stationary solution of mean-field game is given by

$$\bar{\xi} = -A_k^{-1} g_k, \quad \bar{y} = -J_k \sqrt{2A_k^{-1^T} b(-A_k^{-1} g_k)}.$$

Moreover, the constant κ of the mean-field stationary solution in (23) is given by:

$$
\begin{aligned}
\kappa &= -0.5 R_{13}^{-1} [(\bar{y}_1)^-]^2 + 0.5 \Gamma_{13}^{-1} [(\bar{y}_1)^+]^2 + f_1(\bar{\xi}) \\
&= -0.5 R_{23}^{-1} [(\bar{y}_2)^-]^2 + 0.5 \Gamma_{23}^{-1} [(\bar{y}_2)^+]^2 + f_2(\bar{\xi}) \\
&= -0.5 R_{31}^{-1} [(\bar{y}_1)^+]^2 - 0.5 R_{32}^{-1} [(\bar{y}_2)^+]^2 \\
&\quad + 0.5 \Gamma_{31}^{-1} [(\bar{y}_1)^-]^2 + 0.5 \Gamma_{32}^{-1} [(\bar{y}_2)^-]^2 + f_3(\bar{\xi}).
\end{aligned}
$$

□

Remark. Assumption 1 is enforced by the more stringent assumption that $g(\xi)$ belongs to the same quadrant for any ξ, i.e.

$$g(\xi) \in \mathcal{Q}_i, \quad \forall \xi \in [0, 1]^2.$$

In such a case, the mean-field equilibrium is given by

$$\bar{\xi} = -A_k^{-1} g_k, \quad \bar{y} = -J_k \sqrt{2A_k^{-1^T} b(-A_k^{-1} g_k)} \in \mathcal{Q}_k,$$

where $k = 4 - i$ if $i = 1, 2, 3$ or $k = 2$ if $i = 4$, which means that $\mathcal{Q}_k$ is the quadrant opposite to $\mathcal{Q}_i$.

We are ready to prove convergence to a stationary distribution of the mean-field response.

Theorem 6: Assume that $g(\xi)$ is continuous and bounded and satisfying assumption (28). Consider system (19), (20). Given a probability vector x_0 and let a terminal condition ψ, let $(\xi(t), y(t))$ be the solution of (19), (20), with initial-terminal conditions $\xi(-T) = \xi_0$ and $y(T) = g(\xi(T))$. Then, for $T \to \infty$,

$$\xi(0) = \bar{\xi}, \quad y(0) = \bar{y},$$

where $\bar{\xi}$ and $\bar{y}$ are the unique stationary solution of Theorem 3. □

Proof. From the given assumptions, a unique stationary solution exists and the solutions $\xi(t)$, $t \geq -T$, $y(t)$, $t \leq T$, are continuous and bounded. The result follows from Theorem 2, 3, 4, 5. ∎

Remark. The stationary solution can be found by standard shooting technique. Let

$$\xi^{[i]}(t) = \varphi_\xi(t, \xi_0, -T, y^{[i]}(\cdot))$$

be the flow of (20) with initial condition $\xi(-T) = \xi_0$, given $y(t) = y^{[i]}(\cdot)$ and

$$y^{[i+1]}(t) = \varphi_y(t, y_T, \xi^{[i]}(\cdot))$$

be the flow of (19) with final condition y_T at time T. Then the composition

$$\xi^{[i+1]}(t) = \varphi_\xi(t, \xi_0, -T, \varphi_y(., y_T, \xi^{[i]}(\cdot)))$$

gives a sequence of uniformly bounded continuous functions in $[-T, T]$ for which a fixed point exists thanks to the Schauder fixed point theorem, see [42], since the set of continuous functions in $[-T, T]$ is a relatively compact, bounded and convex set, and all functions are uniformly bounded. We know that a unique stationary solution exists, so the sequence converges to $\bar{\xi}$ (and then $y \to \bar{y}$) for $T \to \infty$.

Example 1: Let

$$
\begin{aligned}
R_{13}^{-1} &= 0.4868, \ R_{31}^{-1} = 0.4359, \ R_{23}^{-1} = 0.4468, \\
R_{32}^{-1} &= 0.3063, \Gamma_{13}^{-1} = 0.5085, \Gamma_{31}^{-1} = 0.5108, \\
\Gamma_{23}^{-1} &= 0.8176, \Gamma_{32}^{-1} = 0.7948, \xi_0 = [0.2 \ 0.3]^T, \\
b(\xi) &= \begin{bmatrix} 8 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 & -4 \\ 5 & -1 \end{bmatrix} \xi \in \mathcal{Q}_1, y_T = \begin{bmatrix} 6 \\ -8 \end{bmatrix},
\end{aligned}
$$

the mean-field equilibrium is given by

$$\bar{\xi} = \begin{bmatrix} 0.2741 \\ 0.4647 \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} -2.461 \\ -2.881 \end{bmatrix},$$
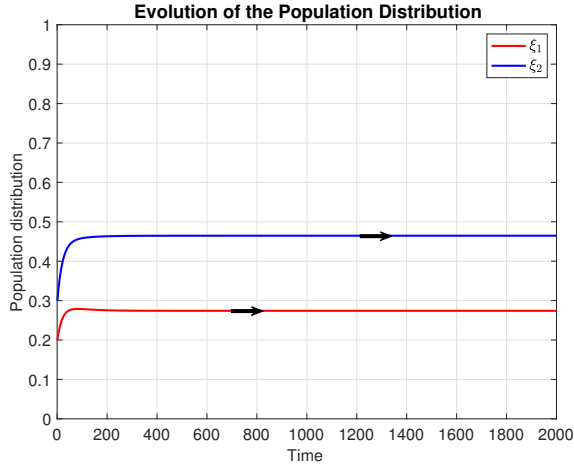
Fig. 1: Forward solution: evolution of the population distribution, with initial condition $\xi_0 = [0.2, 0.3]^T$.
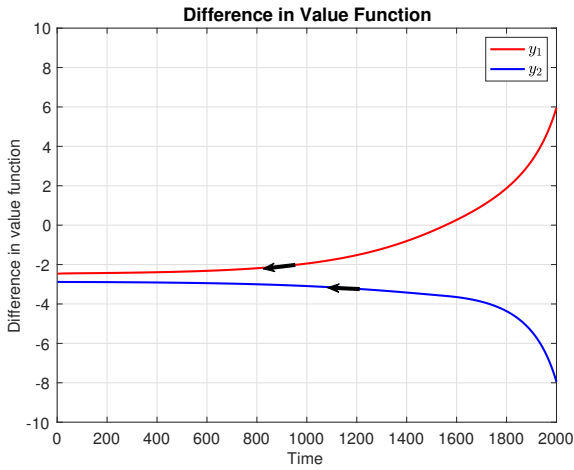

Fig. 2: Backward solution: evolution of the difference in value function $y_1 = v_3 - v_1$ and $y_2 = v_3 - v_2$.

and the costant κ of the mean-field equilibrium point in (23) is $\kappa = 0.5150$. The simulations for the initial transients of ξ (for $t \geq 0$) and y (for $t < T = 2000$) are reported in Figg. 1 and 2, respectively. In particular, Figure 1 shows the forward solution, namely the evolution of the population distribution over time, given an initial condition; Figure 2 shows the backward solution, i.e. the difference in value function over time, given a terminal condition. When 0 is replaced by $-T$ and $T \to \infty$, both ξ and y converge to the stationary values $\bar{\xi}$ and $\bar{y} \in \mathcal{Q}_3$.

In this example, we analyse the mean-field equilibrium and its uniqueness for the system of equations in (19)-(20). Specifically, it can be observed that, after setting the parameters to random values between 0 and 1, and giving an initial condition on the population distribution, namely $\xi = [0.2, 0.3]^T$, and a final condition on the value function $y_T = [6, -8]^T$, we can calculate the constant κ of the mean-field equilibrium. Indeed, the stationary solution is bounded, as stated in Theorem 4, and it is unique in accordance to Theorem 5.
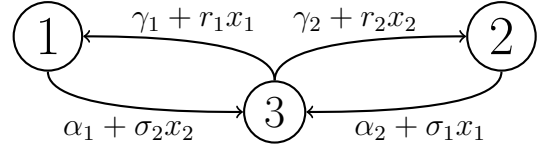

Fig. 3: Markov chain corresponding to system (29).

IV. DECISION-MAKING IN HONEYBEES

In this section, we provide a link between the optimal control approach discussed previously and the cognitive task of nest-site selection in the context of honeybee swarms. First, let us introduce the model corresponding to system (1) in [7] where we consider only the deterministic part as in the following:

$$\begin{cases} \dot{x}_1 = x_3(x_1 r_1 + \gamma_1) - x_1(x_2 \sigma_2 + \alpha_1), \\ \dot{x}_2 = x_3(x_2 r_2 + \gamma_2) - x_2(x_1 \sigma_1 + \alpha_2), \\ \dot{x}_3 = -\dot{x}_1 - \dot{x}_2, \end{cases} \tag{29}$$

where x_i represents state i, and r, γ, σ and α are positive definite. We have replaced ρ with r to avoid confusion with the control ρ in our game model. In (29), the waggle dance is captured by parameter r, and the cross-inhibitory stop signal is captured by σ. Bees can spontaneously choose to commit to one option or to abandon their commitment through parameters γ and α, respectively. The Markov chain corresponding to the above model is depicted in Fig. 3.

In the following we show that, by an appropriate choice of the values of the control and disturbance, system (19) takes the form in (29), given the equilibrium point calculated in (27). Let us first rewrite system (19) as in the following:

$$\dot{\xi} = \text{diag}(y)(\hat{A}_\sigma \xi + g_\sigma x_3), \tag{30}$$

where $\hat{A}_\sigma$ is defined as:

$$\hat{A}_\sigma = \begin{bmatrix} \hat{a}_{11,\sigma} & 0 \\ 0 & \hat{a}_{22,\sigma} \end{bmatrix},$$

and $g_\sigma = [g_{1,\sigma}, g_{2,\sigma}]^T$ consists of the elements as in system (19). In the above, $\hat{a}_{11,\sigma}$ and $\hat{a}_{22,\sigma}$ are the element $a_{11,\sigma}$ and $a_{22,\sigma}$ in the original A_σ without the parameter in the off-diagonal for any quadrant σ, respectively. Therefore, e.g. for $\sigma = 1$ we have $\hat{a}_{11,1} = -\Gamma_{13}^{-1}$ and $\hat{a}_{22,1} = -\Gamma_{23}^{-1}$. As for g_σ, in the same quadrant $\sigma = 1$ we have $g_{1,1} = R_{31}^{-1}$ and $g_{2,1} = R_{32}^{-1}$. We are now ready to show the next result, which brings together the optimal control approach developed so far with the collective decision-making in honeybees. As usual, the vector ξ corresponds to $[x_1, x_2]^T$.

Lemma 1: Assume that the parameters in g_σ depend on the population distribution as in the following:

$$g_{1,\sigma} = -\frac{x_1 r_1 + \gamma_1}{x_2 \sigma_2 + \alpha_1} \hat{a}_{11,\sigma},$$

$$g_{2,\sigma} = -\frac{x_2 r_2 + \gamma_2}{x_1 \sigma_1 + \alpha_2} \hat{a}_{22,\sigma},$$

and let the cost functions $f_i(x)$ be such that

$$\begin{bmatrix} f_1(x) - f_3(x) \\ f_2(x) - f_3(x) \end{bmatrix} = b(\xi) = \frac{A_k^T}{2} \begin{bmatrix} \frac{(x_2 \sigma_2 + \alpha_1)^2}{\hat{a}_{11,k}^2} \\ \frac{(x_1 \sigma_1 + \alpha_2)^2}{\hat{a}_{22,k}^2} \end{bmatrix},$$

where k is a fixed integer in $\{1, 2, 3, 4\}$. Then, under the difference of potential feedback

$$y_1 = -\frac{x_2\sigma_2 + \alpha_1}{\hat{a}_{11,\sigma}}, \quad y_2 = -\frac{x_1\sigma_1 + \alpha_2}{\hat{a}_{22,\sigma}},$$

the population dynamics in (30) coincides with (29) and the equilibrium is given by $\bar{\xi} = -\hat{A}_k^{-1} g_k / (1 - \mathbf{1}^T \hat{A}_k^{-1} g_k)$ as in Theorem 3. $\square$

Proof. The proof follows from substituting the values of g_σ in system (30), from which we obtain:

$$\dot{x}_1 = -(x_2\sigma_2 + \alpha_1)x_1 + (r_1 x_1 + \gamma_1)x_3,$$
$$\dot{x}_2 = -(x_1\sigma_1 + \alpha_2)x_2 + (r_2 x_2 + \gamma_2)x_3,$$

which corresponds to system (29). Let us now consider system (19) as $\dot{\xi} = \operatorname{diag}(y)(A_\sigma \xi + g_\sigma)$ and let us investigate the behavior at the equilibrium in quadrant k. By substituting $A_k = \hat{A}_k - g_k \mathbf{1}^T$ in (23) and from the sum of matrices inversion lemma, we have

$$\bar{\xi} = -\frac{\hat{A}_k^{-1} g_k}{1 - \mathbf{1}^T \hat{A}_k^{-1} g_k}.$$

Then, the equilibrium in (20) coincides with (27), since

$$\bar{y}^{[k]} = -J_k \sqrt{2 A_k^{-1^T} b(\bar{\xi})}$$
$$= -J_k \begin{bmatrix} \frac{\bar{x}_2\sigma_2 + \alpha_1}{|\hat{a}_{11,k}|} \\ \frac{\bar{x}_1\sigma_1 + \alpha_2}{|\hat{a}_{22,k}|} \end{bmatrix} = \begin{bmatrix} \frac{\bar{x}_2\sigma_2 + \alpha_1}{\hat{a}_{11,k}} \\ \frac{\bar{x}_1\sigma_1 + \alpha_2}{\hat{a}_{22,k}} \end{bmatrix}.$$

This concludes the proof. ∎

In light of the above, we find that the mean-field game model studied in Section II yields the evolutionary model for the collective decision-making in honeybee swarms.

V. PERIODIC SOLUTIONS

Motivated by the literature on environmental feedback, we now investigate the case where the parameters depend on the state through the value function. The authors in [36] propose intrinsic and extrinsic dynamics for the evolution of the environment, where the strategy profile of the population has an impact on the environmental variable and the environment also evolves intrinsically on its own. We propose to model the environmental feedback through the value function and its direct dependence on the state. The intuition is that the decision-making in honeybees strongly depends on the waggle dance and cross-inhibitory signalling, which take the form of a control and disturbance in the value function. Therefore, we include these two behavioral traits in the optimal control problem as environmental feedback through the value function. As a consequence of this feedback, the system dynamics yield oscillatory behaviors that result in limit cycles and bifurcations as it is shown in the following.

We introduce a form of environmental feedback by assuming that the penalty coefficients R_{13}, R_{31}, Γ_{13}, Γ_{31}, R_{23}, R_{32}, Γ_{23}, Γ_{32} are functions of variables y_1 and y_2. More formally, we let

$$R_{13}^{-1} = R_{32}^{-1} = \Gamma_{32}^{-1} = \Gamma_{13}^{-1} = |y_1|, \tag{31}$$
$$R_{31}^{-1} = R_{23}^{-1} = \Gamma_{23}^{-1} = \Gamma_{31}^{-1} = |y_2|. \tag{32}$$

Moreover, we also assume that functions $f_i(x)$ (and hence $b(\xi)$) depend only on y. With a slight abuse of notation we have

$$b(y) = My, \quad M = \begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \tag{33}$$

where M is parametrized in μ, which satisfies the left-hand equation above to accommodate the change of sign and the corresponding parametric study across the four quadrants. By so doing, the Hamiltonian switched system is still given by (19), (20) where

$$A_1 = A_3 = \begin{bmatrix} -y_1 - y_2 & -y_2 \\ -y_1 & -y_1 - y_2 \end{bmatrix}, \tag{34}$$

$$A_2 = A_4 = \begin{bmatrix} -y_1 + y_2 & y_2 \\ y_1 & -y_2 + y_1 \end{bmatrix}, \tag{35}$$

$$g_1 = g_3 = \begin{bmatrix} y_2 \\ y_1 \end{bmatrix}, \tag{36}$$

$$g_2 = g_4 = \begin{bmatrix} -y_2 \\ -y_1 \end{bmatrix}. \tag{37}$$

We now study the existence and property of mean-field equilibria as a function of the parameter μ. To this end, let

$$\mu^* = -1.5674.$$

The details are given in the proof of the following theorem in the Appendix.

Theorem 7: Consider the switched Hamiltonian system with the structure given by equations (31)-(37). Given $T > 0$, an initial condition $\xi_0 = \xi(-T)$ for (19), a terminal condition $y(T) = \psi$ for (20), the switching equilibrium points are characterized as a function of the real parameter μ as follows:

- **Case 1**. If $\mu > 0$, the equilibrium is given by the stable zero equilibrium

$$\bar{\xi} = 0, \quad \bar{y} = 0.$$

- **Case 2**. When $\mu \leq 0$, the equilibrium point in 0 is no longer stable and depending on the value of μ, the following holds true:
  - **Case 2.1**. If $\mu^* < \mu \leq 0$, the equilibrium is characterized by a supercritical Hopf bifurcation at $\mu = 0$, and a stable periodic limit cycle exists for $\tilde{y}(t)$ and $\tilde{\xi}(t)$.
  - **Case 2.2**. If $\mu \leq \mu^*$, the system is characterized by a homoclinic bifurcation to nonhyperbolic saddle, for which the periodic limit cycle is no longer stable and four equilibrium points exist, two stable and two saddle points. $\square$

Example 2: With reference to Theorem 7, let us consider each case in the order they appear in the theorem, and therefore let $\mu > 0$ as in *Case 1*, first. A unique mean-field equilibrium exists $\bar{\xi} = 0$, $\bar{y} = 0$, and is stable, in accordance with Theorem 7. Figure 4 depicts the phase portrait of $y(t)$ for $\mu = 0.2$.

Now, we set $\mu = -1$ as in *Case 2.1*. As proved in the Appendix, this case corresponds to the existence of a stable limit cycle. The limit cycle is inscribed in the annulus described by the circles whose radiuses are defined from the
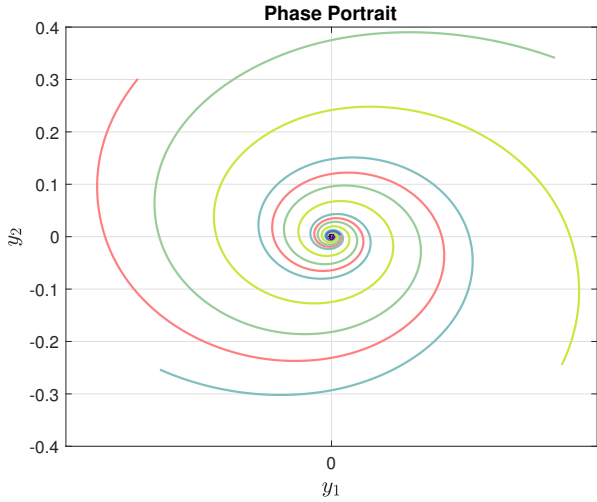
Fig. 4: Phase portrait of y, $\mu = 0.2$.

center to the closest point and the furthest point of the gray area, respectively. The gray area is defined by the candidate Lyapunov function $V = y_1^2 + y_2^2$ specialized in each of the four quadrants. The phase portrait in the plane y_1-y_2 is illustrated in Fig. 5: regardless of the terminal condition on y, all trajectories converge to the stable limit cycle depicted in purple. The rotation is counterclockwise, as indicated by the black arrow. Letting $y(T) = [-5, 6]^T$ and $\xi(0) = [0.2, 0.3]^T$, the forward solution of $\xi(t)$ and the backward solution of $y(t)$ are represented in Figures 6 and 7, respectively.
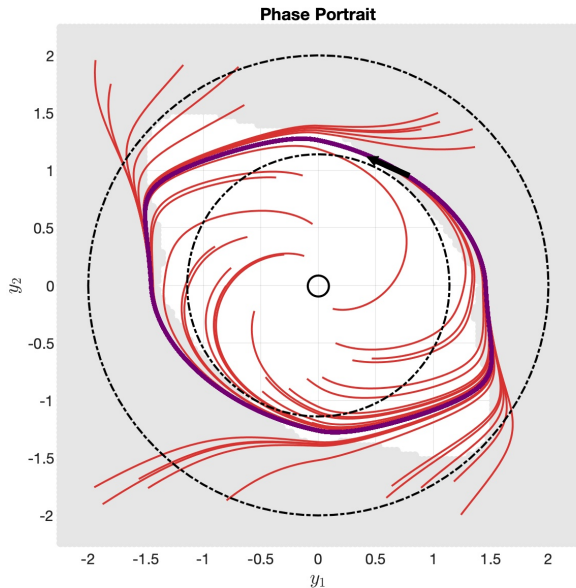

Fig. 5: Phase portrait of y, $\mu = -1$. The gray area is the one defined by the candidate Lyapunov function $V = y_1^2 + y_2^2$ specialized in each of the four quadrants. The limit cycle is found inside the annulus described by the dotted circles.

Finally, we consider $\mu < \mu^*$ as in *Case 2.2*. In this case, two stable equilibria exist, one in the second and one in the fourth quadrant. The phase portrait for $\mu = -2$ corresponding to this
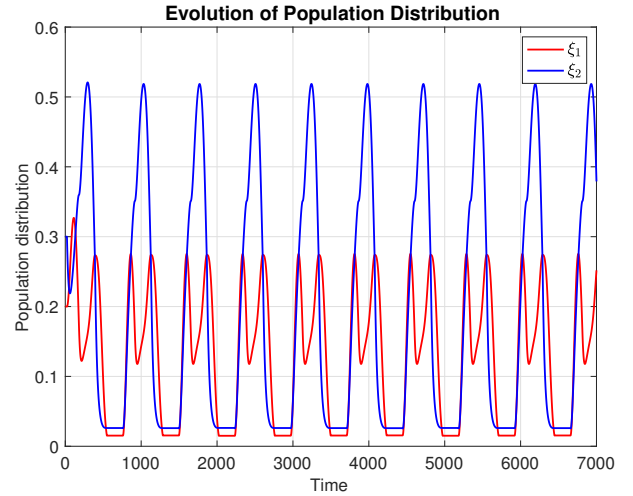

Fig. 6: Forward solution: oscillations of $\xi(t)$, $\mu = -1$.
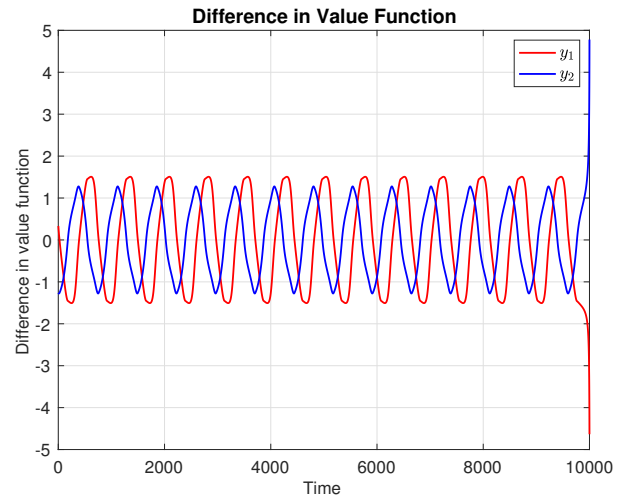
Fig. 7: Backward solution: oscillations of $y(t)$, $\mu = -1$.

case is depicted in Fig. 8. Here, y tends to one of the two stable equilibria, namely $\bar{y}^{[2]} = -\bar{y}^{[4]} = [-2.1415, 1.5787]^T$ in accordance with Theorem 7. The corresponding equilibrium for the population distribution is $\bar{\xi} = [0.1818, 0.7734]^T$, which shows that a lower value of μ prevents deadlocks and allows the population to obtain a large consensus on option 2.

VI. NETWORKED MICROSCOPIC SYSTEM

In the following, we derive a microscopic model where we consider a finite number of players interacting by means of a network topology. This model approximates system (1), of which system (29) represents the evolutionary dynamics corresponding to the collective decision-making in honeybee swarms. The state of player i is represented by the probability across the three states r_i, s_i, z_i. We now introduce the microscopic dynamics in the form of a networked model for collective decision-making. More specifically, we study a population of N players, each corresponding to a node of a network. An edge between two nodes indicates that the corresponding players interact with one another. The interaction

Fig. 8: Phase portrait of y, $\mu = -2$.

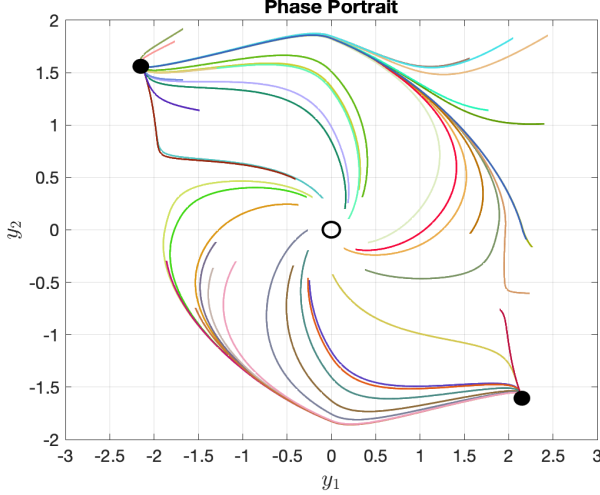topology is described by a fully connected, undirected and complete communication graph $G = \{\mathcal{V}, \mathcal{E}\}$, with adjacency matrix A, i.e. $A = \frac{1}{N-1}(\mathbf{1}_N \mathbf{1}'_N - I_N)$. In the following, let us denote r_i, s_i and z_i the probability that player i is in state 1, 2 and 3, respectively. Due to the dependence of matrix β on the state, we split the linear and constant components as follows: we indicate by β'_{ij} the linear transition rates and have the state explicitly indicated in the equations; we indicate by β''_{ij} the constant transition rates. The model describing the time evolution of the players' state is given by the following system of equations:

$$
\begin{cases}
\dot{r}_i(t) = -\beta'_{13} r_i(t) \sum_{j=1}^{N} a_{ij} s_j(t) + \beta'_{31} z_i(t) \sum_{j=1}^{N} a_{ij} r_j(t) \\
\qquad -\beta''_{13} r_i(t) + \beta''_{31} z_i(t), \\
\dot{s}_i(t) = -\beta'_{23} s_i(t) \sum_{j=1}^{N} a_{ij} r_j(t) + \beta'_{32} z_i(t) \sum_{j=1}^{N} a_{ij} s_j(t) \\
\qquad -\beta''_{23} s_i(t) + \beta''_{32} z_i(t), \\
\dot{z}_i(t) = \beta'_{13} r_i(t) \sum_{j=1}^{N} a_{ij} s_j(t) - \beta'_{31} z_i(t) \sum_{j=1}^{N} a_{ij} r_j(t) \\
\qquad +\beta'_{23} s_i(t) \sum_{j=1}^{N} a_{ij} r_j(t) - \beta'_{32} z_i(t) \sum_{j=1}^{N} a_{ij} s_j(t) \\
\qquad +\beta''_{13} r_i(t) - \beta''_{31} z_i(t) + \beta''_{23} s_i(t) - \beta''_{32} z_i(t).
\end{cases}
\tag{38}
$$

We can rewrite system (38) in vector form as

$$
\begin{cases}
\dot{r}(t) = -\beta'_{13} \mathrm{diag}(r(t)) A s(t) + \beta'_{31} \mathrm{diag}(z(t)) A r(t) \\
\qquad -\beta''_{13} r(t) + \beta''_{31} z(t), \\
\dot{s}(t) = -\beta'_{23} \mathrm{diag}(s(t)) A r(t) + \beta'_{32} \mathrm{diag}(z(t)) A s(t) \\
\qquad -\beta''_{23} s(t) + \beta''_{32} z(t), \\
\dot{z}(t) = +\beta'_{13} \mathrm{diag}(r(t)) A s(t) - \beta'_{31} \mathrm{diag}(z(t)) A r(t) \\
\qquad +\beta'_{23} \mathrm{diag}(s(t)) A r(t) - \beta'_{32} \mathrm{diag}(z(t)) A s(t) \\
\qquad +\beta''_{13} r(t) - \beta''_{31} z(t) + \beta''_{23} s(t) - \beta''_{32} z(t).
\end{cases}
\tag{39}
$$

In the following, we assume that the linear and constant coefficients are coupled through a parameter k as in the following:

$$
\beta'_{31} = k\beta'_{13}, \ \beta''_{31} = k\beta''_{13}, \ \beta'_{32} = k\beta'_{23}, \ \beta''_{32} = k\beta''_{23}, \tag{40}
$$

where k is a parameter that corresponds to a measure of the coupling strength between each pair of parameters. It is possible to formalize a problem for a differential game similar to the one in Section III for system (39). The differential game

converges to the stationary solutions studied in Section III with the main difference that the control parameters are now coupled through parameter k.

In order to prove convergence of system (39), let us first consider the three dimensional system for a single agent (the time dependence has been omitted for conciseness):

$$
\begin{aligned}
\dot{\hat{r}} &= -\beta'_{13}\hat{s} + k\beta'_{13}\hat{z}\hat{r} - \beta''_{13}\hat{r} + k\beta''_{13}\hat{z}, &\tag{41} \\
\dot{\hat{s}} &= -\beta'_{23}\hat{s}\hat{r} + k\beta'_{23}\hat{z}\hat{s} - \beta''_{23}\hat{s} + k\beta''_{23}\hat{z}, &\tag{42} \\
\dot{\hat{z}} &= -\dot{\hat{r}} - \dot{\hat{s}}. &\tag{43}
\end{aligned}
$$

We are now ready to establish the following result.

Theorem 8: Consider system (41), (42), (43) under assumption (40). If

$$
\nu = \frac{\beta''_{13}\beta''_{23}}{\beta'_{13}\beta'_{23}} > \frac{k}{4(k+1)},
$$

there is an unique equilibrium point

$$
\bar{\hat{r}} = \frac{k}{2k+1}, \quad \bar{\hat{s}} = \frac{k}{2k+1}, \quad \bar{\hat{z}} = \frac{1}{2k+1},
$$

and this equilibrium point is asymptotically stable. $\qquad\square$

We can now use the result in Theorem 8 to prove convergence for system (39): more specifically, because of the homogeneity of the population we multiply the system for the single player by $\mathbf{1}_N$ and subtract this to the networked system (39). We can now prove that this difference goes to 0 and therefore the equilibrium point is asymptotically stable.

Theorem 9: Consider the networked model (39) under assumption (40). The following statements hold:
1) If $r(0), s(0), z(0) \in [0\,1]_N$, then $r(t), s(t), z(t) \in [0\,1]_N$ for all $t > 0$.
2) If

$$
\nu = \frac{\beta''_{13}\beta''_{23}}{\beta'_{13}\beta'_{23}} > \frac{k}{4(k+1)},
$$

then the equilibrium point

$$
\bar{r} = \frac{k}{2k+1}\mathbf{1}_N, \quad \bar{s} = \frac{k}{2k+1}\mathbf{1}_N, \quad \bar{z} = \frac{1}{2k+1}\mathbf{1}_N
$$

is asymptotically stable. $\qquad\square$

Remark. The novelty of the above result is that a lower value of the parameter k determines higher value of z at the equilibrium. Thus, the larger the parameter k, the smaller the probability of a generic agent to be in the uncommitted state. To provide a physical interpretation in the context of the decision-making, this means that if the strength of the disruptive signals is k times larger than the force to convince other agents – in the context of honeybees the two are given by the cross-inhibitory signal and the waggle dance, respectively – the probability that an agent is in the uncommitted state is proportionally lower.

Example 3: In this example, we present a set of simulations to corroborate our theoretical results on the microscopic networked model (39). We consider $N = 20$ agents, each with a random initial condition such that $r_i(0) + s_i(0) + z_i(0) = 1$. We set the parameters as:

$$
\begin{aligned}
\beta'_{13} &= 2.5, \ \beta''_{13} = 1.1, \ \beta'_{23} = 2.2, \\
\beta''_{23} &= 1.2, \ \beta'_{31} = 0.625, \ \beta''_{31} = 0.275, \\
\beta'_{32} &= 0.55, \ \beta''_{32} = 0.3, \ k = 0.25.
\end{aligned}
$$

In accordance with Theorem 9, the condition $\nu > k/4(k+1)$ holds true and the swarm, regardless of the initial conditions, converges to the following stable equilibrium point:

$$\bar{r} = 0.1667\,\mathbf{1}_N, \quad \bar{s} = 0.1667\,\mathbf{1}_N, \quad \bar{z} = 0.6664\,\mathbf{1}_N.$$

This scenario is depicted in Fig. 9. Each agent i is represented by the same color: the dashed line for state r, the dotted line for state s and the solid line for state z. As it can be seen from the figure, regardless of the random initial conditions, all agents converge to the same equilibrium point.
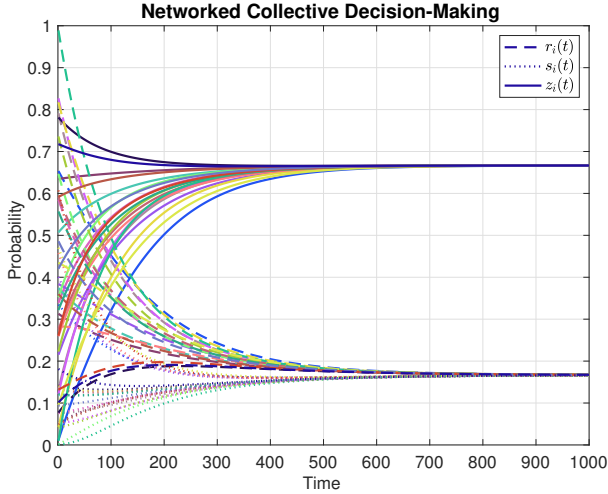

Fig. 9: Evolution of population distribution, $k = 0.25$.

VII. Conclusion

In this paper, we have proposed an optimal control framework to study the mean-field game model for the collective decision-making originating in the context of honeybee swarms. We have reframed this problem in a continuous-time dynamic game framework via switched systems in the form of an *initial-terminal value problem*. We have investigated the corresponding stationary solutions and proved the existence and uniqueness of a mean-field Nash equilibrium. We have then considered the case where parameters depend on the difference between the value function calculated in two states in a form of game-environment feedback and have studied the periodic solutions associated to this case. Furthermore, we have studied the corresponding networked model for a finite number of players where the state of each player represents the probability to choose one of the options. We have modeled the interactions through a network topology and studied how the equilibrium points are affected by a coupling parameter k that links the linear transition rates together, as well as the constant ones. We have also shown the connection between this problem and the initial mean-field model by highlighting the difference that consists in the parameters being linked through k. The plan for future works includes: i) the study of the corresponding master model, in the case of more than two options, and ii) the extension of the results to sliding modes in the form of sliding mean-field equilibrium points.

Appendix

Proof of Theorem 1. Suppose that ρ_i is any control. Suppose also that w_i^* is the optimal disturbance obtained from the robust Hamiltonian in (7), for given ρ_i, and that given the nature of w_i^*, it can be regarded as a constant value for which we can establish the following:

$$
\begin{aligned}
&J_{x_i}^i(\rho_i, w_i, t) \\
&= \mathbb{E}_{i_t=i}^{\rho_i, w_i^*}\left[\psi(i_T, x_{i_T}) + \int_t^T g(\cdot) - \sum_{j \neq i_\tau} w_{i_\tau j}^2 \Gamma_{i_\tau j}\right] d\tau \Big] \\
&= v_i(t) + \mathbb{E}_{i_t=i}^{\rho_i, w_i^*}\left[\int_t^T \left[\frac{dv_{i_\tau}}{dt}(\tau) + (A^\rho v)^{i_\tau}(\tau)\right.\right. \\
&\quad \left.\left. + g(i_\tau, x_{i_\tau}(\tau), \rho_{i_\tau}(\tau)) - \sum_{j \neq i_\tau} w_{i_\tau j}^2 \Gamma_{i_\tau j}\right] d\tau\right].
\end{aligned}
\tag{44}
$$

The second equality above is obtained from the Dynkin formula

$$
\begin{aligned}
&\mathbb{E}_{i_t=i}^{\rho_i, w_i^*}\left[v_{i_T}(T) - v_i(t)\right] = \mathbb{E}_{i_t=i}^{\rho_i, w_i^*}\left[\int_t^T \left[\frac{dv_{i_\tau}}{dt}(\tau)\right.\right. \\
&\quad \left.\left. + (A^\rho v)^{i_\tau}(\tau) d\tau\right]\right],
\end{aligned}
\tag{45}
$$

where the terminal condition is $v_{i_T}(T) = \psi(i_T, x_{i_T})$ and $(A^\rho v)^{i_\tau}(\tau) = \sum_j \rho_{ij}(\tau)[v_j(\tau) - v_{i_\tau}(\tau)]$ is the infinitesimal generator of process i_τ. Then we have that

$$
\begin{aligned}
J_{x_i}^i(\rho_i, w_i, t) \geq\ & v_i(t) + \mathbb{E}_{i_t=i}^{\rho_i, w_i^*}\left[\int_t^T \left[\frac{dv_{i_\tau}}{dt}(\tau)\right.\right. \\
&+ \min_{\mu(\cdot) \in (\mathbb{R}_0^+)^3} \sum_j \mu_j \left[v_j(\tau) - v_{i_\tau}(\tau)\right] \\
&\left.\left. + g(i_\tau, x_{i_\tau}(\tau), \rho_{i_\tau}(\tau)) - \sum_{j \neq i_\tau} w_{i_\tau j}^2 \Gamma_{i_\tau j}\right] d\tau\right].
\end{aligned}
\tag{46}
$$

The above inequality follows from the fact that we minimize over any control $\mu(\cdot) \in (\mathbb{R}_0^+)^3$. Given the assumption that the running cost is concave in the disturbance and that the coefficients are bounded, the RHS of the previous equation can be rewritten as:

$$
\begin{aligned}
J_{x_i}^i(\rho_i, w_i, t) \geq\ & v_i(t) + \mathbb{E}_{i_t=i}^{\rho_i, w_i}\left[\int_t^T \left[\frac{dv_{i_\tau}}{dt}(\tau)\right.\right. \\
&+ \min_{\mu(\cdot) \in (\mathbb{R}_0^+)^3} \max_{w(\cdot) \in (\mathbb{R}_0^+)^3} \sum_j \mu_j \left[v_j(\tau) - v_{i_\tau}(\tau)\right] \\
&\left.\left. + g(i_\tau, x_{i_\tau}(\tau), \rho_{i_\tau}(\tau)) - \sum_{j \neq i_\tau} w_{i_\tau j}^2 \Gamma_{i_\tau j}\right] d\tau\right].
\end{aligned}
\tag{47}
$$

The above equation is obtained from the definition of w_i^*, by maximising over any possible disturbance.

Replacing the minimax term by the robust Hamiltonian as in (7), we obtain:

$$
\begin{aligned}
J_{x_i}^i(\rho_i, w_i, t) &= v_i(t) + \mathbb{E}_{i_t=i}^{\rho_i, w_i^*}\left[\int_t^T \left[\frac{dv_{i_\tau}}{dt}(\tau)\right.\right. \\
&\quad \left.\left. + \mathcal{H}(x_{i_\tau}(\tau), \Delta_{i_\tau} v(\tau), i_\tau)\right] d\tau\right] \\
&= v_i(t).
\end{aligned}
\tag{48}
$$

To conclude the proof, the mean-field response as in (10) exists and it is the value function of the optimal control problem. By differentiating (7) with respect to ρ_i and taking the gradient equal to zero, we have

$$R_i\rho_i + \Delta_i v = 0, \qquad (49)$$

from which we obtain the optimal control in (11), in the general case for any R_{ij}. Similarly, by differentiating (7) with respect to w_i and taking the gradient equal to zero, we have:

$$-\Gamma_i w_i + \Delta_i v = 0, \qquad (50)$$

which yields the optimal adversarial disturbance as in (12), in the general case for Γ_{ij}. This concludes the proof. ∎

Proof of Theorem 2. Let

$$B(k,y) = \left[\begin{array}{cc} \text{diag}(y)(A_k + g_k\mathbf{1}_2^T) & \text{diag}(y)g_k \\ -y^T(A_k + g_k\mathbf{1}_2^T) & -y^Tg_k \end{array} \right].$$

We already know that, since $\sigma(t) = k$, then

$$B(k,y)\bar{x}^{[k]} = 0,$$

with $B(k,y)$ Metzler with $\mathbf{1}_3^T B(k,y) = 0$. Denote by $B_{ij}(k,y)$ the entries of $B(k,y)$ and take as Lyapunov function the relative entropy

$$V(x) = \sum_{i=1}^{3} \frac{x_i}{\bar{x}_i^{[k]}} log(x_i) + x_i - \bar{x}_i^{[k]}.$$

The derivative is

$$\begin{aligned} \dot{v}(x) = & -\sum_{i=1}^{3}\sum_{j\neq i} B_{ij}(k,y)\Phi\left(\frac{x_j\bar{x}_i^{[k]}}{\bar{x}_j^{[k]}x_i}\right)x_i\frac{\bar{x}_j^{[k]}}{\bar{x}_i^{[k]}} \\ & + \sum_{i=1}^{3}\sum_{j=1}^{3} B_{ij}(k,y)\frac{x_i}{\bar{x}_i^{[k]}}\bar{x}_j^{[k]}, \end{aligned}$$

where $\Phi(s) = 1 - s + s\log(s) \geq 0$. Since $B(k,y)\bar{x}^{[k]} = 0$ the last term vanishes and since $B_{ij}(k,y) \geq 0$ for $i \neq j$ we have $\dot{V}(x) \leq 0$. Finally notice that, thanks to the irreducibility of $B(k,y)$, the only possible perturbed trajectory yielding $\dot{V}(x) = 0$ is such that $\Phi(s) = 0$ that means $s = 1$, i.e. $x = \bar{x}^{[k]}$. Asymptotic stability then follows from La Salle, and the proof is concluded. ∎

Proof of Theorem 3. It is easy to see that there exists a unique $k \in \{1,2,3,4\}$ such that $J_k g(\bar{\xi}) \gg 0$. Indeed, if $g(\xi) \in \mathcal{Q}_i$ then k is the index of the quadrant oppositive to the $i - th$ quadrant. Moreover, thanks to the structure of matrix A_k, it also follows that $A_k^{-1^T} b(\bar{\xi}) \gg 0$, so ensuring the existence of a positive equilibrium for $\text{diag}(y)y$. Hence, the equilibrium in the k-th quadrant is recovered by taking the square root and multiplying by $-J_k$. As for the stability, compute the Jacobian of system (20), i.e.

$$H = -A_k^T \text{diag}(\bar{y}^{[k]}).$$

Matrix $-H$ is element-wise negative with positive determinant. This proves backward stability of $\bar{y}^{[k]}$. This concludes the proof. ∎

Proof of Theorem 4. Let $z(t) = |y(T-t)|$, $\gamma(t) = \sigma(T-t)$, $r(t) = \xi(T-t)$, $q_\gamma(t) = J_\gamma b(r(t))$, $Q_\gamma = -1/2J_\gamma A'_\gamma$, $M = \sup_{\xi\in[0\,1]^2} \|b(\xi)\|$. Then,

$$\dot{z}(t) = Q_{\gamma(t)}\text{diag}(z(t))z(t) + q_{\gamma(t)}(t), \quad z(0) = |y_T|.$$

Now notice that the 1-Lozinski measure μ_1 for each γ is strictly negative, i.e. there exists $\alpha > 0$ such that

$$\mu_1(Q_k) \leq -\alpha, \quad k = 1,2,3,4.$$

Therefore

$$\mathbf{1}_2^T \dot{z} \leq -\beta(z)\mathbf{1}_2^T z + \mathbf{1}_2^T q_\gamma,$$

with $\beta(z) = \alpha \min_{i,2} z_i$. Notice that the positive variable $z_i(t)$ cannot be zero in a nonzero length interval, and this means that for any interval $[\tau\, t]$ there exists $\bar{\beta} > 0$ such that $\int_\tau^t \beta(z(\tau))d\tau \geq (t-\tau)\bar{\beta}$. Therefore

$$\mathbf{1}_2^T z(t) \leq e^{-\bar{\beta}t}|y_T| + \frac{M}{\bar{\beta}}(1 - e^{-\bar{\beta}t}).$$

This concludes the proof. ∎

Proof of Theorem 5. By taking the derivatives equal to zero, the equilibrium points $\bar{\xi}$, $\bar{y}$ of (19), (20) must satisfy:

$$\begin{aligned} 0 &= \text{diag}(\bar{y})\left(A_{\sigma(\bar{y})}\bar{\xi} + g_{\sigma(\bar{y})}\right), & (51) \\ 0 &= -\frac{1}{2}A_{\sigma(\bar{y})}^T\text{diag}(\bar{y})\bar{y} + b(\bar{\xi}). & (52) \end{aligned}$$

Assumption 1 is equivalent to $2A_{\sigma(\bar{y})}^{-1T}b(-A_{\sigma(\bar{y})}^{-1}g_{\sigma(\bar{y})}) \gg 0$ so that, $\sigma(\bar{y}) = k$, $\bar{\xi} = -A_k^{-1}g_k$ and $\bar{y} = -J_k\sqrt{2A_k^{-1T}b(-A_k^{-1}g_k)}$. The formula for κ derives directly from (49), letting $\dot{v}_i = 0$. This concludes the proof. ∎

Proof of Theorem 7. Notice that the equation in y is independent of ξ and can be written as

$$\dot{y} = 0.5 \left[\begin{array}{c} (y_1 + y_2)y_1^2 + y_1y_2^2 \\ y_2y_1^2 + (y_1 + y_2)y_2^2 \end{array} \right] + My,$$

if $y_1y_2 > 0$ (quadrants I, III) and

$$\dot{y} = 0.5 \left[\begin{array}{c} (y_1 - y_2)y_1^2 - y_1y_2^2 \\ -y_2y_1^2 + (y_2 - y_1)y_2^2 \end{array} \right] + My,$$

if $y_1y_2 < 0$ (quadrants II, IV).

- **Case 1**: $\mu > 0$. It is clear that $\bar{y} = 0$ is an equilibrium point and that (through a simple computation) there are no other equilibria for $\mu > 0$. The reverse Jacobian is

$$J = \left[\begin{array}{cc} -\mu & -1 \\ 1 & -\mu \end{array} \right],$$

  that is Hurwitz for $\mu > 0$. Moreover, let a candidate Lyapunov function be $V = y_1^2 + y_2^2$. We have

$$\dot{V} = -V^2 - (y_1y_2 + 2\mu)V,$$

  in quadrants I and III and

$$\dot{V} = (y_1y_2 - 2\mu)V - (y_1^2 - y_2^2)^2,$$

  in quadrants II and IV. Therefore, for $\mu > 0$ the Lyapunov function V is a common Lyapunov function in all quadrants so that the equilibrium point $\bar{y} = 0$

is globally asymptotically stable. Correspondingly, the mean-field equilibrium is

$$\bar{\xi} = 0, \ \bar{y} = 0.$$

- **Case 2**: $\mu \leq 0$. The zero equilibrium point is unstable. The point $\mu = 0$ is a supercritical Hopf bifurcation as it can be seen by looking at the Jacobian matrix above for Case 1: when $\mu > 0$, we have two negative real eigenvalues, but at $\mu = 0$, we have two purely imaginary eigenvalues. It is a matter of simple computations to show that all other possible equilibrium points are in quadrants II and IV, characterized by $\bar{y}_2 = \alpha \bar{y}_1$ with $\alpha \in (-1, 0)$ and

$$\bar{y}_1^2 = \frac{\alpha^2 + 1}{\alpha(\alpha^2 - 1)},$$

where α is related to the given μ as follows:

$$\mu = \frac{-\alpha^4 + \alpha^3 + 2\alpha^2 + \alpha - 1}{2\alpha(\alpha^2 - 1)}.$$

The reverse Jacobian in quadrants II, IV is

$$J = \begin{bmatrix} (\alpha + \alpha^2/2 - 3/2)\bar{y}_1^2 - \mu & (1/2 + \alpha)\bar{y}_1^2 - 1 \\ (\alpha + \alpha^2/2)\bar{y}_1^2 + 1 & (1/2 - 3\alpha^2/2 + \alpha)\bar{y}_1^2 - \mu \end{bmatrix},$$

that is Hurwitz for $\alpha \in (-1, -0.544)$. By substituting the value $\alpha = -0.544$ in the above formula for μ, we obtain the critical value $\mu^* = -1.5674$. At this value we have a homoclinic bifurcation to nonhyperbolic saddle for which two foci and two saddles are present and the trajectories converge to one focus depending on the terminal condition, see Fig. 19 in [39]. In the following, we investigate two values of α: the first case is for $\alpha_1 \in (-0.544, 0)$, corresponding to $\mu^* = -1.5674 < \mu \leq 0$, and then the second case is for $\alpha_2 \in (-1, -0.544)$, corresponding to $\mu \leq \mu^*$.

  - **Case 2.1**: $\mu^* = -1.5674 < \mu \leq 0$. As stated in the general Case 2, the only equilibrium point is $\bar{y} = 0$ and it is unstable as the Jacobian matrix at this point has eigenvalues with positive real parts. To prove the existence of a limit cycle, we need to find a closed bounded subset M of the plane such that every trajectory starting in M stays in M for all time. To this end, let us consider the candidate Lyapunov function $V = y_1^2 + y_2^2$ as before and let $M = V \leq c$, with $c > 0$. For quadrants I-III, at the boundary $V = c$ we have:

$$y\nabla V =$$
$$y_1^4 + 2y_1^2 y_2^2 + y_2^4 + y_1 y_2(y_1^2 + y_2^2) + 2\mu(y_1^2 + y_2^2)$$
$$= (y_1^2 + y_2^2)^2 + y_1 y_2(y_1^2 + y_2^2) + 2\mu(y_1^2 + y_2^2)$$
$$\leq (y_1^2 + y_2^2)^2 + (y_1^2 + y_2^2)^2 + 2\mu(y_1^2 + y_2^2)$$
$$= 2c^2 + 2\mu c,$$

where we used the fact that $\|y_1 y_2\| \leq y_1^2 + y_2^2$. A similar calculation for quadrants II-IV yields:

$$y\nabla V = y_1^4 + y_2^4 - y_1 y_2(y_1^2 + y_2^2) + 2\mu(y_1^2 + y_2^2)$$
$$\leq (y_1^2 + y_2^2)^2 + (y_1^2 + y_2^2)^2 + 2\mu(y_1^2 + y_2^2)$$
$$= 2c^2 + 2\mu c,$$

where we used the fact that $y_1^4 + y_2^4 \leq (y_1^2 + y_2^2)^4$ and $\|y_1 y_2\| \leq y_1^2 + y_2^2$. Therefore, by choosing $c \leq -\mu$,

we can ensure that all trajectories are trapped inside M. Therefore, by the Poincaré-Bendixon Theorem there exists a globally stable periodic motion $y(t)$ (with period τ depending on μ) in the positively invariant region M, see [40]. The time-varying equilibrium is given by $y(t)$ and by the unique τ-periodic solution of (19). Trajectories converge to the limit cycle in counterclockwise direction. For $\alpha_1 \in (-0.544, 0)$, two unstable equilibrium points (saddles) exist in the second and fourth quadrants, namely $\bar{y}_1$, $\bar{y}_2 = \alpha_1 \bar{y}_1$.

  - **Case 2.2**: $\mu \leq \mu^*$. The equilibrium point $y = 0$ is unstable. For $\alpha_2 \in (-1, -0.544)$, two stable equilibrium points and two saddle points exist in the second and fourth quadrants, namely $\bar{y}_1$, $\bar{y}_2 = \alpha_2 \bar{y}_1$, for the different values of α solving the equation for μ. In conclusion, for $\mu \leq \mu^*$ there are two locally stable equilibrium points, denoted by $\bar{y}^{[2]}$ in the second quadrant and $\bar{y}^{[4]}$ in the fourth quadrant (depending on $y(T)$). Depending on the values of α, both foci and saddle points are given by

$$\bar{y}^{[2]} = -\bar{y}^{[4]} = -\begin{bmatrix} 1 \\ \alpha_2 \end{bmatrix} \sqrt{\frac{\alpha_2^2 + 1}{\alpha_2(\alpha_2^2 - 1)}},$$

and the associated mean-field equilibrium is given by either $(\bar{\xi}, y^{[2]})$ or $(\bar{\xi}, y^{[4]})$ with

$$\bar{\xi} = -A_2^{-1} g_2 = -A_4^{-1} g_4 = \frac{1}{\alpha_2^2 + 1}\begin{bmatrix} \alpha_2^2 - \alpha_2 - 1 \\ -\alpha_2^2 + 1 - \alpha_2 \end{bmatrix}.$$

Letting $\beta = y_2/y_1$ it follows that

$$\dot{\beta} = \beta^2 + 1,$$

in the first and third quadrants and

$$\dot{\beta} = \beta^2 + 1 + \beta(1 - \beta^2)y_1^2,$$

in the second and third quadrants. This means that $y(t)$ crosses the y_2 axis in anticlockwise direction towards $\bar{y}^{[2]}$ (second quadrant) or $\bar{y}^{[4]}$ (fourth quadrant) corresponding to $\bar{\beta} = 1/\alpha_2$.

This concludes the proof. ∎

Proof of Theorem 8. The equilibriun points of the system satisfy:

$$\begin{bmatrix} -\beta_{13}'' & \beta_{13}'\bar{r} \\ \beta_{23}'\bar{s} & -\beta_{23}'' \end{bmatrix}\begin{bmatrix} \bar{r} - k\bar{z} \\ \bar{s} - k\bar{z} \end{bmatrix} = 0.$$

Hence one equilibrium is $\bar{r} = \bar{s} = k\bar{z} = \frac{k}{2k+1}$. Other equilibria arise if $\nu = \frac{\beta_{13}''\beta_{23}''}{\beta_{13}'\beta_{23}'} = \bar{r}\bar{z}$. In such a case a simple computation shows that

$$\bar{r}^3 + \bar{r}^2(\frac{k+1}{k}\alpha - 1) + \bar{r}(\frac{k+1}{k}\nu - \alpha) + \nu\alpha = 0,$$

i.e.

$$\bar{r}(\bar{r}^2 - \bar{r} + \frac{k+1}{k}\nu) + \alpha(\bar{r}^2\frac{k+1}{k} - \bar{r} + \nu) = 0,$$

where $\alpha = \frac{\beta_{13}''}{\beta_{13}'}$. Standard root locus arguments show that there are no real and positive equilibria if the zeros of $\bar{r}^2 - \bar{r} + \frac{k+1}{k}\nu$ are not real, and this happens if $\nu > \frac{\beta_{13}''\beta_{23}''}{\beta_{13}'\beta_{23}'}$.

Finally, take the linearized system along the above equilibrium point and compute the Jacobian J (reduced to the variations of $\hat{r}$ and $\hat{s}$). It follows

$$
J = - \left[\begin{array}{cc} (1+k)\beta_{13}'' + \frac{k^2}{1+2k}\beta_{13}' & k\beta_{13}'' + k\frac{k+1}{1+2k}\beta_{13}' \\ k\beta_{23} + k\frac{k+1}{1+2k}\beta_{23}'' & (1+k)\beta_{13}'' + \frac{k^2}{1+2k}\beta_{13}' \end{array} \right].
$$

The stability result follows by noticing that J has negative trace and that its determinant $(2k+1)(\beta_{23}''\beta_{13}'' - \frac{k^2}{(1+2k)^2}\beta_{13}'\beta_{23}')$ is positive iff $\nu > \frac{k^2}{(1+2k)^2}$. This condition is enforced by the assumption $\nu > \frac{k}{4(k+1)}$ since $\frac{k^2}{(1+2k)^2} \leq \frac{k}{4(k+1)}$. This concludes the proof. ∎

Proof of Theorem 9. First of all recall from [41] that a nonlinear system $\dot{x} = f(x)$ is positive if and only if $x_i = 0 \longrightarrow f_i(x) \geq 0$ for any x in the boundary of the positive orthant. This property is satisfied by system (38). Moreover, as apparent from (38), $\mathbf{1}_3^T \dot{p}_i(t) = 0$, where $p_i = [r_i\, s_i\, z_i]^T$. This concludes the proof of point (1).

As for point (2), notice first that $A\mathbf{1}_N = \mathbf{1}_N$, and this easily implies that

$$
\bar{r} = \frac{k}{2k+1}\mathbf{1}_N, \quad \bar{s} = \frac{k}{2k+1}\mathbf{1}_N, \quad \bar{z} = \frac{1}{2k+1}\mathbf{1}_N
$$

is an equilibrium point. A cumbersome computation, relying on the fact that $A = \frac{1}{N-1}(\mathbf{1}_N\mathbf{1}_N' - I_N)$, shows that the linearized system can be written as

$$
\left[\begin{array}{c} \delta\dot{r} \\ \delta\dot{s} \\ \mathbf{1}_N'\delta\dot{r} \\ \mathbf{1}_N'\delta\dot{s} \end{array} \right] = \left[\begin{array}{cc} (B - \frac{C}{N-1}) \otimes I_N & \frac{C}{N-1} \otimes \mathbf{1}_N \\ 0 & B + C \end{array} \right] \left[\begin{array}{c} \delta r \\ \delta s \\ \mathbf{1}_N'\delta r \\ \mathbf{1}_N'\delta s \end{array} \right],
$$

where $\delta r = r - \bar{r}$ and similarly for δs, and

$$
B = \left[\begin{array}{cc} -(1+k)(\beta_{13}'\bar{\bar{r}} + \beta_{13}'') & -k(\beta_{13}\bar{\bar{r}} + \beta_{13}'') \\ -k(\beta_{23}'\hat{r} + \beta_{23}'') & -(k+1)(\beta_{23}'\hat{r} + \beta_{23}'') \end{array} \right],
$$

$$
C = \left[\begin{array}{cc} \beta_{13}'\bar{\bar{r}} & -\beta_{13}'\bar{\bar{r}} \\ -\beta_{23}'\hat{r} & \beta_{23'}\hat{r} \end{array} \right].
$$

Matrix $B + C$ coincides with the Jacobian J in the proof of Theorem 8, and it is Hurwitz. Moreover, $B - C/(N-1)$ is also Hurwitz, for any $k > 0$ and any $N > 1$. This concludes the proof. ∎

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