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GAP STATISTICS AND HIGHER CORRELATIONS FOR GEOMETRIC PROGRESSIONS MODULO ONE

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Abstract. Koksma's equidistribution theorem from 1935 states that for Lebesgue almost every $\alpha > 1$, the fractional parts of the geometric progression $(\alpha^n)_{n\geq 1}$ are equidistributed modulo one. In the present paper we sharpen this result by showing that for almost every $\alpha > 1$, the correlations of all finite orders and hence the normalized gaps of $(\alpha^n)_{n\geq 1}$ mod 1 converge to the Poissonian model, thereby resolving a conjecture of the two first named authors. While an earlier approach used probabilistic methods in the form of martingale approximation, our reasoning in the present paper is of an analytic nature and based upon the estimation of oscillatory integrals. This method is robust enough to allow us to extend our results to a natural class of sub-lacunary sequences.

1. Introduction

A sequence $(\vartheta_n)_{n\geq 1} \subseteq [0,1)$ is called uniformly distributed (or equidistributed) if each test interval $I \subseteq [0,1)$ contains asymptotically its "fair share" of points, that is, $(\vartheta_n)_{n\geq 1}$ is equidistributed when

$$\frac{\#\left\{n \leq N : \vartheta_n \in I\right\}}{N} \underset{N\to\infty}{\longrightarrow} \lambda\left(I\right)$$

for all intervals $I \subseteq [0,1)$, where λ denotes the Lebesgue measure. A sequence $(\vartheta_n)_{n\geq 1}$ of numbers in $\mathbb{R}$ is called uniformly distributed modulo one if the sequence of fractional parts $(\{\vartheta_n\})_{n\geq 1}$ is uniformly distributed in $[0,1)$. The classical theory of uniform distribution modulo one dates back to the early twentieth century, when Weyl [33] laid its foundations in his famous paper of 1916.

One of the basic results in the area is Koksma's equidistribution theorem [20], which states that for λ-almost every $\alpha > 1$, the sequence corresponding to the geometric progression $(\alpha^n)_{n\geq 1}$ is uniformly distributed modulo one. Such sequences with a "typical" value of α have been famously proposed by Knuth in his monograph *The art of computer programming* [19] as examples of sequences showing strong pseudorandomness properties. Koksma's equidistribution theorem has been extended to so-called complete uniform distribution by Niederreiter and Tichy [24], and quantitative equidistribution estimates were obtained in [1]. A version of Koksma's equidistribution theorem for self-similar measures was proved in [4]. Describing the behaviour of $(\alpha^n)_{n\geq 1}$ for specific values of α is a challenging problem. A well known and open problem due to Mahler asks for the range of $(\{\xi(3/2)^n\})_{n\geq 1}$, where $\xi > 0$ is a real parameter. For more on this topic, and the study of the sequence $(\alpha^n)_{n\geq 1}$ modulo one, we refer the reader to [5, 7, 8, 12] and the references therein.

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While the classical notion of equidistribution modulo one addresses the "large-scale" behaviour of the fractional parts of a sequence (counting the number of points in *fixed* intervals), the study of the *fine-scale* statistics of sequences modulo one, i.e. statistics on the scale of the mean gap $1/N$, has attracted growing attention in recent years. Among the most popular fine-scale statistics are the k-point correlations and the nearest-neighbour gap distribution, which are defined as follows.

Let $\vartheta = (\vartheta_n)_{n\geq 1} \subseteq \mathbb{R}$ be a sequence, and let $k \geq 2$ be an integer. Let $\mathcal{B}_k = \mathcal{B}_k(N)$ denote the set of integer k-tuples $(x_1, \ldots, x_k)$ such that all components are in the range $\{1, \ldots, N\}$ and no two components are equal. For a compactly supported function $f : \mathbb{R}^{k-1} \to \mathbb{R}$, the k-point correlation sum $R_k(f, \vartheta, N)$ is defined to be

$$R_k(f, \vartheta, N) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{\mathbf{x}\in\mathcal{B}_k} \sum_{\mathbf{m}\in\mathbb{Z}^{k-1}} f\left(N\left(\Delta(\mathbf{x}, \vartheta) - \mathbf{m}\right)\right) \tag{1.1}$$

where $\Delta(\mathbf{x}, \vartheta)$ denotes the difference vector

$$\Delta(\mathbf{x}, \vartheta) = \left(\vartheta_{x_1} - \vartheta_{x_2}, \vartheta_{x_2} - \vartheta_{x_3}, \ldots, \vartheta_{x_{k-1}} - \vartheta_{x_k}\right) \in \mathbb{R}^{k-1}. \tag{1.2}$$

Let $C_c^\infty(\mathbb{R}^{k-1})$ denote the space of real-valued, smooth, compactly supported functions on $\mathbb{R}^{k-1}$. If

$$\lim_{N\to\infty} R_k(f, \vartheta, N) = \int_{\mathbb{R}^{k-1}} f(\mathbf{x})\, \mathrm{d}\mathbf{x}$$

for all $f \in C_c^\infty(\mathbb{R}^{k-1})$ (equivalently, if $R_k(1_\Pi, \vartheta, N) \to \mathrm{vol}(\Pi)$ as $N \to \infty$ for all axis-parallel boxes Π, where 1_Π is the indicator function of Π), then we say that the k-point correlation of $(\{\vartheta_n\})_{n\geq 1}$ is "Poissonian". This notion alludes to the fact that such behaviour is in accordance with the (almost sure) behaviour of a Poisson process with intensity one.

To define the distribution of the so-called level spacings or nearest-neighbour gaps, i.e. gaps between consecutive elements of $(\{\vartheta_n\})_{n\geq 1}$, we need to consider the *reordered* elements

$$\vartheta_{(1)}^N \leq \vartheta_{(2)}^N \leq \cdots \leq \vartheta_{(N)}^N \leq \vartheta_{(N+1)}^N,$$

which we obtain as a reordering of $\{\vartheta_1\}, \ldots, \{\vartheta_{N+1}\}$. Assume that the limit $N \to \infty$ of the function

$$G(s, \vartheta, N) = \frac{1}{N} \#\left\{n \leq N : \ N\left(\vartheta_{(n+1)}^N - \vartheta_{(n)}^N\right) \leq s\right\}$$

exists for all $s \geq 0$. Then the limit function $G(s)$ is called the asymptotic distribution function of the level spacings (or, alternatively, of the nearest-neighbour gaps) of $(\{\vartheta_n\})_{n\geq 1}$. We say that the level spacings are *Poissonian* when $G(s) = 1 - e^{-s}$, which is in agreement with the well-known fact that the waiting times in the Poisson process are exponentially distributed.

The k-point correlation of order $k = 2, 3, \ldots$, is also called the pair correlation, triple correlation, etc. Poissonian behaviour of these local statistics can be seen as a pseudorandomness property, since a sequence $X_1, X_2, \ldots$ of independent, identically distributed random variables with uniform distribution on $[0, 1)$ will almost surely have Poissonian correlations/gap distributions. Note that equidistribution is also traditionally seen as a pseudorandomness property, albeit on a "global" rather than on a "local" level.

Recently, the first two authors of the present paper proved that $(\{\alpha^n\})_{n\geq 1}$ has Poissonian pair correlation for almost all $\alpha > 1$; see [2]. This is a refinement of Koksma's equidistribution theorem mentioned earlier, since it is known that a sequence with Poissonian pair correlations is necessarily equidistributed [3, 13, 23]. In [2] it was conjectured that for almost all $\alpha > 1$, the k-point correlation of $(\{\alpha^n\})_{n\geq 1}$ should also be Poissonian for all $k \geq 2$, and that as a consequence the level spacings of $(\{\alpha^n\})_{n\geq 1}$ are Poissonian as well. The main purpose of the present paper is to prove this conjecture.

Theorem 1.1. *For almost every $\alpha > 1$, the k-point correlation of $(\{\alpha^n\})_{n\geq 1}$ is Poissonian for all $k \geq 2$.*

It is known that if the k-point correlation is Poissonian for all $k \geq 2$, then the level spacings are also Poissonian (see Appendix A of [21]). Thus as a direct consequence of Theorem 1.1 we obtain that for almost all $\alpha > 1$, the level spacings of $(\{\alpha^n\})_{n\geq 1}$ are Poissonian. The same principle applies to other ordered statistics, such as the second-to-nearest neighbour gaps etc.; whenever all k-point correlations are Poissonian, then these ordered statistics also behave in accordance with the Poissonian model.

We deduce Theorem 1.1 from a more general result which, due to the robustness of our method, comes at essentially no extra cost. This more general result is the following.

Theorem 1.2. *Let $(a_n)_{n\geq 1}$ be an increasing sequence of positive real numbers such that*

$$\lim_{n\to\infty} \frac{a_n}{\log n} = \infty, \tag{1.3}$$

and such that

$$a_{n+1} - a_n \geq n^{-C} \tag{1.4}$$

for some $C > 0$ for all sufficiently large n. Then for almost every $\alpha > 0$, the k-point correlation of the sequence $(\{e^{\alpha a_n}\})_{n\geq 1}$ is Poissonian for all $k \geq 2$.

Theorem 1.1 follows upon letting $a_n = n$ in Theorem 1.2, and observing that the map $\alpha \mapsto e^\alpha$ from $(0, \infty)$ to $(1, \infty)$ preserves measure zero sets. We remark that in Theorem 1.2, the sequence $(e^{\alpha a_n})_{n\geq 1}$ can very well be a sequence of sub-exponential growth, that is

$$\lim_{n\to\infty} \frac{e^{\alpha a_{n+1}}}{e^{\alpha a_n}} = 1,$$

and still have a gap statistic following the Poissonian model for almost all $\alpha > 0$, e.g. by taking $a_n = \sqrt{n}$, $a_n = (\log(n + 1))^2$, $a_n = (\log\log(n + 2)) \log(n + 1)$ etc.

To put our results into perspective, we mention some earlier related results. Fundamental work on the correlations of sequences in the unit interval was carried out by Rudnick, Sarnak and Zaharescu; see for example [25, 26, 27]. Recently, the last two authors of the present paper proved [31] that for every $k \geq 2$, the k-point correlation of $(\{n^\alpha\})_{n\geq 1}$ is Poissonian for almost all $\alpha > 4k^2 - 4k - 1$; in the notation of Theorem 1.2 this corresponds to $a_n = \log n$, so that assumption (1.3) fails to hold. All these metric results rely, broadly speaking, on sufficiently fast concentration of the k-point correlation around its mean; this approach was already used by Sarnak [29] in 1997. As a general principle, proving Poissonian behaviour of the k-point correlation of a sequence becomes increasingly difficult when k becomes large. For example, it is known [25] that $(\{n^2\alpha\})_{n\geq 1}$ has Poissonian pair correlation for almost all α; the same is conjectured to be true for the triple correlation (and probably all higher correlations), but only partial results exist in this direction [30]. This is due to a blow-up of the variance of the triple correlation of $(\{n^2\alpha\})_{n\geq 1}$, which turns out to be large due to massive contributions from small intervals centred at rational points with small denominator — "major arcs" in the language of the Hardy–Littlewood circle method.

On the other hand, for rapidly growing sequences (e.g. lacunary sequences) one expects the contribution of the major arcs to the variance of the k-point correlation to be benign. In fact, Rudnick and Zaharescu [28] proved that for a lacunary sequence of integers $(a_n)_{n\geq 1}$, i.e. a sequence satisfying $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, for almost all α, the k-point correlation of $(\{a_n\alpha\})_{n\geq 1}$ is Poissonian for all $k \geq 2$. The variance of the k-point correlations of $(\{a_n\alpha\})_{n\geq 1}$, which plays a key role in [28], can be expressed as a sum of oscillatory integrals. Due to the 1-periodicity of

$z \mapsto \exp(2\pi i z)$ the oscillatory integrals contribute to this sum only when certain Diophantine equations are satisfied. Similarly, in this manuscript we also seek to bound the variance of the k-point correlation via a sum of oscillatory integrals (that arise naturally on the frequency side through Poisson summation). However, in marked difference to the work of Rudnick and Zaharescu [28], our oscillatory integrals do not simplify to Diophantine equations. Thus we need to develop methods to estimate the oscillatory integrals, individually, to ultimately bound the sum in question.

Results of a non-metric nature are particularly sparse. A marked exception is the sequence $(\{\sqrt{n}\})_{n\geq 1}$, for which the level spacings distribution is *not* Poissonian, as was shown by Elkies and McMullen using methods from ergodic theory [10]; somewhat surprisingly, the pair correlation is, in fact, Poissonian [9]. Furthermore Eskin, Margulis and Mozes [11] proved (via a quantitative solution to the Oppenheim conjecture in a particularly difficult case) that the values of certain binary quadratic forms give rise to a sequence with Poissonian pair correlation. Recently Lutsko, Sourmelidis, and the third author [22] established (via Fourier analysis) that $(\{\alpha n^{\theta}\})_{n\geq 1}$ has Poissonian pair correlation if $\alpha > 0$ and $\theta \in (0, 14/41)$. It is conjectured that the sequence $(\{n^2\alpha\})_{n\geq 1}$ has Poissonian pair correlation for every α which cannot be too well approximated by rational numbers, but for this problem only partial results exist [14, 32].

Additionally, there has been considerable interest in the level spacing of the residues modulo q, as $q \to \infty$ along certain naturally arising sequences. Hooley [15, 16, 17, 18] established that the distribution of the spacings between elements co-prime to q converges to the Poissonian model if the mean spacing $q/\phi(q) \to \infty$. Furthermore, Cobeli and Zaharescu [6] showed that the spacings between primitive roots modulo a prime p are Poissonian, provided that the mean spacing $p/\phi(p-1)$ tends to infinity. Kurlberg and Rudnick [21] proved that the level spacing of quadratic residues converges to the Poissonian model as long as q is taken from an increasing sequence of square-free numbers so that the number of prime factors $\omega(q) \to \infty$.

2. Outline of the argument

For the remaining part of this manuscript, we will only be dealing with the sequence $\vartheta(\alpha) = (e^{\alpha a_n})_{n\geq 1}$; we shall simply write $\Delta(\mathbf{x}, \vartheta)$ instead of $\Delta(\mathbf{x}, \vartheta(\alpha))$, and $R_k(f, \alpha, N)$ instead of $R_k(f, \vartheta(\alpha), N)$.

The strategy to prove Theorem 1.2 is in the spirit of [31] and will now be detailed. To begin with, we restrict our attention to intervals of the special form

$$\mathcal{J} = \mathcal{J}(A) \stackrel{\text{def}}{=} [A, A+1], \qquad (A > 0) \tag{2.1}$$

which will remain fixed throughout the proof. It is certainly enough to demonstrate that for each $k \geq 2$ and $A > 0$ the assertion of Theorem 1.2 holds, for almost every $\alpha \in \mathcal{J}$. Note that $\mathcal{J}$, equipped with Borel sets and Lebesgue measure, forms a probability space, so it is natural to speak about expectations, variances, etc., of real-valued functions defined on $\mathcal{J}$.

To prove Theorem 1.2, we show via a variance estimate that R_k concentrates around its mean value $\int_{-\infty}^{\infty} f(\mathbf{x})\, \mathrm{d}\mathbf{x}$ with a reasonable error term; Theorem 1.2 will then follow from a routine argument (see [31, Proposition 7.1]). Using Poisson summation, we can phrase the variance estimate in terms of oscillatory integrals of the form:

$$I(\mathbf{u}, \mathbf{t}) \stackrel{\text{def}}{=} \int_{\mathcal{J}} e(\phi(\mathbf{u}, \mathbf{t}, \alpha))\, \mathrm{d}\alpha \tag{2.2}$$

where $e(z) \stackrel{\text{def}}{=} e^{2\pi i z}$, and the *phase function* ϕ is given by

$$\phi(\mathbf{u},\mathbf{t},\alpha) \stackrel{\text{def}}{=} \sum_{i\le 2k} u_i e^{\alpha a_{t_i}} \qquad \mathbf{u}=(u_1,\dots,u_{2k}), \quad \mathbf{t}=(t_1,\dots,t_{2k}). \tag{2.3}$$

Recall the notation $\mathcal{B}_k = \{(x_1,\dots,x_k)\in\{1,\dots,N\}^k : x_i \neq x_j \text{ for all } i\neq j\le k\}$. There are additional constraints on the integer vectors $\mathbf{u},\mathbf{t}$ which naturally arise from the analysis. More precisely, fixing $\varepsilon>0$, we will have $\mathbf{u}=(\mathbf{v},\mathbf{w})$, $\mathbf{t}=(\mathbf{x},\mathbf{y})$ where $\mathbf{x},\mathbf{y}\in\mathcal{B}_k$, and $\mathbf{v},\mathbf{w}\in\mathcal{U}_k^{\varepsilon}$, where

$$\mathcal{U}_k^{\varepsilon} = \mathcal{U}_k^{\varepsilon}(N) \stackrel{\text{def}}{=} \{\mathbf{u}=(u_1,\dots,u_k)\in\mathbb{Z}^k : 1\le \|\mathbf{u}\|_\infty \le 2N^{1+\varepsilon},\ u_1+\cdots+u_k=0\}. \tag{2.4}$$

Our desired variance estimate can, after a simple computation, be phrased as a bound for an average of these integrals. It will then be shown that

$$V_k(N,\mathcal{J},\varepsilon) \stackrel{\text{def}}{=} \frac{1}{N^{2k}} \sum_{\mathbf{z}=(\mathbf{u},\mathbf{t})\in\left(\mathcal{U}_k^{\varepsilon}\right)^2\times\mathcal{B}_k^2} |I(\mathbf{u},\mathbf{t})| = O(N^{-1+\varepsilon}). \tag{2.5}$$

To prove such a bound, we use a variant of Van der Corput's lemma which requires us to guarantee that at each point $\alpha\in\mathcal{J}$ at least some derivative of ϕ with respect to α is large. Such a "repulsion" property is captured by the function

$$\mathrm{Van}_\ell\phi(\alpha) \stackrel{\text{def}}{=} \max_{i\le\ell} |\phi^{(i)}(\alpha)|, \tag{2.6}$$

and we shall derive an acceptable lower bound on $\mathrm{Van}_\ell\phi$, uniformly throughout $\mathcal{J}$.

This repulsion principle (see Lemma 4.2 below) is the driving force behind the argument, and the only part of the proof where assumptions (1.3) and (1.4) are used. Moreover, this way of reasoning is robust, and the arithmetic that we require is quite simple and essentially just the structure of the real numbers plus quantitative growth and spacing conditions. A technical complication which often arises in the study of k-point correlation sums (see e.g. [31, 28]) is that we have to deal with "degenerate" configurations where $\mathbf{u}$ and $\mathbf{t}$ are such that some of the terms in the function $\phi(\mathbf{u},\mathbf{t},\alpha)$ vanish; this will be handled by a combinatorial argument (see Proposition 5.5 below).

3. Preliminaries

In this section we collect the tools that we will use later and introduce further notation.

3.1. **Notation.** Throughout the rest of this manuscript the implied constants may depend on the sequence $(a_n)_{n\ge1}$ from the statement of Theorem 1.2, as well as on $k, f, \mathcal{J}, \varepsilon, \eta$, and we shall not indicate this dependence explicitly. The dependence on any other parameter will be indicated. The Bachmann–Landau O symbol, or interchangeably the Vinogradov symbols $\ll$ and $\gg$, have their usual meaning. Throughout the manuscript, k is a fixed integer satisfying $k\ge2$.

3.2. **Oscillatory integrals.** The bulk of our work is concerned with understanding the magnitude of the one-dimensional oscillatory integrals

$$I(\phi,\mathcal{J}) \stackrel{\text{def}}{=} \int_{\mathcal{J}} e(\phi(\alpha))\,\mathrm{d}\alpha$$

where $\phi:\mathcal{J}\to\mathbb{R}$ is a C^∞-function, the so called *phase function*. The phase functions we are required to understand are of the shape $\phi(\alpha)=\phi(\mathbf{u},\mathbf{t},\alpha)$ as in (2.3). We need the following variant of Van der Corput's lemma:

Lemma 3.1. *Let $\phi : \mathcal{J} \to \mathbb{R}$ be a C^∞-function. Fix $\ell \geq 1$, and suppose that $\phi^{(\ell)}(\alpha)$ has at most C zeros, and that the inequality $\mathrm{Van}_\ell \phi(\alpha) \geq \lambda > 0$ holds throughout the interval $\mathcal{J}$. Then the bound*

$$I(\phi, \mathcal{J}) \ll_{\ell,C} \lambda^{-1/\ell}$$

holds when $\ell \geq 2$, or when $\ell = 1$ and ϕ' is monotone on $\mathcal{J}$.

Proof. This can be found in [31, Lemma 3.3]. □

Lemma 3.1 requires a bound on the number of zeros for the derivatives of ϕ. For this we prove the following which is a very minor modification of [31, Lemma 4.3].

Lemma 3.2. *Let $\psi(\alpha) = \sum_{i \leq \ell} u_i e^{\alpha x_i}$ for $\mathbf{u} = (u_1, \dots, u_\ell) \in \mathbb{R}^\ell_{\neq 0}$ and $\mathbf{x} = (x_1, \dots, x_\ell) \in \mathbb{R}^\ell$ such that $x_1 < \cdots < x_\ell$. Then ψ has at most $\ell - 1$ zeros in $\mathbb{R}$.*

Proof. We argue by induction on ℓ. For $\ell = 1$ the correctness of the statement is clear. Assume that the lemma is true for $\ell - 1$ ($\ell \geq 2$), and let

$$\psi(\alpha) = \sum_{i \leq \ell} u_i e^{\alpha x_i}.$$

The zeros of ψ are exactly the zeros of the function

$$\tilde{\psi}(\alpha) = \sum_{i \leq \ell - 1} \tilde{u}_i e^{\alpha \tilde{x}_i} + 1,$$

where $\tilde{u}_i = \frac{u_i}{u_\ell}$, and $\tilde{x}_i = x_i - x_\ell$ ($1 \leq i \leq \ell - 1$), since $\psi(\alpha) = u_\ell e^{\alpha x_\ell} \tilde{\psi}(\alpha)$. Moreover,

$$\tilde{\psi}'(\alpha) = \sum_{i \leq \ell - 1} v_i e^{\alpha \tilde{x}_i},$$

where $v_i = \tilde{u}_i \tilde{x}_i$ ($1 \leq i \leq \ell - 1$).

Clearly, the numbers $v_1, \dots, v_{\ell-1}$ are nonzero, and the $\tilde{x}_1, \dots, \tilde{x}_{\ell-1}$ are distinct. Therefore, by the induction hypothesis, $\tilde{\psi}'$ has at most $\ell - 2$ zeros. Hence, by Rolle's theorem, $\tilde{\psi}$ has at most $\ell - 1$ zeros, completing the proof. □

4. The repulsion principle

Lemma 4.1. *Let ℓ be a positive integer. Let $\gamma > 0$. Let $0 < x_1 < x_2 < \ldots < x_\ell$ be real numbers such that $x_{i+1} - x_i \geq \gamma$ for $1 \leq i \leq \ell - 1$. Then the matrix*

$$M = M(x_1, \dots, x_\ell) = \begin{pmatrix} x_1 & \dots & x_\ell \\ \vdots & \ddots & \vdots \\ x_1^\ell & \dots & x_\ell^\ell \end{pmatrix} \tag{4.1}$$

is invertible and the operator norm $\|\cdot\|_\infty$ of its inverse satisfies

$$\left\| M^{-1} \right\|_\infty \ll_\ell x_\ell^{\ell-1} x_1^{-1} \left(\frac{1}{\gamma} \right)^{\ell-1}.$$

Proof. The conclusion is trivial when $\ell = 1$, so we will assume that $\ell \geq 2$. The matrix M is the transpose of a scaled Vandermonde matrix; the entry m_{ij} of its inverse M^{-1} is given by (see, e.g. [19, Ex. 40])

$$m_{ij} = (-1)^{j-1} \frac{\sum\limits_{\substack{1 \leq m_1 < \cdots < m_{\ell-j} \leq \ell, \\ m_1, \dots, m_{\ell-j} \neq i}} x_{m_1} \cdots x_{m_{\ell-j}}}{x_i \prod\limits_{\substack{1 \leq m \leq \ell, \\ m \neq i}} (x_m - x_i)}.$$

Hence

$$|m_{ij}| \ll_\ell x_\ell^{\ell-1} x_1^{-1} \left(\frac{1}{\gamma}\right)^{\ell-1} \tag{4.2}$$

for all $1 \le i, j \le \ell$. It is well-known that the maximum norm $\|\cdot\|_M$ (maximal absolute value of a matrix entry) dominates the operator norm $\|\cdot\|_\infty$, i.e. we have $\|\cdot\|_\infty \ll_\ell \|\cdot\|_M$. Thus, (4.2) gives

$$\left\|M^{-1}\right\|_\infty \ll_\ell x_\ell^{\ell-1} x_1^{-1} \left(\frac{1}{\gamma}\right)^{\ell-1},$$

as desired. □

As a consequence, we are now able to prove the enunciated repulsion principle. In the statement of the following lemma, as throughout the proof, $(a_n)_{n\ge1}$ is the sequence from the statement of Theorem 1.2. Recall that by assumption $(a_n)_{n\ge1}$ satisfies (1.3) and (1.4), which will be used in the proof of the lemma. Recall also the definition of $\phi(\mathbf{u}, \mathbf{t}, \alpha)$ in (2.3) and the definition of Van_ℓ in (2.6).

Lemma 4.2 (Repulsion principle)**.** *Let ℓ be a positive integer such that $\ell \le 2k$. Let $\mathbf{u} \in \mathbb{Z}^\ell_{\neq 0}$, and let $\mathbf{t} = (t_1, \dots, t_\ell) \in \mathbb{N}^\ell$ be such that $t_1 < \cdots < t_\ell$. Then for any (arbitrarily large) $\eta > 0$,*

$$\min_{\alpha\in\mathcal{J}} \mathrm{Van}_\ell\left(\phi(\mathbf{u}, \mathbf{t}, \alpha)\right) \gg t_\ell^\eta. \tag{4.3}$$

The implied constant in (4.3) depends on η, the sequence (a_n), the interval $\mathcal{J}$ and the parameter k, which throughout the proof are assumed to be fixed.

Proof. Let $\alpha \in \mathcal{J}$. To make the underlying structure more transparent, we denote $\boldsymbol{\tau} = (\partial_\alpha^j \phi(\mathbf{u}, \mathbf{t}, \alpha))_{j=1,\dots,\ell}$, $\mathbf{w} = (u_i e^{\alpha a_{t_i}})_{i=1,\dots,\ell}$, and $M = M(a_{t_1}, \dots, a_{t_\ell})$ as in (4.1). Then

$$\mathrm{Van}_\ell\left(\phi(\mathbf{u}, \mathbf{t}, \alpha)\right) = \|\boldsymbol{\tau}\|_\infty,$$

and

$$\boldsymbol{\tau} = M\mathbf{w}. \tag{4.4}$$

Note that we have $a_{t_{i+1}} - a_{t_i} \gg t_\ell^{-C}$ for some fixed positive constant C by assumption (1.4). Thus by Lemma 4.1 and (4.4) we have

$$\|\mathbf{w}\|_\infty \le \left\|M^{-1}\right\|_\infty \|\boldsymbol{\tau}\|_\infty \ll a_{t_\ell}^{\ell-1} t_\ell^{C(\ell-1)} \|\boldsymbol{\tau}\|_\infty$$

(we have used the bound $a_{t_1}^{-1} \ll a_1^{-1} \ll 1$). Now note that

$$\|\mathbf{w}\|_\infty \ge |u_\ell| e^{\alpha a_{t_\ell}} \ge e^{\alpha a_{t_\ell}}.$$

Combining the two equations above we obtain

$$e^{\alpha a_{t_\ell}} \ll a_{t_\ell}^{\ell-1} t_\ell^{C(\ell-1)} \|\boldsymbol{\tau}\|_\infty. \tag{4.5}$$

Recall that the interval $\mathcal{J}$ and therefore α are bounded away from 0 by assumption, so that $a_{t_\ell}^{\ell-1} \ll e^{\frac{\alpha a_{t_\ell}}{2}}$, and hence (4.5) gives

$$\|\boldsymbol{\tau}\|_\infty \gg e^{\frac{\alpha a_{t_\ell}}{2}} t_\ell^{-C(\ell-1)}. \tag{4.6}$$

By assumption (1.3) it follows that $e^{\frac{\alpha a_{t_\ell}}{2}} \gg t_\ell^{\eta+C(\ell-1)}$. Inserting this into (4.6) gives the required bound (4.3). □

As a corollary, we get the required bound for the integral $I(\mathbf{u}, \mathbf{t})$ (recall the notation (2.2)).

Corollary 4.3. *Let ℓ be a positive integer such that $\ell \leq 2k$. Let $\mathbf{u} \in \mathbb{Z}^{\ell}_{\neq 0}$, and let $\mathbf{t} = (t_1, \dots, t_\ell) \in \mathbb{N}^{\ell}$ be such that $t_1 < \cdots < t_\ell$. Then for any (arbitrarily large) $\eta > 0$,*

$$I(\mathbf{u}, \mathbf{t}) \ll t_\ell^{-\eta}. \tag{4.7}$$

Proof. This is an immediate consequence of Lemma 3.1, Lemma 3.2 and Lemma 4.2. □

5. Variance estimates

We begin this section with our definition of the variance of the k-point correlation sum $R_k(f, \alpha, N)$ with respect to $\alpha \in \mathcal{J}$. Recall that $\mathcal{B}_k = \mathcal{B}_k(N)$ is the set of integer k-tuples $(x_1, \dots, x_k)$ such that $1 \leq x_i \leq N$ for all $i = 1, \dots, k$ and such that no two components x_i are equal.

Definition 5.1. The variance of the k-point correlation sum $R_k(f, \alpha, N)$ with respect to the interval $\mathcal{J}$ is defined as

$$\mathrm{Var}\left(R_k(f, \cdot, N), \mathcal{J}\right) \stackrel{\mathrm{def}}{=} \int_{\mathcal{J}} \left(R_k(f, \alpha, N) - C_k(N) \int_{\mathbb{R}^{k-1}} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \right)^2 \mathrm{d}\alpha,$$

where

$$C_k(N) \stackrel{\mathrm{def}}{=} \frac{\#\mathcal{B}_k}{N^k} = \left(1 - \frac{1}{N}\right) \cdots \left(1 - \frac{k-1}{N}\right). \tag{5.1}$$

The reason for the combinatorial factor (5.1) will be apparent in the proof below.

The goal of this section is to show that the variance $\mathrm{Var}\left(R_k(f, \cdot, N), \mathcal{J}\right)$ decays polynomially in N:

Proposition 5.2. *For all $\varepsilon > 0$, we have*

$$\mathrm{Var}\left(R_k(f, \cdot, N), \mathcal{J}\right) = O(N^{-1+\varepsilon}).$$

The first routine step will be to express R_k in terms of an exponential sum. Fix $\varepsilon > 0$, and set

$$\mathcal{N}_{k-1}^{\varepsilon} = \mathcal{N}_{k-1}^{\varepsilon}(N) = \left\{ \mathbf{n} \in \mathbb{Z}^{k-1} : \ 1 \leq \|\mathbf{n}\|_\infty \leq N^{1+\varepsilon} \right\}.$$

For the statement of the following lemma, recall the definition of the difference vector $\boldsymbol{\Delta}(\mathbf{x}, \alpha) = \boldsymbol{\Delta}(\mathbf{x}, \vartheta(\alpha))$ in (1.2).

Lemma 5.3. *Let $\varepsilon > 0$. For all $\eta > 0$, we have*

$$\begin{aligned} R_k(f, \alpha, N) = C_k(N) \int_{\mathbb{R}^{k-1}} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ + \frac{1}{N^k} \sum_{\mathbf{x} \in \mathcal{B}_k} \sum_{\mathbf{n} \in \mathcal{N}_{k-1}^{\varepsilon}} \widehat{f}\left(\frac{\mathbf{n}}{N}\right) e\left(\langle \boldsymbol{\Delta}(\mathbf{x}, \alpha), \mathbf{n} \rangle\right) + O(N^{-\eta}) \end{aligned} \tag{5.2}$$

as $N \to \infty$.

Proof. By the Poisson summation formula,

$$R_k(f, \alpha, N) = \frac{1}{N^k} \sum_{\mathbf{x} \in \mathcal{B}_k} \sum_{\mathbf{n} \in \mathbb{Z}^{k-1}} \widehat{f}\left(\frac{\mathbf{n}}{N}\right) e\left(\langle \boldsymbol{\Delta}(\mathbf{x}, \alpha), \mathbf{n} \rangle\right).$$

Formula (5.2) now easily follows by separating the zero-th term and using the fact that the Fourier coefficients of any $f \in C_c^\infty(\mathbb{R}^{k-1})$ decay to zero faster than the reciprocal of any polynomial, see the proof of [31, Lemma 3.4]. □

Given $\mathbf{n} \in \mathbb{Z}^{k-1}$, we define the vector $\mathbf{h}(\mathbf{n}) = (h_1(\mathbf{n}), \dots, h_k(\mathbf{n})) \in \mathbb{Z}^k$ by the rule

$$h_i(\mathbf{n}) \stackrel{\text{def}}{=} \begin{cases} n_1, & \text{if } i = 1, \\ n_i - n_{i-1}, & \text{if } 2 \le i \le k-1, \\ -n_{k-1}, & \text{if } i = k. \end{cases}$$

This definition is motivated by the identity

$$\langle \Delta(\mathbf{x}, \alpha), \mathbf{n} \rangle = \phi(\mathbf{h}(\mathbf{n}), \mathbf{x}, \alpha). \tag{5.3}$$

Note that the linear map $\mathbf{n} \mapsto \mathbf{h}(\mathbf{n})$ is injective. Moreover, it satisfies the bound

$$\|\mathbf{h}(\mathbf{n})\|_\infty \le 2\|\mathbf{n}\|_\infty \tag{5.4}$$

and the relation

$$\sum_{i=1}^{k} h_i(\mathbf{n}) = 0. \tag{5.5}$$

Let

$$\mathcal{U}_k^\varepsilon = \mathcal{U}_k^\varepsilon(N) = \left\{ \mathbf{u} = (u_1, \dots, u_k) \in \mathbb{Z}^k : 1 \le \|\mathbf{u}\|_\infty \le 2N^{1+\varepsilon},\ u_1 + \dots + u_k = 0 \right\},$$

and note that the relations (5.4), (5.5) imply that $\mathbf{h}(\mathbf{n}) \in \mathcal{U}_k^\varepsilon$ whenever $\mathbf{n} \in \mathcal{N}_{k-1}^\varepsilon$.

Lemma 5.4. *Let $\varepsilon > 0$. For all $\eta > 0$, we have*

$$\operatorname{Var}(R_k(f, \cdot, N), \mathcal{J}) \ll V_k(N, \mathcal{J}, \varepsilon) + N^{-\eta} \tag{5.6}$$

as $N \to \infty$, where $V_k(N, \mathcal{J}, \varepsilon)$ is given in (2.5).

Proof. By Lemma 5.3, for all $\tilde{\eta} > 0$ we have

$$\operatorname{Var}(R_k(f, \cdot, N), \mathcal{J}) = \int_{\mathcal{J}} \Big(N^{-k} \sum_{\mathbf{x} \in \mathcal{B}_k} \sum_{\mathbf{n} \in \mathcal{N}_{k-1}^\varepsilon} \widehat{f}\left(\frac{\mathbf{n}}{N}\right) e\left(\langle \Delta(\mathbf{x}, \alpha), \mathbf{n} \rangle\right) + O(N^{-\tilde{\eta}}) \Big)^2 \, \mathrm{d}\alpha.$$

Expanding the square and taking $\tilde{\eta}$ sufficiently large, the bound $\widehat{f} \ll 1$ readily implies that for all $\eta > 0$,

$$\operatorname{Var}(R_k(f, \cdot, N), \mathcal{J}) = I + O\left(N^{-\eta}\right)$$

where

$$I = N^{-2k} \sum_{\substack{\mathbf{x}, \mathbf{y} \in \mathcal{B}_k, \\ \mathbf{n}, \mathbf{m} \in \mathcal{N}_{k-1}^\varepsilon}} \widehat{f}\left(\frac{\mathbf{n}}{N}\right) \widehat{f}\left(\frac{\mathbf{m}}{N}\right) \int_{\mathcal{J}} e\left(\langle \Delta(\mathbf{x}, \alpha), \mathbf{n} \rangle + \langle \Delta(\mathbf{y}, \alpha), \mathbf{m} \rangle \right) \mathrm{d}\alpha.$$

By the identity (5.3) and by the injectivity of the map $\mathbf{n} \mapsto \mathbf{h}(\mathbf{n})$ we conclude that

$$I \ll N^{-2k} \sum_{\substack{\mathbf{x}, \mathbf{y} \in \mathcal{B}_k, \\ \mathbf{n}, \mathbf{m} \in \mathcal{N}_{k-1}^\varepsilon}} \left| \int_{\mathcal{J}} e\left(\phi(\mathbf{h}(\mathbf{n}), \mathbf{x}, \alpha) + \phi(\mathbf{h}(\mathbf{m}), \mathbf{y}, \alpha) \right) \mathrm{d}\alpha \right| \ll V_k(N, \mathcal{J}, \varepsilon)$$

which gives the claimed bound. □

We will now bound $V_k(N, \mathcal{J}, \varepsilon)$ using a combinatorial argument. Combined with Lemma 5.4, this will give Proposition 5.2.

Proposition 5.5. *For all $\varepsilon > 0$, we have*

$$V_k(N, \mathcal{J}, \varepsilon) = O(N^{-1+(2k-1)\varepsilon}).$$

Proof. Let

$$[k] \stackrel{\text{def}}{=} \{1, \dots, k\}.$$

Let $\mathcal{I}_1, \mathcal{I}_1', \mathcal{I}_2, \mathcal{I}_2', \mathcal{I}_3, \mathcal{I}_3' \subseteq [k]$ be (possibly empty) sets of indices. Fixing

$$\boldsymbol{\tau} \stackrel{\text{def}}{=} (\mathcal{I}_1, \mathcal{I}_1', \mathcal{I}_2, \mathcal{I}_2', \mathcal{I}_3, \mathcal{I}_3'),$$

we denote by $\mathcal{V}_k^{\varepsilon}(\boldsymbol{\tau})$ the set of vectors $(\mathbf{u}, \mathbf{t}) = ((\mathbf{v}, \mathbf{w}), (\mathbf{x}, \mathbf{y})) \in (\mathcal{U}_k^{\varepsilon})^2 \times \mathcal{B}_k^2$ for which

$$\begin{aligned}
\{i \in [k] : \exists_{j(i) \in [k]} : x_i = y_{j(i)}\} &= \mathcal{I}_1, \\
\{j \in [k] : \exists_{i(j) \in [k]} : x_{i(j)} = y_j\} &= \mathcal{I}_1', \\
\{i \in \mathcal{I}_1 : v_i + w_{j(i)} = 0, \text{where } j(i) \text{ is s.t. } x_i = y_{j(i)}\} &= \mathcal{I}_2, \\
\{j \in \mathcal{I}_1' : v_{i(j)} + w_j = 0, \text{where } i(j) \text{ is s.t. } x_{i(j)} = y_j\} &= \mathcal{I}_2', \\
\{i \in [k] \setminus \mathcal{I}_1 : v_i = 0\} &= \mathcal{I}_3, \\
\{j \in [k] \setminus \mathcal{I}_1' : w_j = 0\} &= \mathcal{I}_3'.
\end{aligned}$$

If $\mathcal{V}_k^{\varepsilon}(\boldsymbol{\tau})$ is non-empty, then clearly $\#\mathcal{I}_1 = \#\mathcal{I}_1'$ and $\#\mathcal{I}_2 = \#\mathcal{I}_2'$. Amongst the list of $2k$ variables $x_1, \dots, x_k, y_1, \dots, y_k$, exactly $2k - \#\mathcal{I}_1$ distinct variables appear (to see this, recall that by the definition of $\mathcal{B}_k$ all numbers $x_1, \dots, x_k$ are distinct, and similarly all numbers $y_1, \dots, y_k$ are distinct). Consequently, if we group similar terms in the corresponding phase function we have

$$\begin{aligned}
\phi(\mathbf{u}, \mathbf{t}, \alpha) &= v_1 e^{\alpha x_1} + \dots + v_k e^{\alpha x_k} + w_1 e^{\alpha y_1} + \dots + w_k e^{\alpha y_k} \\
&= \sum_{i \in [k] \setminus (\mathcal{I}_1 \cup \mathcal{I}_3)} v_i e^{\alpha x_i} + \sum_{j \in [k] \setminus \left(\mathcal{I}_1' \cup \mathcal{I}_3'\right)} w_j e^{\alpha y_j} + \sum_{i \in \mathcal{I}_1 \setminus \mathcal{I}_2} \left(v_i + w_{j(i)}\right) e^{\alpha x_i},
\end{aligned} \tag{5.7}$$

and the number of non-vanishing terms is

$$l \stackrel{\text{def}}{=} 2k - \#\mathcal{I}_1 - \#\mathcal{I}_2 - \#\mathcal{I}_3 - \#\mathcal{I}_3'.$$

Now let us consider the constraints on the variables $v_1 \dots, v_k, w_1, \dots, w_k$:

- The constraints $v_i = 0$ $(i \in \mathcal{I}_3)$ and $v_1 + \dots + v_k = 0$ (recall that $\mathbf{v} \in \mathcal{U}_k^{\varepsilon}$) determine $\#\mathcal{I}_3 + 1$ of the variables $v_1, \dots, v_k$ in terms of the other variables; note that $\mathbf{v} \neq (0, \dots, 0)$, so that $\#\mathcal{I}_3 < k - 1$.
- The constraints $w_j = 0$ $(j \in \mathcal{I}_3')$ and $w_j = -v_{i(j)}$ $(j \in \mathcal{I}_2')$ determine $\#\mathcal{I}_2' + \#\mathcal{I}_3'$ of the variables $w_1, \dots, w_k$ in terms of the variables v_i.
- Since $w \in \mathcal{U}_k^{\varepsilon}$, we also have the constraint $w_1 + \dots + w_k = 0$ which is either contained in the previous constraints (this happens if and only if $l = 0$), or determines one more variable.

To conclude, the constraints on the variables $v_1, \dots, v_k, w_1, \dots, w_k$ determine at least

$$(\#\mathcal{I}_3 + 1) + (\#\mathcal{I}_2' + \#\mathcal{I}_3')$$

many of these variables. Thus there is a number m with

$$m \leq 2k - \#\mathcal{I}_3 - \#\mathcal{I}_2' - \#\mathcal{I}_3' - 1 \tag{5.8}$$

such that we can assign a total number of m variables out of $v_1, ..., v_k, w_1, ..., w_k$ to be *independent*, in the sense that the values of all these independent variables together uniquely determine the values of all the other (dependent) variables. The choice of which variables are to be considered independent is not unique (there are $O(1)$ many possibilities), but once such a choice is made, the independent variables together uniquely determine the dependent ones. In other words, if we relabel the independent variables as $u_1, ..., u_m$, then given the values

of $u_1, ..., u_m$ there exists a unique $\mathbf{u}^*(u_1, \dots, u_m) \in (\mathcal{U}_k^\epsilon)^2$ so that all constraints captured by $\mathcal{I}_3, \mathcal{I}_2'$ and $\mathcal{I}_3'$ are satisfied. We now proceed via a case analysis based upon the value of l to obtain a uniform upper bound for $\sum_{(\mathbf{u},\mathbf{t})\in\mathcal{V}_k^\varepsilon(\boldsymbol{\tau})} |I(\mathbf{u},\mathbf{t})|$.

Case 1. Assume that $l = 0$. Then $\phi(\mathbf{u},\mathbf{t},\alpha) = 0$ so that $I(\mathbf{u},\mathbf{t}) = 1$. By the above considerations we may conclude that

$$\begin{aligned}\sum_{(\mathbf{u},\mathbf{t})\in\mathcal{V}_k^\varepsilon(\boldsymbol{\tau})} |I(\mathbf{u},\mathbf{t})| &\ll \sum_{\substack{|u_1|,\dots,|u_m|\le 2N^{1+\varepsilon}\\ \mathbf{u}^*(u_1,\dots,u_m)\in(\mathcal{U}_k^\epsilon)^2}} \sum_{\substack{1\le t_1,\dots,t_{2k-\#\mathcal{I}_1}\le N\\ t_i\neq t_j}} 1\\ &\ll N^{m(1+\varepsilon)+2k-\#\mathcal{I}_1} \le N^{(2k-1)(1+\varepsilon)}.\end{aligned}$$

In the final line we used (5.8) and the fact that $l = 0$.

Case 2. Assume that $l \ge 1$. In this case we relabel the distinct variables x_i, y_j appearing on the r.h.s. of (5.7) by $t_1, t_2, \dots, t_l$. We also relabel by $s_1, \dots, s_r$ the

$$r \stackrel{\text{def}}{=} \#\mathcal{I}_2 + \#\mathcal{I}_3 + \#\mathcal{I}_3' \tag{5.9}$$

variables x_i, y_j from our list of distinct variables which do not appear on the r.h.s. of (5.7) because their corresponding exponentials $e^{\alpha s_i}$ have zero coefficients. Moreover we denote by $\mathbf{t}^*(t_1, \dots, t_l, s_1, \dots, s_r)$ the unique element of $\mathcal{B}_k^2$ determined by $t_1, \dots, t_l$, $s_1, \dots, s_r$, and the conditions imposed by $\boldsymbol{\tau}$. Note that by Corollary 4.3, we always have

$$I(\mathbf{u}^*(u_1, \dots, u_m), \mathbf{t}^*(t_1, \dots, t_l, s_1, \dots, s_r)) \ll |\max_i t_i|^{-\eta}$$

for any $\eta > 0$. By the above considerations we may conclude that

$$\begin{aligned}&\sum_{(\mathbf{u},\mathbf{t})\in\mathcal{V}_k^\varepsilon(\boldsymbol{\tau})} |I(\mathbf{u},\mathbf{t})|\\ &\ll \sum_{\substack{|u_1|,\dots,|u_m|\le 2N^{1+\varepsilon}\\ \mathbf{u}^*(u_1,\dots,u_m)\in(\mathcal{U}_k^\epsilon)^2}} \sum_{\substack{1\le s_1,\dots,s_r\le N\\ s_i\neq s_j}} \sum_{\substack{1\le t_1,\dots,t_l\le N\\ t_i\neq t_j, t_i\neq s_j}} |I(\mathbf{u}^*(u_1,\dots,u_m), \mathbf{t}^*(t_1,\dots,t_l,s_1,\dots,s_r))|\\ &\ll \sum_{\substack{|u_1|,\dots,|u_m|\le 2N^{1+\varepsilon}\\ \mathbf{u}^*(u_1,\dots,u_m)\in(\mathcal{U}_k^\epsilon)^2}} \sum_{1\le s_1,\dots,s_r\le N} \sum_{1\le t_1,\dots,t_l\le N} |\max_i t_i|^{-\eta}\\ &\ll \sum_{|u_1|,\dots,|u_m|\le 2N^{1+\varepsilon}} \sum_{1\le s_1,\dots,s_r\le N} 1 \ll N^{m(1+\varepsilon)+r} \le N^{(2k-1)(1+\varepsilon)}\end{aligned}$$

where in the last line we used (5.8), (5.9), and $\#\mathcal{I}_2 = \#\mathcal{I}_2'$.

Combining the above cases, we have shown that for any value of l we always have

$$\sum_{(\mathbf{u},\mathbf{t})\in\mathcal{V}_k^\varepsilon(\boldsymbol{\tau})} |I(\mathbf{u},\mathbf{t})| \ll N^{(2k-1)(1+\varepsilon)}.$$

Therefore summing over all $O(1)$ configurations $\boldsymbol{\tau}$, we conclude that

$$V_k(N, \mathcal{J}, \varepsilon) = \frac{1}{N^{2k}} \sum_\tau \sum_{(\mathbf{u},\mathbf{t})\in\mathcal{V}_k^\varepsilon(\boldsymbol{\tau})} |I(\mathbf{u},\mathbf{t})| \ll N^{-1+(2k-1)\varepsilon}.$$

This completes our proof. □

6. Proof of Theorem 1.2

Theorem 1.2 can be deduced from Proposition 5.2 following a standard argument whose proof in a fairly general setting was given in [31].

Proposition 6.1 ([31, Proposition 7.1])**.** *Fix $k \geq 2$, $J \subset \mathbb{R}$ a bounded interval, and a sequence $c_k(N)$ such that $c_k(N) \to 1$ as $N \to \infty$. Let $(\vartheta_n(\alpha))_{n\geq 1}$ $(\alpha \in J)$ be a parametrised family of sequences such that the map $\alpha \mapsto \vartheta_n(\alpha)$ is continuous for each fixed $n \geq 1$. Assume that there exists $\rho > 0$ such that for all $f \in C_c^\infty(\mathbb{R}^{k-1})$*

$$\int_J \left(R_k\left(f, \alpha, N\right) - c_k\left(N\right) \int_{\mathbb{R}^{k-1}} f\left(\mathbf{x}\right)\, d\mathbf{x} \right)^2 \mathrm{d}\alpha = O(N^{-\rho})$$

as $N \to \infty$. Then for almost all $\alpha \in J$, the sequence $(\{\vartheta_n(\alpha)\})_{n\geq 1}$ has Poissonian k-point correlation.

Proof. Poissonian k-point correlation is first established along a polynomially sparse subsequence N_m using the Borel-Cantelli lemma. This is extended to Poissonian k-point correlation along the full sequence by a simple sandwiching argument, using the fact that $\lim_{m\to\infty} N_{m+1}/N_m = 1$. For the full details see [31]. □

Proof of Theorem 1.2. Theorem 1.2 follows upon letting $\vartheta_n(\alpha) = e^{\alpha a_n}$, $J = \mathcal{J}$, $c_k(N) = C_k(N)$ (recall (5.1)) and $\rho = -1 + \varepsilon$ in Proposition 6.1. □

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