The modularity of random graphs on the hyperbolic plane

Chellig, Jordan; Fountoulakis, Nikolaos; Skerman, Fiona

[Link to publication on Research at Birmingham portal](#)

The modularity of random graphs on the hyperbolic plane

JORDAN CHELLIG AND NIKOLAOS FOUNTOULAKIS[†]

School of Mathematics, University of Birmingham, B15 2TT, UK
[†]Corresponding author. Email: n.fountoulakis@bham.ac.uk

AND

FIONA SKERMAN

Department of Mathematics, Uppsala University, Uppsala 75106, Sweden

Edited by: Piet Van Mieghem

Modularity is a quantity which has been introduced in the context of complex networks in order to quantify how close a network is to an ideal modular network in which the nodes form small interconnected communities that are joined together with relatively few edges. In this article, we consider this quantity on a probabilistic model of complex networks introduced by Krioukov *et al.* (2010, *Phys. Rev. E*, **82**, 036106). This model views a complex network as an expression of hidden popularity hierarchies (i.e. nodes higher up in the hierarchies have more global reach), encapsulated by an underlying hyperbolic space. For certain parameters, this model was proved to have typical features that are observed in complex networks such as power law degree distribution, bounded average degree, clustering coefficient that is asymptotically bounded away from zero and ultra-small typical distances. In the present work, we investigate its modularity and we show that, in this regime, it converges to one in probability.

Keywords: random geometric graphs, hyperbolic plane, complex networks, modularity.

1. Introduction

Granovetter, in his pioneering analysis of social networks [1], pointed out that a fundamental feature of social networks is the distinction between weak links and strong links. These strengths reflect the intensity of interaction between two individuals, which may be dependent on measures such as the frequency of interaction. As Granovetter remarked, an individual is more likely to interact with other individuals through the strong links. This is expressed in terms of structural features of the social network, whereby individuals belong to communities, tightly knit by strong links, and these communities are typically joined through weak links. These ideas postulate that a fundamental characteristic of social networks is the existence of communities, or modules, within such a network. These are mutually disjoint subsets of nodes/individuals which have many edges connecting them but are joined to other modules by few edges.

Identifying such a partition in a given social network or any other complex network is computationally challenging. But before we set out to find algorithms that give even an approximate solution to this problem, one needs to quantify what is a good partition of the node set of a given network. Such a quantification was given by Newman and Girvan [2] and is called the *modularity score* of a given partition. The highest modularity score among all partitions is called the *modularity* of a network (cf. Section 2.2 for the precise definition). The most popular algorithms used to cluster large network data use the modularity score as a quality function for partitions (see e.g. [3]). In this article, we investigate

the modularity of a recent model of complex networks in which a network is sampled as a geometric random graph on the hyperbolic plane.

1.1 *The KPKBV model: a geometric framework for complex networks*

Krioukov *et al.* [4] introduced a model of random geometric graphs on the hyperbolic plane as a model of complex networks, which we abbreviate as the *KPKBV model* after its inventors. This is based on the assumption that the geometry of the hyperbolic plane can accommodate the hidden hierarchy of a complex network and its intrinsic inhomogeneity. Their basic assumption is that the hierarchies that are present in a complex network induce a tree-like structure, and this suggests that there is an underlying geometry of a complex network which is the hyperbolic.

There are several representations of the standard hyperbolic plane H^2_{-1} of curvature -1. In this article, we shall use the Poincaré unit disc representation, which is simply the open disc of radius one, that is, $\{(u, v) \in \mathbb{R}^2 \; : \; u^2 + v^2 < 1\}$, which is equipped with the hyperbolic metric: $4 \frac{du^2 + dv^2}{(1 - u^2 - v^2)^2}$. This is a standard formulation of the hyperbolic plane. In particular, a suitable integration of the metric shows that the length of a circle of (hyperbolic) radius r (centred at the origin) is $2\pi \sinh(r)$, whereas the area of this circle (centred at the origin) is $2\pi(\cosh(r) - 1)$. Hence, a fundamental difference with the Euclidean plane is that volumes grow exponentially.

The KPKBV model introduced by Krioukov *et al.* [4] yields a random geometric graph on H^2_{-1}. Consider the Poincaré disc representation of the hyperbolic plane H^2_{-1}. The random graph will have n vertices and this is the parameter we take asymptotics with respect to. Let $\nu > 0$ be a fixed constant and let $R = R(n) > 0$ satisfy $n = \nu e^{R/2}$.

Consider the disc $\mathcal{D}_R$ of hyperbolic radius R centred at the origin of the Poincaré disc (i.e. the set of points of the Poincaré disc at hyperbolic distance at most R from its origin).

We take a random set of points of size n that are the outcomes of the *i.i.d.* random variables $v_1, \ldots, v_n$ taking values on $\mathcal{D}_R$. (We will be referring to the random variables v_i as vertices, meaning their values on $\mathcal{D}_R$.) More specifically, assume that v_1 has *polar* coordinates (r, θ). The angle θ is uniformly distributed in $(0, 2\pi]$ and the probability density function of r, which we denote by $\rho_n(r)$, is determined by a parameter $\alpha > 0$ and is equal to

$$\rho_n(r) = \begin{cases} \alpha \frac{\sinh(\alpha r)}{\cosh(\alpha R) - 1}, & \text{if } 0 \le r \le R \\ 0, & \text{otherwise} \end{cases} . \tag{1.1}$$

The aforementioned formulae for the area and the length of a circle of a given radius imply that if we set $\alpha = 1$, the distribution described in (1.1) is the uniform distribution on $\mathcal{D}_R$ (under the hyperbolic metric). For general $\alpha > 0$ Krioukov *et al.* [4] called this the *quasi-uniform* distribution on $\mathcal{D}_R$. Let us remark that in fact this is the uniform distribution on a disc of hyperbolic radius R within $H^2_{-\alpha^2}$ (the hyperbolic plane that has curvature $-\alpha^2$).

Given the point process $V_n = \{v_1, \ldots, v_n\}$ on $\mathcal{D}_R \subset H^2_{-1}$ and the fixed parameters α and ν we define the random graph $\mathcal{G}(n; \alpha, \nu)$ on the point-set of V_n, where two distinct points form an edge if and only if they are within (hyperbolic) distance R from each other. Figure 1 (below) shows the ball of radius R around a point $p \in \mathcal{D}_R$, denoted by $B(p; R)$. Thus, any point/vertex of $\mathcal{G}(n; \alpha, \nu)$ that falls inside the shaded region becomes connected to p.
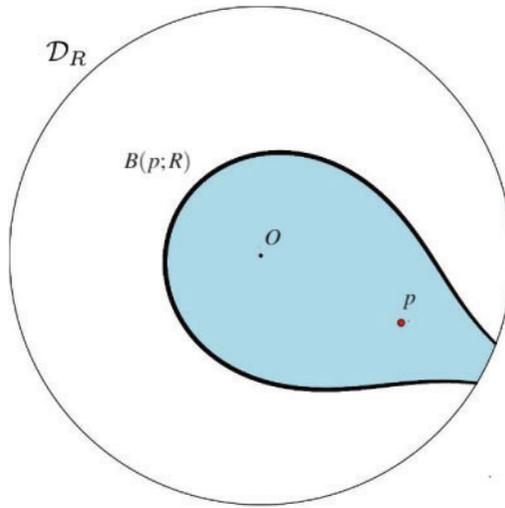
FIG. 1. The ball of radius R centred at point p, within $\mathcal{D}_R$.

2. Typical properties of the KPKBV model

For $\alpha \in (1/2, \infty)$, Krioukov *et al.* [4] show that the tails of the distribution of the degrees in $\mathcal{G}(n; \alpha, \nu)$ follow a power law with exponent $2\alpha + 1$. This was verified rigorously by Gugelman *et al.* in [5]. Thus, when $\alpha \in (1/2, 1)$ the exponent is between 2 and 3. There has been experimental evidence that this is indeed the case in a number of networks that emerge in applications (the survey [6] contains a comprehensive list of such examples). Krioukov *et al.* [4] also observe that the average degree of $\mathcal{G}(n; \alpha, \nu)$ is also tuned by the parameter ν for $\alpha \in (1/2, \infty)$. This was also proved by Gugelman *et al.* [5]. They showed that the average degree tends to $8\alpha^2 \nu / \pi (2\alpha - 1)^2$ in probability. However, when $\alpha \in (0, 1/2]$, the average degree tends to infinity as $n \to \infty$. Thus, in this sense, the regime $\alpha \in (1/2, \infty)$ corresponds to the so-called *thermodynamic regime* in the context of random geometric graphs on the Euclidean plane [7].

Gugelman *et al.* [5] also showed $\mathcal{G}(n; \alpha, \nu)$ has *clustering coefficient* that is a.a.s. bounded away from 0. More precise results about the scaling of the local clustering coefficient in terms of the degrees of the vertices were obtained by Stegehuis *et al.* [8]. More recently in [9], convergence in probability of the clustering coefficient to an explicitly determined constant was derived.

When α is small, the density ρ_n induces more points near the origin and one may expect increased graph connectivity there. In [10], Bode *et al.* proved that $\alpha = 1$ is the critical point for the emergence of a giant component in $\mathcal{G}(n; \alpha, \nu)$. (Let us abbreviate as *a.a.s.* the term 'asymptotically almost surely' which in our context means 'with probability that approaches 1 as $n \to \infty$'—see below for the precise definition.) When $\alpha \in (0, 1)$, the fraction of the vertices contained in the largest component is bounded away from 0 a.a.s. [10], whereas if $\alpha \in (1, \infty)$, the largest component is sub-linear in n a.a.s (see Figure 2). For $\alpha = 1$, the component structure depends on ν. If ν is large enough, then a giant component exists a.a.s., but if ν is small enough, then a.a.s. all components have sub-linear size [10].

The above results were strengthened in [11]. In that paper, it was shown that the fraction of vertices which belong to the largest component converges in probability to a certain constant which depends on α and ν. More specifically, when $\alpha = 1$, it turns out that there exists a critical value $\nu_0 \in (0, \infty)$ such
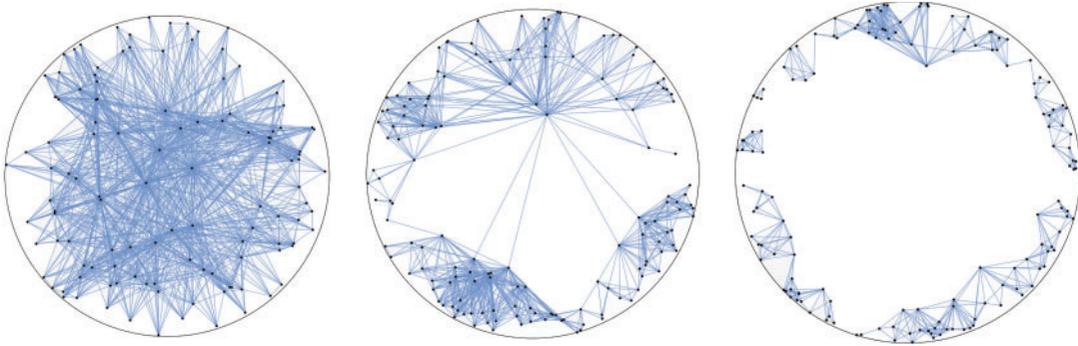
FIG. 2. Three samples of $\mathcal{G}(n; \alpha, \nu)$ with a fixed $n = 150$, $\nu = 2$ and variable α. From left to right: $\alpha = 0.6$, $\alpha = 1$ and $\alpha = 1.8$

that when ν crosses ν_0 a giant component emerges a.a.s. The papers [12] and [13] consider the size of the second largest component. Therein, it is shown that when $\alpha \in (0, 1)$ the second largest component has polylogarithmic order a.a.s.

The connectivity of $\mathcal{G}(n; \alpha, \nu)$ was considered by Bode *et al.* in [14]. There, it is shown that for $\alpha < 1/2$ the random graph $\mathcal{G}(n; \alpha, \nu)$ is a.a.s. connected, it is disconnected for $\alpha > 1/2$. When $\alpha = 1/2$, it turns out that the probability of connectivity converges to a certain constant which is given explicitly in [14].

The a.a.s. disconnectedness of $\mathcal{G}(n; \alpha, \nu)$ for $\alpha > 1/2$ follows easily from the a.a.s. existence of isolated vertices. Recently, asymptotic distributional properties of the number of isolated as well as the extreme points in the poissonisation of $\mathcal{G}(n; \alpha, \nu)$ were derived in [15]. (A point is called extreme, when it is not connected to any other point of larger radius.) The authors showed that the former satisfies a central limit theorem when $\alpha > 1$, but it does not when $\alpha < 1$. However, the number of extreme points satisfies a central limit theorem for any $\alpha > 1/2$. This is due to the fact that the number of isolated vertices is sensitive on the existence of a few vertices close to the centre of $\mathcal{D}_R$. Those a.a.s. appear when $1/2 < \alpha < 1$. On the other hand, extreme points involve only local dependencies.

Bounds on the diameter of $\mathcal{G}(n; \alpha, \nu)$ were derived in [12] and [16]. Therein, polylogarithmic upper bounds on the diameter are shown. These were improved by Müller and Staps [17] who deduced a logarithmic upper bound on the diameter. Furthermore, in [18] it is shown that for $\alpha \in (1/2, 1)$ the largest component has doubly logarithmic typical distances and it forms what is called an *ultra-small world*.

2.1 *Poissonisation of the KPKBV model*

In this paper, we will work on the Poissonization of the above model. Recall that $\mathcal{D}_R$ was defined to be the disc of hyperbolic radius R around the origin O of the Poincaré disc representation of the hyperbolic plane of curvature -1. Here, the vertex set is the point-set of a Poisson point process on $\mathcal{D}_R$ with intensity

$$n \frac{1}{2\pi} \rho_n(r) dr d\theta.$$

We denote it by $P_{\alpha, \nu, n}$. We also denote by $\kappa_{\alpha, \nu, n}$ the Borel measure on $\mathcal{D}_R$ given by

$$\kappa_{\alpha, \nu, n}(S) = \frac{1}{2\pi} \int_S \rho_n(r) dr d\theta,$$

for any Borel-measurable set S. Hence, the number of points that $P_{\alpha,\nu,n}$ has inside S is distributed as $\mathrm{Po}\left(n \cdot \kappa_{\alpha,\nu,n}(S)\right)$. Moreover, the numbers of points in any finite collection of pairwise disjoint Borel-measurable subsets of $\mathcal{D}_R$ are independent Poisson-distributed random variables.

We will define the random graph whose vertex set is the set of points of $P_{\alpha,\nu,n}$ in $\mathcal{D}_R$. As in $\mathcal{G}(n; \alpha, \nu)$, two vertices/points of $P_{\alpha,\nu,n}$ are adjacent if and only if their hyperbolic distance is at most R. We denote the resulting graph by $\mathcal{P}(n; \alpha, \nu)$.

2.2 *The modularity of $\mathcal{P}(n; \alpha, \nu)$*

The notion of modularity was introduced by Newman and Girvan in [2]. For a graph $G = (V, E)$ with $m \geq 1$ edges, define the *modularity score* associated with the partition $\mathcal{A}$ of the vertex set V to be

$$\mathrm{mod}_{\mathcal{A}}(G) = \sum_{A \in \mathcal{A}} \left(\frac{e(A)}{m} - \left(\frac{\mathrm{vol}(A)}{2m} \right)^2 \right),$$

where $e(A)$ denotes the number of edges within part A and $\mathrm{vol}(A) = \sum_{v \in A} \deg(v)$ denotes the volume of A, that is, the sum of the degrees of the vertices in A.

For graphs G without edges define $\mathrm{mod}_{\mathcal{A}}(G) = 0$. Note that the definition of modularity extends naturally to weighted graphs and is often used in the weighted form in applications. The term $e(A)$ becomes the sum of the weights of edges in A and the degree of a vertex $\deg(v)$ is the sum of the weights of the edges incident to v. This sum is effectively a comparison between the given network G and a random network with the same degree sequence. The first term $\frac{1}{m} \sum_{A \in \mathcal{A}} e(A)$ is the probability that a randomly chosen edge of G will lie inside one of the parts, whereas the term $\sum_{A \in \mathcal{A}} (\mathrm{vol}(A)/2m)^2$ represents the probability that a random edge lies in one of the parts in a uniformly random graph with the same degree distribution as G. On one extreme if there were no edges between the parts of $\mathcal{A}$, then $\frac{1}{m} \sum_{A \in \mathcal{A}} e(A) = 1$. If $\mathcal{A}$ consists of a large number of parts that are comparable in volume, then the second term $\sum_{A \in \mathcal{A}} (\mathrm{vol}(A)/2m)^2$ is small. Hence, such a highly modular partition will have a modularity score close to 1. With $\mathcal{P}(V)$ denoting the set of all partitions of V the *modularity of graph G* is then

$$\mathrm{mod}(G) = \max\{\mathrm{mod}_{\mathcal{A}}(G) \; : \; \mathcal{A} \in \mathcal{P}(V)\}.$$

The set $\mathcal{P}(V)$ includes the trivial partition $\{V\}$ placing all vertices into the same part. Note that the modularity score of $\{V\}$ is zero for any graph. Hence for any graph $0 \leq \mathrm{mod}(G) < 1$ with values near 1 taken to indicate a high level of community structure and values near 0 taken to indicate a lack of community structure. Newman [19] determined the modularity of several examples of complex networks, not only social, finding them ranging between 0.3 and 0.8. Among these examples, higher modularity (> 0.7) was found in the social network of co-authorship among scientists working on condensed matter.

Brandes *et al.* [20] showed that finding the modularity of a given graph is NP-hard. Further it was established by Dinh *et al.* that it is NP-hard to approximate modularity to within any constant factor [21]. However, community detection in networks has been a central theme in network science. Newman [22] used modularity to design a spectral algorithm for community detection in a given network. A popular algorithm, the *Louvain method*, is an iterative clustering technique using the modularity function to compare candidate partitions [23]. Methods that utilize random walks on the network in order to trace its community structure are the *Infomap algorithm* [24] and the *Walktrap algorithm* [25]. The *Label propagation algorithm* [26] on the other hand uses an approach that is similar to majority dynamics where

the states of the nodes are labels that indicate different communities. More recently, Wang *et al.* [27] came up with another method which uses the notion of *local expansion*. We further note that in [27] give a comparison of these methods on a number of networks that emerge in a variety of settings. According to Table 3 therein, typically, the modularity of these networks, as it is evaluated by these algorithms, is far from 1 (we see values ranging from 0.39 to 0.94).

For binomial random graphs from the $G(n, p)$ model, where on a set of n vertices, each pair is included as an edge independently with probability p, there is a transition the typical behaviour of $\mathrm{mod}(G(n, p))$ that is determined by np. In particular, the third author together with McDiarmid showed [28] that when $np \leq 1 + o(1)$, then $\mathrm{mod}(G(n, p))$ is concentrated around 1, but when np exceeds and is bounded away from 1, then it scales like $(np)^{-1/2}$. They have also shown [29] that for random d-regular graphs of bounded degree, it is bounded away from 0 and 1 with high probability and scales approximately like $1/\sqrt{d}$ when d is large. Recently, Lichev and Mitsche [30] showed that for $d = 3$ the modularity exceeds $2/3$ (confirming a conjecture of McDiarmid and Skerman) and is below 0.8 with high probability. They further considered the modularity of random graphs having a given degree distribution with bounded maximum degree.

The relation between topological properties of a graph and its modularity was explored in [29, 31]. In particular, Lasoń and Sulkowska [31] showed that the class of graphs with an excluded minor and sub-linear maximum degree have modularity that approaches 1 as their number of vertices grows.

The main theorem of this article is that with high probability the modularity of $\mathcal{P}(n; \alpha, \nu)$ is close to 1.

THEOREM 2.1 For any $\alpha > 1/2$ and $\nu > 0$, we have

$$\mathrm{mod}(\mathcal{P}(n; \alpha, \nu)) \to 1,$$

as $n \to \infty$, in probability.

Unlike the $G(n, p)$ model, the modularity of $\mathcal{P}(n; \alpha, \nu)$ approaches 1 as $n \to \infty$, without any dependence on the average degree or the existence of a giant component. Very recently, Kovács and Palla consider an empirical approach to community structure in hyperbolic networks. In particular, they show that almost optimal modularity scores (greater than 0.95) can be achieved, using simulations of the hyperbolic random graph; however, they work in a range of parameters where the resulting random graph exhibits a high clustering coefficient and large α. In light of this, our results show that for sufficiently large networks this dependency on parameters is no longer required to achieve optimal modularity.

In the context of community driven models, Zuev *et al.* [32] introduced a model (*preferential geometric attachment*) which encapsulates the notion of an underlying hyperbolic space of hierarchies where the angle of each vertex is not selected uniformly at random, but it follows a distribution which is biased towards sectors that are more densely populated. This aims at modelling the notion of *homophily* through a preferential attachment mechanism: a newly arrived node is more likely to be located in a region/community of larger size. In this model, the notion of *soft communities* is considered in which nodes form a community if they are close to each other in terms of their relative angle. This model rectifies the lack of community structure that is a exhibited in the *popularity–similarity–optimization* model that was introduced by Papadopoulos *et al.* [33]. Later on Muscoloni and Cannistraci [34] introduced the *non-uniform* popularity-similarity-optimization model which is also a strengthening of the model of Zuev *et al.* This model gives the opportunity to fix the number and the size of communities and tune their

mixing. Furthermore, the authors use in [35] this model as a benchmark for the algorithms we described above.

On the other hand, the classic *preferential attachment model* of Barabási and Albert [36] has modularity which is bounded away from 1. In particular, Ostroumova Prokhorenkova *et al.* [37] observed that this is bounded away from 1 with high probability (in the version of the model where each newly arriving vertex is attached to $m \geq 2$ existing vertices—in fact, they show that for $m \geq 3$ the modularity is at most 15/16 with high probability). This is not the case for the *spatial preferential attachment model* which was introduced by Aiello *et al.* [38]. This is a geometric version of the preferential attachment model where vertices arrive one at a time at the d-dimensional unit cube and are allocated uniformly therein. They are attached to existing vertices if they are within the *sphere of influence* of the latter. The size of this is proportional to their degree. Ostroumova Prokhorenkova *et al.* [37] showed that the modularity of the resulting random graph approaches 1 with high probability as the number of vertices tends to infinity.

Notation

We now introduce some notation which we use throughout out proofs. If $\mathcal{E}_n$ is an event on the probability space $(\Omega_n, \mathbb{P}_n, \mathcal{F}_n)$, for each $n \in \mathbb{N}$, we say that sequence $\mathcal{E}_n$ *occurs asymptotically almost surely (a.a.s.)* if $\mathbb{P}_n(\mathcal{E}_n) \to 1$ as $n \to \infty$. In our context, we will be using the term *a.a.s.* for the sequence of probability spaces of the random graphs $\mathcal{P}(n; \alpha, \nu)$.

2.3 *Approximating a ball around a point—geometric notation*

The main lemma in this section provides a useful (almost) characterization of two vertices being within hyperbolic distance R, given their radii. The lemma reduces a statement about hyperbolic distances to a statement about the relative angle between two points. Let us first introduce some notation. For a point $p \in \mathcal{D}_R$, we let $\theta(p) \in (-\pi, \pi]$ be the angle $p\hat{O}s$ between p and a (fixed) reference point $s \in \mathcal{D}_R$ (moving from s to p in the anti-clockwise direction). For $\theta, \theta' \in (-\pi, \pi]$, we set

$$|\theta - \theta'|_\pi = \min\{|\theta - \theta'|, 2\pi - |\theta - \theta'|\} \in [0, \pi].$$

For two points $p, p' \in \mathcal{D}_R$ we denote by $\theta(p, p') \in [0, \pi]$ their relative angle:

$$\theta(p, p') = |\theta(p) - \theta(p')|_\pi.$$

Also, for $p \in \mathcal{D}_R$ we let $y(p)$ denote the *defect radius* of p in $\mathcal{D}_R$. In other words, if $r(p)$ is the radius (the hyperbolic distance of p from O), then $y(p) = R - r(p)$. The following lemma gives a characterization of what it is to have hyperbolic distance at most R in terms of the relative angle between two points. For r, r' such that $r + r' > R$, let $\theta_R(r, r') \in (-\pi, \pi]$ be such that if two points p, p' with $r(p) = r$ and $r(p') = r'$ have $\theta(p, p') = \theta_R(r, r')$ iff $d_H(p, p') = R$. In other words, $\theta_R(r, r')$ is the relative angle of two points of radii r and r', respectively, which are at distance R (see Figure 3).

Also, we set $T_R(y, y') = 2 \cdot e^{-R/2} e^{\frac{1}{2}(y+y')}$, for $y, y' \in [0, R]$. The following lemma is a consequence of Lemma 28 in [11].
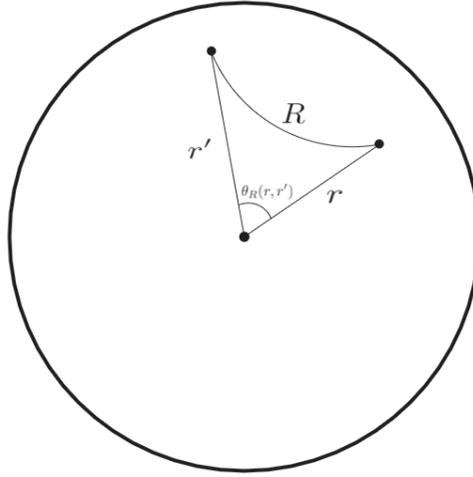
FIG. 3. An example of $\theta_R(r, r')$ for two points at a distance of r and r' from the centre

LEMMA 2.2 Let $\kappa \in (0, 1)$. For any $\delta > 0$ and any n sufficiently large, uniformly for any $p, p' \in \mathcal{D}_R$ with $y(p) + y(p') \leq \kappa R$ the following holds

$$\left| \frac{\theta_R(r(p), r(p'))}{T_R(y(p), y(p'))} - 1 \right| < \delta.$$

Proof. Lemma 28 in [11] states that there exists a constant $K > 0$ such that, for every $\varepsilon > 0$ and for R sufficiently large, the following holds: with

$$\Delta(r, r') := \frac{1}{2} e^{R/2} \arccos \left((\cosh r \cosh r' - \cosh R) / \sinh r \sinh r' \right),$$

For every $r, r' \in [\varepsilon R, R]$ with $r + r' > R$ we have that

$$e^{\frac{1}{2}(y+y')} - K e^{\frac{3}{2}(y+y')-R} \leq \Delta(r, r') \leq e^{\frac{1}{2}(y+y')} + K e^{\frac{3}{2}(y+y')-R}, \tag{2.1}$$

where $y := R - r, y' := R - r'$. The above is a consequence of the hyperbolic law of cosines. Take two points p and p' inside $\mathcal{D}_R$ having radii r and r', respectively, and consider the triangle pOp'. If the distance between p and p' is R, then the angle opposite that side is $\theta_R(r, r')$ and satisfies:

$$\cosh(R) = \cosh r \cosh r' - \sinh r \sinh r' \cos(\theta_R(r, r')).$$

Therefore,

$$\theta_R(r, r') = 2 e^{-R/2} \Delta(r, r').$$

So multiplying (2.1) by $2e^{-R/2}$ we get

$$T_R(y, y') - K e^{\frac{3}{2}(y+y'-R)} \leq \theta_R(r, r') \leq T_R(y, y') + K e^{\frac{3}{2}(y+y'-R)},$$

or

$$1 - \frac{K}{2}e^{y+y'-R} \leq \frac{\theta_R(r,r')}{T_R(y,y')} \leq 1 + \frac{K}{2}e^{y+y'-R}.$$

But $y + y' \leq \kappa R$ whereby $y + y' - R \leq (\kappa - 1)R$. Since $\kappa < 1$, the lemma follows provided that n is sufficiently large. □

For a point $p \in \mathcal{D}_R$, we let $B(p;R)$ denote the set of points in $\mathcal{D}_R$ of hyperbolic distance at most R from p. We further define

$$\check{B}_{\kappa,\delta}(p) := \{p' \in \mathcal{D}_R \ : \ y(p') + y(p) \leq \kappa R, \theta(p,p') < (1+\delta)T_R(y(p),y(p'))\}.$$

We can think of $\check{B}_{\kappa,\delta}(p)$ to be an approximation to the ball $B(p;R)$. Essentially $B(p;R)$ is a subset of $\check{B}_{\kappa,\delta}(p)$, and in subsequent calculations $\check{B}_{\kappa,\delta}(p)$ is easier to work with. Let $\mathcal{A}_r := \mathcal{D}_R \setminus \mathcal{D}_r$ denote the annulus of the disc $\mathcal{D}_R$ which consists of all points of defect radius at most $R - r$. The above lemma implies that for any $\kappa \in (0,1), \delta > 0$ and any n sufficiently large we have

$$\check{B}_{\kappa,-\delta}(p) \subset B(p;R) \cap \mathcal{A}_{(1-\kappa)R+y(p)} \subset \check{B}_{\kappa,\delta}(p); \tag{2.2}$$

hence, the set $\check{B}_{\kappa,\delta}(p)$ includes all points in $B(p;R)$ of defect radius at most $\kappa R - y(p)$. Furthermore, the following holds and will be useful later on during our second moment calculations.

CLAIM 2.3 If $\kappa \in (0,1)$ and $\delta > 0$, then for any n sufficiently large whenever $\theta(p,p') > 4(1+\delta)e^{-(1-\kappa)R/2}$ for points $p, p' \in \mathcal{D}_R$ with $y(p), y(p') < R/2$, we have

$$\left(B(p;R) \cap \mathcal{A}_{(1-\kappa)R+y(p)}\right) \cap \left(B(p';R) \cap \mathcal{A}_{(1-\kappa)R+y(p')}\right) = \varnothing.$$

Proof. By (2.2), it is sufficient to show that

$$\check{B}_{\kappa,\delta}(p) \cap \check{B}_{\kappa,\delta}(p') = \varnothing.$$

Suppose not and let $p'' \in \check{B}_{\kappa,\delta}(p) \cap \check{B}_{\kappa,\delta}(p')$. Then by the definition of $\check{B}_{\kappa,\delta}(p)$ we have

$$\theta(p,p'') < 2(1+\delta)e^{R/2}e^{(y(p)+y(p''))/2} < 2(1+\delta)e^{-R/2+\kappa R/2}.$$

Similarly,

$$\theta(p',p'') < 2(1+\delta)e^{R/2}e^{(y(p')+y(p''))/2} < 2(1+\delta)e^{-R/2+\kappa R/2}.$$

So,

$$\theta(p,p') \leq \theta(p,p'') + \theta(p',p'') < 4(1+\delta)e^{-(1-\kappa)R/2}.$$

This concludes the proof of the claim. □

Another result, that will be useful later on, provided a bound on the expected number of point of $P_{\alpha,v,n}$ inside $\check{B}_{\kappa,\delta}(p)$.

CLAIM 2.4 For any $\kappa \in (1/2, 1)$ and $\delta \in (-1, 1)$, uniformly for any $p \in \mathcal{D}_R$ with $y(p) \leq R/2$ we have

$$\mathbb{E}\left(|P_{\alpha,v,n} \cap \check{B}_{\kappa,\delta}(p)|\right) = \Theta(e^{y(p)/2}).$$

Proof. We firstly observe that uniformly for all $0 < \rho < R$:

$$\frac{\alpha \sinh(\alpha\rho)}{\cosh(\alpha R) - 1} = \Theta\left(\frac{e^{\alpha\rho}}{e^{\alpha R} - 1}\right) = \Theta\left(e^{-\alpha(R-\rho)}\right).$$

Hence, we may calculate the following

$$\begin{aligned}
\mathbb{E}&\left(|P_{\alpha,v,n} \cap \check{B}_{\kappa,\delta}(p)|\right) = \\
&= n\frac{1+\delta}{2\pi} \cdot 2e^{-R/2+y(p)/2} \cdot \int_{(1-\kappa)R+y(p)}^{R} e^{(R-\varrho)/2} \frac{\alpha \sinh(\alpha\varrho)}{\cosh(\alpha R) - 1} d\varrho \\
&= \Theta(1) \cdot e^{y(p)/2} \int_{(1-\kappa)R+y(p)}^{R} e^{(1/2-\alpha)(R-\varrho)} d\varrho \\
&= \Theta(1) \cdot e^{y(p)/2} \int_{0}^{\kappa R-y(p)} e^{(1/2-\alpha)y} dy \overset{\alpha > 1/2}{=} \Theta(e^{y(p)/2}). \qquad \square
\end{aligned}$$

Furthermore, since $|P_{\alpha,v,n} \cap \check{B}_{\kappa,\gamma}(p)|$ follows the Poisson distribution, the above claim also yields that for any $\kappa \in (1/2, 1)$ and $\delta > 0$,

$$\mathbb{E}\left(|P_{\alpha,v,n} \cap \check{B}_{\kappa,\delta}(p)|^2\right) = O(e^{y(p)}), \tag{2.3}$$

uniformly for any $p \in \mathcal{D}_R$ with $y(p) \leq R/2$.

2.3.1 *Projecting $\mathcal{D}_R$ onto $\mathbb{R}^2$* To simplify our calculations, we will transfer our analysis from $\mathcal{D}_R$ to $\mathbb{R}^2$. In particular, we will make use of a mapping that was introduced in [11] and reduces our model to a percolation model on $\mathbb{R}^2$. Our result could in principle be proved without the use of it, but the proofs would be much heavier. This is achieved using a local approximation of the hyperbolic metric as given in Lemma 2.2. For a point $p \in \mathcal{D}_R$, let $(\theta(p), y(p)) \in (-\pi, \pi] \times [0, R]$ denote its angle with respect to a reference point and its defect radius, respectively. We define the map $\Phi : \mathcal{D}_R \to \mathcal{B} = (-\frac{\pi}{2}e^{R/2}, \frac{\pi}{2}e^{R/2}] \times [0, R]$, mapping a point $p = (\theta(p), y(p)) \in \mathcal{D}_R$ to a point $(x(p), y(p)) \in \mathcal{B}$

$$\theta(p) \mapsto x(p) := \frac{1}{2}\theta(p)e^{R/2} \text{ and } y(p) \mapsto y(p).$$
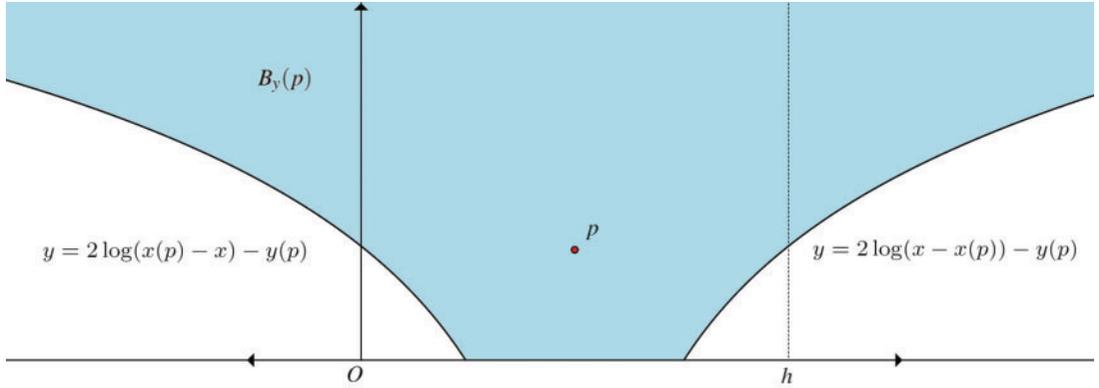
FIG. 4. The ball $B_y(p)$.

For simplicity, we set $I := I(R) := \frac{\pi}{2}e^{R/2}$. The map Φ projects the process $P_{\alpha,\nu,n}$ to a point process on $\mathcal{B}$. We will approximate this process with the Poisson point process on $\mathcal{B}$ having intensity

$$\frac{2\nu}{\pi}\alpha e^{-\alpha y}dxdy.$$

For any measurable subset $S \subseteq \mathcal{B}$, we set $\mu_{\alpha,\beta}(S) = \beta \int_S e^{-\alpha y}dxdy$, with $\beta = \frac{2\nu\alpha}{\pi}$. We denote this Poisson process by $P_{\alpha,\beta}$. The analogue of the relative angle between points in $\mathcal{D}_R$ is defined as follows. For $x, x' \in (-I, I]$, we let

$$|x - x'|_\mathcal{B} := \min\left\{|x - x'|, 2I - |x - x'|\right\}.$$

For a positive real number $y < R$, we set $\mathcal{B}(y) := (-\frac{\pi}{2}e^{R/2}, \frac{\pi}{2}e^{R/2}] \times [0, y]$; thus, $\mathcal{B}(R) = \mathcal{B}$. We define the random graph $\mathcal{B}_y(n; \alpha, \nu)$ with vertex set the point set of $P_{\alpha,\beta} \cap \mathcal{B}(y)$, and for any distinct $p, p' \in P_{\alpha,\beta}$, the vertices p, p' are adjacent if and only if

$$|x(p) - x(p')|_\mathcal{B} < e^{(y(p)+y(p'))/2}.$$

We define the ball around a point $p \in \mathcal{B}(y)$ as $B_y(p) = \{p' \in \mathcal{B}(y) : |x(p) - x(p')|_\mathcal{B} < e^{\frac{1}{2}(y(p)+y(p'))}\}$. Thus, for a point $p \in P_{\alpha,\beta}$, the neighbourhood of p in the random graph $\mathcal{B}_y(n; \alpha, \nu)$ is $B_y(p) \cap P_{\alpha,\beta} \setminus \{p\}$. Figure 4 shows the neighbourhood around a point $p \in \mathcal{B}(y)$. Thus any point lying in the region bounded by the x-axis and the two log curves will be connected to p. The rectangular region bounded by the axis and the dotted line represents a single box in our partition, see Section 5.

3. Mapping $\mathcal{P}(n; \alpha, \nu)$ into $\mathcal{B}$ and the Proof of Theorem 2.1

To prove Theorem 2.1, it suffices to consider a subgraph of $\mathcal{P}(n; \alpha, \nu)$ which contains most edges of it. To this end, we use Lemma 5.1 from [28].

LEMMA 3.1 Let $G = (V, E)$ be a graph with $|E| \geq 1$, let E_0 be a nonempty subset of E. For $E' = E \backslash E_0$, let $G' = (V, E')$. Then

$$|\text{mod}(G) - \text{mod}(G')| < 2|E_0|/|E|.$$

We will show the following lemma.

LEMMA 3.2 For every $\varepsilon > 0$, there exists $y_\varepsilon > 0$ such that a.a.s.

$$\text{vol}(\text{P}_{\alpha,\nu,n} \cap \mathcal{D}_{R-y_\varepsilon}) \leq \varepsilon e(\mathcal{P}(n; \alpha, \nu)).$$

For a positive real number $y < R$, let $\mathcal{P}_{\leq y}(n; \alpha, \nu)$ denote the subgraph of $\mathcal{P}(n; \alpha, \nu)$ induced by the points of $\text{P}_{\alpha,\nu,n}$ having defect radius at most y. (The subgraph $\mathcal{P}_{>y}(n; \alpha, \nu)$ is defined analogously.) As the number of edges incident to points in $\text{P}_{\alpha,\nu,n} \cap \mathcal{D}_{R-y_\varepsilon}$ is at most $\text{vol}(\text{P}_{\alpha,\nu,n} \cap \mathcal{D}_{R-y_\varepsilon})$, the above two results imply that for every $\varepsilon > 0$ there exists $y_\varepsilon > 0$ such that a.a.s.

$$|\text{mod}(\mathcal{P}(n; \alpha, \nu)) - \text{mod}(\mathcal{P}_{\leq y_\varepsilon}(n; \alpha, \nu))| < 2\varepsilon.$$

Thereby, to prove Theorem 2.1 it suffices to show that

$$\text{mod}(\mathcal{P}_{\leq y_\varepsilon}(n; \alpha, \nu)) \to 1, \tag{3.1}$$

as $n \to \infty$ in probability. To show this, we will couple the random graph $\mathcal{P}_{\leq y_\varepsilon}(n; \alpha, \nu)$ with the random graph $\mathcal{B}_{y_\varepsilon}(n; \alpha, \nu)$.

LEMMA 3.3 (Lemmas 27 and 30 in [11]) There is a coupling between the point processes $\text{P}_{\alpha,\beta}$ and $\text{P}_{\alpha,\nu,n}$ such that a.a.s. on the coupling space $\Phi(\text{P}_{\alpha,\nu,n}) = \text{P}_{\alpha,\beta}$. Furthermore, a.a.s. on the coupling space for any distinct $p, p' \in \text{P}_{\alpha,\nu,n}$ with $y(p), y(p') \leq R/4$ we have $d_H(p, p') \leq R$ if and only if $\Phi(p') \in B(\Phi(p); R)$.

The above lemma implies that there is a coupling between the processes $\text{P}_{\alpha,\beta}$ and $\Phi(\text{P}_{\alpha,\nu,n})$ on $\mathcal{B}$ such that for any fixed $y > 0$ a.a.s., on this coupling space, the two point-sets coincide and moreover the random graph $\mathcal{B}_y(n; \alpha, \nu)$ is isomorphic to $\mathcal{P}_{\leq y}(n; \alpha, \nu)$.

So, we can deduce (3.1) from the following theorem.

THEOREM 3.4 For any $\alpha > 1/2$, $\nu > 0$ and any fixed $y > 0$, we have

$$\text{mod}(\mathcal{B}_y(n; \alpha, \nu)) \to 1,$$

as $n \to \infty$ in probability.

4. Some general properties of the modularity of a graph

Let $G = (V, E)$ be a graph. For $A, B \subset V$, let $\bar{A}$ denote $V \setminus A$ and, for disjoint A, B, let $e(A, B)$ denote the number of edges with one end vertex in A and the other in B. Let $\mathcal{A}$ be a partition of V. It will sometimes be helpful to talk separately of the edge-contribution, also called *coverage*

$$\text{mod}_{\mathcal{A}}^E(G) = \frac{1}{m} \sum_{A \in \mathcal{A}} e(A) = 1 - \frac{1}{2m} \sum_{A \in \mathcal{A}} e(A, \bar{A}),$$

and the *degree tax*

$$\text{mod}_{\mathcal{A}}^D(G) = \frac{1}{(2m)^2} \sum_{A \in \mathcal{A}} \text{vol}(A)^2.$$

The following lemma provides a lower bound on $\text{mod}_{\mathcal{A}}(G)$ with respect to the parameters of a given partition $\mathcal{A}$.

LEMMA 4.1 Let G be a graph with m edges. Suppose the partition $\mathcal{A} = \{A_1, \ldots, A_k\}$ has the property that for each $1 \leq i \leq k$,

$$e(A_i, \bar{A}_i) \leq \varepsilon m \quad \text{and} \quad |\text{vol}(A_i) - 2m/k| \leq 2m\delta,$$

then

$$\text{mod}_{\mathcal{A}}(G) \geq 1 - \frac{k\varepsilon}{2} - \frac{1}{k} - k\delta^2.$$

Proof. Define δ_i to be such that $\text{vol}(A_i) = (1/k + \delta_i)2m$ and note that $\sum_i \delta_i = 0$ and by assumption $\forall i$ $|\delta_i| \leq \delta$. We may now bound the degree tax of $\mathcal{A}$,

$$\text{mod}_{\mathcal{A}}^D(G) = \frac{1}{4m^2} \sum_{i=1}^k \text{vol}(A_i)^2 = \sum_{i=1}^k \left(\frac{1}{k} + \delta_i\right)^2 \leq \frac{1}{k} + k\delta^2.$$

The edge contribution of $\mathcal{A}$ is $\text{mod}_{\mathcal{A}}^E(G) = 1 - \sum_i e(A_i, \bar{A}_i)/2m \geq 1 - k\varepsilon/2$ and thus we have our required bound. □

5. The modularity of $\mathcal{B}_r(n; \alpha, \nu)$.

In this section, we prove Theorem 3.4. We shall make use of the following identity which is an application of the (multivariate) *Campbell–Mecke* formula (see for example Theorem 4.4 [39]): for a Poisson point process $\mathcal{P}$ on a measurable space S with intensity ρ and a measurable non-negative function $h : S^k \times \mathcal{N} \to \mathbb{R}$, where $\mathcal{N}$ is the set of all locally finite collections of points in S, we have

$$\mathbb{E}\left(\sum_{\substack{x_1,\ldots,x_k \in \mathcal{P} \\ \forall i,j,x_i \neq x_j}} h(x_1, \ldots, x_k, \mathcal{P} \setminus \{x_1, \ldots, x_k\})\right) \tag{5.1}$$
$$= \int_S \cdots \int_S \mathbb{E}\left(h(x_1, \ldots, x_r, \mathcal{P} \cup \{x_1, \ldots, x_k\})\right) \rho(x_1) \cdots \rho(x_k) dx_1 \cdots dx_k.$$

We set out to show that for any fixed $y > 0$, we have $\text{mod}(\mathcal{B}_y(n; \alpha, \nu)) \to 1$ in probability as $n \to \infty$. To this end, we will use Lemma 4.1 on a specific partition of the vertex set of $\mathcal{B}_y(n; \alpha, \nu)$. More specifically, we consider a partition of the box $\mathcal{B}_y = (-I, I] \times [0, y]$ into $2t$ boxes $B_i := (i \cdot hI, (i+1) \cdot hI] \times [0, y]$, for

$i = -1/h, \ldots, 1/h-1$, where $h = 1/t$ with $t \in \mathbb{N}$. Given this partition of the box $\mathcal{B}_y$, we let $A_i = P_{\alpha,\beta} \cap B_i$, for $i = -t, \ldots, t-1$. With $\mathcal{A} = \{A_{-t}, \ldots, A_{t-1}\}$, we will show that a.a.s.

$$\text{mod}_{\mathcal{A}}(\mathcal{B}_y(n; \alpha, \nu)) \geq 1 - 4h - o(1). \tag{5.2}$$

Therefore, for $\varepsilon > 0$, we take $t \in \mathbb{N}$ to be such that $4h = 4/t < \varepsilon/2$ and deduce that a.a.s.

$$\text{mod}(\mathcal{B}_y(n; \alpha, \nu)) \geq 1 - \varepsilon.$$

Let us now proceed with the proof of (5.2). Firstly, note that since the random variables $\text{vol}(A_i)$ are identically distributed, with m denoting the number of edges of the random graph $\mathcal{B}_y(n; \alpha, \nu)$, we have

$$\mathbb{E}(\text{vol}(A_i)) = \frac{1}{2t}\mathbb{E}\left[\text{vol}(P_{\alpha,\beta} \cap \mathcal{B}_y)\right] = \frac{\mathbb{E}(m)}{t}. \tag{5.3}$$

We will use a second moment argument to show that a.a.s. for each $i = -t, \ldots, t-1$, we have

$$2m\left(\frac{h}{2} - 3h^2\right) \leq \text{vol}(A_i) \leq 2m\left(\frac{h}{2} + 3h^2\right). \tag{5.4}$$

Furthermore, we will show the following.

CLAIM 5.1   There exists a constant C (depending on y) such that a.a.s.

$$\mathbb{E}\left(e(A_1, \overline{A}_1)\right) < C.$$

By the union bound, this implies that a.a.s. for all $i = -t, \ldots, t-1$

$$e(A_i, \overline{A}_i) < \log n.$$

Since a.a.s. $m = \Omega(n)$, we can then deduce (5.2) by applying Lemma 4.1 with $\varepsilon = \log^2 n/n$, $\delta = 3h^2$ and $k = t = 1/h$. We will deduce (5.4) from Chebyschev's inequality having shown that both the expectation and the variance of $\text{vol}(A_i)$ are of order n.

CLAIM 5.2   We have

$$\mathbb{E}(\text{vol}(A_1)) = \Theta(n) \text{ and } \text{Var}(\text{vol}(A_1)) = O(n).$$

Since the random variables $\text{vol}(A_i)$ are identically distributed, the first part of the above claim together with (5.3) imply that $\mathbb{E}(m) = \Theta(n)$ too. Furthermore, Chebyschev's inequality implies that a.a.s.

$$2\mathbb{E}(m)\left(\frac{h}{2} - h^2\right) \leq \text{vol}(A_1) \leq 2\mathbb{E}(m)\left(\frac{h}{2} + h^2\right).$$

In turn, the union bound implies that a.a.s. for all $i = -t, \ldots, t-1$, we have

$$2\mathbb{E}(m)\left(\frac{h}{2} - h^2\right) \leq \text{vol}(A_i) \leq 2\mathbb{E}(m)\left(\frac{h}{2} + h^2\right). \tag{5.5}$$

Furthermore, a.a.s $m \geq \mathbb{E}(m)(1-h^3)$. Indeed, we have by Chebyshev's inequality that for each $i = -t, \ldots, t-1$

$$\mathbb{P}\left[|\mathrm{vol}(A_i) - \mathbb{E}(\mathrm{vol}(A_i))| > h^3 \mathbb{E}(\mathrm{vol}(A_i))\right] \leq \frac{\mathrm{Var}(\mathrm{vol}(A_i))}{h^6 \mathbb{E}(\mathrm{vol}(A_i))^2} \overset{Claim\ 5.2}{=} o(1).$$

Hence, by the union bound, we have that a.a.s $\mathrm{vol}(A_i) \geq (1-h^3)\mathbb{E}(\mathrm{vol}(A_i))$ for all $i = -t, \ldots, t-1$. Therefore by the Handshaking Lemma, $\sum_{i=-t}^{t-1} \mathrm{vol}(A_i) = 2m$ whereby

$$2m = \sum_{-t \leq i \leq t-1} \mathrm{vol}(A_i) \geq \sum_{-t \leq i \leq t-1} (1-h^3)\mathbb{E}(\mathrm{vol}(A_i)) = 2\mathbb{E}(m)(1-h^3)$$

and,

$$2m = \sum_{-t \leq i \leq t-1} \mathrm{vol}(A_i) \leq \sum_{-t \leq i \leq t-1} (1+h^3)\mathbb{E}(\mathrm{vol}(A_i)) = 2\mathbb{E}(m)(1+h^3).$$

From the above, we deduce (5.4) since a.a.s. for all $i = -t, \ldots, t-1$

$$\mathrm{vol}(A_i) \leq \left(\frac{h}{2} + h^2\right)(1-h^3)^{-1}m \leq \left(\frac{h}{2} + h^2\right)(1+h^2)m \leq \left(\frac{h}{2} + 3h^2\right)m,$$

provided that $t \geq 2$ (so that $h^2 > h^3 + h^5$ which is equivalent to $1 > h + h^3$ and holds if $h \leq 1/2$), and

$$\mathrm{vol}(A_i) \geq \left(\frac{h}{2} - h^2\right)(1+h^3)^{-1}m \geq \left(\frac{h}{2} - h^2\right)(1-h^3)m \geq \left(\frac{h}{2} - 3h^2\right)m.$$

Proof of Claim 5.1. Firstly, let us point out that if a point $p \in A_1$ is far from the boundary of B_1, then it does not contribute to $e(A_1, \overline{A}_1)$. To quantify this, let us recall that for another $p' \in \mathcal{B}(y)$, if $|x(p') - x(p)|_{\mathcal{B}} > e^{\frac{1}{2}(y(p)+y)}$, then $p' \notin B_y(p)$. Since $y(p) \leq y$ as well, we can further conclude that for any point $p' \in \mathcal{B}(y)$, if $|x(p') - x(p)|_{\mathcal{B}} > e^y$, then $p' \notin B_y(p)$.

Hence, the only points $p \in A_1$ that may contribute to $e(A_1, \overline{A}_1)$ are such that $0 \leq x(p) < e^y$ or $hI - e^y \leq x(p) < hI$. Let $A_1^{(1)}$ denote the set of the former and $A_1^{(2)}$ the set of the latter. Hence,

$$\mathbb{E}\left(e(A_1, \overline{A}_1)\right) \leq \mathbb{E}\left(\mathrm{vol}(A_1^{(1)})\right) + \mathbb{E}\left(\mathrm{vol}(A_1^{(2)})\right) = 2 \cdot \mathbb{E}\left(\mathrm{vol}(A_1^{(1)})\right),$$

where the last equality holds since the random variables $\mathrm{vol}(A_1^{(1)})$ and $\mathrm{vol}(A_1^{(2)})$ are identically distributed. For a finite set of points P and a point $p \in P$, we let $\deg(p; P) = |B_y(p) \cap P \setminus \{p\}|$. Now, we apply the Campbell–Mecke formula (5.1) and get

$$\mathbb{E}\left(\mathrm{vol}(A_1^{(1)})\right) = \mathbb{E}\left(\sum_{p \in \mathrm{P}_{\alpha,\beta} \cap A_1^{(1)}} \deg(p; \mathrm{P}_{\alpha,\beta})\right)$$

$$\overset{(5.1)}{=} \beta \cdot \int_0^y \int_0^{e^y} \mathbb{E}\left(\deg((x_0, y_0)); \mathrm{P}_{\alpha,\beta} \cup \{(x_0, y_0)\}\right) \cdot e^{-\alpha y_0} dx_0 dy_0.$$

But $\mathbb{E}\left(\deg((x_0, y_0)); \mathrm{P}_{\alpha,\beta} \cup \{(x_0, y_0)\}\right) = |B_y((x_0, y_0)) \cap \mathrm{P}_{\alpha,\beta}| = O(1)$, uniformly over all $x_0 \in (0, e^y]$ and $y_0 \in [0, y]$. So

$$\mathbb{E}\left(\mathrm{vol}(A_1^{(1)})\right) = O(1) \cdot \int_0^y \int_0^{e^y} e^{-\alpha y_0} dx_0 dy_0 = O(1). \qquad \Box$$

Proof of Claim 5.2. We will calculate $\mathbb{E}(\mathrm{vol}(A_1))$ with the use of the Campbell–Mecke formula (5.1):

$$\begin{aligned}
\mathbb{E}(\mathrm{vol}(A_1)) &= \mathbb{E}\left(\sum_{p \in A_1 \cap \mathrm{P}_{\alpha,\beta}} \deg(p; \mathrm{P}_{\alpha,\beta} \cup \{p\})\right) \\
&= \beta \cdot \int_0^{hI} \int_0^y \mathbb{E}\left(\deg((x_0, y_0); \mathrm{P}_{\alpha,\beta} \cup \{(x_0, y_0)\})\right) e^{-\alpha y_0} dy_0 dx_0 \\
&= \beta h I \cdot \int_0^y \mathbb{E}\left(\deg((0, y_0); \mathrm{P}_{\alpha,\beta} \cup \{(0, y_0)\})\right) e^{-\alpha y_0} dy_0, \qquad (5.6)
\end{aligned}$$

since $\mathrm{P}_{\alpha,\beta}$ is homogeneous on the *x*-coordinate and $\deg((x_0, y_0); \mathrm{P}_{\alpha,\beta} \cup \{(x_0, y_0)\})$ are identically distributed with respect to x_0. Now,

$$\begin{aligned}
\mathbb{E}\left(\deg((0, y_0); \mathrm{P}_{\alpha,\beta} \cup \{(0, y_0)\})\right) &= 2\beta \cdot \int_0^y e^{(y_0 + y_0')/2} e^{-\alpha y_0'} dy_0' \\
&\overset{\alpha > 1/2}{=} \frac{2\beta}{\alpha - 1/2} e^{y_0/2} \left(1 - e^{-y(\alpha - 1/2)}\right). \qquad (5.7)
\end{aligned}$$

We substitute the integrand in (5.6) with the above expression and get

$$\begin{aligned}
\mathbb{E}(\mathrm{vol}(A_1)) &= h I \frac{2\beta^2}{\alpha - 1/2} \left(1 - e^{-y(\alpha - 1/2)}\right) \cdot \int_0^y e^{y_0/2 - \alpha y_0} dy_0 \\
&= 2h I \left[\frac{\beta}{\alpha - 1/2} \left(1 - e^{-y(\alpha - 1/2)}\right)\right]^2 = \Theta(n).
\end{aligned}$$

We calculate $\mathrm{Var}(\mathrm{vol}(A_1)) = \mathbb{E}\left(\mathrm{vol}(A_1)^2\right) - (\mathbb{E}(\mathrm{vol}(A_1)))^2$. For convenience, let $p_0 = (x_0, y_0)$ and similarly $p_0' = (x_0', y_0')$. With the use of the Campbell–Mecke formula (5.1). We write

$$\begin{aligned}
\mathbb{E}\left(\mathrm{vol}(A_1)^2\right) &= \mathbb{E}\left(\sum_{p, p' \in \mathrm{P}_{\alpha,\beta} \cap B_1} \deg(p; \mathrm{P}_{\alpha,\beta}) \cdot \deg(p'; \mathrm{P}_{\alpha,\beta})\right) \overset{(5.1)}{=} \\
&\int_0^y \int_0^{hI} \int_0^y \int_0^{hI} \mathbb{E}\left(\deg(p_0; \mathrm{P}_{\alpha,\beta} \cup \{p_0, p_0'\}) \times \right. \qquad (5.8) \\
&\left. \deg(p_0'; \mathrm{P}_{\alpha,\beta} \cup \{p_0, p_0'\})\right) \cdot e^{-\alpha y_0} e^{-\alpha y_0'} dx_0' dy_0' dx_0 dy_0.
\end{aligned}$$

We will now argue that for the majority of the pairs of points $p_0, p_0' \in B_1$, the expectation that is inside this integral factorizes. Suppose without loss of generality that $x_0 < x_0'$. In this case, $B_y(p_0) \cap B_y(p_0') = \varnothing$ if

and only if $x'_0 - x_0 > e^{(y'_0+y)/2} + e^{(y_0+y)/2}$. So, if this is the case, the random variables $\deg(p_0; P_{\alpha,\beta} \cup \{p_0, p'_0\})$ and $\deg(p_0; P_{\alpha,\beta} \cup \{p_0, p'_0\})$ are independent.

For given $y_0, y'_0 \in [0, y]$, we let

$$S(y_0, y'_0) = \{(x_0, x'_0) \in (0, hI] \times (0, hI] \ : \ 0 < x'_0 - x_0 \le e^{(y'_0+y)/2} + e^{(y_0+y)/2}\}.$$

With this definition, we split the quadruple integral in (5.8) in the following way:

$$\int_0^y \int_0^y \int_{(0,hI]\times(0,hI]\setminus S(y_0,y'_0)} \mathbb{E}\left(\deg(p_0) \cdot \deg(p'_0); P_{\alpha,\beta} \cup \{p_0, p'_0\}\right) \times$$
$$e^{-\alpha y_0} e^{-\alpha y'_0} dx_0 dx'_0 dy_0 dy'_0$$
$$+ \int_0^y \int_0^y \int_{S(y_0,y'_0)} \mathbb{E}\left(\deg(p_0) \cdot \deg(p'_0); P_{\alpha,\beta} \cup \{p_0, p'_0\}\right) \times$$
$$e^{-\alpha y_0} e^{-\alpha y'_0} dx_0 dx'_0 dy_0 dy'_0. \tag{5.9}$$

If $(x_0, x'_0) \in (0, hI] \times (0, hI] \setminus S(y_0, y'_0)$, then the random variables $\deg(p_0; P_{\alpha,\beta} \cup \{p_0, p'_0\})$ and $\deg(p'_0; P_{\alpha,\beta} \cup \{p_0, p'_0\})$ are independent. In the first integral, the integrand is

$$\mathbb{E}\left(\deg(p_0) \cdot \deg(p'_0); P_{\alpha,\beta} \cup \{p_0, p'_0\}\right)$$
$$= \mathbb{E}\left(\deg(p_0); P_{\alpha,\beta} \cup \{p_0, p'_0\}\right) \cdot \mathbb{E}\left(\deg(p'_0); P_{\alpha,\beta} \cup \{p_0, p'_0\}\right)$$
$$= \mathbb{E}\left(\deg(p_0); P_{\alpha,\beta} \cup \{p_0\}\right) \cdot \mathbb{E}\left(\deg(p'_0); P_{\alpha,\beta} \cup \{p'_0\}\right).$$

Therefore, we can bound the first integral in (5.9) as follows:

$$\int_0^y \int_0^y \int_{(0,hI]\times(0,hI]\setminus S(y_0,y'_0)} \mathbb{E}\left(\deg(p_0) \cdot \deg(p'_0); P_{\alpha,\beta} \cup \{p_0, p'_0\}\right) \times$$
$$e^{-\alpha y_0} e^{-\alpha y'_0} dx_0 dx'_0 dy_0 dy'_0$$
$$= \int_0^y \int_0^y \int_{(0,hI]\times(0,hI]\setminus S(y_0,y'_0)} \mathbb{E}\left(\deg(p_0); P_{\alpha,\beta} \cup \{p_0\}\right) \times$$
$$\mathbb{E}\left(\deg(p'_0); P_{\alpha,\beta} \cup \{p'_0\}\right) e^{-\alpha y_0} e^{-\alpha y'_0} dx_0 dx'_0 dy_0 dy'_0$$
$$\le \int_0^y \int_0^y \int_{(0,hI]\times(0,hI]} \mathbb{E}\left(\deg(p_0); P_{\alpha,\beta} \cup \{p_0\}\right) \cdot \mathbb{E}\left(\deg(p'_0); P_{\alpha,\beta} \cup \{p'_0\}\right) \times$$
$$e^{-\alpha y_0} e^{-\alpha y'_0} dx_0 dx'_0 dy_0 dy'_0$$
$$= \left(\int_0^y \int_0^{hI} \mathbb{E}\left(\deg(p_0); P_{\alpha,\beta} \cup \{p_0\}\right) e^{-\alpha y_0} dx_0 dy_0\right)^2.$$

But by the Campbell–Mecke formula (5.1), the latter is

$$(\mathbb{E}\left(\text{vol}(A_1)\right))^2 = \left(\int_0^y \int_0^{hI} \mathbb{E}\left(\deg(p_0); P_{\alpha,\beta} \cup \{p_0\}\right) e^{-\alpha y_0} dx_0 dy_0\right)^2.$$

Now, let us consider the second integral in (5.9). We first consider the following claim:

CLAIM 5.3 Let $y > 0$ be a constant. Then for every $y_0, y_0' \in [0, y]$ and $(x_0, x_0') \in S(y_0, y_0')$, we have

$$\mathbb{E}\left(\deg(p_0) \cdot \deg(p_0'); P_{\alpha,\beta} \cup \{p_0, p_0'\}\right) = O(1).$$

Proof. By recalling that $p_0 = (x_0, y_0)$ and $p_0' = (x_0', y_0')$, we note by the Cauchy–Schwarz inequality

$$\mathbb{E}\left(\deg(p_0) \cdot \deg(p_0'); P_{\alpha,\beta} \cup \{p_0, p_0'\}\right) \leq \mathbb{E}\left(\deg(p_0)^2; P_{\alpha,\beta} \cup \{p_0, p_0'\}\right)^{1/2} \times$$
$$\mathbb{E}\left(\deg(p_0')^2; P_{\alpha,\beta} \cup \{p_0, p_0'\}\right)^{1/2}.$$

Furthermore, as conditioning on a single point in the process can only change the square of the degree by a constant factor, we have that

$$\mathbb{E}\left(\deg(p_0)^2; P_{\alpha,\beta} \cup \{p_0, p_0'\}\right) = \Theta(\mathbb{E}\left(\deg(p_0)^2; P_{\alpha,\beta} \cup \{p_0\}\right)).$$

As the intensity measure is homogeneous on the x co-ordinate, it suffices for us to show that for any $y_0 \in [0, y]$,

$$\mathbb{E}\left(\deg(0, y_0)^2; P_{\alpha,\beta} \cup \{(0, y_0)\}\right) = O(1).$$

Indeed, we note that as a consequence of equation (5.7) and the definition of the point process, the random variable $\deg(0, y_0)$ on the probability space of the point process $P_{\alpha,\beta} \cup \{(0, y_0)\}$ follows the Poisson distribution $\text{Po}\left(\frac{2\beta}{\alpha - 1/2} e^{y_0/2} \left(1 - e^{-y(\alpha - 1/2)}\right)\right)$. Set λ to be the parameter of this Poisson distribution.

Since y is fixed and $y_0 \leq y$ we have $\lambda = O(1)$. Using the second moment of the Poisson distribution, we have:

$$\mathbb{E}\left(\deg(0, y_0)^2; P_{\alpha,\beta} \cup \{(0, y_0)\}\right) = \lambda^2 + \lambda = O(1). \qquad \square$$

We now return our attention to the second integral in (5.9),

$$\int_0^y \int_0^y \int_{S(y_0, y_0')} \mathbb{E}\left(\deg(p_0) \cdot \deg(p_0'); P_{\alpha,\beta} \cup \{p_0, p_0'\}\right) \times$$

$$e^{-\alpha y_0} e^{-\alpha y_0'} dx_0 dx_0' dy_0 dy_0'$$

$$= O(1) \cdot \int_0^y \int_0^y \int_{S(y_0, y_0')} e^{-\alpha y_0} e^{-\alpha y_0'} dx_0 dx_0' dy_0 dy_0'$$

$$= O(1) \cdot \int_0^y \int_0^y \int_0^{hI} \int_{x - 2e^y}^{x + 2e^y} e^{-\alpha y_0} e^{-\alpha y_0'} dx_0 dx_0' dy_0 dy_0'$$

$$= O(1) \int_0^y \int_0^y \int_0^{hI} e^{-\alpha y_0} e^{-\alpha y_0'} dx_0' dy_0 dy_0'$$

$$= O(n).$$

Thus, we conclude that

$$\mathbb{E}\left(\mathrm{vol}(A_1)^2\right) \leq (\mathbb{E}\left(\mathrm{vol}(A_1)\right))^2 + O(n),$$

whereby

$$\mathrm{Var}(\mathrm{vol}(A_1)) = O(n). \qquad \square$$

6. Proof of Lemma 3.2

Here, we return to the probability space associated with the random graph $\mathcal{P}(n; \alpha, \nu)$. In particular, we will work with a subset of the point process $\mathrm{P}_{\alpha,\nu,n}$ on $\mathcal{D}_R$, which we denote by $\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)}$: we set $\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)} = \mathrm{P}_{\alpha,\nu,n} \setminus \mathcal{D}_{\delta R}$, for some $\delta \in (0, 1)$. In other words, $\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)}$ is $\mathrm{P}_{\alpha,\nu,n}$ but without the points inside the disc $\mathcal{D}_{\delta R}$. The reason for working with this process is that it is hard to bound the degrees of the points of $\mathrm{P}_{\alpha,\nu,n}$ which may appear close to the centre of $\mathcal{D}_R$. However, we can show that the two processes coincide a.a.s. provided that δ is small enough.

CLAIM 6.1 If $\delta < 1 - 1/(2\alpha)$, then a.a.s.

$$\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)} = \mathrm{P}_{\alpha,\nu,n}.$$

Proof. This follows from a simple first moment argument. Indeed,

$$\mathbb{E}\left(|\mathrm{P}_{\alpha,\nu,n} \cap \mathcal{D}_{\delta R}|\right) = n \cdot \kappa_{\alpha,\nu,n}(\mathcal{D}_{\delta R}) = n \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\delta R} \rho_n(r) dr d\theta.$$

But

$$\int_0^{\delta R} \rho_n(r) dr = \int_0^{\delta R} \frac{\alpha \sinh(\alpha r)}{\cosh(\alpha R) - 1} dr = \frac{\cosh(\alpha \delta R) - 1}{\cosh(\alpha R) - 1} = O(n^{-2\alpha(1-\delta)}).$$

Therefore,

$$\mathbb{E}\left(|\mathrm{P}_{\alpha,\nu,n} \cap \mathcal{D}_{\delta R}|\right) = O(n^{1-2\alpha(1-\delta)}).$$

So, if $\delta < 1 - 1/(2\alpha)$, then the exponent is negative and this expected value is $o(1)$. $\qquad \square$

Note that $1 - 1/(2\alpha) < 1$, as $\alpha > 1/2$. Furthermore, note that the definition of $\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)}$ allows for both processes to be defined on the same probability space, thus being naturally coupled. The intensity measure of $\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)}$ is $n \cdot \kappa_{\alpha,\nu,n}(\cdot \setminus \mathcal{D}_{\delta R})$. For the moment, we shall assume that $\delta < 1 - 1/(2\alpha)$, so that the conclusion of Claim 6.1 holds.

For a point $p \in \mathcal{D}_R$ and a finite set of points $P \subset \mathcal{D}_R$, we set $\deg(p; P) = |B(p; R) \cap P \setminus \{p\}|$. For $0 \leq y_1 < y_2 \leq R$, we let $\mathcal{A}_{y_1,y_2} \subset \mathcal{D}_R$ denote the annulus inside $\mathcal{D}_R$ consisting of those points in $\mathcal{D}_R$ having defect radius between y_1 and y_2. We set

$$X_{y_1,y_2}(P) = \sum_{p \in P \cap \mathcal{A}_{y_1,y_2}} \deg(p; P).$$

Clearly, for any $0 < y < R$ on the event $\{P_{\alpha,\nu,n} = P^{(>\delta R)}_{\alpha,\nu,n}\}$, we have

$$\mathrm{vol}(P_{\alpha,\nu,n} \cap \mathcal{D}_{R-y}) = X_{y,R}(P^{(>\delta R)}_{\alpha,\nu,n})$$

and,

$$e(\mathcal{P}(n;\alpha,\nu)) = \frac{1}{2}X_{0,R}(P^{(>\delta R)}_{\alpha,\nu,n}).$$

So, on $\{P_{\alpha,\nu,n} = P^{(>\delta R)}_{\alpha,\nu,n}\}$, if $\mathrm{vol}(P_{\alpha,\nu,n} \cap \mathcal{D}_{R-y}) > \varepsilon e(\mathcal{P}(n;\alpha,\nu))$, for some $\varepsilon > 0$, then

$$X_{y,R}(P^{(>\delta R)}_{\alpha,\nu,n}) > \frac{\varepsilon}{2}X_{0,R}(P^{(>\delta R)}_{\alpha,\nu,n}). \tag{6.1}$$

We will give a general result on the concentration of the sum $X_{y,R}(P_{\alpha,\nu,n})$, parametrized by y. We will show the following.

LEMMA 6.2 For any fixed $y \geq 0$, we have

$$\frac{X_{y,R}(P^{(>\delta R)}_{\alpha,\nu,n})}{\mathbb{E}\left(X_{y,R}(P^{(>\delta R)}_{\alpha,\nu,n})\right)} \to 1,$$

as $n \to \infty$ in probability.

Furthermore, we show that $\mathbb{E}\left(X_{y,R}(P^{(>\delta R)}_{\alpha,\nu,n})\right)$ decays exponentially in y.

LEMMA 6.3 For any $0 \leq y < R/4$ and any n sufficiently large, we have

$$\mathbb{E}\left(X_{y,R}(P^{(>\delta R)}_{\alpha,\nu,n})\right) \leq 2e^{-(\alpha-1/2)y} \cdot \mathbb{E}\left(X_{0,R}(P^{(>\delta R)}_{\alpha,\nu,n})\right).$$

The above two lemmas imply that a.a.s.

$$X_{y,R}(P^{(>\delta R)}_{\alpha,\nu,n}) \leq 3e^{-(\alpha-1/2)y}X_{0,R}(P^{(>\delta R)}_{\alpha,\nu,n}).$$

If we set $y = y_\varepsilon := \frac{1}{\alpha-1/2} \cdot \log(6/\varepsilon)$, it follows from (6.1) that

$$\mathbb{P}(e(\mathcal{P}_{>y_\varepsilon}(n;\alpha,\nu)) > \varepsilon e(\mathcal{P}(n;\alpha,\nu))) \leq \mathbb{P}(X_{y_\varepsilon,R}(P_{\alpha,\nu,n}) > \frac{\varepsilon}{2}X_{0,R}(P^{(>\delta R)}_{\alpha,\nu,n}))$$

$$= \mathbb{P}(X_{y_\varepsilon,R}(P_{\alpha,\nu,n}) > 3e^{-(\alpha-1/2)y_\varepsilon}X_{0,R}(P^{(>\delta R)}_{\alpha,\nu,n})) = o(1).$$

This concludes the Proof of Lemma 3.2, assuming Lemmas 6.2 and 6.3. We now proceed with the proofs of these two lemmas.

Proof of Lemma 6.3. We begin with an upper bound on the expected value of $X_{y,R}(P_{\alpha,v,n}^{(>\delta R)})$. Note that for $S < R$ we have $X_{y,S}(P_{\alpha,v,n}^{(>\delta R)}) \leq X_{y,R}(P_{\alpha,v,n}^{(>\delta R)})$. So we can bound

$$
0 \leq X_{y,R}(P_{\alpha,v,n}^{(>\delta R)}) - X_{y,R/2}(P_{\alpha,v,n}^{(>\delta R)}) \leq 2 \cdot |\{p \in P_{\alpha,v,n}^{(>\delta R)} \cap \mathcal{D}_{R/2}\}|^2
$$
$$
+ \sum_{p \in P_{\alpha,v,n}^{(>\delta R)} \cap \mathcal{A}_{R/2,(1-\delta)R}} \deg(p; P_{\alpha,v,n}^{(>\delta R)} \cap \mathcal{A}_{0,R/2}). \tag{6.2}
$$

We will show that the right-hand side is sub-linear a.a.s.

CLAIM 6.4 $\mathbb{E}\left(X_{y,R}(P_{\alpha,v,n}^{(>\delta R)})\right) - \mathbb{E}\left(X_{y,R/2}(P_{\alpha,v,n}^{(>\delta R)})\right) = o(n).$

Proof. The expected value of the first term on the right-hand side of (6.2) is

$$
\mathbb{E}\left(|\{p \in P_{\alpha,v,n}^{(>\delta R)} \cap \mathcal{D}_{R/2}\}|\right) = n \cdot \kappa_{\alpha,v,n}(\mathcal{D}_{R/2})
$$
$$
< n \cdot \frac{\alpha}{2\pi} \int_0^{R/2} \int_{-\pi}^{\pi} \frac{\sinh(\alpha r)}{\cosh(\alpha R) - 1} d\theta dr
$$
$$
= n \cdot \frac{\cosh(\alpha R/2) - 1}{\cosh(\alpha R) - 1} = O(n^{1-\alpha}) \stackrel{\alpha > 1/2}{=} o(n^{1/2}).
$$

Since this random variable is Poisson-distributed, the expected value of its square is proportional to the square of its expected value. Thereby,

$$
\mathbb{E}\left(|\{p \in P_{\alpha,v,n}^{(>\delta R)} \cap \mathcal{D}_{R/2}\}|^2\right) = o(n). \tag{6.3}
$$

We now bound the expected value of the last term in (6.2), using the Campbell–Mecke formula (5.1):

$$
\mathbb{E}\left(\sum_{p \in P_{\alpha,v,n}^{(>\delta R)} \cap \mathcal{A}_{R/2,(1-\delta)R}} \deg(p; P_{\alpha,v,n}^{(>\delta R)} \cap \mathcal{A}_{0,R/2})\right)
$$
$$
= n \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{\delta R}^{R/2} \mathbb{E}\left(\deg((\varrho,\theta); (P_{\alpha,v,n}^{(>\delta R)} \cup \{(\varrho,\theta)\}) \cap \mathcal{A}_{0,R/2})\right) \rho_n(\varrho) d\varrho d\theta. \tag{6.4}
$$

For a point $p = (\varrho,\theta) \in \mathcal{D}_R$ (here ϱ is the radius of p), we set $h_\kappa(p) := \kappa R - R + \varrho$. We will use the upper bound which is a consequence of (2.2): for $\kappa \in (0,1)$ and $\gamma \in (0,1)$ and for n sufficiently large

$$
\deg((\varrho,\theta); (P_{\alpha,v,n}^{(>\delta R)} \cup \{(\varrho,\theta)\}) \cap \mathcal{A}_{0,R/2}) \leq |\check{B}_{\kappa,\gamma}((\varrho,\theta)) \cap \mathcal{A}_{0,R/2} \cap P_{\alpha,v,n}^{(>\delta R)}|
$$
$$
+ |B^{\uparrow}((\varrho,\theta); P_{\alpha,v,n}^{(>\delta R)})|,
$$

where

$$
B^{\uparrow}((\varrho,\theta); P) := \{p \in P \;:\; y(p) > h_\kappa((\varrho,\theta))\}.
$$

Thereby,

$$\mathbb{E}\left(\deg((\varrho,\theta);(\mathrm{P}_{\alpha,v,n}^{(>\delta R)}\cup\{(\varrho,\theta)\})\cap\mathcal{A}_{0,R/2})\right)\leq$$
$$\mathbb{E}\left(|\check{B}_{\kappa,\gamma}((\varrho,\theta))\cap\mathcal{A}_{0,R/2}\cap\mathrm{P}_{\alpha,v,n}^{(>\delta R)}|\right)+\mathbb{E}\left(|B^{\uparrow}((\varrho,\theta);\mathrm{P}_{\alpha,v,n}^{(>\delta R)})|\right). \tag{6.5}$$

Now, the first term on the right-hand side of (6.5) can be bounded as follows:

$$\mathbb{E}\left(|\check{B}_{\kappa,\gamma}((\varrho,\theta))\cap\mathcal{A}_{0,R/2}\cap\mathrm{P}_{\alpha,v,n}^{(>\delta R)}|\right) \tag{6.6}$$

$$=n\frac{1+\gamma}{2\pi}\cdot 2e^{-R/2+(R-\varrho)/2}\cdot\int_{R/2}^{R}e^{(R-z)/2}\frac{\alpha\sinh(\alpha z)}{\cosh(\alpha R)-1}dz$$

$$=\Theta(1)\cdot e^{(R-\varrho)/2}\int_{R/2}^{R}e^{(1/2-\alpha)(R-z)}dz$$

$$=\Theta(1)\cdot e^{(R-\varrho)/2}\int_{0}^{R/2}e^{(1/2-\alpha)z}dz=\Theta(e^{(R-\varrho)/2}).$$

Note that this bound is uniform over all $\varrho\in(\delta R,R/2)$. Substituting it in (6.4) we get

$$\mathbb{E}\left(\sum_{p\in\mathrm{P}_{\alpha,v,n}^{(>\delta R)}\cap\mathcal{A}_{R/2,(1-\delta)R}}|\check{B}_{\kappa,\gamma}((\varrho,\theta))\cap\mathcal{A}_{0,R/2}\cap\mathrm{P}_{\alpha,v,n}^{(>\delta R)}|\right)$$

$$=O(n)\int_{-\pi}^{\pi}\int_{\delta R}^{R/2}e^{(R-\varrho)/2}\rho_{n}(\varrho)d\varrho d\theta$$

$$=O(n)\int_{\delta R}^{R/2}e^{(R-\varrho)/2}e^{-\alpha(R-\varrho)}d\varrho$$

$$\stackrel{\alpha>1/2}{=}O(n)e^{-(\alpha-1/2)R/2}=o(n). \tag{6.7}$$

For the second term, we have:

$$\mathbb{E}\left(|B^{\uparrow}((\varrho,\theta);\mathrm{P}_{\alpha,v,n}^{(>\delta R)})|\right)=n\cdot\kappa_{\alpha,v,n}(\{p\ :\ y(p)>\kappa R-R+\varrho\})$$

$$=n\cdot\frac{\alpha}{2\pi}\int_{0}^{2R-\kappa R-\varrho}\int_{-\pi}^{\pi}\frac{\sinh(\alpha r)}{\cosh(\alpha R)-1}d\theta dr$$

$$=n\cdot\frac{\cosh(\alpha(2R-\kappa R-\varrho))-1}{\cosh(\alpha R)-1}=O(n\cdot e^{\alpha(R(1-\kappa)-\varrho)}), \tag{6.8}$$

uniformly over all $R/2<\varrho<R-y$. Therefore,

$$n\cdot\frac{1}{2\pi}\int_{\delta R}^{R/2}\int_{-\pi}^{\pi}\mathbb{E}\left(|B^{\uparrow}((\varrho,\theta);\mathrm{P}_{\alpha,v,n}^{(>\delta R)})|\right)\rho_{n}(\varrho)d\theta d\varrho$$

$$=\quad O(n^{2})\cdot e^{\alpha R(1-\kappa)}\cdot\int_{\delta R}^{R/2}e^{-\alpha\varrho}\frac{\sinh(\alpha\varrho)}{\cosh(\alpha R)-1}d\varrho$$

$$\overset{\sinh(x)\le e^x}{\le} O(n^2) \cdot e^{\alpha R(1-\kappa)} \cdot \int_{\delta R}^{R/2} e^{-\alpha\varrho} \frac{e^{\alpha\varrho}}{\cosh(\alpha R) - 1} d\varrho$$

$$= O(n^2) \cdot e^{\alpha R(1-\kappa)-\alpha R} \int_{\delta R}^{R/2} d\varrho$$

$$= O(R) \cdot n^2 e^{-\alpha\kappa R} \overset{\alpha\ge 1/2}{=} O(R) \cdot n^{2(1-\alpha\kappa)} \overset{\alpha\ge 1/2}{=} o(n),$$

provided that $1 - \kappa$ is sufficiently small (depending on α). $\qquad\square$

We can now consider $\mathbb{E}\left(X_{y,R/2}(P_{\alpha,v,n}^{(>\delta R)})\right)$. Applying the Campbell–Mecke identity (5.1) to the point process $P_{\alpha,v,n}^{(>\delta R)}$ on $\mathcal{D}_R$ with intensity measure $n \cdot \kappa_{\alpha,v,n}(\cdot \setminus \mathcal{D}_{\delta R})$, we have

$$\mathbb{E}\left(X_{y,R/2}(P_{\alpha,v,n}^{(>\delta R)})\right) = \mathbb{E}\left(\sum_{p\in P_{\alpha,v,n}^{(>\delta R)}\cap\mathcal{A}_{y,R/2}} \deg(p; P_{\alpha,v,n}^{(>\delta R)})\right)$$

$$= n \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{R/2}^{R-y} \mathbb{E}\left(\deg((\varrho,\theta); (P_{\alpha,v,n}^{(>\delta R)} \cup \{(\varrho,\theta)\}))\right) \rho_n(\varrho) d\varrho d\theta. \qquad (6.9)$$

Now, we bound the degree of (ϱ,θ) inside $\mathcal{A}_{y,R/2}$ with respect to the point process $P_{\alpha,v,n}^{(>\delta R)} \cup \{(\varrho,\theta)\}$ with the use of Lemma 2.2.

We apply (2.2) with κ sufficiently close to 1. For $\gamma \in (0, 1)$, and any finite subset $P \subset \mathcal{D}_R$ we bound

$$|\check{B}_{\kappa,-\gamma}((\varrho,\theta)) \cap P| \le \deg((\varrho,\theta); P) \le |B^{\uparrow}((\varrho,\theta); P)| + |\check{B}_{\kappa,\gamma}((\varrho,\theta)) \cap P|,$$

with

$$B^{\uparrow}((\varrho,\theta); P) := \{p \in P : y(p) > h_{\kappa}((\varrho,\theta))\}.$$

Let us set

$$X_{y_1,y_2}^{(\kappa,\gamma)}(P) = \sum_{p\in P\cap\mathcal{A}_{y_1,y_2}} |\check{B}_{\kappa,\gamma}(p) \cap P \setminus \{p\}|.$$

For the expected value of the first term, we use the calculation in (6.8) which holds uniformly over all $R/2 < \varrho < R - y$:

$$n \cdot \frac{1}{2\pi} \int_{R/2}^{R-y} \int_{-\pi}^{\pi} \mathbb{E}\left(B^{\uparrow}((\varrho,\theta); (P_{\alpha,v,n}^{(>\delta R)} \cup \{(\varrho,\theta)\}) \cap \mathcal{D}_{R-y})\right) \rho_n(\varrho) d\theta d\varrho = o(n) \qquad (6.10)$$

as in the proof of the previous claim, provided that $1 - \kappa$ is sufficiently small (depending on α).

Therefore,

$$0 \le \mathbb{E}\left(X_{y,R/2}(P_{\alpha,v,n}^{(>\delta R)})\right) - \mathbb{E}\left(X_{y,R/2}^{(\kappa,\gamma)}(P_{\alpha,v,n}^{(>\delta R)})\right) = o(n). \qquad (6.11)$$

Now, for any real γ such that $|\gamma| \in (0, 1)$, we have

$$
\mathbb{E}\left(|\check{B}_{\kappa,\gamma}((\varrho, \theta)) \cap \mathrm{P}_{\alpha,\nu,n}^{(>\delta R)}|\right)
$$

$$
= n \cdot \frac{\alpha}{2\pi} \cdot (1 + \gamma)2e^{-R/2} \cdot e^{\frac{1}{2}(R-\varrho)} \int_{2R-\kappa R-\varrho}^{R} e^{\frac{1}{2}(R-z)} \frac{\sinh(\alpha z)}{\cosh(\alpha R) - 1} dz.
$$

Note that since $\varrho > R/2$, for n sufficiently large we have

$$
\left| \frac{\rho_n(z)}{e^{-\alpha(R-z)}} - 1 \right| = \left| \frac{1}{e^{-\alpha(R-z)}} \cdot \frac{\alpha \sinh(\alpha z)}{\cosh(\alpha R) - 1} - 1 \right| < |\gamma|.
$$

For real quantities a, b, c, d, with $c, d > 0$, we write that $a = d(b \pm c)$ if $d(b - c) \leq a \leq d(b + c)$. So by the above inequality, the last integral is bounded, for n sufficiently large, as

$$
\int_{2R-\kappa R-\varrho}^{R} e^{\frac{1}{2}(R-z)} \frac{\sinh(\alpha z)}{\cosh(\alpha R) - 1} dz = \frac{(1 \pm |\gamma|)}{\alpha} \cdot \int_{2R-\kappa R-\varrho}^{R} e^{(\frac{1}{2}-\alpha)(R-z)} dz. \tag{6.12}
$$

By applying the fact that $\varrho > R/2$ and $1 - \kappa$ is sufficiently small (hence κ is bounded away from $1/2$), we can compute the right-hand integral as follows:

$$
\int_{2R-\kappa R-\varrho}^{R} e^{(\frac{1}{2}-\alpha)(R-z)} dz = \int_{0}^{\kappa R+\varrho-R} e^{(\frac{1}{2}-\alpha)z} dz
$$

$$
\overset{\alpha > 1/2}{=} \frac{1}{(\alpha - 1/2)} \cdot \left(1 - e^{(\frac{1}{2}-\alpha)(\kappa R+\varrho-R)}\right)
$$

$$
= \frac{1}{(\alpha - 1/2)}(1 - o(1)).
$$

Therefore by substituting this expression into (6.12), and taking n to be sufficiently large for any $\varrho > R/2$

$$
\int_{2R-\kappa R-\varrho}^{R} e^{\frac{1}{2}(R-z)} \frac{\sinh(\alpha z)}{\cosh(\alpha R) - 1} dz = \frac{(1 \pm 2|\gamma|)}{\alpha(\alpha - 1/2)}.
$$

By substituting (6.12) and recalling that $\nu = ne^{-R/2}$, and setting $C_{\alpha,\nu} = \nu/(\pi(\alpha - 1/2))$, it follows that uniformly for all $\varrho \in [R/2, R - y]$ and $\theta \in (-\pi, \pi]$ we have:

$$
\frac{\mathbb{E}\left(|\check{B}_{\kappa,\gamma}((\varrho, \theta)) \cap \mathrm{P}_{\alpha,\nu,n}^{(>\delta R)}|\right)}{e^{\frac{1}{2}(R-\varrho)}} = (1 \pm 2|\gamma|)^2 C_{\alpha,\nu}.
$$

Therefore, by the Campbell–Mecke formula (5.1) we get:

$$\mathbb{E}\left(X_{y,R/2}^{(\kappa,\gamma)}(\mathrm{P}_{\alpha,v,n}^{(>\delta R)})\right) = \frac{n}{2\pi}\int_{R/2}^{R-y}\int_{-\pi}^{\pi}\mathbb{E}\left(|\check{B}_{\kappa,\gamma}((\varrho,\theta))\cap \mathrm{P}_{\alpha,v,n}^{(>\delta R)}|\right)\rho_n(\varrho)d\theta d\varrho$$

$$= (1\pm 2|\gamma|)^2 C_{\alpha,v}\cdot\frac{n}{2\pi}\int_{R/2}^{R-y}\int_{-\pi}^{\pi}e^{\frac{1}{2}(R-\varrho)-\alpha(R-\rho)}d\theta d\varrho. \tag{6.13}$$

Again, we turn our attention to the right-hand integral, as $\alpha > 1/2$ and $y < R/4$, we have the following:

$$\int_{R/2}^{R-y}\int_{-\pi}^{\pi}e^{\frac{1}{2}(R-\varrho)-\alpha(R-\rho)}d\theta d\varrho = 2\pi\int_{R/2}^{R-y}e^{(1/2-\alpha)(R-\varrho)}d\varrho$$

$$= 2\pi\int_{y}^{R/2}e^{(1/2-\alpha)z}dz$$

$$= \frac{2\pi}{\alpha-1/2}e^{-(\alpha-1/2)y}\left(1-e^{-(\alpha-1/2)(R/2-y)}\right)$$

$$= \frac{2\pi}{\alpha-1/2}e^{-(\alpha-1/2)y}(1-o(1)),$$

uniformly over $y < R/4$.

Substituting the above into (6.13), and taking n sufficiently large and setting $C_{\alpha,v}' = C_{\alpha,v}/(\alpha-1/2)$, we have the following:

$$\mathbb{E}\left(X_{y,R/2}^{(\kappa,\gamma)}(\mathrm{P}_{\alpha,v,n}^{(>\delta R)})\right) = n(1\pm 3|\gamma|)^2 C_{\alpha,v}' e^{-(\alpha-1/2)y}.$$

So (6.9) and (6.11) yield, for sufficiently large n

$$\mathbb{E}\left(X_{y,R/2}(\mathrm{P}_{\alpha,v,n}^{(>\delta R)})\right) = n(1\pm 4|\gamma|)^2\cdot C_{\alpha,v}' e^{-(\alpha-1/2)y}.$$

Combining this with Claim 6.4 we deduce the following result: for $\gamma\in(-1,1)$, and n sufficiently large, we have for all $0\le y < R/4$,

$$n(1-5|\gamma|)^2 C_{\alpha,v}' e^{-(\alpha-1/2)y}\le \mathbb{E}\left(X_{y,R}(\mathrm{P}_{\alpha,v,n}^{(>\delta R)})\right)\le n(1+5|\gamma|)^2 C_{\alpha,v}' e^{-(\alpha-1/2)y}. \tag{6.14}$$

By applying (6.14), we bound the following ratio: for $|\gamma|$ chosen small enough such that $(1+5|\gamma|)/(1-5|\gamma|) < \sqrt{2}$ and n sufficiently large: for all $0\le y < R/4$,

$$\frac{X_{y,R}(\mathrm{P}_{\alpha,v,n}^{(>\delta R)})}{X_{0,R}(\mathrm{P}_{\alpha,v,n}^{(>\delta R)})}\le\frac{n(1+5|\gamma|)^2 C_{\alpha,v}' e^{-(\alpha-1/2)y}}{n(1-5|\gamma|)^2 C_{\alpha,v}'} = \frac{(1+5|\gamma|)^2}{(1-5|\gamma|)^2}e^{-(\alpha-1/2)y}$$

$$\le 2e^{-(\alpha-1/2)y}. \qquad\Box$$

Proof of Lemma 6.2. Since

$$0\le \mathbb{E}\left(X_{y,R}(\mathrm{P}_{\alpha,v,n}^{(>\delta R)})\right) - \mathbb{E}\left(X_{y,R/2}(\mathrm{P}_{\alpha,v,n}^{(>\delta R)})\right) = o(n),$$

but $\mathbb{E}\left(X_{y,R}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})\right) = \Theta(n)$ (for fixed $y > 0$) to show the concentration of $X_{y,R}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})$ around its expected value, it suffices to show that

$$\frac{X_{y,R/2}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})}{\mathbb{E}\left(X_{y,R/2}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})\right)} \to 1$$

as $n \to \infty$, in probability.

We decompose this random variable as follows:

$$X_{y,R/2}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)}) = X_{y,\log R}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)}) + X_{\log R,R/2}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)}).$$

By applying the upper bound of (6.14) with $y = \log R$, we deduce that

$$\mathbb{E}\left(X_{\log R,R/2}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})\right) = o(n).$$

For $p \in \mathcal{D}_R$, we set

$$\deg_{>h_\kappa(p)}(p;P) := |\{p' \in (P \setminus \{p\}) \cap B(p;R) \ : \ y(p') > h_\kappa(p)\}|$$

and

$$\deg_{\leq h_\kappa(p)}(p;P) := |\{p' \in (P \setminus \{p\}) \cap B(p;R) \ : \ y(p') \leq h_\kappa(p)\}|.$$

Hence, we express

$$X_{y,\log R}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)}) = \sum_{p \in \mathrm{P}_{\alpha,\nu,n}^{(>\delta R)} \cap \mathcal{A}_{y,\log R}} \deg_{>h_\kappa(p)}(p;\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})$$
$$+ \sum_{p \in \mathrm{P}_{\alpha,\nu,n}^{(>\delta R)} \cap \mathcal{A}_{y,\log R}} \deg_{\leq h_\kappa(p)}(p;\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)}).$$

Note that $\deg_{>h_\kappa(p)}(p;P) \leq |B^\uparrow(p;P)|$. So, by (6.10), the first term has

$$\mathbb{E}\left(\sum_{p \in \mathrm{P}_{\alpha,\nu,n}^{(>\delta R)} \cap \mathcal{A}_{y,\log R}} \deg_{>h_\kappa(p)}(p;\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})\right) = o(n).$$

For any $\kappa \in (0,1)$ and any finite set $P \subset \mathcal{D}_R$, set

$$X_{y,\log R}^{(\kappa)}(P) := \sum_{p \in P \cap \mathcal{A}_{y,\log R}} \deg_{\leq h_\kappa(p)}(p;P).$$

Therefore,

$$\mathbb{E}\left(X_{y,\log R}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})\right) = \mathbb{E}\left(X_{r,\log R}^{(\kappa)}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})\right) + o(n).$$

In turn,

$$\mathbb{E}\left(X_{y,\log R}^{(\kappa)}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})\right) = \Theta(n).$$

too. Hence, to show its concentration around its expected value, it suffices to show that $X_{y,\log R}^{(\kappa)}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})$ is concentrated around its expected value: as $n \to \infty$

$$\frac{X_{y,\log R}^{(\kappa)}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})}{\mathbb{E}\left(X_{y,\log R}^{(\kappa)}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})\right)} \to 1, \tag{6.15}$$

in probability. Since the expected value scales linearly in n, (6.15) will follow if we show that

$$\mathrm{Var}\left(X_{y,\log R}^{(\kappa)}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})\right) = o(n^2).$$

6.1 *Second-moment calculations*

To bound the variance of $X_{y,\log R}^{(\kappa)}(\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})$, we will use Claim 2.3: we set $t_{\kappa,\gamma,R} := 4(1+\gamma)e^{-(1-\kappa)R/2}$ and write $\mathcal{A}_{y,\log R}^2$ for the product $\mathcal{A}_{y,\log R} \times \mathcal{A}_{y,\log R}$.

We apply the Campbell–Mecke formula (5.1)

$$\frac{(2\pi)^2}{n^2}\mathbb{E}\left(\left(\sum_{p \in \mathrm{P}_{\alpha,\nu,n}^{(>\delta R)} \cap \mathcal{A}_{y,\log R}} \deg_{<h_\kappa(p)}(p; \mathrm{P}_{\alpha,\nu,n}^{(>\delta R)})\right)^2\right)$$

$$= \int_{\mathcal{A}_{y,\log R}^2} \mathbb{E}\left(\deg_{<h_\kappa((\varrho,\theta))}((\varrho,\theta)) \cdot \deg_{<h_\kappa((\varrho',\theta'))}((\varrho',\theta'));\right.$$
$$\left.\mathrm{P}_{\alpha,\nu,n}^{(>\delta R)} \cup \{(\varrho,\theta),(\varrho',\theta')\}\right) \rho_n(\varrho)\rho_n(\varrho')d\theta'd\varrho'd\theta d\varrho$$

$$= \int_{\mathcal{A}_{y,\log R}^2} \mathbb{E}\left(\deg_{<h_\kappa((\varrho,\theta))}((\varrho,\theta)) \cdot \deg_{<h_\kappa((\varrho',\theta'))}((\varrho',\theta')) \times\right.$$
$$\left.\mathbf{1}_{|\theta-\theta'|_\pi \le t_{\kappa,\gamma,R}}; \mathrm{P}_{\alpha,\nu,n}^{(>\delta R)} \cup \{(\varrho,\theta),(\varrho',\theta')\}\right) \rho_n(\varrho)\rho_n(\varrho')d\theta'd\varrho'd\theta d\varrho$$

$$+ \int_{\mathcal{A}_{y,\log R}^2} \mathbb{E}\left(\deg_{<h_\kappa((\varrho,\theta))}((\varrho,\theta)) \cdot \deg_{<h_\kappa((\varrho',\theta'))}((\varrho',\theta')) \times\right.$$
$$\left.\mathbf{1}_{|\theta-\theta'|_\pi > t_{\kappa,\gamma,R}}; \mathrm{P}_{\alpha,\nu,n}^{(>\delta R)} \cup \{(\varrho,\theta),(\varrho',\theta')\}\right) \rho_n(\varrho)\rho_n(\varrho')d\theta'd\varrho'd\theta d\varrho.$$

Recall that for $r > 0$, we defined $\mathcal{A}_r = \mathcal{A}_{0,R-r}$. To bound the second integral, let us observe that by Claim 2.3, if $|\theta - \theta'|_\pi > t_{\kappa,\gamma,R}$, then

$$\left(B_R((\varrho,\theta)) \cap \mathcal{A}_{R-h_\kappa((\varrho,\theta))}\right) \cap \left(B_R((\varrho',\theta')) \cap \mathcal{A}_{R-h_\kappa((\varrho',\theta'))}\right) = \varnothing.$$

So, the random variable $\deg_{<h_\kappa((\varrho,\theta))}((\varrho,\theta); P^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho,\theta),(\varrho',\theta')\})$ and the random variable $\deg_{<h_\kappa((\varrho',\theta'))}((\varrho',\theta'); P^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho,\theta),(\varrho',\theta')\})$ are independent. Thus, we can write

$$
\int_{\mathcal{A}^2_{y,\log R}} \mathbb{E}\left(\deg_{<h_\kappa((\varrho,\theta))}((\varrho,\theta)) \cdot \deg_{<h_\kappa((\varrho',\theta'))}((\varrho',\theta')) \times \right.
$$
$$
\left. \mathbf{1}_{|\theta-\theta'|_\pi > t_{\kappa,\gamma,R}}; P^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho,\theta),(\varrho',\theta')\}\right) \rho_n(\varrho)\rho_n(\varrho')d\theta'd\varrho'd\theta d\varrho
$$
$$
= \int_{\mathcal{A}^2_{y,\log R}} \mathbb{E}\left(\deg_{<h_\kappa((\varrho,\theta))}((\varrho,\theta)); P^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho,\theta)\}\right) \times
$$
$$
\mathbb{E}\left(\deg_{<h_\kappa((\varrho',\theta'))}((\varrho',\theta')); P^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho',\theta')\}\right)
$$
$$
\mathbf{1}_{|\theta-\theta'|_\pi > t_{\kappa,\gamma,R}} \cdot \rho_n(\varrho)\rho_n(\varrho')d\theta'd\varrho'd\theta d\varrho
$$
$$
\leq \int_{\mathcal{A}^2_{y,\log R}} \mathbb{E}\left(\deg_{<h_\kappa((\varrho,\theta))}((\varrho,\theta)); P^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho,\theta)\}\right) \times
$$
$$
\mathbb{E}\left(\deg_{<h_\kappa((\varrho',\theta'))}((\varrho',\theta')); P^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho',\theta')\}\right) \rho_n(\varrho)\rho_n(\varrho')d\theta'd\varrho'd\theta d\varrho
$$
$$
= \left(\int_{\mathcal{A}_{y,\log R}} \mathbb{E}\left(\deg_{<h_\kappa((\varrho,\theta))}((\varrho,\theta)); P^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho,\theta)\}\right) \cdot \rho_n(\varrho)d\varrho d\theta\right)^2
$$
$$
= \frac{(2\pi)^2}{n^2}\mathbb{E}\left(\sum_{p \in P^{(>\delta R)}_{\alpha,\nu,n} \cap \mathcal{A}_{y,\log R}} \deg_{<h_\kappa(p)}(p)\right)^2,
$$

by the Campbell–Mecke formula (5.1).

For the first integral, we bound the product of the degrees by the sum of their squares:

$$
\deg_{<h_\kappa((\varrho,\theta))}((\varrho,\theta)) \cdot \deg_{<h_\kappa((\varrho',\theta'))}((\varrho',\theta')) \leq
$$
$$
\deg^2_{<h_\kappa((\varrho,\theta))}((\varrho,\theta)) + \deg^2_{<h_\kappa((\varrho',\theta'))}((\varrho',\theta')).
$$

So, by symmetry, we bound the first integral as follows:

$$
\int_{\mathcal{A}^2_{y,\log R}} \mathbb{E}\left(\deg_{<h_\kappa((\varrho,\theta))}((\varrho,\theta)) \cdot \deg_{<h_\kappa((\varrho',\theta'))}((\varrho',\theta')) \times \right.
$$
$$
\left. \mathbf{1}_{|\theta-\theta'|_\pi \leq t_{\kappa,\gamma,R}}; P^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho,\theta),(\varrho',\theta')\}\right) \rho_n(\varrho)\rho_n(\varrho')d\theta'd\varrho'd\theta d\varrho
$$
$$
\leq 2 \cdot \int_{\mathcal{A}^2_{y,\log R}} \mathbb{E}\left(\deg^2_{<h_\kappa((\varrho,\theta))}((\varrho,\theta); P^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho,\theta)\})\right) \cdot \mathbf{1}_{|\theta-\theta'|_\pi \leq t_{\kappa,\gamma,R}} \times
$$
$$
\rho_n(\varrho)\rho_n(\varrho')d\theta'd\varrho'd\theta d\varrho
$$
$$
= 4t_{\kappa,\gamma,R}\left(\int_{\mathcal{A}_{y,\log R}} \mathbb{E}\left(\deg^2_{<h_\kappa((\varrho,\theta))}((\varrho,\theta); P^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho,\theta)\})\right) \rho_n(\varrho)d\theta d\varrho\right) \times
$$

$$\left(\int_{\mathcal{A}^2_{y,\log R}} \rho_n(\varrho') d\theta' d\varrho' \right). \tag{6.16}$$

But by (2.2), we have

$$\deg_{<h_\kappa((\varrho,\theta))}((\varrho,\theta); \mathrm{P}^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho,\theta)\}) \le |\mathrm{P}^{(>\delta R)}_{\alpha,\nu,n} \cap \check{B}_{\kappa,\gamma}(p)|.$$

So by (2.3), we have

$$\mathbb{E}\left(\deg^2_{<h_\kappa((\varrho,\theta))}((\varrho,\theta); \mathrm{P}^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho,\theta)\})\right) = O(e^{R-\varrho}),$$

uniformly over all $R - \log R < \varrho < R - y$. Therefore,

$$\int_{\mathcal{A}_{y,\log R}} \mathbb{E}\left(\deg^2_{<h_\kappa((\varrho,\theta))}((\varrho,\theta); \mathrm{P}^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho,\theta)\})\right) \rho_n(\varrho) d\theta d\varrho$$

$$= O(1) \cdot \int_{R-\log R}^{R-y} e^{R-\varrho} \frac{\sinh(\alpha\varrho)}{\cosh(\alpha R) - 1} d\varrho$$

$$= O(1) \cdot \int_{R-\log R}^{R-y} e^{(R-\varrho)(1-\alpha)} d\varrho$$

$$= O(1) \cdot \int_{y}^{\log R} e^{(1-\alpha)z} dz \overset{\alpha > 1/2}{=} O(1) \cdot R^{1/2}. \tag{6.17}$$

Furthermore,

$$\int_{\mathcal{A}_{y,\log R}} \rho_n(\varrho') d\theta' d\varrho' = 2\pi \frac{\cosh(\alpha(R-y)) - \cosh(\alpha(R-\log R))}{\cosh(\alpha R) - 1} = O(1). \tag{6.18}$$

Using (6.17) and (6.18) into (6.16), we get

$$\int_{\mathcal{A}^2_{y,\log R}} \mathbb{E}\left(\deg_{<h_\kappa((\varrho,\theta))}((\varrho,\theta)) \cdot \deg_{<h_\kappa((\varrho',\theta'))}((\varrho',\theta')) \times \right.$$

$$\left. \mathbf{1}_{|\theta-\theta'|_\pi \le t_{\kappa,\gamma,R}}; \mathrm{P}^{(>\delta R)}_{\alpha,\nu,n} \cup \{(\varrho,\theta),(\varrho',\theta')\}\right) \rho_n(\varrho) \rho_n(\varrho') d\theta' d\varrho' d\theta d\varrho$$

$$= O(1) \cdot t_{\kappa,\gamma,R} R^{1/2} = O(1) \cdot e^{-(1-\kappa)R/2} R^{1/2}$$

$$= O(1) \cdot n^{-(1-\kappa)} R^{1/2}.$$

Therefore, we obtain

$$\mathbb{E}\left(\left(\sum_{p \in \mathrm{P}^{(>\delta R)}_{\alpha,\nu,n} \cap \mathcal{A}_{y,\log R}} \deg_{<h_\kappa(p)}(p)\right)^2\right)$$

$$\leq \mathbb{E}\left(\sum_{p\in \mathrm{P}_{\alpha,v,n}^{(>\delta R)}\cap \mathcal{A}_{y,\log R}} \deg_{<h_\kappa(p)}(p)\right)^2 + O(1)\cdot n^{2-(1-\kappa)}R^{1/2}.$$

Rearranging the above, we get

$$\mathrm{Var}\left(\sum_{p\in \mathrm{P}_{\alpha,v,n}^{(>\delta R)}\cap \mathcal{A}_{y,\log R}} \deg_{<h_\kappa(p)}(p)\right) = O(1)\cdot n^{1+\kappa}R^{1/2} = o(n^2).$$

$\square$

7. Discussion

In this article, we have considered the modularity value of the KPKVB model of a random graph on the hyperbolic plane. We have shown that for all $\alpha > 1/2$ and $v > 0$ we have that $\mathrm{mod}(\mathcal{P}(n;\alpha,v)) \to 1$ as $n \to \infty$ in probability. The partition we consider is that of dividing the Poincaré disc into a constant number of equal sectors. We show that the modularity of this partition is closely related to the box partition given in $\mathcal{B}_y(n;\alpha,v)$. Following from this, we observe that for any $\varepsilon > 0$ a.a.s the modularity of $\mathcal{B}_y(n;\alpha,v)$ is at least $1 - \varepsilon$ and thus $\mathrm{mod}(\mathcal{P}(n;\alpha,v)) \to 1$, as $n \to \infty$, in probability.

As we have now shown that the modularity value of the KPKVB model tends to 1 in probability, a natural question is the order of $1 - \mathrm{mod}(G)$. This is also referred to as the *modularity deficit* of the graph [28]. The modularity deficit of a partition, $1 - \mathrm{mod}_\mathcal{A}(G)$, quantifies how much a given partition $\mathcal{A}$ differs from optimal modularity. Trees on n nodes with maximum degree Δ have modularity deficit $O(\Delta^{\frac{1}{2}}n^{-\frac{1}{2}})$ [29], while the torus graph on n nodes in dimension d has deficit $O(n^{-1/2d})$ [40]. While we deduce that the modularity deficit of the sector division in the KPKVB model can be made arbitrarily small, it is open to determine whether we can explicitly express the rate of convergence asymptotically. It is also open to determine for a given growth rate, whether we can exhibit a partition that possesses such a deficit.

A modular community structure is characterized by a vertex partition where edge density within parts is much greater than expected, while density between parts is much smaller. While a high a modularity value (>0.3) can be indicative of an underlying modular community structure, a high value alone does not guarantee that such a community structure exists. This tends to occur in sparse networks. For example, in regimes where the average degree is bounded, the Erdős–Rényí random graph can exhibit a high modularity value in probability, without possessing a modular community structure [28].

In the case of the KPKBV model, the high modularity may be a consequence of the tree-like structure of the random graph. Generally, trees with sub-linear maximum degree demonstrate an almost optimal modularity value; see [29]. Here, the term "tree-like" does not refer to the lack of short cycles (in fact, the presence of clustering implies that there are many short cycles with high probability). It refers to the existence of a hierarchy on the set of vertices of the random graph, which resembles the natural hierarchy that a rooted tree exhibits. Let us note that as a consequence of the negative curvature of hyperbolic space, tangential distances in the Poincaré disc expand exponentially with the respect to the radial distance from the centre. Pairs of vertices near the boundary of the disc are much less likely to connect, as they must possess a much smaller relative angle for this to happen. In contrast, vertices near the centre have relatively high degree, as the balls of radius R around them cover almost all of the disc. This means that the communities tend to have an underlying hierarchical structure, where the communities are formed from the mutual descendants of nodes with larger defect radii. Each part of the sector partition

tends to capture a large proportion of one of these rooted sub-trees; therefore, this may suggest why the modularity score of the sector partition tends to one, in probability.

Acknowledgements

Funding

REFERENCES

1. GRANOVETTER, M. (1973) The strength of weak ties. *Am. J. Sociol.*, **78**, 1360–1380.
2. NEWMAN, M. & GIRVAN, M. (2004) Finding and evaluating community structure in networks. *Phys. Rev. E*, **69**, 026113.
3. LANCICHINETTI, A. & FORTUNATO, S. (2011) Limits of modularity maximization in community detection. *Phys. Rev. E*, **84**, 066122.
4. KRIOUKOV, D., PAPADOPOULOS, F., KITSAK, M., VAHDAT, A. & BOGUÑÁ, M. (2010) Hyperbolic geometry of complex networks. *Phys. Rev. E*, **82**, 036106.
5. GUGELMANN, L., PANAGIOTOU, K. & PETER, U. (2021) Random hyperbolic graphs: degree sequence and clustering. *Proceedings of the 39th International Colloquium Conference on Automata, Languages, and Programming - Volume Part II* (A. Czumaj, K. Mehlhorn, A. Pitts, R. Wattenhofer eds). pp. 573–585.
6. ALBERT, R. & BARABÁSI, A.-L. (2002) Statistical mechanics of complex networks. *Rev. Mod. Phys.*, **74**, 47–97.
7. PENROSE, M. D. (2003) *Random Geometric Graphs*, volume 5 of *Oxford Studies in Probability*. Oxford: Oxford University Press.
8. STEGEHUIS, C., VAN DER HOFSTAD, R. & VAN LEEUWAARDEN, J. S. H. (2019) Scale-free network clustering in hyperbolic and other random graphs. *J. Phys. A Math. Theor.*, **52**, 295101.
9. FOUNTOULAKIS, N., VAN DER HOORN, P., MÜLLER, T. & SCHEPERS, M. (2021) Clustering in a hyperbolic model of complex networks. *Electron. J. Probab.*, **26**, 1–132.
10. BODE, M., FOUNTOULAKIS, N. & MÜLLER, T. (2015) On the largest component of a hyperbolic model of complex networks. *Electron. J. Comb.*, **22**, P3.24.
11. FOUNTOULAKIS, N. & MÜLLER, T. (2018) Law of large numbers in a hyperbolic model of complex networks. *Ann. Appl. Probab.*, **28**, 607–650.
12. KIWI, M. A. & MITSCHE, D. (2015) A bound for the diameter of random hyperbolic graphs. *Proceedings of the Twelfth Workshop on Analytic Algorithmics and Combinatorics, ANALCO 2015, San Diego, CA, USA, January 4, 2015* (R. Sedgewick and M. D. Ward eds.) pp. 26–39.
13. KIWI, M. A. & MITSCHE, D. (2019) On the second largest component of random hyperbolic graphs. *SIAM J. Discrete Math.*, **33**, 2200–2217.
14. BODE, M., FOUNTOULAKIS, N. & MÜLLER, T. (2016) The probability of connectivity in a hyperbolic model of complex networks. *Random. Struct. Algor.*, **49**, 65–94.
15. FOUNTOULAKIS, N. & YUKICH, J. (2020) Limit theory for isolated and extreme points in hyperbolic random geometric graphs. *Electron. J. Probab.*, **25**, 1–51.
16. FRIEDRICH, T. & KROHMER, A. (2018) On the diameter of hyperbolic random graphs. *SIAM J. Discrete Math.*, **32**, 1314–1334.
17. MÜLLER, T. & STAPS, M. (2019) The diameter of KPKVB random graphs. *Adv. Appl. Probab.*, **51**, 358–377.

18. ABDULLAH, M., BODE, M. & FOUNTOULAKIS, N. (2017) Typical distances in a geometric model for complex networks. *Internet Math.*, **1**, 1–39.

19. NEWMAN, M. (2006a) Finding community structure in networks using the eigenvectors of matrices. *Phys. Rev. E*, **74**, 036104.

20. BRANDES, U., DELLING, D., GAERTLER, M., GÖRKE, R., HOEFER, M., NIKOLOSKI, Z. & WAGNER, D. (2007) On finding graph clusterings with maximum modularity. *Proceedings of the 33rd International Workshop on Graph-theoretic Concepts in Computer Science* (A. Brandstädt, D. Kratsch, H. Müller eds). Lecture Notes in Computer Science. pp. 121–132.

21. DINH, T., LI, X. & THAI, M. (2015) Network clustering via maximizing modularity: approximation algorithms and theoretical limits. *2015 IEEE International Conference on Data Mining* (C. Aggarwal, Z.-H. Zhou, A. Tuzhilin, H. Xiong, and X. Wu eds.). pp. 101–110.

22. NEWMAN, M. (2006) Modularity and community structure in networks. *Proc. Natl. Acad. Sci. USA*, **103**, 8577–8582.

23. BLONDEL, V., GUILLAUME, J., LAMBIOTTE, R. & LEFEBVRE, E. (2008) Fast unfolding of communities in large networks. *J. Stat. Mech.: Theory Exp.*, **2008**, P10008.

24. ROSVALL, M. & BERGSTROM, C. (2011) Multilevel compression of random walks on networks reveals hierarchical organisation in large integrated systems. *PLoS One*, **6**, e18209.

25. PONS, P. & M., L. (2006) Computing communities in large networks using random walks. *J. Graph Algorithms Appl.*, **10**, 191–218.

26. RAGHAVAN, U., ALBERT, R. & KUMARA, S. (2007) Near linear time algorithm to detect community structures in large-scale networks. *Phys. Rev. E*, **76**, 36106.

27. WANG, X., LIU, G., JIANHUA LI, J. & NEES, J. (2017) Locating structural centers: a density-based clustering method for community detection. *PLoS One*, **12**, e0169355.

28. MCDIARMID, C. & SKERMAN, F. (2020) Modularity of Erdős–Rényi random graphs. *Random. Struct. Algor.*, **57**, 211–243.

29. MCDIARMID, C. & SKERMAN, F. (2018) Modularity of regular and tree-like graphs. *J. Complex Netw.*, **4**, 596–619.

30. LICHEV, L. & MITSCHE, D. (2020) On the modularity of 3-regular graphs and random graphs with given degree sequences. *arXiv preprint*, `2007.15574`, 44 pages.

31. LASOŃ, M. & SULKOWSKA, M. (2021), Modularity of minor-free graphs. Combinatorics, Optimization and Algorithms for Telecommunications (211142); Institute of Mathematics of the Polish Academy of Sciences; Wrocław University of Science and Technology. 2021, Report:hal-03462570

32. ZUEV, K., BOGUÑÁ, M., BIANCONI, G. & KRIOUKOV, D. (2015) Emergence of soft communities from geometric preferential attachment. *Sci. Rep.*, **5**, 9421.

33. PAPADOPOULOS, F., KITSAK, M., SERRANO, M., BOGUÑÁ, M. & KRIOUKOV, D. (2012) Popularity versus similarity in growing networks. *Nature*, **489**, 537–540.

34. MUSCOLONI, A. & CANNISTRACI, C. (2018) A nonuniform popularity-similarity optimization (nPSO) model to efficiently generate realistic complex networks with communities. *New J. Phys.*, **20**, 052002.

35. MUSCOLONI, A. & CANNISTRACI, C. (2018) Leveraging the nonuniform PSO network model as a benchmark for performance evaluation in community detection and link prediction. *New J. Phys.*, **20**, 063022.

36. BARABÁSI, A.-L. & ALBERT, R. (1999) Emergence of scaling in random networks, *Science* **286**, 509–512.

37. OSTROUMOVA PROKHORENKOVA, L., PRAŁAT, P. & RAIGORODSKII, A. (2017) Modularity in several random graph models. Electronic Notes in Discrete Mathematics, vol. **61**. pp. 947–953.

38. AIELLO, W., BONATO, A., COOPER, C., JANSSEN, J., & PRAŁAT. P. (2009) A spatial web graph model with local influence regions, *Internet Math.*, **5**, 175–196.

39. LAST, G. & PENROSE, M. (2018) Lectures on the Poisson Process. *IMS Textbooks*. Cambridge: Cambridge University Press.

40. DE MONTGOLFIER F., SOTO M., & VIENNOT L. (2011) Asymptotic modularity of some graph classes. *Algorithms and Computation. ISAAC 2011.* (T. Asano, S. Nakano, Y. Okamoto, O. Watanabe eds). Lecture Notes in Computer Science, vol. **7074**, Berlin, Heidelberg: Springer, pp. 435–444.